GENERALIZED ROUGH SET MODELS

M.Sc Thesis

by

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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **GENERALIZED ROUGH SET MODELS** in the partial fulfillment of the requirements for the award of the degree of **MASTER OF SCIENCE** and submitted in the **DISCIPLINE OF MATHEMATICS, Indian Institute of Technology Indore**, is an authentic record of my own work carried out during the time period from July 2019 to May 2020 under the supervision of **Dr. Md Aquil Khan**, Associate Professor, Discipline of Mathematics, IIT Indore.

The matter presented in this thesis by me has not been submitted for the award of any other degree of this or any other institute.

Ashok kumar Mallick 02/07/2020

Signature of the student with date

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This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

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ABSTRACT

Rough set theory is an important tool for dealing with uncertain and vague data. In this project work, we study some different approaches of generalized rough sets, like generalized rough sets induced by an arbitrary binary relation, generalized rough sets based on neighborhood systems and the covering based rough sets. We further explore the properties and structure of rough set model induced by an arbitrary binary relation. We also investigate the relationship between Alexandrov space and generalized approximation space induced by an arbitrary binary relation. A one-to-one correspondence between the class of generalized approximation spaces based on pre-order relations and the class of Alexandrov spaces are given. In addition, several counter-examples are provided to indicate counter connections.

Contents

Chapter 1

Introduction and Basic Concepts

1.1 Introduction

Rough set theory is a mathematical tool to handle vagueness and uncertainty in data analysis, and has arisen the interest of experimenters and practitioners in various fields of science and technology. The basic idea of approximation space was proposed during the early 1980s by Pawlak [2, 3]

In Pawlak's rough set theory equivalence (indiscernibility) relation is a primitive concept. However, equivalence relations are too restrictive for many applications. To avoid this issue, several generalizations of rough set models are proposed by different researchers and practitioners. For example, rough set model is extended to arbitrary binary relation [6, 7], covering based generalized rough set model [8, 9, 10, 11], neighborhood system based rough set model [13, 16] etc. Some researchers even extended classical rough sets to fuzzy sets [17], Boolean algebras [19] and fuzzy lattices [18].

Various kinds of approximations in generalized rough set model based on arbitrary binary relation were studied. For instance, Y.Y. Yao [7] defined a new type of generalized rough set model based on binary relation and Z. Pei [5] explore the topological point of view of this type of rough sets. In our work, we explore the structure and properties of some generalized rough set models like, covering based rough set model, neighborhood system based rough sets. We establish a close relation between generalized approximation space induced by an arbitrary binary relation and Alexandrov space.

The thesis is structured as follows: First, we present some basic definitions and Pawlak's classical rough set model with its properties. Chapter 2 presents various generalizations of rough set theory and their properties. In Chapter 3, we investigate the relationship between generalized approximation spaces induced by an arbitrary binary relation and Alexandrov spaces.

1.2 Basic Concepts

In this section, we give some basic concepts of rough set theory which are required for our work. We use the symbol $\emptyset$ to denote empty set throughout this work.

Definition 1 (Relation)**.** Let W be a non-empty set. By a binary relation Θ on W, we mean a subset of $W \times W$.

In this article, we will work only with binary relations and therefore we will call them simply as a relation. Usually we write $a\Theta b$ for $(a, b) \in \Theta$. Both the notations will be used in this thesis.

Definition 2. Let Θ be a relation on W. Then Θ is called

- *serial* if for every $u \in W$, we have $a\Theta b$ for some $b \in W$.
- *reflexive* if $u\Theta u$ for all $u \in W$.
- *symmetric* if $u\Theta v$ implies $v\Theta u$ for all $u, v \in W$.
- *transitive* if $u\Theta v$ and $v\Theta w$ implies $u\Theta w$ for all $u, v, w \in W$.

Θ is called a *pre-order* (relation) if Θ is both pre-order; if Θ is both reflexive and symmetric then it is called a *tolerance* relation. If Θ is reflexive, symmetric and transitive, then we say that Θ is an equivalence relation.

Definition 3 (Pawlak Approximation Space [2, 3])**.** By an *approximation space*, we mean a tuple (W, Θ) where W[1] is a non-empty set of objects and Θ is an equivalence relation (called the *indiscernibility* relation) on W.

The objects belonging to the same equivalence class of the relation Θ are indistinguishable with respect to the information provided by Θ. Therefore, having 'complete information' about the domain W is identified with the case when Θ is the identity relation on W. A concept given by a subset Z of W, may not, in general, be expressible in terms of equivalence classes using the set-theoretic operations of union, intersection and complementation. Thus, we approximate Z using the notions of lower and upper approximations defined as follows.

Definition 4. Let (W, Θ) be an approximation space and $X \subseteq W$. The *lower approximation* of X, denoted as $\underline{X}_\Theta$, and *upper approximation* of Z, denoted as $\overline{Z}_\Theta$, are defined as follows. Let us use $\Theta(x)$ to denote the set $\{y \in W : x\Theta y\}$.

$$\underline{Z}_\Theta = \{x \in W : \Theta(x) \subseteq Z\};$$
$$\overline{Z}_\Theta = \{x \in W : \Theta(x) \cap Z \neq \emptyset\}.$$

Given an approximation space (W, Θ) and a subset $Z \subseteq W$, the domain W is divided into three disjoint sets viz. $\underline{Z}_\Theta$, $B_\Theta(Z) := \overline{Z}_\Theta \setminus \underline{Z}_\Theta$ and $(\overline{Z}_\Theta)^c := W \setminus \overline{Z}_\Theta$. The elements of $\underline{Z}_\Theta$, $B_\Theta(Z)$ and $(\overline{Z}_\Theta)^c$ are called *positive*, *boundary/undecidable* and *negative* elements of Z, respectively. As mentioned above, if $B_\Theta(Z) \neq \emptyset$, then we cannot define Z in terms of equivalence classes using the set-theoretic operations of union, intersection and complementation. In such a case, Z is called *rough*. Z is called *definable* if $B_\Theta(Z) = \emptyset$.

