SYMMETRY, MATRIX LIE GROUPS THEIR LIE ALGEBRAS AND REPRESENTATIONS

M.Sc Thesis

by

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DISCIPLINE OF MATHEMATICS

INDIAN INSTITUTE OF TECHNOLOGY INDORE

KHANDWA ROAD, SIMROL, INDORE-453552 INDIA

JULY 2020

SYMMETRY, MATRIX LIE GROUPS THEIR LIE ALGEBRAS AND REPRESENTATIONS

A THESIS

Submitted in partial fulfillment of the

requirements for the award of the degree

of

Master of Science

by

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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **Symmetry, Matrix Lie Groups their Lie Algebras and Representations** in the partial fulfillment of the requirements for the award of the degree of **MASTER OF SCIENCE** and submitted in the **DISCIPLINE OF MATHEMATICS, Indian Institute of Technology Indore**, is an authentic record of my own work carried out during the time period from July 2019 to May 2020 under the supervision of Dr. Ashisha Kumar, Assistant Professor, Discipline of Mathematics, IIT Indore.

The matter presented in this thesis by me has not been submitted for the award of any other degree of this or any other institute.

Harishkishnani

Signature of the student with date

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This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

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ACKNOWLEDGEMENTS

I would like to express my sincere gratitude to my thesis supervisor Dr. Ashisha Kumar for his patience, motivation and enthusiasm in helping me to complete this thesis.

I heartily thank Director, IIT Indore for providing healthy atmosphere and magnificent facilities that lead to the successful completion of the thesis.

I extend my thanks to Dean, Academic Affairs to consider the request to submit the thesis electronically in extended dates because I got stuck at home due to recent development of COVID-19 in the country.

Sincere gratitude is also extended to DPGC Convener, Discipline of Mathematics and the PSPC members, Dr. Sk. Safique Ahmad and Dr. Vijay Kumar Sohani for their valuable remarks and kind suggestions.

I am grateful to Dr. Md. Aquil Khan, Head-Discipline of Mathematics, for his academic support and lab facilities.

Last, but not the least, I would like to thank my colleagues for their encouragement and support and those who have helped me directly or indirectly during this thesis.

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ABSTRACT

In this project work, we first become familiar with rigid motions and symmetry, discuss a result stating all possibilities for a finite subgroup of SO_3. We further study free groups to understand the structure of finite groups using Todd Coxeter Algorithm.

The matrix Lie group, matrix exponential map and its existence have also been discussed. Further, we have studied the Lie algebras of the matrix Lie groups. In the last chapter, we study some examples of representations of the matrix Lie groups and their Lie algebras. The last result is about the completely reducibility of a compact matrix Lie group.

Contents

Chapter 1

Rigid Motions and Symmetry

All the definitions and theorems discussed in this chapter can be found in [MA].

Definition 1 (Rigid motions)**.** A distance preserving map m from $\mathbb{R}^n$ to $\mathbb{R}^n$ is called a **rigid motion** i.e. it is a map having the following property: For any two points X and Y in $\mathbb{R}^n$, we have

$$|Y - X| = |m(Y) - m(X)|.$$

This map carries triangles to congruent triangles, therefore, it preserves shapes and angles in general. If we take the set M_n as the set containing all the rigid motions of $\mathbb{R}^n$, then this set is a group with the group operation as composition of maps known as the **group of motions**.

Theorem 1. *If m is a rigid motion on $\mathbb{R}^n$, then it is nothing but a orthogonal operator composed with a translation. So, every rigid motion is of the form*

$$m(X) = PX + a$$

where P is an orthogonal matrix and a is a fixed vector.

Proof. We can assume $a = m(0)$. Let t' denotes the translation map. We have $t'_{-a}(a) = 0$, so $t'_{-a}m$ fixes zero and we can easily prove that every rigid motion fixing zero is a left multiplication by an orthogonal matrix P. So, we have $t'_{-a}m(X) = PX$ for all $X \in \mathbb{R}^n$. Thus, $m(X) = PX + a$. □

Definition 2 (Symmetry)**.** A symmetry of a subset F of a plane is a rigid motion carrying F to itself.

Suppose M is the set containing all the rigid motions of the plane. Then M is classified into two parts:

1. **The orientation preserving motion** of the plane is the motion by which the plane is not fliped.

2. **The orientation reversing motion** of the plane is the motion that do flip the plane over.

A further classification can be done as:

1. Orientation preserving motions:

   (i) **Translations**: It is a parallel motion of the plane by a vector $b : x \to x + b$.

   (ii) **Rotation**:[VP]. It is a motion in which the plane is rotated by some non-zero angle θ about a fixed point.

2. Orientation reversing motions:

   (i) **Reflection**: This motion reflects the plane about a fixed line l.

   (ii) **Glide reflection**: It includes reflection about some line l and then translating by a non-zero vector b, parallel to l.

Definition 3. Group Operations: An operation of a group G on the set S is a map

$$\phi : G \times S \to S,$$

such that the element $\phi(g, s)$ is denoted by gs and satisfy the following two properties:

1. If 1 is the identity of G, then $1s = s$ for all $s \in S$.

2. $(gg')s = g(g's)$ for all $g, g' \in G$ and $s \in S$.

A G**-set** is a set S that is equipped with an operation of some group G.

Definition 4. For an element s of a G-set S. The **orbit** is defined as the set

$$O_s = \{s' \in S : s' = gs \text{ for some } g \in G\} = \{gs : g \in G\}.$$

The orbits are equivalence classes for the relation

$$s' \sim s \text{ if } s' = gs \text{ for some } g \in G.$$

Thus, S is a union of disjoint orbits. A transitive operation of G on the set S is an operation which has only one orbit in S .

Definition 5. For an element $s \in S$, the stabilizer of s is the subgroup of G of elements that fixes s and is denoted by G_s. Thus,

$$G_s = \{g \in G : gs = s\}$$

Observation:- If S is a G-set, $s \in S$ and let the stabilizer of s be H and orbit of s is O_s, then there exists a natural bijective map

$$\frac{G}{H} \xrightarrow{\psi} O_s$$

given by

$$gH \to gs.$$

1.1 Counting Formula

Let $s \in S$, H_s denotes the stabilizer of s and O_s be its orbit. Then, by using the above observation we have,

$$\text{(order of } G\text{)=(order of orbit)(order of stabilizer)}$$

i.e.,

$$|G| = |G_s||O_s|.$$

Example:- Rotations of a regular dodecahedron- Let G denotes the group of all orientation preserving symmetries of a regular dodecahedron D, then G must contain all the rotational symmetries of D.

Consider S to be the set of all faces of D. Then, for a fixed face s of D, the stabilizer is the group of all rotations by multiples of the angle $2\pi/5$ about a perpendicular passing through the center of the face s. Thus, G_s has order 5. We can see that G acts transitively on S and as the total number of faces in D is 12, therefore, $|O_s| = 12$ and hence order of G is equal to $5 \times 12 = 60$.

Similarly, If we take the set S to be the set of all vertices, then for a fixed vertex $s \in S$, we have $|G_s| = 3$ and $|O_s| = 20$. Thus, we get $|G| = 3 \times 20 = 60$. Similar calculations done by taking the set S as the set of all edges of D yields $|G| = 60$.

Thus, the G has 60 elements.

1.2 Finite subgroups of the rotational group

Theorem 2. *If G is a finite subgroup of SO_3, then G has only these possibilities:*

1. *C_k: group of all rotations about a line l by multiples of $2\pi/k$ which turns out to be cyclic group of order k;*
2. *D_k: group of all symmetries of a regular k-gon that is the dihedral group;*
3. *group of all 60 rotations of a regular dodecahedron, known as the icosahedral group;*
4. *group of all 12 rotations of a regular tetrahedron, known as the tetrahedral group;*
5. *group of all 24 rotations of a cube, known as the octahedral group.*

Proof. Let N be the order of a finite subgroup G of SO_3, then every non-identity element, g of G is a rotation about a unique line, say l. Thus, exactly two points on the unit sphere S in $\mathbb{R}^3$ are fixed by g. These are the points of intersection of the line l and the unit sphere S. These points are known as **poles** of g. The group G operates on P.

