Behaviour of quantum fields in curved Spacetime

M.Sc. Thesis

By

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DISCIPLINE OF PHYSICS

INDIAN INSTITUTE OF TECHNOLOGY INDORE

June 2020

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A THESIS

Submitted in partial fulfillment of the requirements for the award of the degree
of
Master of Science

by

Pravesh C. Awasthi

DISCIPLINE OF PHYSICS
INDIAN INSTITUTE OF TECHNOLOGY INDORE

June 2020

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I hereby certify that the work which is being presented in the thesis entitled **"Behaviour of quantum fields in curved Space-time"** in the partial fulfillment of the requirements for the award of the degree of **MASTER OF SCIENCE** and submitted in the **DISCIPLINE OF PHYSICS, Indian Institute of Technology Indore**, is an authentic record of my own work carried out during the time period from july 2019 to june, 2020 under the supervision of **Dr. Manavendra Mahato**, Associate Professor.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other institute.

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This is to certify that the above statement made by the candidate is correct to the best of my/our knowledge.

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Signature of PSPC Member #1
Date: 19/06/2020

ACKNOWLEDGEMENT

First of all, I would like to express my deepest gratitude to my supervisor Dr. Manvendra Mahato not only for his support and guidance throughout the project but also for allowing me the opportunity to undertake this exciting project. He has introduced me to the wonderful world of theoritical physics. I am thankful to him for sharing his valuable time.
I would like to thank my PSPC committee members Dr. Ankhi Roy and Dr. Bhargav Vaidya for their support and motivation. I am very grateful to the Physics Department of IIT Indore for providing me the required facilities.
I would like to express my humble gratitude to my family and friends for their blessing and support.

pravesh c. Awasthi

Dedicated to my family

ABSTRACT

It is believed that all physical fields should be described at fundamental level by the general framework of quantum field theory. But we do not have any successfull theory of quantum gravity, which describes the gravity in the quantum domain. Therefore, to study the influence of the gravitational field on quantum phenomena we quantize the matter fields in the usual way while the gravitational field is treated as the fixed background. Classically gravity is described by the *theory of general relativity* as the curvature of spacetime. So in order to understand the quantum aspect of gravity one is lead to the subject of quantum field theory in background curved spacetime, which is the subject of this work.

In curved spacetime, in general, we do not have any preferred set of basis modes. Even our very notion of particle and vacuum is not very well defined, they depend on the set of modes (in other words they are observer-dependent). Theory predicts some interesting phenomenon eg. Casimir effect, Unruh effect. We see that the frequency modes get modified in the presence of boundary or in presence of non-trivial geometry of spacetime. Vacuum is the most simple yet most bizzare. The Casimir effect is regarded as one of the most striking manifestations of vacuum fluctuations in quantum field theory. The Casimir stress for spherical shell turns out to be repulsive with certain constraints, which can be related to the expansion of the universe.

Contents

List of Figures

Chapter 1

Introduction

There are four fundamental forces in nature. Standard model explains three of the fundamental forces in the framework of quantum field theory but fails to describe the gravity. The two major advances of the 20th century physics : quantum mechanics and General relativity, has changed our view towards nature. These theories work very well in their respectie areas and have been experimentally verified. We understand gravity at the classical level as the curvature of spacetime described by Einstein's field equation. But to find the solution of Einstein's equation, in general, is tedious. Blackhole is the prediction of Einstien's equation, which is now experimentally verified.

However, the main problem is to find a consistent framework for quantum gravity that unifies general relativity with quantum mechanics. There are many approaches to such a theory, but no one is fully successful. String theory offers a route towards such a theory, it predicts the existence of a vertex operator corresponding to a massless spin two particle. On the other hand, the unexpected similarities between the properties of blackhole and the laws of thermodynamics have gave us a new way to understand the gravity. If the blackhole has entropy then in quantum theory it must be an ensemble of some kind of microstates in quantum Hilbert space as defined by Boltzmann law. This approach leads to the involvement of number theory and different kinds of dualities in string theory.

here we proceed with adopting Einstien's general theory of relativity as a description of gravity and quantize the material field as usual.

1.1 Quantum fields in curved spacetime

There exist a number of way for formulating a quantum field theory from a classical field theory which is described in terms of Lagrangian or Hamiltonian formulation. Howeve, the field which describes the gravity is sufficiently different from other classical field theories, the essential difference between the general relativity and other classical theories appears to be the dual role-played by the metric $g_{\mu\nu}$ as both the quantity which describes the dynamical aspect of gravity and the quantity which describes the background spacetime structure. Thus, it would appear that to quantize the dynamical degrees of freedom of the gravitational field, one must also give a quantum mechanical description of spacetime structure. It is convenient to first study the Quantization of the scalar and another vector fields in curved spacetime, for which we know how to quantize them in Minkowski space.