Example 1. *Let* (W, Θ) *be a classical approximation space, where*

- $W = \{1, 2, 3 \cdots, 10\}$*;*
- *For all* $x, y \in W$*,* $(x, y) \in \Theta$ *if and only if either both* x *and* y *are even or both* x *and* y *are odd.*

[1] Pawlak considered W to be finite in his original definition.

Let $Y = \{1, 3, 4, 5, 7, 9, 10\}$. *One can verify that* $\underline{Y}_\Theta = \{1, 3, 5, 7, 9\}$ *and* $\overline{Y}_\Theta = W$.

Proposition 1 ([2, 3])**.** *Consider an approximation space* (W, Θ)*. Let* $U, V \in 2^W$*. Then the following holds.*

1. $\underline{U}_\Theta \subseteq U \subseteq \overline{U}_\Theta$
2. $\underline{\emptyset}_\Theta = \overline{\emptyset}_\Theta = \emptyset$; $\underline{W}_\Theta = \overline{W}_\Theta = W$
3. $\underline{(U \cap V)}_\Theta = \underline{U}_\Theta \cap \underline{V}_\Theta$
4. $\overline{(U \cup V)}_\Theta = \overline{U}_\Theta \cup \overline{V}_\Theta$
5. $U \subseteq V$ *implies* $\underline{U}_\Theta \subseteq \underline{V}_\Theta$
6. $U \subseteq V$ *implies* $\overline{U}_\Theta \subseteq \overline{V}_\Theta$
7. $\overline{(U \cap V)}_\Theta \subseteq \overline{U}_\Theta \cap \overline{V}_\Theta$
8. $\underline{(U \cup V)}_\Theta \supseteq \underline{U}_\Theta \cup \underline{V}_\Theta$
9. $\underline{U}_\Theta = (\overline{U^c}_\Theta)^c$
10. $\overline{U}_\Theta = (\underline{U^c}_\Theta)^c$
11. $\underline{(\underline{U}_\Theta)}_\Theta = \overline{(\underline{U}_\Theta)}_\Theta = \underline{U}_\Theta$
12. $\overline{(\overline{U}_\Theta)}_\Theta = \underline{(\overline{U}_\Theta)}_\Theta = \overline{U}_\Theta$

Definition 5 (Topological Space)**.** A *topological space* is defined as a tuple (W, τ), where W is a set and τ is a family of subsets of W satisfying the following conditions:

- $\emptyset \in \tau$ and $W \in \tau$.
- τ is closed under finite intersection.
- τ is closed under arbitrary union.

The elements of τ are termed *open sets*. The complements of elements of τ are termed closed sets.

Example 2. *Let W be any set and τ be the set of all subsets of W. One can easily verify that, (W, τ) is a topological space.*

Definition 6 (Alexandrov Space [5])**.** A topological space (W, τ) is called an Alexandrov Space if τ is closed under arbitary intersection.

Example 3. *Let (W, τ) be a topological space and τ be the set of all subsets of W. One can easily verify (W, τ) is an Alexandrov space.*

Definition 7. Suppose (W, τ) is a topological space and $Y \subseteq W$. The interior of Y, denoted as $\underline{Y}_\tau$, and the closure of Y, denoted as $\overline{Y}_\tau$, are defined as follows.

$\underline{Y}_\tau := \{x \in W :$ there exists a $X \in \tau$ such that $x \in X$ and $X \subseteq Y\}$;

$\overline{Y}_\tau := \{x \in W : X \cap Y \neq \emptyset$ for all $X \in \tau$ with $x \in X\}$.

Example 4. *Let (W, τ) be a topological space, where*

- $W = \{a, b, c, d\}$
- $\tau = \{\emptyset, \{b\}, \{b, c\}, \{a, b, c\}, W\}$

Let $A = \{b, d\}$ be a subset of W. One can verify that $\underline{A}_\tau = \{b\}$ and $\overline{A}_\tau = \{b, c, d\}$.

Chapter 2

Review of Generalized Rough Set Models

In literature, one can find many generalization of the classical rough set theory due to demands from various practical applications. In this chapter, we briefly study a few generalized rough set models viz. generalization based on arbitrary binary relation, covering based rough set models, generalized rough sets based on neighborhood systems.

2.1 Rough Set Model Based on Arbitrary Binary Relation

In Pawlak's classical rough set model, the relation Θ is considered to be an equivalence relation. Thus, a natural generalization is obtained by relaxing the requirement of Θ to be an equivalence. Therefore, in literature, one can find various types of useful generalizations by considering different types of Θ. For example, tolerance approximation space is studied in [18, 19], where the relation is taken to be reflexive and symmetric. In these generalized approximation spaces, the following natural extension of Pawlak's classical notions of lower and upper approximations are considered. Let $\Theta(x) := \{y \in W : (x, y) \in \Theta\}$.