Stabilizer for a particular pole p contains only the rotations in G about the line l, passing through the origin and the pole p. Thus, the stabilizer for the pole p is a cyclic group generated by the rotation of smallest angle θ. We can see that if order of the stabilizer is r_p, then θ must be equal to $2\pi/r_p$. As, p is a pole, therefore, it is fixed by some non-identity element of G as well. Thus, it's stabilizer G_p has more then one element.

By counting formula,

$$|G_p||O_p| = |G|.$$

If n_p denotes the number of elements that an orbit O_p has, then

$$r_p n_p = N.$$

In G, there are $r_p - 1$ elements with p as a pole. If p and p' are in the same orbit, then by counting formula, their stabilizers also have the same number of elements. Also, every element in G has 2 poles except the identity. Thus, we have

$$\sum_{p \in P} (r_P - 1) = 2N - 2,$$

which is equal to

$$\sum_{i}^{n} n_i(r_i - 1) = 2N - 2$$

where the sum varies over the disjoint orbits, namely $O_1, O_2, \ldots, O_n$ and $n_i = |O_i|$ and $r_i = |G_p|$ for any $p \in O_i$.

As, $N = n_i r_i$ for all i, therefore, we can divide both sides of the above equation by N to get the formula

$$2 - \frac{2}{N} = \sum_{i}^{n} (1 - \frac{1}{r_i}).$$

The left side of this equation is less then 2 and since, each $r_i > 1$, therefore, each term in the right hand side is at least $\frac{1}{2}$. So, the number of orbits cannot exceed 3.

Now, we will list all the possibilities.

1. **One orbit:-** If we have only one orbit, then the above formula gives
$$2 - \frac{2}{N} = 1 - \frac{1}{r}.$$
The left hand side of this equation is more then or equal to 1 and $1 - \frac{1}{r} < 1$, therefore, this case is ruled out.

2. **Two orbits:-** In this case, we have
$$2 - \frac{2}{N} = \left(1 - \frac{1}{r_1}\right) + \left(1 - \frac{1}{r_2}\right)$$
this is equivalent to
$$\frac{2}{N} = \frac{1}{r_1} + \frac{1}{r_2}.$$
As, $r_i \leq N$ for each i, therefore, this can happen only if $r_1 = r_2 = N$ which implies, $n_1 = n_2 = 1$. Hence, each element of the group G fixes only two poles p and p'. Thus, G is a cyclic group C_N containing exactly the rotations about the axis l passing through these two poles p and p'.

3. **Three orbits:-** In this case our formula gives
$$\frac{2}{N} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} - 1.$$
Let us assume that $r_i's$ are in increasing order. Then

Case 1:- If all $r_i's$ are greater then or equal to 3, then the right hand side is ≤ 0, which is not possible. So, $r_1 = 2$.

Case 2:- If $r_1 = r_2 = 2$, then $N = 2r_3$ which implies $n_3 = 2$. So, the third orbit contains only two poles say p and p'. Thus, every element g of G either fixes both of these poles or interchange them. So, all the elements of G are either rotations about the line $l = (p, p')$ or reflections by the angle π about a line l' perpendicular to l. Thus, G turns out to be the group of all rotations that fixes a regular r-gon. So, G is the dihedral group D_r in this case.

Case 3:- If only one r_i is 2, then the following cases cannot occur:
(i) $r_1 = 2, r_2 \geq 4, r_3 \geq 4$ since
$$\frac{1}{2} + \frac{1}{4} + \frac{1}{4} - 1 = 0.$$
(ii) $r_1 = 2, r_2 = 3, r_3 \geq 6$ since
$$\frac{1}{2} + \frac{1}{3} + \frac{1}{6} - 1 = 0.$$
So, we are left with the following three possibilities:-

(a) $(r_1, r_2, r_3) = (2, 3, 3)$, $(n_1, n_2, n_3) = (6, 4, 4)$, $N = 12$;

(b) $(r_1, r_2, r_3) = (2, 3, 4)$, $(n_1, n_2, n_3) = (12, 8, 6)$, $N = 24$;

(c) $(r_1, r_2, r_3) = (2, 3, 5),\ (n_1, n_2, n_3) = (30, 20, 12),\ N = 60.$

Now, only the above cases are to be analysed. We will discuss the third case only because the conclusions for the rest two cases can be obtained similarly.

In the third case, we have $(n_1, n_2, n_3) = (30, 20, 12)$. Then, O_3 has 12 poles. Let p be one of them and let $q \in O_2$ such that q is nearest to p. Now, the stabilizer of p has order 5 and it operates on O_2. Thus, there are five elements (images of q) which are nearest to p. These five elements forms the vertices for a regular pentagonal. The 12 pentagonals so obtained yields a regular dodecahedron.

Similarly, the other two cases can be analysed to get the final conclusions as:

(a) The poles in O_2 forms a regular tetrahedron, and thus, G is a group of rotations of a regular tetrahedron.

(b) The poles in O_2 forms a regular octahedron, and thus, G is a group of rotations of a regular octahedron.

□

1.3 Free Group

Definition 6. Let S be an arbitrary set of symbols (may be finite or infinite), say $S = \{a, b, c, \dots\}$, then a **word** is a finite set of strings of symbols of S, in which repetition of symbols is allowed. For example:- ba, aaa, bbb abb are all words.

Two words are composed by juxtaposition:

$$aaa,\ ba \to aaaba;$$

The empty word, which is denoted by 1, is the identity element for this associative law. Now, to get the inverses, let us modify our set S.

Let S' be the set of symbols containing all the symbols from S and the symbols a^{-1} for each $a \in S$

$$S' = \{a, a^{-1}, b, b^{-1}, c, c^{-1}, \dots\}.$$

Let the set of words from the set S' be denoted by W'. If a word $w \in W'$ has the following form:

$$\dots x^{-1}x \dots \quad \textit{or} \quad \dots xx^{-1} \dots$$

for any $x \in S$, then we will cancel two symbols $x^{-1}x$ to have a word with reduced length. If all such cancellation have been done for a word and there is no scope of further cancellations, then the form so obtained is called a **reduced form** of that word.

Two words w and w' are said to be **equivalent**, and we are able to write $w \sim w'$, if w and w' yields the same reduced form after cancellations. This turns out to be an equivalence relation. We can see that the product of two equivalent words are equivalent and inverse of class of $w = xy \dots z$ is the class of $z^{-1} \dots y^{-1}x^{-1}$. So, the set F of all equivalence classes of words form a group under the compositions induced from the set W'. This group is called the **free group** on the set S.

1.3.1 Generators and relations

Theorem 3 (Mapping property of the free groups). *Let $S = \{a, b, c, \dots\}$ be a non-empty set, F be the free group on it and G be a group, then any map $f : S \to G$ can be extended to a group homomorphism $\psi : F \to G$ such that a word in $S' = \{a, a^{-1}, b, b^{-1}, \dots\}$ is sent to the corresponding product of elements $\{f(a), f(a)^{-1}, f(b), f(b)^{-1}, \dots\}$ in G.*

Proof. This rule defines a map on S' with the equivalent words being sent to the same product in G. The map defined is a homomorphism as the multiplication in F is given by juxtaposition. □

If the above map ψ is surjective, then S is said to **generate** the group G. In this case, the elements of S are called **generators** of G. Also, by first theorem of isomorphism, if $Ker(\psi) = N$, then G is isomorphic to $\frac{F}{N}$. The elements of the group N are called **relations** among the generators. If $N = 1$, then ψ is an isomorphism and G is called a free group too.

Definition 7. A set R of **Defining relations** for the group G, is a subset of N such that N is the smallest normal subgroup containing R.

A group G generated by elements $x_1, \ldots, x_n$, with defining relations $r_1, \ldots, r_m$ is denoted by $< x_1, \ldots, x_n; r_1, \ldots, r_m >$.

If the set of generators and all the defining relations are known then we can compute in F/N and hence in isomorphic group G as well.