1.1.1 Scalar field quantization in curved spacetime

In order to study the behavior of quantum fields in curved space, we start with Lagrangian density [1]

$$\mathcal{L}_{\mathcal{M}} = \frac{1}{2}\sqrt{-g(x)}\left[g^{\mu\nu}(x)\nabla\mu\Phi(x)\nabla\nu\Phi(x) - m^2\Phi^2(x) - \xi R(x)\Phi^2(x)\right] \quad (1.1)$$

where m is the mass of field quanta of scalar field $\phi(x)$. The coupling between the scalar field and the gravitational field is presented by $\xi R\phi^2$, where ξ is a dimensionless factor and $R(x)$ is the **Ricci scalar curvature.** Setting the variation of the action with respect to ϕ equal to zero yields the scalar field equation

$$\left[\Box_x + m^2 + \xi R(x)\right]\phi(x) = 0$$

the field ϕ may be expanded as:

$$\phi(x) = \sum_i \left[\hat{a}_i f_i(x) + \hat{a}_i{}^\dagger f_i{}^\star(x)\right] \tag{1.2}$$

where in Minkowski space, $f_i(x^\mu) = \left[2\omega(2\pi)^{n-1}\right]^{-\frac{1}{2}} e^{ik_\mu x^\mu}$

The inner product is generalized to,(for a spacelike hypersurface Σ)

$$(\phi_1, \phi_2) = -\iota \int_\Sigma (\phi_1 \nabla_\mu \phi_2{}^* - \nabla_\mu \phi_1 \phi_2{}^*) \left[-g_\Sigma(x)\right]^{\frac{1}{2}} d\Sigma^\mu$$

where $d\Sigma^\mu = n^\mu d\Sigma$, with n^μ a future directed unit vector orthogonal hypersurface Σ and $d\Sigma$ is the volume element in Σ.

There is a difference here how we treat the quantum fields in any spacetime. In Minkowski space, there is a natural set of modes.The vector $\frac{\partial}{\partial t}$ is a **Killing vector** in this space, and the modes are eigenfunctions of this killing vector. However, in curved spacetime, the **Poincare group** is no longer a symmetry of spacetime. Indeed, in general , there will be no Killing vectors at all with which to define positive frequency modes. There is no preferred set of modes in curved spacetime. Field ϕ may be expanded in another complete orthonormal set of modes.

$$\phi(x) = \sum_i \left[\hat{b}_i g_i(x) + \hat{b}_i{}^\dagger g_i{}^\star(x)\right] \tag{1.3}$$

1.1.2 Bogolubov transformation

The two observers corresponding to different set of modes will disagree on the number of particles observed. It is convenient to expand each set of modes (**from equation 6 and 7**)in terms of other

$$g_i = \sum_i (\alpha_{ij} f_j + \beta_{ij} f_j{}^\star)$$

conversely

$$f_i = \sum_i (\alpha_{ji}{}^\star g_j - \beta_{ji} g_j{}^\star)$$

These transformations from one set of basis into another is known as **Bogolubov transformation.** The matrices α_{ij} and β_{ij} are called **Bogolubov coefficients.**

$$\alpha_{ij} = (g_i, f_j)$$

$$\beta_{ij} = -(g_i, f_J{}^{\star})$$

These Bogolubov coefficient can be used to transform between the operators corresponding to different set of modes,

$$\hat{a_i} = \sum_i (\alpha_{ji}\hat{b_j} + \beta_{ji}{}^{\star}\hat{b_j}{}^{\dagger})$$

$$\hat{b_i} = \sum_i (\alpha_{ij}{}^{\star}\hat{a_j} - \beta_{ij}{}^{\star}\hat{a_j}{}^{\dagger})$$

Expectation value of the g number operator $\hat{n_{gi}} = b_i{}^{\dagger}b_i$ in f-vacuum :

$$\langle 0_f | \hat{n_{gi}} | 0_f \rangle = \sum_i |\beta_{ij}|^2$$

1.2 Concept of particle : Particle detectors

The concept of particle and vacuum do not generally have universal significance, there is an essential observer dependent quality about them. One is still free to assert the presence of particles, but without specifying the state of motion of the detector, the concept is not very useful, even in Minkowski space.

We define a particle as being something which is "detected" by a "particle detector".

We shall treat a model of particle detector due to Unruh (1976) and Dewitt (1979). It consists of an idealized point particle with internal energy levels labeled by the energy E, coupled via a monopole interaction with a scalar field ϕ.