Property	serial	reflexive	symmetric	transitive
$\underline{X}_\Theta = (\overline{X^c}_\Theta)^c$	hold	hold	hold	hold
$\overline{X}_\Theta = (\underline{X^c}_\Theta)^c$	hold	hold	hold	hold
$\underline{W}_\Theta = W$	hold	hold	hold	hold
$\overline{\emptyset}_\Theta = \emptyset$	hold	hold	hold	hold
$\underline{(X \cap Y)}_\Theta = \underline{X}_\Theta \cap \underline{Y}_\Theta$	hold	hold	hold	hold
$\overline{(X \cup Y)}_\Theta = \overline{X}_\Theta \cup \overline{Y}_\Theta$	hold	hold	hold	hold
$\underline{\emptyset}_\Theta = \emptyset$	hold	hold	does not hold	does not hold
$\overline{W}_\Theta = W$	hold	hold	does not hold	does not hold
$\underline{X}_\Theta \subseteq X \subseteq \overline{X}_\Theta$	does not hold	hold	does not hold	does not hold
$X \subseteq Y$ implies $\overline{X}_\Theta \subseteq \overline{Y}_\Theta$	hold	hold	hold	hold
$X \subseteq Y$ implies $\underline{X}_\Theta \subseteq \underline{Y}_\Theta$	hold	hold	hold	hold
$\overline{(X \cap Y)}_\Theta \subseteq \overline{X}_\Theta \cap \overline{Y}_\Theta$	hold	hold	hold	hold
$\underline{(X \cup Y)}_\Theta \supseteq \underline{X}_\Theta \cup \underline{Y}_\Theta$	hold	hold	hold	hold

Table 2.1: Properties of lower and upper approximation operator

Then for $X \subseteq W$, $\underline{X}_\Theta := \{x \in W : \Theta(x) \subseteq X\}$ and $\overline{X}_\Theta := \{x \in W : \Theta(x) \cap X \neq \emptyset\}$.

Table 2.1 lists the properties of lower and upper approximation operator when Θ is serial, reflexive, symmetric and transitive.

Let us consider two binary relations Θ_1 and Θ_2 on W. It is natural to ask if there is any relationship between the approximations with respect to the relations Θ_1 and Θ_2. We have the following theorem on this.

Theorem 1 ([6]). Let Θ_1 and Θ_2 be two binary relations on W. Then the following conditions are equivalent.

- $\Theta_1 \subseteq \Theta_2$.
- $\Theta_1(x) \subseteq \Theta_2(x), x \in W$.
- $\underline{X}_{\Theta_1} \supseteq \underline{X}_{\Theta_2}, X \subseteq W$.
- $\overline{X}_{\Theta_1} \subseteq \overline{X}_{\Theta_2}, X \subseteq W$.

2.2 Rough Set Models Based on Covering

In this section, we study rough sets based on covering and discuss the properties of lower and upper approximation operators induced by covering. Further, we see the relationship between classical approximation space and covering approximation space.

Definition 8 (Covering [8])**.** Let W be a non empty set of objects and $\mathcal{D} = \{D_n \mid n \in I\}$ be a collection of subsets of W. If $\bigcup_{n\in I} D_n = W$, then $\mathcal{D}$ is called a covering of W. The tuple $(W, \mathcal{D})$ is called a covering approximation space.

Remark 1. It is not difficult to see that every partition of a set gives a covering of that set.

We may ask that is there any connection between the classical approximation space and the covering approximation space defined on a set.
Let (W, Θ) be a classical approximation space, then Θ divides the set W into disjoint equivalence classes. Let S be the set of all equivalence classes of W generated by Θ. Then S forms partition for W. Since each partition of W is a covering, (W, S) is a covering approximation space.

But it is not always possible to obtain a classical approximation space from covering approximation space. It is possible only when the members of covering are disjoint. Let $(W, \mathcal{D})$ be a covering approximation space and the members of $\mathcal{D}$ are disjoint to each other. Let $\mathcal{D} = \{D_n \mid n \in I\}$, where I denotes the index set. Define,

$$x\Theta y \text{ if and only if } x, y \in D_n \text{ for all } n \in I.$$

Clearly, Θ is an equivalence relation. Thus (W, Θ) is a classical approximation space.

In literature one can find various types of rough sets determined by coverings (cf. e.g. [10]). Here, we present a few of them. Consider the following definition.

Definition 9. Consider a covering approximation space $(W, \mathcal{D})$. The neighbourhood of $x \in W$, denoted as $N(x)$, is defined as follows:

$$N(x) = \bigcap\{D_n \in \mathcal{D} : x \in D_n\}.$$

Let $X \subseteq W$, the lower and upper approximations of X, denoted as $\underline{X}_{\mathcal{D}}$ and $\overline{X}_{\mathcal{D}}$ are defined as follows:

$$\underline{X}_{\mathcal{D}} := \bigcup\{D_n : D_n \in \mathcal{D} \text{ and } D_n \subseteq X\};$$
$$\overline{X}_{\mathcal{D}} := \underline{X}_{\mathcal{D}} \bigcup\{N(x) : x \in X \setminus \underline{X}_{\mathcal{D}}\}.$$

Note that we can give another representation of $\overline{X}_{\mathcal{D}}$, i.e.,

$$\overline{X}_{\mathcal{D}} = \bigcup\{N(x) : x \in X\}.$$

Note that the above notion of lower and upper approximations are identical with the Pawlak's authoritative notion of lower and upper approximations, when $\mathcal{D}$ is nothing but a partition of W.

Proposition 2 ([8])**.** *Consider a covering approximation space* $(W, \mathcal{D})$ *and let* $U, V \in 2^W$*. Then we have the following.*

1. $\underline{U}_{\mathcal{D}} \subseteq U \subseteq \overline{U}_{\mathcal{D}}$.

2. $\overline{W}_{\mathcal{D}} = W$.

3. $\underline{\emptyset}_{\mathcal{D}} = \overline{\emptyset}_{\mathcal{D}} = \emptyset$.

4. $\overline{(U \cup V)}_{\mathcal{D}} = \overline{U}_{\mathcal{D}} \cup \overline{V}_{\mathcal{D}}$.

5. $U \subseteq V \;\Rightarrow \underline{U}_{\mathcal{D}} \subseteq \underline{V}_{\mathcal{D}}$.

6. $U \subseteq V \;\Rightarrow \overline{U}_{\mathcal{D}} \subseteq \overline{V}_{\mathcal{D}}$.