1.4 Todd Coxeter Algorithm

If H is a subgroup of a finite group G then this algorithm is a method of counting the cosets of H in G without knowing it's order and it also determines the operation of G on the set of cosets. Since, every operation on an orbit is similar to an operation on cosets, this algorithm is a method of describing any group operation the set of cosets of a subgroup.

Let the group G be defined as

$$G =< x_1, \ldots, x_n; r_1, \ldots, r_m >$$

with generators $x_1, \ldots, x_n$ and the defining relations $r_1, \ldots, r_m$. So, G is realized as the group $\frac{F}{N}$, where F is the free group on the set $\{x_1, \ldots, x_n\}$ and the smallest normal subgroup containing $\{r_1, \ldots, r_m\}$ is N. We also assume that the images in G of the set of words

$$\{h_1, h_2, \ldots, h_s\}$$

in the free group F generate H.

The algorithm works on the following set of rules:

1. The operation of each generator is a permutation.

2. The operation on cosets is transitive.

3. The generators of the subgroup H fixes H.

4. Relations operate trivially.

Let us consider one example to make the idea more clear. Let G be a group defined as

$$G =< x, y, z; x^3, y^2, z^2, xyz >$$

and $H =< z >$, or H is a cyclic subgroup which is generated by the element z. Now, we aim to determine the operations on cosets, which is a permutation representation.

Let the indices $1, 2, 3, \dots$ denote the cosets, with 1 denoting H. We will define a new index only if an action cannot be determined by using the previous information.

By rule 3, $1z = 1$. We don't know how x acts on 1. So, we assign a new index, say $2 = 1x$ and similarly $1x^2 = 2x = 3$. Now, as x^3 is a relation, therefore, $1x^3 = 3x = 1$. So, we have

$$1 \xrightarrow{x} 2 \xrightarrow{x} 3 \xrightarrow{x} 1.$$

Now, the operation of y is to be determined. As we also don't know how y acts on H, thus $1y = 4$ and $1y^2 = 4y = 1$. So, we have

$$1 \xrightarrow{y} 4 \xrightarrow{y} 1.$$

As, we have defined the operations by x and y, therefore, let us try to find the information about xyz. We have $1x = 2$ but we don't know how y acts on 2. Let $2y = 5$. So, we have $1xy = 2y = 5$. As, xyz is a relation, therefore, we have $1xyz = 5z = 1$. So, we have,

$$1 \xrightarrow{x} 2 \xrightarrow{y} 5 \xrightarrow{z} 1.$$

Now, by rule 1, since $1z = 1 = 5z$, so we have $1 = 5$. Hence, we have $2y = 1$ but we already have $4y = 1$ leading to $4 = 2$. So, we have only 3 indices now and we get the following table:

	x	x	x	y	y	z	z	x	y	z
1	2	3	1	2	1	1	1	2	1	1
2	3	1	2	1	2		2	3		2
3	1	2	3		3		3	1	2	3

The table gives $2z = 3$. So, $3z = 2$ and then we have $3y = 3$ and in this way all the blank spaces in the table is determined. Thus, we have

$$x = (1\ 2\ 3),\ \ y = (1\ 2),\ \ z = (2\ 3).$$

So, there are 3 indices and hence, there are 3 cosets of H in G and H has order 2. So, G has order 6. The 3 permutations listed above can be shown to generate the group S_3. So, G is isomorphic to S_3.

Example:-2. Let T denotes the tetrahedral group that is the of all rotational symmetries of a regular tetrahedron. We know that the T has 12 elements. Let x and y denotes the counter clockwise rotations by $2\pi/3$ about the center of a face and about a vertex of that face respectively. If z denotes the rotation about an edge by an angle π. Then, we have $yx = z$ and the relations

$$x^3 = 1,\ \ y^3 = 1, z^2 = yxyx = 1$$

hold in T.

Now, we will show that these relations are enough to define the group completely. So, we can take the group G as

$$G =< x,\ y;\ x^3,\ y^3,\ yxyx > .$$

As, x and y generate T. So, we can define a surjective homomorphism $\phi : G \to T$. Now, to show that ϕ is injective it is enough to show that order of G is 12. If we take subgroup H to be the subgroup generated by y, by doing similar calculations as in the last example, we get the resulting table as:

	x	x	x	y	y	y	y	x	y	x
1	2	3	1	1	1	1	1	2	3	1
2	3	1	2	3	4	2	3	1	1	2
3	1	2	3	4	2	3	4	4	2	3
4	4	4	4	2	3	4	2	3	4	4

Thus, we have

$$x = (1\ 2\ 3),\ \ y = (2\ 3\ 4).$$

So, order of H is 3 and it has index 4 in G. So, order of G is 12. Thus, T is isomorphic to G which is isomorphic to A_4.

Example:-3. Just modify the relations given in the last example as

$$G =< x,\ y;\ x^3,\ y^3,\ yxy^2x = 1 >,$$

and let the subgroup H is the one generated by y. Note that $y^2 = y^{-1}$, so we have the modified table as:

	x	x	x	y	y	y	y	x	y^{-1}	x
1	2	3	1	1	1	1	1	2	3	1
2			2			2	3	1	1	2

The remaining entries are obtained by working from the right. It is clear that $2y^{-1} = 3$, which implies $3y = 2$ and as $2y = 3$, therefore $3y^2 = 3$ and $3y^3 = 2$ but $y^3 = 1$, so we have $3 = 2$ which implies $1 = 2 = 3$. So, $H = G$ and x is some power of y, in fact $x = y^3 = 1$. Hence, G turns out to be a cyclic group of order 3. This example shows that just a small change in one of the relation may reduce or increase the order of G.

G can also be defined as:

$$< x;\ x^3 > .$$

We can define G by the following form as well,

$$< x,\ y;\ x^3,\ y^3,\ xy^2x > .$$

So, we can have more than one way to define a group G.

Chapter 2

Matrix Lie Groups

In this Chapter, we will see what is a Matrix Lie Group, some of it's examples and counter examples. All the definitions and theorems discussed in this chapter and in successive chapters can be found in [BH].

Definition 8. Let $M_n(\mathbb{C})$ be denoted as the space of all matrices of order $n \times n$ with complex entries and $GL(n;\mathbb{C})$ be the subset $M_n(\mathbb{C})$ which consists of all invertible matrices. With respect to the operation of matrix multiplication, $GL(n;\mathbb{C})$ turns out to be a group known as the **general linear group**.

Definition 9. A sequence of matrices A_m in $M_n(\mathbb{C})$ is said to converge to a matrix A if

$$(A_m)_{ij} \to (A)_{ij}, \quad \text{for all } 1 \leq i, j \leq n.$$

Definition 10 (Matrix Lie Group)**.** A subgroup G of $GL(n;\mathbb{C})$ is known as a matrix Lie group if whenever there is a sequence matrices A_m in G which converges to A then either A is not invertible or $A \in G$.

2.0.1 Counterexample

1. Let G be a subgroup of the group $GL(2;\mathbb{C})$ consisting of matrices with rational entries. We observe that the sequence A_n in G defined as

$$A_n = \begin{bmatrix} \left(1+\frac{1}{n}\right)^n & \left(1+\frac{1}{n}\right)^n - 1 \\ 1 & 1 \end{bmatrix}$$

converges to

$$A = \begin{bmatrix} e & e-1 \\ 1 & 1 \end{bmatrix},$$

which is invertible but not in G. Thus, though G is a subgroup of $GL(2;\mathbb{C})$, it is not closed as a subset, hence not a matrix Lie group.

2.0.2 Examples

The following are some important examples of the matrix Lie groups.

1. $GL(n;\mathbb{R})$ or $GL(n;\mathbb{C})$ are trivial examples of matrix Lie groups.

2. **The special linear groups** $SL(n;\mathbb{R})$ **and** $SL(n;\mathbb{C})$

   As the determinant is a continuous function therefore a sequence of matrices with determinant 1 has to converge to a matrix of determinant 1. Therefore, $SL(n;\mathbb{R})$ and $SL(n;\mathbb{C})$ are the matrix Lie groups.