1.2.1 Transition probability and response function

To understand how the particle detector detects the presence of particles, let the particle detector moves along the world line $x^\mu(\tau)$, where τ is the detector's proper time. The detector-field interaction is described by the interaction Lagrangian [2]

$$\mathcal{L}_{in} = cm(x)\phi[x(\tau)]$$

where c is a small coupling constant and m is the monopole moment of detector. Suppose the field ϕ is in the vacuum state $|0_M\rangle$. For a general trajectory detector will undergo a transition $E > E_0$, also the field will make transition to a arbitrary state $|\psi\rangle$. For small c amplitude for this transition may be given by first order perturbation theory as

$$ic \langle E, \psi| \int_{-\infty}^{\infty} m(\tau)\phi[x(\tau)]d\tau \, |0_M, E_0\rangle$$

Using time evolution of $m(\tau)$ and $H_0 |E\rangle = E |E\rangle$, the transition amplitude factorize to give

$$\iota c \langle E| m(0) |E_0\rangle \int_{-\infty}^{\infty} e^{i(E-E_0)\tau} \langle \psi| \phi(x) |0_M\rangle \, d\tau$$

We calculate the transition probability to all possible E and ψ, obtained by the squaring the modulus of **Eq. (8)**, and summing over E and the complete set ψ

$$c^2 \sum_E | \langle E| m(0) |E_0\rangle |^2 \mathcal{F}(E - E_0)$$

where

$$\mathcal{F}(E) = \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau' e^{-iE(\tau-\tau')} G^+(x(\tau), x(\tau')) \tag{1.4}$$

The function $\mathcal{F}(E)$, is known as the **Response function** of detector, which is independent of the internal details of the detector, and is determined by the positive frequency **Wightman Green function G^+**. It represents the bath of the 'particles' that the detector effectively experience as a result of its motion. The remaining factor represents the **selectivity** of the detector.

In Minkowski space the system is invariant under time translation in the detector's reference frame, so we can write

$$G^+(x(\tau), x(\tau')) = g(\Delta\tau)$$

$$\Delta\tau = \tau - \tau'$$

The transition probability per unit proper time in this case is modified as

$$c^2 \sum_E |\langle E| m(0) |E_0\rangle|^2 \int_{-\infty}^{\infty} d(\Delta\tau) e^{i(E-E_0)\Delta\tau} G^+(\Delta\tau) \tag{1.5}$$

1.2.2 Detector moving along a hyperbolic trajectory

Considering that the detector moves along a hyperbolic trajectory in the (t,z) plane:

$$z^2 = t^2 + \alpha^2$$

$$x = y = 0$$

where α is a constant. Detector accelerates uniformly with acceleration α^{-1} in the frame of detector. The detector's proper time τ is given as

$$t = \alpha \sinh\left(\frac{\tau}{\alpha}\right)$$

. Considering the field massless, the positive frequency Wightman function is evaluated as

$$D^+(x, x^{'}) = -\frac{1}{4\pi^2}\left[(t - t^{'} - \iota\epsilon)^2 - |\vec{x} - \vec{x}^{'}|^2\right] \tag{1.6}$$

For this trajectory we obtain:

$$D^{+}(\Delta\tau) = -(4\pi^2)^{-1} \sum_{k=-\infty}^{\infty} (\Delta\tau - 2\iota\epsilon + 2\pi\iota\alpha k)^{-2}$$

Substituting this into **Eq. (9)** and using the Fourier transform with the help of a contour integral yields the transition probability per unit time [2]

$$\frac{c^2}{2\pi} \sum_{E} \frac{(E - E_0)| \langle E| \, m(0) \, |E_0\rangle \, |^2}{e^{2\pi(E-E_0)\alpha} - 1}$$

In this expression we get a term like Planck factor **(Bose-Einstein's Distribution)** $\left[e^{2\pi(E-E_{\rangle}\alpha} - 1\right]^{-1}$, which indicates that the equilibrium between the accelerated detector and the ϕ field in the state $|0_M\rangle$ is the same as that would have been achieved when the detector remained unaccelerated, but immersed in a bath of thermal radiation at the temperature

$$T = \frac{1}{2\pi\alpha k_B} = \frac{acceleration}{2\pi k_B}$$

where k_B is Boltzmann's constant.

In other words we can say that the vacuum Green function for a uniformly accelerated detector is the same as the thermal Green function for an inertial detector.