7. $\underline{(\underline{U}_{\mathcal{D}})}_{\mathcal{D}} = \underline{U}_{\mathcal{D}}$.

8. $\overline{(\overline{U}_{\mathcal{D}})}_{\mathcal{D}} = \overline{U}_{\mathcal{D}}$.

2.3 Rough Set Models based on Neighborhood Systems

In this section, we discuss rough sets based on neighborhood system. Let W be a universe of discourse and $\mathfrak{P}(W)$ be the power set of W. A map $n : W \longrightarrow \mathfrak{P}(W)$ is called neighborhood operator. Each element $x \in W$ is associated with a subset $n(x) \subseteq W$, called a neighborhood of x. Note that a neighborhood of x may or may not contain x.

Definition 10 (Neighborhood System [12, 13])**.** A neighborhood system of an object $x \in W$, denoted as $NS(x)$, is a nonempty family of neighborhoods of x. The neighborhood system of W is denoted by $NS(W)$ and defined as:

$$NS(W) = \{NS(x) : x \in W\}.$$

Further, we study the following properties of neighborhood system.

$$\begin{aligned}
\text{Serial} &: \text{ for any } x \in W \text{ and } n(x) \in NS(x), n(x) \text{ is nonempty.} \\
\text{Reflexive} &: \text{ for any } x \in W \text{ and } n(x) \in NS(x), x \in n(x). \\
\text{Symmetric} &: \text{ for any } x, y \in W, n(x) \in NS(x) \text{ and } n(y) \in NS(y); \\
& x \in n(y) \Rightarrow y \in n(x). \\
\text{Transitive} &: \text{ for any } x, y, z \in W, n(y) \in NS(y) \text{ and } n(z) \in NS(z), \\
& x \in n(y) \text{ and } y \in n(z) \Rightarrow x \in n(z).
\end{aligned}$$

Definition 11. Let $NS(W)$ be a neighborhood system of W. If for any $n_1(x),\ n_2(x) \in NS(x),\ \exists n_3(x) \in NS(x)$ such that

$$n_3(x) \subseteq n_1(x) \cap n_2(x),$$

then $NS(W)$ is called weak-unary neighborhood system of W.

Definition 12. Let $NS(W)$ be a neighborhood system of W. $NS(W)$ is called weak-transitive neighborhood system of W if for any $x \in W$ and $n(x) \in NS(x),\ \exists n_1(x) \in NS(x)$ satisfying that for any $y \in n_1(x),\ \exists$ an $n(y) \in NS(y)$ such that $n(y) \subseteq n(x)$.

Let $NS(W)$ be a neighborhood system of W. Let $X \subseteq W$, the lower approximation of X ($\underline{X}_{NS}$) and the upper approximation of X ($\overline{X}_{NS}$) are defined as follows:

$$\underline{X}_{NS} = \{x \in W : \exists n(s) \in NS(x), n(x) \in X\},$$
$$\overline{X}_{NS} = \{x \in W : \forall n(s) \in NS(x), n(x) \cap X \neq \emptyset\}.$$

Let W be a universe of discourse and $NS(W)$ be a neighborhood system of W. Let $X, Y \in \mathfrak{P}(W)$. Then the lower and upper approximation operators satisfy the following properties.

Proposition 3 ([13]).

$\overline{\emptyset}_{NS} = \emptyset.$

$\underline{W}_{NS} = W.$

$X \subseteq Y$ *implies* $\underline{X}_{NS} \subseteq \underline{Y}_{NS}.$

$X \subseteq Y$ *implies* $\overline{X}_{NS} \subseteq \overline{Y}_{NS}.$

$\underline{X}_{NS} = (\overline{X^c}_{NS})^c.$

$\overline{X}_{NS} = (\underline{X^c}_{NS})^c.$

Moreover, if $NS(W)$ is serial, then

- $\underline{\emptyset}_{NS} = \emptyset.$
- $\overline{W}_{NS} = W.$

If $NS(W)$ is reflexive, then

- $\underline{X}_{NS} \subseteq X.$
- $X \subseteq \overline{X}_{NS}.$

If $NS(W)$ is symmetric, then

- $X \subseteq \underline{(\overline{X}_{NS})}_{NS}.$
- $\overline{(\underline{X}_{NS})}_{NS} \subseteq X.$

If $NS(W)$ is transitive, then

- $\underline{X}_{NS} \subseteq \underline{(\underline{X}_{NS})}_{NS}$.
- $\overline{(\overline{X}_{NS})}_{NS} \subseteq \overline{X}_{NS}$.

If $NS(W)$ is weak-unary, then

- $\overline{(X \cup Y)}_{NS} = \overline{X}_{NS} \cup \overline{Y}_{NS}$.
- $\underline{(X \cap Y)}_{NS} = \underline{X}_{NS} \cap \underline{Y}_{NS}$.

Proposition 4 ([13]). *The following properties are equivalent:*

1. $NS(W)$ *is weak-transitive.*
2. $\underline{X}_{NS} \subseteq \underline{(\underline{X}_{NS})}_{NS}$.
3. $\overline{(\overline{X}_{NS})}_{NS} \subseteq \overline{X}_{NS}$.
4. $\underline{x}_{NS} \subseteq \underline{(\underline{x}_{NS})}_{NS}$ $\forall x \in W$ *and* $n(x) \in NS(x)$.

We may ask that is there any connection between the lower and upper approximations induced by arbitrary binary relation and the lower and upper approximations induced by neighborhood system.

Let (W, Θ) be a generalized approximation space, One can define the neighborhood approximation space (W, NS), where $NS := \{X \subseteq W : \Theta(x) \subseteq X\}$. Let $Y \subseteq W$, then we get $\underline{Y}_{NS} = \underline{Y}_{\Theta}$ and $\overline{Y}_{NS} = \overline{Y}_{\Theta}$.