3. **The orthogonal group** $O(n)$ **and special orthogonal group** $SO(n)$

   Let A_m be a sequence in $O(n)$ such that $A_m \to A$. Since $A_m^{tr} A_m = I$ and $A_m^{tr} A_m \to A^{tr} A$, hence $A^{tr} A = I$ and $A \in O(n)$ and thus we conclude that the orthogonal group satisfies the Definition 10. Further, using the continuity of determinant function as in the above example $SO(n)$ also turns out to be a matrix Lie group.

   In the similar lines we can show that **the complex orthogonal groups**, $O(n,\mathbb{C})$ and $SO(n,\mathbb{C})$ are matrix Lie groups.

4. Further, since complex conjugation of matrices is also preserved under taking limits, we can easily observe that the **unitary group** $U(n)$ and **special unitary group** $SU(n)$ are also matrix Lie groups.

5. **The Heisenberg group H**

The Heisenberg group is the group of all 3×3 real matrices of the form

$$A = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}, \tag{2.1}$$

which is easily seen to be closed under limits, hence H is a matrix Lie group.

2.0.3 Compactness of matrix Lie groups

Definition 11. We say that a matrix Lie group G is **compact** if:

1. for every sequence A_m in G such that $A_m \to A$, then $A \in G$, and
2. there is a constant C such that modulus of every entry of any given matrix $A \in G$, is less than C, thus
$$|A_{ij}| \leq C \quad \text{for all } A \in G \text{ and } 1 \leq i, j \leq n.$$

Examples and counterexamples

The groups $U(n)$, $SU(n)$, $O(n)$ and $SO(n)$ are compact matrix Lie groups while the groups H, $SL\,(n;\mathbb{R})$ and $SL\,(n;\mathbb{C})$ are not compact.

2.0.4 Connectedness of matrix Lie groups

Definition 12. A matrix Lie group G is called connected if it is path connected. Thus, G is connected if for any two points A and B in G, there exists a continuous map ϕ from $[0,1]$ to G such that $\phi(0) = A$ and $\phi(1) = B$.

It can be easily shown that the connected component of a matrix Lie group G, containing the identity, is also a subgroup of G.

Examples and counterexamples of connected matrix Lie groups

The groups $U(n)$, $SU(n)$, $GL\,(n;\mathbb{C})$ and $SL\,(n;\mathbb{C})$ are connected while the group $GL\,(n;\mathbb{R})$ is not connected.

Simple connectedness of matrix Lie groups

Definition 13. A connected matrix Lie group G is called **simply connected** if every continuous map $\phi : [0, 1] \to G$ such that $\phi(0) = \phi(1)$, there exists a continuous map $\Phi : [0, 1] \times [0.1] \to G$ such that

1. $\phi(s, 0) = \Phi(s, 1)$ for all $1 \leq s \leq 1$,

2. $\Phi(0, t) = \phi(t)$ for all $t \in [0, 1]$, and

3. $\Phi(1, t) = \Phi(1, 0)$ for all $t \in [0, 1]$.

For example, $SU(2)$ is simply connected and $SO(2)$ is not simply connected.

2.0.5 Homomorphism and Isomorphism

Definition 14. Suppose G and H are two matrix Lie groups. A homomorphism $\Phi : G \to H$ is said to be a **Lie group homomorphism**, if Φ is continuous.

Further, if Φ is a group isomorphism between G and H, then it is said to be **Lie group isomorphism** if both Φ and Φ^{-1} are continuous maps.

There exists a two-to-one Lie group homomorphism from $SU(2)$ onto $SO(3)$.

Chapter 3

Matrix Lie Algebras and the Exponential Mapping

In this chapter, the focus is on associated matrix Lie algebras of some well known matrix Lie groups. Some of the results, which are stated without proof, can be found in [BH].

Definition 15 (The Matrix Exponential)**.** For a $n \times n$ complex or real matrix X, the exponential of X that is e^X is defined as

$$e^X = \sum_{m=0}^{\infty} \frac{X^m}{m!} \ . \tag{3.1}$$

This series is convergent and the function e^X is continuous.

Some elementary results related to matrix exponential:-

For any arbitrary $n \times n$ matrices X and Y, matrix exponential satisfies the following properties:-

1. $(e^X)^* = e^{X^*}$.
2. $e^0 = I$.
3. $e^{(\alpha+\beta)X} = e^{\alpha X} + e^{\beta X}$ for all α and β in $\mathbb{C}$
4. e^X is invertible and $(e^X)^{-1} = e^{-X}$
5. If $XY = YX$, then $e^X e^Y = e^Y e^X = e^{X+Y}$.

6. If A is an invertible matrix, then we have $e^{AXA^{-1}} = Ae^X A^{-1}$.

7. $\|e^X\| \leq e^{\|X\|}$.

Proof. The properties 1,2 and 6 are easy to see by the definition of the function e^X and the property 7 can be proved in the process of proving that the series of e^X converges. Now 3 and 4 follows directly from 5. So, we will prove only the fifth one.

$$e^X e^Y = \left(I + X + \frac{X^2}{2!} + \frac{X^3}{3!} + \dots\right)\left(I + Y + \frac{Y^2}{2!} + \frac{Y^3}{3!} + \dots\right).$$

By rearranging the terms, we get

$$\begin{aligned} e^X e^Y &= \sum_{n=0}^{\infty}\sum_{l=0}^{n} \frac{X^n}{n!}\frac{Y^{n-l}}{(n-l)!} \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{l=0}^{n}\frac{n!}{l!(n-l)!}X^l Y^{n-l} \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}(X+Y)^n \\ &= e^{X+Y}. \end{aligned}$$

□

3.1 The Matrix Logarithm

Definition 16. If P is a $n \times n$ complex matrix, then the matrix logarithm of P is defined by the function

$$\log P = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(P-I)^n}{n}, \tag{3.2}$$

wherever the series converges.

Theorem 4. *The matrix logarithm is well defined for all the $n \times n$ complex matrices P such that $\|P - I\| < 1$. In this domain it is a continuous function and*

$$e^{\log P} = P.$$

Moreover, for all X such that $\|X\| < \log 2, \|e^X - I\| < 1$ we have

$$\log e^X = X.$$

Proof. As $\|(P-I)^n\| \leq \|(P-I)\|^n$ and the complex function $\log z$ has radius of convergence equal to 1, therefore, the given series is absolutely convergent for all P with $\|(P-I)^n\| < 1$ and we can also show that in this domain, the function is continuous.

Now, one can prove that $e^{\log P} = P$ in this domain by first looking at the case when P is diagonalizable and then proving it for any arbitrary matrix by constructing a sequence of diagonalizable matrices converging to P.

The last part follows directly by the properties of $\log z$ function. $\square$

Theorem 5. *There exists a constant K such that for all $n \times n$ matrices P with $\|P\| < \frac{1}{2}$,*

$$\|\log(I+P) - P\| \leq K\|P\|^2.$$

Proof. Since

$$\log(I+P) - P = \textstyle\sum_{n=2}^{\infty}(-1)^{n+1}\frac{P^n}{n} = P^2\sum_{n=2}^{\infty}(-1)^{n+1}\frac{P^{n-2}}{n}.$$

so that

$$\|\log(I+P) - P\| \leq \|P\|^2 \sum_{n=2}^{\infty} \frac{(\frac{1}{2})^{n-2}}{n} \leq K\|P\|^2$$

for some constant K. $\square$

Theorem 6 (Lie Product Formula)**.** *For any two complex matrices X and Y of order n, we have*

$$e^{X+Y} = \lim_{m\to\infty}(e^{\frac{X}{m}}e^{\frac{Y}{m}})^m.$$

The proof uses basic properties of exponential and logarithm function and easy to conclude.

Next theorem is also stated without proof as the proof is easy to write by using some linear algebra results.

Theorem 7. *For any $A \in M_n(\mathbb{C})$, we have*

$$det(e^A) = e^{trace(A)}.$$

Definition 17. A **one-parameter subgroup** of $GL(n;\mathbb{C})$ is a function defined as, $P : \mathbb{R} \to GL(n;\mathbb{C})$ such that

1. $P(0) = I$,

2. P is continuous,

3. $P(u+v) = P(u)P(v)$ for all $u, v \in \mathbb{R}$.