1.2.3 Cosmological particle creation: an example

To see how particle creation can occur in a expanding (or contracting) spacetime with Minkowskian in and out regions, let us consider a simple example. Considering a two-dimensional Robertson-Walker universe with line element

$$ds^2 = dt^2 - a^2(t)dx^2$$

Introducing new time parameter η (conformal time) defined by $d\eta = \frac{dt}{a}$, the metric becomes

$$ds^2 = C(\eta)(d\eta^2 - dx^2)$$

where $C(\eta) = a^2(\eta)$ is the 'conformal scale factor'. The form of the metric here is conformal to Minkowski space.

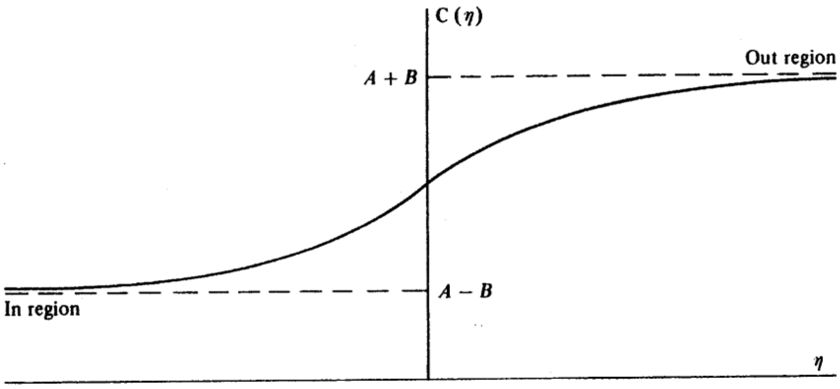

Figure 1.1: *The conformal scale factor C(η) represents an asymptotically static universe that undergoes a period of smooth expansion*

Taking $C(\eta) = A + B\tanh(\rho\eta)$, A, B, ρ are constants.
So that in the far past and future the spacetime looks like the Minkowskian since

$$C(\eta) \to A \pm B, \qquad\qquad \eta \to \pm\infty$$

This spacetime has the spatial translation symmetry, so separating the variable in scalar mode function as

$$u_k(\eta, x) = (2\pi)^{-\frac{1}{2}} e^{\iota kx} \chi_k(\eta)$$

Substituting $u_k(\eta, x)$ in scalar field equation with $\xi = 0$, we obtain an ordinary differential equation:

$$\frac{d^2}{d\eta^2}\chi_k(\eta) + (k^2 + C(\eta)m^2)\chi_k(\eta) = 0$$

This equation can be solved in terms of hypergeometric function. [2]

$$u_k{}^{in}(\eta, x) = (4\pi\omega_{in})^{-\frac{1}{2}} \exp\{\iota kx - \iota\omega_+\eta - (\iota\omega_-/\rho)\ln[2\cosh(\rho\eta)]\}$$
$$F_1(1 + (\iota\omega_-/\rho), \iota\omega_-/\rho; 1 - (\iota\omega_{in}/\rho); \frac{1}{2}(1 + \tanh(\rho\eta)))$$

$$u_k{}^{out}(\eta,x) = (4\pi\omega_{out})^{-\frac{1}{2}} \exp\{\iota kx - \iota\omega_+\eta - (\iota\omega_-/\rho)\ln\left[2\cosh(\rho\eta)\right]\}$$
$$F_1(1+(\iota\omega_-/\rho), \iota\omega_-/\rho; 1-(\iota\omega_{out}/\rho); \frac{1}{2}(1+\tanh(\rho\eta)))$$

The normalized modes which behaves like the positive frequency Minkowski space modes in remote past ($\eta \to -\infty$) and future ($\eta \to +\infty$)are

$$u_k{}^{in}(\eta,x) \to (4\pi\omega_{in})^{-\frac{1}{2}} e^{\iota\vec{k}.\vec{x}-\omega_{in}\eta}$$

$$u_k{}^{out}(\eta,x) \to (4\pi\omega_{out})^{-\frac{1}{2}} e^{\iota\vec{k}.\vec{x}-\omega_{out}\eta}$$

Where

$$\omega_{in} = \left[k^2 + m^2(A-B)\right]^{\frac{1}{2}}$$

$$\omega_{out} = \left[k^2 + m^2(A+B)\right]^{\frac{1}{2}}$$

$$\omega_\pm = \frac{1}{2}(\omega_{out} \pm \omega_{in})$$

Bogolubov coefficient is non-vanishing. We can write

$$u_k{}^{in}(\eta,x) = \alpha_k u_K{}^{out}(\eta,x) + \beta_k u_{-k}{}^{out\star}(\eta,x)$$

Where

$$|\alpha_k|^2 = \frac{\sinh^2(\pi\omega_+/\rho)}{\sinh(\pi\omega_{in}/\rho)\sinh(\pi\omega_{out}/\rho)}$$

$$|\beta_k|^2 = \frac{\sinh^2(\pi\omega_-/\rho)}{\sinh(\pi\omega_{in}/\rho)\sinh(\pi\omega_{out}/\rho)}$$

If the quantum field resides in the state $|0,in\rangle$, the unaccelerated particle detectors in the out region ($\eta \to +\infty$) will register the presence of quanta. We can therefore describe this quantum development as the creation of particles in the mode k as the consequence of the cosmic expansion.