Also we ask is their any connection between covering approximation space and generalized approximation space based on neighborhood system. Let (W, C) be a covering approximation space. Let $x \in W$. Define,

$$N^C(x) = \{C_i : x \in C_i \text{ and } C_i \in C\}$$

be the neighborhood system of x. Then we obtain neighborhood approximation space (W, N^C). We relate the covering and neighborhood based rough set model in the following proposition.

Proposition 5. *1. Neighborhood system (W, N^C) satisfies the following.*

- $x \in X \in N_C(y) \to X \in N^C(x)$;
- $X \in N^C(x) \to x \in X$;
- $N^C(x) \neq \emptyset, \forall x \in W$.

2. *$\underline{X}_{NS} = \underline{X}_C$ and $\overline{X}_{NS} = \overline{X}_C$, where X_C and X^C are defined in section 2.*

3. *Given a neighborhood system (W, N) satisfying all the three properties mention in first point, then there exists a covering C of W such that $N^C = N$.*

2.4 Generalized Rough Sets Induced by Core of Neighborhood Systems

This section presents rough set model induced by the core of neighborhood systems.

Definition 13. Consider a relation Θ on a set W and $u \in W$. The *right* and the *left neighborhood* of u induced by Θ, denoted as $RN(u)$ and $LN(u)$, respectively, are defined as follows:

$$RN(u) = \{v \in W : u\Theta v\},$$
$$LN(u) = \{y \in W : v\Theta u\}.$$

Definition 14 (Core Neighborhood [14, 15])**.** Consider a relation Θ on a set W and $u \in W$. The core neighborhood of u induced by Θ, denoted as $CN(u)$, is defined as the set $\{v \in W : RN(v) = RN(u) \text{ and } LN(v) = LN(u)\}$.

Let Θ be a relation on a non-empty set W. In [14, 15]], the following four kinds of core of neighborhood system are defined.

1. The core of right neighborhood $(CN_R(u))$:
   $CN_R(u) = \{v \in W : RN(u) = RN(v)\}$.

2. The core of left neighborhood $(CN_L(u))$:
   $CN_L(u) = \{v \in W : LN(u) = LN(v)\}$.

3. The core of union neighborhood $(CN_U(u))$:
   $CN_U(u) = CN_R(v) \cup CN_L(u)$.

4. The core of intersection neighborhood $(CN_I(u))$:
   $CN_I(u) = CN_R(u) \cap CN_L(u)$.

Definition 15. Consider a relation Θ on a non-empty set W. Suppose CN_H is the core of neighborhood systems where $H \in \{R, U, L, I\}$. Then, by a CN_H approximation space we mean the tuple (W, Θ, CN_H).

Consider a CN_J approximation space (W, Θ, CN_H) and $X \subseteq W$. Then the CN_H-lower approximation and the CN_H-upper approximation of X are defined as follows:

$$\underline{X}_{CN_H} = \bigcup\{CN_H(u) : CN_H(u) \subseteq X\},$$
$$\overline{X}_{CN_H} = \bigcap\{CN_H(u) : CN_H(u) \cap X \neq \emptyset\}.$$

Proposition 6 ([16])**.** *Consider a CN_H approximation space (W, Θ, CN_H) and $Y, X \subseteq W$. Then, CN_H lower approximation and CN_H upper approximation satisfy the following properties:*

1. $\underline{X}_{CN_H} \subseteq X \subseteq \overline{X}_{CN_H}$.

2. $\underline{\emptyset}_{CN_H} = \overline{\emptyset}_{CN_H} = \emptyset$; $\underline{W}_{CN_H} = \overline{W}_{CN_H} = W$.

3. $\underline{(X \cap Y)}_{CN_H} = \underline{X}_{CN_H} \cap \underline{Y}_{CN_H}$.

4. $\overline{(X \cup Y)}_{CN_H} = \overline{X}_{CN_H} \cup \overline{Y}_{CN_H}$.

5. $X \subseteq Y \Rightarrow \underline{X}_{CN_H} \subseteq \underline{Y}_{CN_H}$.

6. $X \subseteq Y \Rightarrow \overline{X}_{CN_H} \subseteq \overline{Y}_{CN_H}$.

7. $\overline{(X \cap Y)}_{CN_H} \subseteq \overline{X}_{CN_H} \cap \overline{Y}_{CN_H}$.

8. $\underline{(X \cup Y)}_{CN_H} \supseteq \underline{X}_{CN_H} \cup \underline{Y}_{CN_H}$.

9. $\underline{X}_{CN_H} = (\overline{X^c}_{CN_H})^c$.

10. $\overline{X}_{CN_H} = (\underline{X^c}_{CN_H})^c$.

11. $\underline{(\underline{X}_{CN_H})}_{CN_H} = \overline{(\underline{X}_{CN_H})}_{CN_H} = \underline{X}_{CN_H}$.

12. $\overline{(\overline{X}_{CN_H})}_{CN_H} = \underline{(\overline{X}_{CN_H})}_{CN_H} = \overline{X}_{CN_H}$.

Remark 2. From above proposition, we observe that in this approach of generalization, the approximation operators satisfy all the properties of Pawlak's approximation operators. Therefore, we say that this is the ideal generalization of rough sets among all the generalizations discussed in this project work.

2.5 Variable Precision Rough Set Model

In this section we briefly discuss variable precision rough set model (VPRS-model) [20]. This model uses a generalization of approximation operators obtained by putting a majority inclusion relation in place of inclusion relation in the definition of Pawlak's approximation operators.