Theorem 8. *For every one-parameter subgroup P of $GL(n;\mathbb{C})$, there is a unique complex matrix X of order n such that*

$$P(u) = e^{uX}.$$

3.2 The Lie Algebra of a Matrix Lie Group

Definition 18. For matrix Lie group G, it's Lie algebra is defined as the set of all matrices X for which e^{sX} is in G for all $s \in \mathbb{R}$. The Lie algebra of G is denoted by $\mathbf{g}$.

Lie algebras of some well-known matrix Lie groups

1. It is easy to see that the Lie algebra of $GL(n;\mathbb{C})$ is the set of all complex matrices of order n, denoted by $\mathbf{gl}(n;\mathbb{C})$ and the Lie algebra of $GL(n;\mathbb{R})$ is the set of all real matrices of order n, denoted by $\mathbf{gl}(n;\mathbb{R})$.

2. The Lie algebra of $SL(n;\mathbb{C})(\backslash SL(n;\mathbb{R}))$ is the space of all complex ($\backslash$ real) matrices of order n having trace equal to zero and is denoted by $\mathbf{sl}(n;\mathbb{C})(\ \backslash\mathbf{sl}(n;\mathbb{R})$.

3. The Lie algebra of the matrix Lie group $U(n)$ is the space of all matrices of order n such that $X^* = -X$ and is denoted by $u(n)$ and thus the Lie algebra of the group $SU(n)$ is the space of all complex matrices of order n such that $X^* = -X$ with $trace(X) = 0$ and is denoted by $su(n)$.

4. The Lie algebra of the group $O(n)$ is same as the Lie algebra of $SO(n)$ and that is the set of all real matrices of order n such that $X^{tr} = -X$ and is denoted by $so(n)$.

5. The Lie algebra of the Heisenberg group H is the set of all upper triangular real matrices of order 3.

Some Properties of Lie algebras

For a matrix Lie group G with it's Lie algebra $\acute{\mathbf{g}}$, we have

1. If $X \in \acute{\mathbf{g}}$, then e^X belongs to the connected component of G which contains identity.

2. If X is in $\acute{\mathbf{g}}$ and P is in G, then $PXP^{-1} \in \acute{\mathbf{g}}$.

3. If X and Y are elements of $\acute{\mathbf{g}}$, then we have $X+Y \in \acute{\mathbf{g}}$, $XY - YX \in \acute{\mathbf{g}}$ and $sX \in \acute{\mathbf{g}}$ for all $s \in \mathbb{R}$. Thus $\acute{\mathbf{g}}$ turns out to be a real linear subspace of $M_n(\mathbb{C})$.

Definition 19. Let P and Q be two matrices of order n. The **bracket** of P and Q is denoted by $[P,Q]$ and defined as

$$[P,Q] = PQ - QP.$$

Theorem 9. *Let $\acute{\mathbf{g}}$ and $\acute{\mathbf{h}}$ be the Lie algebras of the matrix Lie groups G and H respectively. If $\Phi : G \to H$ be a Lie group homomorphism, then is a unique real linear transformation $\phi : \acute{\mathbf{g}} \to \acute{\mathbf{h}}$ with the property*

$$\Phi(e^X) = e^{\phi(X)}$$

for all $X \in G$. This map ϕ satisfies the following properties:-

1. *$\phi(PXP^{-1}) = \Phi(P)\phi(X)\Phi(P)^{-1}$, for all $X \in \acute{\mathbf{g}}, P \in G$.*

2. *$\phi([X,Y]) = [\phi(X), \phi(Y)]$, for all $X, Y \in \acute{\mathbf{g}}$.*

3. *$\phi(X) = \frac{d}{dt}\Phi(e^{(tX)})|_{t=0}$.*

Definition 20. A linear map ϕ between two matrix Lie algebras $\acute{\mathbf{g}}$ and $\acute{\mathbf{h}}$ is called a **Lie algebra homomorphism** if it preserve the bracket. More precisely, a linear map $\phi : \acute{\mathbf{g}} \to \acute{\mathbf{h}}$ is Lie algebra homomorphism if

$$\phi([X,Y]) = [\phi(X), \phi(Y)], \text{ for all } X, Y \in \acute{\mathbf{g}}.$$

Definition 21 (The Adjoint Mapping)**.** Let $\acute{\mathbf{g}}$ be the Lie algebra of the matrix Lie group G. For a given $A \in G$ the adjoint mapping is denoted by Ad_A and defined to be an invertible linear map $Ad_P : \acute{\mathbf{g}} \to \acute{\mathbf{g}}$ as

$$Ad_P(X) = PXP^{-1}.$$

Theorem 10. *Let $\acute{g}$ be the Lie algebra of the matrix Lie group G. Let $\boldsymbol{GL(\acute{g})}$ denotes the group of all invertible linear transformations of $\acute{g}$ and regarded as a matrix Lie group. Then the map $P \to Ad_P$ is a Lie group homomorphism of G into $\boldsymbol{GL(\acute{g})}$ and for every $P \in G$*

$$Ad_P[X, Y] = [Ad_P(X), Ad_P(Y)] \quad \text{for all } X, Y \in \acute{g}.$$

If $ad : \acute{g} \to \boldsymbol{GL(\acute{g})}$ is the associated Lie algebra map, then for all $X, Y \in \acute{g}$

$$ad_X(Y) = [X, Y].$$

Definition 22. The exponential mapping for a matrix Lie group G, is the restriction of matrix exponential to the Lie algebra $\acute{\mathbf{g}}$ of G.

The exponential mapping can be used to prove the following results

Theorem 11. *Let G be a connected matrix Lie group with its Lie algebra g. If $A \in G$, then A can be written as*

$$A = e^{X_1} e^{X_2} \dots e^{X_l}$$

for some $X_1, X_2, \dots X_l$ in g.

Theorem 12. *Let H and K be two matrix Lie groups. If Φ_1 and Φ_2 are Lie group homomorphisms of H into K and ϕ_1 and ϕ_2 are the associated Lie algebra homomorphisms. If H is connected and $\phi_1 = \phi_2$, then $\Phi_1 = \Phi_2$.*

Definition 23 (Lie algebra)**.** A finite-dimensional real or complex Lie algebra is a finite-dimensional real or complex vector space $\acute{\mathbf{g}}$, with a map $[\cdot, \cdot]$ from $\acute{\mathbf{g}} \times \acute{\mathbf{g}}$ into $\acute{\mathbf{g}}$ such that

1. $[X, Y] = -[Y, X]$ for all $X, Y \in \acute{\mathbf{g}}$.

2. $[\cdot, \cdot]$ is bilinear.

3. $[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0$ for all $X,Y,Z\in\acute{\mathbf{g}}$.

Property 3 is known as **Jacobi Identity**.

Example:-$M_n(\mathbb{R})$ with respect to the bracket operation $[P,Q]=PQ-QP$ is a real Lie algebra.

A **subalgebra** $\acute{\mathbf{h}}$ of a real or complex Lie algebra $\acute{\mathbf{g}}$ is a subspace of $\acute{\mathbf{g}}$ which is closed with respect to the bracket operation.

For the Lie algebras $\acute{\mathbf{g}}$ and $\acute{\mathbf{h}}$, a **Lie algebra homomorphism** is a linear map $\phi:\acute{\mathbf{g}}\to\acute{\mathbf{h}}$ such that $\phi([X,Y])=[\phi(X),\phi(Y)]$ for all $X.Y\in\acute{\mathbf{g}}$ and a **Lie algebra isomorphism** is a one-to-one, onto Lie algebra homomorphism.

Definition 24 (ad map)**.** For a Lie algebra $\acute{\mathbf{g}}$ with $X\in\acute{\mathbf{g}}$, define a linear map $ad_X:\acute{\mathbf{g}}\to\acute{\mathbf{g}}$ by

$$ad_X(Y)=[X,Y].$$

Thus, ad (the map $X\to ad_X$) is a linear map from $\acute{\mathbf{g}}$ into $\mathbf{gl}(\acute{\mathbf{g}})$, the space of linear operators on $\acute{\mathbf{g}}$.

Theorem 13. *If $\acute{\mathbf{g}}$ is a Lie algebra, then for all $X,Y\in\acute{\mathbf{g}}$*

$$ad_{[X,Y]}=ad_Xad_Y-ad_Yad_X=[ad_X,ad_Y].$$

Thus the map $ad:\acute{\mathbf{g}}\to\mathbf{gl}(\acute{\mathbf{g}})$ is a Lie algebra Homomorphism.

Complexification of real Lie Algebras

Definition 25. Let V be a finite-dimensional vector space. The complexification of V is a complex vector space, denoted by $V_{\mathbb{C}}$, is the space of formal linear combinations v_1+iv_2, for $v_1,v_2\in V$ with a condition that $i(v_1+iv_2)=-v_2+iv_1$.

It can be shown easily that if $g_{\mathbb{C}}$ is the complexification of a finite-dimensional real Lie algebra g, then the bracket operation on g can be uniquely extended to a $g_{\mathbb{C}}$ making it a complex Lie algebra and this complex Lie algebra $g_{\mathbb{C}}$ is known as complexification of the real Lie algebra g.

Chapter 4

Representations of the Matrix Lie Groups and their Lie Algebras

4.1 Introduction

In this chapter, we will first define finite dimensional complex and real representations for matrix Lie groups and Lie algebras. Then invariant space, irreducible representation, and intertwining map are defined. We see a relation between the representations of matrix Lie groups and their Lie algebras. Next, we explore some examples of representations like: the adjoint representation, some representations of $SU(2)$ and corresponding representations of its Lie algebra $su(2)$. Further, we see that every compact matrix Lie group is completely reducible. Certain facts and results stated without proof can found in [BH] and the references therein.

4.2 Representations

Definition 26. A Lie group homomorphism

$$\Pi : G \to GL(n; \mathbb{C})$$

or a Lie group homomorphism

$$\Pi : G \to GL(V),$$

where V is a finite dimensional complex vector space, is called a finite-dimensional complex representation of the matrix group G. The dimension of V is called the dimension of the representation Π. Similarly a **finite dimensional real representation** of a matrix Lie group can be defined.

Definition 27. A Lie algebra homomorphism π of $\acute{\mathbf{g}}$ into $\mathbf{gl}(n;\mathbb{C})$ or into $GL(V)$ for a finite dimensional complex vector space V is called a **finite-dimensional complex representation** of the Lie algebra $\acute{\mathbf{g}}$
Similarly a **finite dimensional real representation** of a Lie algebra can be defined.

A representation is called **faithful** if it is a one-one homomorphism.

Definition 28. If Π is a finite dimensional representation of a matrix Lie group G, acting on a vector space V then an **invariant subspace** W of V is a subspace of V such that $\Pi(P)w$ is in W for all $w \in W$ and for all $P \in G$.

An **irreducible** representation is one which have no non-trivial invariant subspaces. Analogously, these terms can be defined for Lie algebras.

Definition 29. Let Π_1 be a representation of a matrix Lie group G acting on the vector space V_1, and let Π_2 be a representation acting on some other space V_2 then an **intertwining map** of representations is a linear map $\phi : V_1 \to V_2$ such that

$$\phi(\Pi_1(P)v) = \Pi_2(P)\phi(v).$$

for all $P \in G$ and for all $v \in V_1$. In the same way the **intertwining map** of representations of a Lie algebra can be defined.

An invertible intertwining map of representations is called an **equivalence** of representations. The representations are called equivalent if there exists an intertwining isomorphism between the two spaces V_1 and V_2.

Theorem 14. *For a matrix Lie group G with the Lie algebra $\acute{\mathbf{g}}$, if Π is a representation acting on the vector space V, then we have a unique representation π of $\acute{\mathbf{g}}$ that acts on the same vector space and satisfies*

$$\Pi(e^X) = e^{\pi(X)}$$

for all X in $\acute{\mathbf{g}}$. Also, the representation π is such that

$$\pi(X) = \frac{d}{dt}\Pi(e^{tX})\bigg|_{t=0}$$

π also satisfies

$$\pi(PXP^{-1}) = \Pi(P)\pi(X)\Pi(P)^{-1}$$

for all X in $\acute{\mathbf{g}}$ and all P in G.

Proof. By using theorem 9, we can say that for any Lie group homomorphism

$$\Phi : G \to K,$$

we have an associated Lie algebra homomorphism

$$\pi : \acute{\mathbf{g}} \to \acute{\mathbf{k}}.$$

Now we can take K to be equal to $GL(V)$ and Φ to be equal to Π. As, we know that the Lie algebra of $GL(V)$ is denoted by $\mathbf{gl}(V)$, the associated Lie algebra homomorphism that is π maps $\acute{\mathbf{g}}$ to $\mathbf{gl}(V)$ and thus, it is a representation of $\acute{\mathbf{g}}$.

The described properties of the map π are satisfied using the theorem 9. □

Theorem 15. *Let $\acute{\mathbf{g}}$ be the Lie algebra of a connected matrix Lie group G.*

1. *If Π is a representation of G with associated representation π of $\acute{\mathbf{g}}$ then π is irreducible iff Π is irreducible.*

2. *If Π_1 and Π_2 are representation of G with associated representation π_1 and π_2 of $\acute{\mathbf{g}}$ then π_1 π_2 are equivalent iff Π_1 Π_2 are equivalent.*

Proof. The proof of second is similar to the first, we will prove the first only.

Let Π be irreducible and W be a subspace of V, invariant under $\pi(X)$ for all X in $\acute{\mathbf{g}}$. Our aim is to show that either W is $\{0\}$ or V.

Since G is connected and $P \in G$, we can write P as

$$P = e^{X_1 \cdots\cdots} e^{X_m}$$

for some $X_1, X_2, \ldots, X_m \in \acute{\mathbf{g}}$. As, the space W is invariant under $\pi(X_i)$, so it will be invariant under $\exp(\pi(X_i))$ as well and then under

$$\Pi(P) = \Pi(e^{X_1 \cdots\cdots} e^{X_m}) = \Pi(e^{X_1}) \ldots \Pi(e^{X_m}) = e^{\pi(X_1)} \ldots e^{\pi(X_m)}.$$

Thus, W is invariant under $\Pi(P)$ for each such $P \in G$. Therefore, W is either $\{0\}$ or whole of V. Thus, π is irreducible.

Now, suppose that π is irreducible and W is an invariant subspace under $\Pi(P)$ for all $P \in G$. Then, by the definition of *exp* function, we can see that W is invariant under $\Pi(exp(sX))$ for all X in $\acute{\mathbf{g}}$ and for all $s \in \mathbb{R}$. Therefore W is invariant under

$$\pi(X) = \frac{d}{ds}\Pi(e^{sX})\bigg|_{s=0}.$$

Hence, irreducibility of π implies that W is either $\{0\}$ or V, proving the irreducibility of Π.

□

Using definitions the following result can be easily proved.

Theorem 16. *If $\acute{\boldsymbol{g}}_{\mathbb{C}}$ is the complexification of a real Lie algebra $\acute{\boldsymbol{g}}$, then we have a unique extension for every finite dimensional complex representation π of $\acute{\boldsymbol{g}}$ to a complex linear representation of $\acute{\boldsymbol{g}}_{\mathbb{C}}$, which is also denoted by π and defined as*

$$\pi(X + iY) = \pi(X) + i\pi(Y)$$

for all X and Y in $\acute{\boldsymbol{g}}$.
Also, π is an irreducible representation as a representation of $\acute{\boldsymbol{g}}$ iff it is an irreducible representation as a representation of $\acute{\boldsymbol{g}}_{\mathbb{C}}$.