Definition 16 ([20])**.** Consider an approximation space (W, Θ) with finite domain. The rough membership function $g : W \times 2^W \longrightarrow [0, 1]$ is given as follows:

$$g(u, v) = \frac{|[u] \cap Z|}{[u]}, \ u \in W, \ Z \subseteq W.$$

For $\alpha \in [0, \frac{1}{2})$, the majority inclusion relation $\subseteq^\alpha$ is given as follows:

$$U \subseteq^\beta V \text{ if and only if } D(U, V) \leq \alpha,$$

where $D(U, V) = 1 - \frac{|U \cap V|}{|U|}$, if $|U| > 0$; otherwise, $D(U, V) = 0$.
Note that $D([u]_\Theta, U) = 1 - g([u]_\Theta, U)$, g being the rough membership function.
Let $U \subseteq W$. Then by using $\subseteq^\alpha$, the α-lower approximation $\underline{\Theta}_\alpha U$ of U and

α-upper approximation $\overline{\Theta}_\alpha U$ of a U is defined as follows:

$$\underline{\Theta}_\alpha U := \{u \in W : [u]_\Theta \subseteq^\alpha U\} = \{u \in W : g([u]_\Theta, U) \geq 1 - \alpha\},$$

$$\overline{\Theta}_\alpha U := \{x \in W : D([u]_\Theta, U) < 1 - \alpha\} = \{u \in W : g([u]_\Theta, U) > \alpha\}$$

One can easily show that $\underline{\Theta}_0 U = \underline{U}_\Theta$ and $\overline{\Theta}_0 U = \overline{U}_\Theta$.

Note that for infinite domain W the definitions of the membership function, α-lower and α-upper approximations may not be well defined. To deal with this problem probabilistic approximation space is introduced.

Chapter 3

Rough Set Theory and Alexandrov Space

This chapter deals with the relationship between Alexandrov space and approximation space. We investigate the connection between the notion of lower/upper approximations based on approximation spaces with interior/closure operators defined on topological spaces. Throughout this chapter, we will write generalized approximation space to mean generalized approximation space based on an arbitrary binary relation.

Let (W, Θ) be a generalized approximation space. For $X \subseteq W$, we define the sets X^{Θ} and X_{Θ} as follows:

$$X^{\Theta} := \{y \in W : x\Theta y \text{ for some } x \in X\};$$

$$X_{\Theta} := \{y \in W : y\Theta x \text{ for some } x \in X\}.$$

Definition 17. Let $F := (W, \Theta)$ be a generalized approximation space and $X \subseteq W$. X is called an *upset* of F if $X^{\Theta} \subseteq X$. X is called a *downset* of F if $X_{\Theta} \subseteq X$.

Given a generalized approximation space $F := (W, \Theta)$, we define $\tau_{\Theta} \subseteq 2^W$ as follows:

$$\tau_{\Theta} := \{X \subseteq W : X \text{ is an upset of } F\}.$$

As shown by the following result, τ_{Θ} is obtained as a topology for W.

Theorem 2. Let $F := (W, \Theta)$ be a generalized approximation space. Then (W, τ_Θ) is an Alexandrov space.

Proof. Note that $\emptyset^\Theta = \emptyset$ and hence $\emptyset^\Theta \subseteq \emptyset$. Obviously, we also have $W^\Theta \subseteq W$. Therefore, $\emptyset, W \in \tau_\Theta$.
Next, suppose $X_i \in \tau_\Theta$ for each $i \in \Lambda$, where Λ is an index set. We need to show that $\bigcup_{i\in\Lambda} X_i \in \tau_\Theta$, that is, $(\bigcup_{i\in\Lambda} X_i)^\Theta \subseteq \bigcup_{i\in\Lambda} X_i$. Let $y \in (\bigcup_{i\in\Lambda} X_i)^\Theta$. Then we obtain $(z, y) \in \Theta$ for some $z \in \bigcup\limits_{i\in\Lambda} X_i$. This implies $z \in X_k$ for some k. Using the definition of X_k^Θ, we obtain $y \in X_k^\Theta$. Since X_k is an upset, we obtain $y \in X_k$ and hence $y \in \bigcup_{i\in\Lambda} X_k$.
Now, suppose $X_i \in \tau_\Theta$ for each $i \in \Lambda$, where Λ is an index set. We need to show that $\bigcap_{i\in\Lambda} X_i \in \tau_\Theta$, that is, $(\bigcap_{i\in\Lambda} X_i)^\Theta \subseteq \bigcap_{i\in\Lambda} X_i$. Let $y \in (\bigcap_{i\in\Lambda} X_i)^\Theta$. Then there exists an element $z \in \bigcap_{i\in\Lambda} X_i$ such that $(z, y) \in \Theta$. This implies that $y \in X_i^\Theta$ for each $i \in \Lambda$. Since $X_i \in \tau_\Theta$ for each i, it follows that $y \in X_i$ for each $i \in \Lambda$. This gives $y \in \bigcap_{i\in\Lambda} X_i$, as required. □

Once we have Theorem 2, a natural question arises about the relationship between (i) $\underline{Y}_\Theta$ and $\underline{Y}_{\tau_\Theta}$ and (ii) $\overline{Y}_\Theta$ and $\overline{Y}_{\tau_\Theta}$, where $Y \subseteq W$. In general, $\underline{Y}_\Theta \neq \underline{Y}_{\tau_\Theta}$ (and $\overline{Y}_\Theta \neq \overline{Y}_{\tau_\Theta}$) as shown by the following example.