Definition 30. Let $\acute{\mathbf{H}}$ be a Hilbert space and $U(\acute{\mathbf{H}})$ be group of all unitary operators on $\acute{\mathbf{H}}$. Let G be a matrix Lie group. Then a unitary representation of G is a homomorphism $\Pi : G \to U(\acute{\mathbf{H}})$ if the given continuity condition is satisfied:
If $P_n, P \in G$ with $P_n \to P$, then

$$\Pi(P_n)v \to \Pi(P)v$$

for all $V \in \acute{\mathbf{H}}$.

It is an important question that why we should study representations?

If Π is a faithful representation of a matrix Lie group G, then the collection of operators $\{\Pi(A) : A \in G\}$ is a group and it turns out to be isomorphic to G itself. Hence, we obtain a way to view G as a group of invertible operators. Despite this important reason, the idea behind studying representation theory is not limited to this only.

Another important reason is that a representation can be viewed as an action of a given group on a linear space. Some such kind of actions are common in many branches of physics and mathematics. It is desirable to understand these actions in some better way.

In case a system possess symmetry, then with the knowledge of the representations of the group of its symmetries, one can work with the system in more efficient and simple way.

For an example, if a differential equation in $\mathbb{R}^3$, possess rotational symmetry, then the solution space remains invariant under the action of $SO(3)$, and hence the representations of rotation group help to understand the solutions.

4.3 Some Examples of Representations

1. **The standard representation**

   If we take any matrix Lie group G, then by its definition, it is a subgroup of $GL(n;\mathbb{C})$. So, the inclusion map of G into $GL(n;\mathbb{C})$ will be a representation of G. This is called the **standard representation** of G. Thus, the standard representation of $SO(3)$ is a representation in which it acts on $\mathbb{R}^3$ in the usual way and for $SU(2)$, the standard representation is the representation in which it acts in the usual way on $\mathbb{C}^2$.

   Now, for any matrix Lie group G, it's Lie algebra $\acute{\mathbf{g}}$ is a subalgebra of

$\mathbf{gl}(n;\mathbb{R})$ or $\mathbf{gl}(n;\mathbb{C})$. Thus, in this case, the inclusion map will work as a **standard representation**.

2. **The trivial representation**

   Consider complex vector space $\mathbb{C}$ of dimension one, then for any matrix Lie group G, the **trivial representation** of G is defined as: $\Pi : G \to GL(1;\mathbb{C})$ such that

   $$\Pi(P) = I$$

   for all P in G. As, the dimension of $\mathbb{C}$ is one, therefore it has no nontrivial subspaces. Thus, Π is an irreducible representation of G.

   Now, for a Lie algebra $\acute{\mathbf{g}}$, the **trivial representation** can be defined in the most obvious way as $\pi : \acute{\mathbf{g}} \to \mathbf{gl}(1;\mathbb{C})$ such that

   $$\pi(X) = 0$$

   for all X in $\acute{\mathbf{g}}$.

3. **The adjoint representation**

   We know that for a matrix Lie group G with the Lie algebra $\acute{\mathbf{g}}$, the adjoint mapping is a map defined as:

   $$Ad : G \to GL(\acute{\mathbf{g}})$$

   given by the formula

   $$Ad_P(X) = PXP^{-1}.$$

   Since, Ad is a Lie group homomorphism, we can consider it as a representation of the matrix Lie group G which acts on it's Lie algebra $\acute{\mathbf{g}}$. So, we can call the map Ad as the **adjoint representation** of G.

   Similarly, for a Lie algebra $\acute{\mathbf{g}}$, the adjoint map is defined as:

   $$ad : \acute{\mathbf{g}} \to \mathbf{gl}(\acute{\mathbf{g}})$$

   given by the formula

   $$ad_X(Y) = [X, Y].$$

   As, ad is a Lie algebra homomorphism, therefore, it is a representation of the Lie algebra $\acute{\mathbf{g}}$ and it is known as **the adjoint representation**.

4. **Some representations of** $SU(2)$

Let V_n be the space of all homogeneous polynomials in two complex variables (z_1 and z_2) having total degree n. So, V_n will contain all the polynomials of the form:

$$f(z_1, z_2) = a_0 {z_1}^n + a_1 {z_1}^{n-1} z_2 + a_2 {z_1}^{n-2} {z_2}^2 + \cdots + a_n {z_2}^n$$

where z_1, z_2 are complex variables and a_j's are arbitrary complex constants.

Any element U of $SU(2)$ acts on $\mathbb{C}^2$ as a linear transformation. Let the order pair (z_1, z_2) in $\mathbb{C}^2$ is denoted by z. Then a linear transformation $\Pi_m(U)$ can be defined on the space V_n as:

$$[\Pi_n(U)f](z) = f(U^{-1}z).$$

By the definition of f, we have,

$$[\Pi_n(U)f](z_1, z_2) = \sum_{k=0}^{n} a_k (U_{11}^{-1} z_1 + U_{12}^{-1} z_2)^{n-k} (U_{21}^{-1} z_1 + U_{22}^{-1} z_2)^k.$$

If we expand the right hand side of this formula, then we observe that $\Pi_n(U)f$ is a homogeneous polynomial having degree n. So, $\Pi_n(U)$ is map from V_n into V_n. We also have

$\Pi_n(U_1)[\Pi_n(U_2)f](z) = [\Pi_n(U_2)f]({U_1}^{-1}z) = f({U_2}^{-1}{U_2}^{-1}z) = \Pi_n(U_1U_2)f(z).$

Hence, Π_n is a representation of $SU(2)$.

Now, let us compute the Lie algebra representation π_n corresponding to the representation Π_n. By theorem 14, we know that

$$\pi_n(X) = \frac{d}{dt}\Pi_n(e^{tX})\bigg|_{t=o}.$$

By using the definition of $\Pi_n(U)$, we have

$$(\pi_n(X)f)(z) = \frac{d}{dt} f(e^{-tX}z)\bigg|_{t=0}.$$

Let us now assume that $z(t)$ is a curve in $\mathbb{C}^2$ defined by the formula

$$z(t) = e^{-tX}z.$$

So, $z(0) = z$ and $z(t) = (z_1(t), z_2(t))$, with $z_j(t) \in \mathbb{C}$. Hence, by the

chain rule, we have,
$$\pi_n(X)f = \frac{\partial f}{\partial z_1}\frac{dz_1}{dt}\bigg|_{t=0} + \frac{\partial f}{\partial z_2}\frac{dz_2}{dt}\bigg|_{t=0}.$$
Since, $\frac{dz}{dt}\big|_{t=0} = -Xz$, therefore, we have
$$\pi_n(X)f = -\frac{\partial f}{\partial z_1}(X_{11}z_1 + X_{12}z_2) - \frac{\partial f}{\partial z_2}(X_{21}z_1 + X_{22}z_2).$$
By theorem 16, we know that every finite dimensional complex representation of a Lie algebra (here we consider the Lie algebra **su(2)** of $SU(2)$) extends uniquely to a complex linear representation of it's complexification.

The complexification of $\mathbf{su}(2)$ is $\mathbf{sl}(2;\mathbb{C})$ (up to isomorphism). So, π_n is extended to a representation of $\mathbf{sl}(2;\mathbb{C})$, which we will denote by π_n itself.

For example, consider the matrix
$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \in \mathbf{sl}(2;\mathbb{C}).$$
Then $(\pi_n(H)f)(z)$ is defined as:
$$(\pi_n(H)f)(z) = -\frac{\partial f}{\partial z_1}z_1 + \frac{\partial f}{\partial z_2}z_2.$$
So,
$$\pi_n(H) = -z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2}.$$

Now, if we apply $\pi_n(H)$ to a basis element ${z_1}^k{z_2}^{n-k}$, then we get
$$\pi_n(H){z_1}^k{z_2}^{n-k} = -k{z_1}^k{z_2}^{n-k} + (n-k){z_1}^k{z_2}^{n-k} = (n-2k){z_1}^k{z_2}^{n-k}.$$

So, each basis element ${z_1}^k{z_2}^{n-k}$ is an eigenvector for $\pi_n(H)$ with eigenvalue $(n-2k)$. Hence, $\pi_n(H)$ is diagonalizable.

If $X, Y \in \mathbf{sl}(2;\mathbb{C})$ are two elements given by
$$X = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$
and
$$Y = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then we have,

$$\pi_n(X) = -z_2 \frac{\partial}{\partial z_1}, \; \pi_n(Y) = -z_1 \frac{\partial}{\partial z_2}$$

Then,

$$\pi_n(X){z_1}^k{z_2}^{n-k} = -k{z_1}^{k-1}{z_2}^{n-k+1},$$

$$\pi_n(Y){z_1}^k{z_2}^{n-k} = (k-n){z_1}^{k+1}{z_2}^{n-k-1}.$$

Theorem 17. *π_n is an irreducible representation of $\boldsymbol{sl}(2;\mathbb{C})$.*

Proof. Let W be a non-zero invariant subspace of V_n, then, we have to show that $W = V_n$.

Let w be a non-zero vector in W, then we can write w uniquely in the following form:

$$w = a_0{z_1}^n + a_1{z_1}^{n-1}z_2 + a_2{z_1}^{n-2}{z_2}^2 + \cdots + a_n{z_2}^n$$

and since, w is non-zero, therefore, at least one of the $a_k's$ is nonzero. Assume that k_0 is the largest value of k such that a_k is nonzero. Then, consider

$$\pi_n(X)^{k_0}w.$$

As, each application of $\pi_n(X)$ reduces the power of z_1 by 1, therefore, $\pi_n(X)^{k_0}$ makes all the terms in w equal to zero except the term $a_{k_0}z_1^{k_0}z_2^{n-k_0}$ and on this term $\pi_n(X)^{k_0}$ acts as:

$$\pi_n(X)^{k_0}(a_{k_0}z_1^{k_0}z_2^{n-k_0}) = k_0!(-1)^{k_0}a_{k_0}{z_2}^n.$$

So, $\pi_n(X)^{k_0}w$ is nothing but a nonzero multiple of ${z_2}^n$. As, W is an invariant subspace, therefore, it must contain this nonzero multiple of ${z_2}^n$ and hence, it must contain ${z_2}^n$.

Now, by the definition of $\pi_n(Y)$, $\pi_n(Y)^k{z_2}^n$ is a nonzero multiple of ${z_1}^k{z_2}^{n-k}$. Therefore,W must contain ${z_1}^k{z_2}^{n-k}$ for all $0 \le k \le n$. As, these elements form a basis for V_n, therefore, this implies that $W = V_n$. Hence, the theorem is proved. □

4.4 Complete Reducibility

Definition 31 (Direct sum)**.** Let $\Pi_1, \Pi_2, \ldots, \Pi_m$ be representations of a matrix Lie group G, which act on the vector spaces $V_1, V_2, \ldots, V_m$ respectively. The **direct sum** of these representations is a representation, denoted by $\Pi_1 \oplus \cdots \oplus \Pi_m$, of G which acts on the vector space $V_1 \oplus V_2 \oplus \cdots \oplus V_m$, such that for all A in G,

$$[\Pi_1 \oplus \cdots \oplus \Pi_m(A)](v_1, \ldots, v_m) = (\Pi_1(A)v_1, \ldots, \Pi_m(A)v_m).$$

In the same way, direct sums of representations for Lie algebras are defined.

Definition 32 (Completely Reducible Representation)**.** Let Π be a finite-dimensional representation of a matrix Lie group G. Then Π is said to be a **completely reducible** representation of the group G if it is isomorphic to a representation of G, which is the direct sum of finitely many irreducible representations of G.

Similarly, one can define Completely Reducible Representation of a Lie algebra.

Definition 33. A Lie algebra or a Lie group is known to possess **complete reducibility property**, if all of its finite dimensional representations are completely reducible.

Theorem 18. *Every finite dimensional unitary representation of a matrix Lie group G, which acts on a finite dimensional real or complex Hilbert space V is completely reducible.*

Proof. Suppose Π acts on a finite dimensional Hilbert space V and let $\langle , \rangle$ be the inner product on the space V. Assume that $U \subset V$ is an invariant subspace, then V is equal to the direct sum of U and $U^\perp$.

Since, Π is a unitary representation, therefore for all A in G we have $\Pi(A^{-1}) = \Pi(A)^{-1} = \Pi(A)^*$. Further, if $u \in U$ and $v \in U^\perp$, then

$$\langle \Pi(A)v, u \rangle = \langle v, \Pi(A)^* u \rangle = \langle v, \Pi(A^{-1})u \rangle = \langle v, u' \rangle = 0.$$

Hence, $U^\perp$ is also an invariant subspace.

Now, suppose that V is not irreducible, then there exists a non-trivial subspace U of V such that $V = U \oplus U^\perp$, where both U and $U^\perp$ are invariant subspaces. Then either we have U as an irreducible space or it can further be decomposed as an orthogonal direct sum of invariant subspaces, similar condition is true for $U^\perp$.

This process of splitting may be continued till we get irreducible pieces. The termination of the process is guaranteed because V is finite dimensional. Hence, at the end we get irreducible invariant subspaces, whose direct sum is V itself. □

Theorem 19. *Every finite group satisfies complete reducibility property.*

Proof. Let Π be a representation of G which acts a space V. We aim to show that there is an inner product on V, with respect to which Π turns out to be a unitary representation on V, Hence the result will follow from previous theorem. Let $\langle , \rangle$ be the inner product on V. Now define a new inner product $\langle , \rangle_G$ on V as:

$$\langle v_1, v_2 \rangle_G = \sum_{h \in G} \langle \Pi(h)v_1, \Pi(h)v_2 \rangle.$$

If $g \in G$, then

$$\langle \Pi(g)v_1, \Pi(g)v_2 \rangle_G = \sum_{h \in G} \langle \Pi(h)\Pi(g)v_1, \Pi(h)\Pi(g)v_2 \rangle = \sum_{h \in G} \langle \Pi(hg)v_1, \Pi(hg)v_2 \rangle.$$

Since, h varies over whole of G, we have

$$\langle \Pi(g)v_1, \Pi(g)v_2 \rangle_G = \langle v_1, v_2 \rangle_G,$$

hence, Π is a unitary representation. □

Definition 34. A nonzero measure μ on the Borel σ-algebra in the matrix Lie group G, is said to be a **left Haar measure** on G, if it locally finite and left translation invariant.

Theorem 20. *Every compact matrix Lie group G possess the complete reducibility property.*

Proof. It is known that every matrix Lie group has a left Haar measure and upto multiplication by a constant there is only one such Haar measure. Further since the group is compact, the left Haar measure is finite.

Suppose Π is a finite-dimensional representation of a compact group G, which acts on a vector space V and let $\langle , \rangle$ be the inner product on V, then define a new inner product $\langle , \rangle_G$ on V as:

$$\langle v_1, v_2 \rangle_G = \int_G \langle \Pi(g)v_1, \Pi(g)v_2 \rangle d\mu(g),$$

where μ is a left Haar measure. If $h \in G$, then we have

$$\begin{aligned} \langle \Pi(h)v_1, \Pi(h)v_2 \rangle_G &= \int_G \langle \Pi(g)\Pi(h)v_1, \Pi(g)\Pi(h)v_2 \rangle d\mu(g) \\ &= \int_G \langle \Pi(gh)v_1, \Pi(gh)v_2 \rangle d\mu(g) = \langle v_1, v_2 \rangle_G. \end{aligned}$$

Thus Π is a unitary representation with respect to the new inner product. Therefore, it is completely reducible by Theorem 18. □

Chapter 5

Conclusion

In this thesis, we worked with the matrix Lie groups which are the subgroups of general linear groups. Lie groups, in general is a very vast topic and need a lot of pre-requirements but matrix Lie groups are special kinds of Lie groups which requires only some basic knowledge of Algebra, Topology and Analysis and are thus easy to understand.

The area of representation theory is a very vast one and whatever we studied in these chapters is just an attempt to become familiar with some examples of representations and so that we can go for some more advanced study related to representation theory in the coming future. We would like to study the representations of SU_3, nilpotent Lie groups and semisimple Lie algebras.

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