Example 5. *Let (W, Θ) be a generalized approximation space, where*

- $W = \{1, 2, 3, ..., 9, 10\}$*;*
- *Fora all $x, y \in W$, $(x, y) \in \Theta$ if and only if $\mid x - y \mid < 2$.*

Note that Θ is reflexive but not transitive.
Claim 1: Let U be a non-empty subset of W such that $U \neq W$. Then, there exists a $y \in W \setminus U$ such that $y + 1 \in U$ or $y - 1 \in U$.
Let us prove the claim. Suppose y be the smallest among the elements of W such that $y \notin U$. Since $U \neq W$, such y exists. If $y \geq 2$, then $y - 1 \in U$ and hence we are done. So, let $y = 1$, that is, $1 \notin U$. Since $U \neq \emptyset$, this guarantees the existence of a z such that $z \in W \setminus U$ and $z + 1 \in U$. This completes the proof of the claim.

Claim 2: Let U be a non-empty subset of W such that $U \neq W$. Then $U^{\Theta} \not\subseteq U$. *Claim 1 guarantees the existence of a $y \in W \setminus U$ such that $y+1 \in U$ or $y-1 \in U$. This shows that $y \in U^{\Theta}$, but $y \notin U$.*
It follows from Claim 2 that $\tau_{\Theta} := \{W, \emptyset\}$. Let $Y = \{1,4,5,6,7,8\}$. Thus, we obtain $\underline{Y}_{\tau_{\Theta}} = \emptyset$. One can also verify that $\underline{Y}_{\Theta} = \{5,6,7\}$. Therefore, $\underline{Y}_{\Theta} \neq \underline{Y}_{\tau_{\Theta}}$.

In general, as shown in Example 5, we do not have $\underline{Y}_{\Theta} = \underline{Y}_{\tau_{\Theta}}$ (and $\overline{Y}_{\Theta} = \overline{Y}_{\tau_{\Theta}}$), but we have the following result.

Theorem 3. Let $F := (W, \Theta)$ be a generalized approximation space and $Y \subseteq W$. Then $\underline{Y}_{\tau_{\Theta}} \subseteq \underline{Y}_{\Theta}$ and $\overline{Y}_{\Theta} \subseteq \overline{Y}_{\tau_{\Theta}}$.

Proof. We provide a proof of $\underline{Y}_{\tau_{\Theta}} \subseteq \underline{Y}_{\Theta}$. We can similarly prove $\overline{Y}_{\Theta} \subseteq \overline{Y}_{\tau_{\Theta}}$. Obviously we have the result if $Y = \emptyset$. So, let $Y \neq \emptyset$. Let $x \in \underline{Y}_{\tau_{\Theta}}$. It is sufficient to show that $\Theta(x) \subseteq Y$. So, suppose $y \in \Theta(x)$ and we show that $y \in Y$. This gives $(x,y) \in \Theta$. Since $x \in \underline{Y}_{\tau_{\Theta}}$, there exists a $U \in \tau_{\Theta}$ such that $x \in U$ and $U \subseteq Y$. Therefore, since $x \in U$, from $(x,y) \in \Theta$, we obtain $y \in U^{\Theta}$. But, since $U \in \tau_{\Theta}$, we have $U^{\Theta} \subseteq U$ and hence $y \in U$. Finally, since $U \subseteq Y$, we get $y \in Y$ as required. □

In the case when Θ is pre-order, we obtain $\underline{Y}_{\Theta} = \underline{Y}_{\tau_{\Theta}}$ and $\overline{Y}_{\Theta} = \overline{Y}_{\tau_{\Theta}}$ as shown by the following result.

Theorem 4. Let (W, Θ) be a generalized approximation space where Θ is a pre-order relation. Then for each $Y \subseteq W$, we obtain $\underline{Y}_{\Theta} = \underline{Y}_{\tau_{\Theta}}$ and $\overline{Y}_{\Theta} = \overline{Y}_{\tau_{\Theta}}$

Proof. We provide a proof of $\underline{Y}_{\Theta} = \underline{Y}_{\tau_{\Theta}}$. One can similarly prove $\overline{Y}_{\Theta} = \overline{Y}_{\tau_{\Theta}}$. From Theorem 3, we have $\underline{Y}_{\tau_{\Theta}} \subseteq \underline{Y}_{\Theta}$ and hence it remains to show that $\underline{Y}_{\Theta} \subseteq \underline{Y}_{\tau_{\Theta}}$. So, let $x \in \underline{Y}_{\Theta}$ and we show that there exists a $U \in \tau_{\Theta}$ such that $x \in U$ and $U \subseteq Y$. Note that $x \in \Theta(x)$ ($\because$ Θ is reflexive) and $\Theta(x) \subseteq Y$. Therefore, it is enough to show that $U \in \tau_{\Theta}$, that is, $\Theta(x)$ is an upset. So, let $y \in (\Theta(x))^{\Theta}$ and we show that $y \in \Theta(x)$. Since $y \in (\Theta(x))^{\Theta}$, there exists a $z \in \Theta(x)$ such that $(z, y) \in \Theta$. Now, using the transitivity of Θ, we obtain $y \in \Theta(x)$. This completes the proof. $\square$

In Example 5, we have seen that the reflexivity of Θ alone is not enough to give the conclusion of Theorem 4. The next example shows that transitivity of Θ alone is also not enough to give us $\overline{Y}_{\Theta} \neq \overline{Y}_{\tau_{\Theta}}$.

Example 6. *Let (W, Θ) be a generalized approximation space, where*

- $W = \{1, 2, 3, ..., 9, 10\}$*;*
- *Fora all $x, y \in W$, $(x, y) \in \Theta$ if and only if $y|x$ and $y \neq x$.*

Let $Y = \{3, 4, 5, 6, 7, 8, 9\}$. One can verify that $\overline{Y}_{\Theta} = \{6, 8, 9, 10\}$ and $\overline{Y}_{\tau_{\Theta}} = \{3, 4, 5, 6, 7, 8, 9, 10\}$.

Let $\mathcal{S}$ and $\mathcal{A}$ be the class of all generalized approximation spaces and the class of all Alexandrov spaces, respectively. Let $\mathcal{S}_{rt}$ be the class of all generalized approximation spaces with pre-order relations. Consider the mapping

$$\varphi : \mathcal{S} \longrightarrow \mathcal{A}$$

such that $\varphi(F) = F_{\varphi}$, where $F = (W, \Theta)$ and $F_{\varphi} := (W, \tau_{\Theta})$. Here, it is pertinent to ask the following questions:

Q1. Is φ surjective?

Q2. Is φ injective?

The answer to Q1 is Yes. In fact, we have the following.

Theorem 5. The mapping $\varphi|_{\mathcal{S}_{rt}} : \mathcal{S}_{rt} \longrightarrow \mathcal{A}$ is a surjective map.

Proof. Let $(W, \tau) \in \mathcal{A}$. We need to find a pre-order relation Θ on W such that $\tau_{\Theta} = \tau$. We define Θ as follows:

$$(x, y) \in \Theta \text{ if and only if } x \in \overline{\{y\}}_{\tau}. \tag{3.1}$$

It is not difficult to verify that

$$(x, y) \in \Theta \text{ if and only if for all } U \in \tau \text{ with } x \in U, \text{ we have } y \in U. \tag{3.2}$$

It is not difficult to see that Θ is pre-order. We need to show that $\tau = \tau_\Theta$ to complete the proof.

Let us first prove $\tau \subseteq \tau_\Theta$. Let $U \in \tau$ and we show that $U^\Theta \subseteq U$.

Let $y \in U^\Theta$. Then there exists a $x \in U$ such that $(x, y) \in \Theta$. Therefore, from (3.1), we obtain $y \in U$ as required.

Next, we show that $\tau_\Theta \subseteq \tau$. Let $U \in \tau_\Theta$, that is, $U^\Theta \subseteq U$. We need to show that $U \in \tau$. Let us take an arbitrary $x \in U$. Note that $x \in \Theta(x)$ ($\because$ Θ is reflexive) and $\Theta(x) \subseteq U$ ($\because U^\Theta \subseteq U$). Therefore, it is enough to show that $\Theta(x) \in \tau$ to complete the proof. Let us prove it.

For each $y \in \Theta(x)$, we define the set

$$U_y^* := \bigcap_{V \in \tau \text{ and } y \in V} V.$$

Note that $y \in U_y^*$ and $U_y^* \in \tau$. We also have $U_y^* \subseteq \Theta(x)$. In fact,

$$\begin{aligned} & z \in U_y^* \\ \Longrightarrow\ & z \in \Theta(y) \\ \Longrightarrow\ & z \in \Theta(x) \quad (\because y \in \Theta(x) \text{ and } \Theta \text{ is transitive}) \end{aligned}$$

Thus, it follows that $\Theta(x) \in \tau$. $\square$

Let us return to Q2. The answer to Q2 is no as shown by the following example.

Example 7. *Consider the generalized approximation spaces $F := (W, \Theta)$ and $F' := (W, \Theta')$, where*

- $W = \{1, 2, \ldots, 10\}$;
- $x\Theta y$ *if and only if* $\mid x - y \mid < 2$;
- $x\Theta' y$ *if and only if* $x \neq y$.

As shown in Example 5, we obtain $\tau_\Theta := \{\emptyset, W\}$. It is also not difficult to verify that $\tau_{\Theta'} := \{\emptyset, W\}$. Thus, $F \neq F'$, but $\varphi(F) = \varphi(F')$.

We end this section with the remark that the map $\varphi|_{\mathcal{S}_{rt}} : \mathcal{S}_{rt} \longrightarrow \mathcal{A}$ is injective as show below.

Theorem 6. $\varphi|_{\mathcal{S}_{rt}} : \mathcal{S}_{rt} \longrightarrow \mathcal{A}$ is injective.

Proof. Let $F_1, F_2 \in \mathcal{S}_{rt}$ be such that $\tau_{\Theta_1} = \tau_{\Theta_2}$. Let $F_1 := (W, \Theta_1)$ and $F_2 := (W, \Theta_2)$. We need to show that $\Theta_1 = \Theta_2$ to complete the proof. If possible, let $\Theta_1 \neq \Theta_2$. Without loss of generality, let us assume that $(x, y) \in \Theta_1$, but $(x, y) \notin \Theta_2$.
As shown in the proof of Theorem 4, $\Theta_2(x)$ is an upset of F_2. That is, $\Theta_2(x) \in \tau_{\Theta_2}$. Also note that $(\Theta_2(x))^{\Theta_1} \nsubseteq \Theta_2(x)$ as $y \in (\Theta_2(x))^{\Theta_1}$ but $y \notin \Theta_2(x)$. Thus, $\Theta_2(x) \notin \tau_{\Theta_1}$. This gives $\tau_{\Theta_1} \neq \tau_{\Theta_2}$, a contradiction. Hence $\Theta_1 = \Theta_2$. $\square$

From Theorems 5 and 6 it is evident that there is a 1-1 correspondence between Alexandroff spaces and generalized approximation spaces with reflexive and transtive relations. Further, from Theorem 4 it follows that lower/upper approximation operators in generalized approximation spaces with reflexive and transtive relations can be identified with the interior/closure operators in the Alexandroff spaces.

CONCLUSION

In this project work, we have studied the relationship between covering approximation space and classical approximation space. We have also discussed the connection between the notion of Pawlak's lower and upper approximations with the notion of lower and upper approximations induced by covering. The relationship between generalized approximation space and Alexandrov space is also studied. We noted that there is a 1-1 correspondence between Alexandroff spaces and generalized approximation spaces with pre-order relations. Further, it is shown that lower/upper approximation operators in generalized approximation spaces with pre-order relations can be identified with the interior/closure operators in the Alexandroff spaces.

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