Chapter 2

Effects of a non-trivial topology in quantum field theory : Cylindrical spacetime

In order to understand the behaviour of quantum fields in curved spacetime, we look for the effect of non-trivial topology in a locally flat spacetime. Considering the $R^1 * S^1$ two dimensional cylindrical spacetime with compactified spatial sections. In this spacetime the spatial points x and $x + L$ are identical, where L is the periodicity length (circumference of the universe) and the length element is given as

$$ds^2 = dt^2 - dx^2$$

The field modes in this case are given by the discrete set

$$U_k = (2L\omega)^{-\frac{1}{2}} e^{\iota(kx - \omega t)})$$

where

$$k = \frac{2\pi n}{L}, n = 0, \pm 1, \pm 2, \pm 3, ...$$

For the massless case, $\omega = |k|$,the components of stress - tensor in this two dimensional space are calculated as

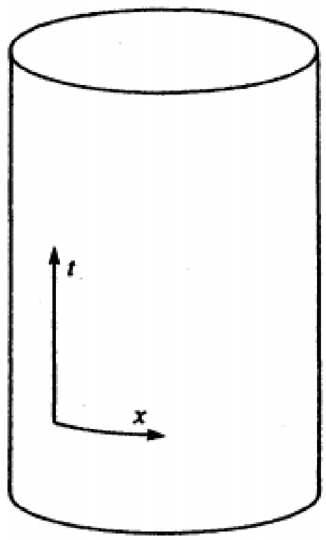

Fig. 10. Two-dimensional spacetime with compact spatial sections ($R^1 \times S^1$). The circumference of the cylinder is L.

[2]

$$T_{xx} \quad = T_{tt} = \tfrac{1}{2}(\partial_t\phi)^2 + \tfrac{1}{2}(\partial_X\phi)^2 \tag{2.1}$$

$$T_{tx} \qquad\quad = T_{xt} = \partial_t\phi\partial_x\phi \tag{2.2}$$

The vacuum associated with these discrete modes $|0_L\rangle$ is different from usual Minkowski space vacuum $|0\rangle$ with the property $|0_L\rangle \to |0\rangle$ as $L \to \infty$. The vacuum expectation value of the stress - tensor is calculated as

$$\langle 0_L| \, T_{tt} \, |0_L\rangle = \left(\frac{1}{2L}\right) \sum_{n=-\infty}^{\infty} |ke| = \left(\frac{2\pi}{L^2}\right) \sum_{n=0}^{\infty} n \tag{2.3}$$

which is clearly diversing. Here in the case of cylindrical space spatial section is compactified, applying normal ordering in this fock space we obtain

$$\langle 0_L| : T_{tt} : |0_L\rangle = \langle 0_L| \, T_{tt} \, |0_L\rangle - \lim_{L'\to\infty} \langle 0_{L'}| \, T_{tt} \, |0_{L'}\rangle = -\frac{\pi}{6L^2} \tag{2.4}$$

The cloud of negative vacuum energy is distributed uniformly throughout the $R^1 * S^1$ universe with total energy $-\frac{\pi}{6L}$.
Here we have used the boundary condition (i.e. $u_k(t,x) = u_k(t,x+L)$). If

we use antiperiodic boundary conditions $u_k(t,x) = (-1)^n u_k(t,x+L)$, the scalar field is referred to as a twisted field. The vacuum energy in case of twisted scalar field is calculated in a similar way [2]

$$\rho = \langle 0_L| : T_{tt} : |0_L\rangle = \frac{\pi}{12L^2}$$

For the twisted scalar field the value of vacuum energy is $-\frac{1}{2}$ of that for untwisted fields. Here in this case we have seen that spatially bounded space suffers the same ultraviolet divergence properties as one observe in usual Minkowski space.Now we want to observe the behaviour of these quantum fields in presence of the boundary.

Chapter 3

Boundary effects in field theory in curved space time : Casimir effect

The Casimir effect, discovered more than 60 years ago by Casimir (1948), is one of the most direct manifestations of the existence of zero- point vacuum oscillations. Casimir effect in a essence is the polarization of the vacuum by boundary conditions or geometry, which alter the zero-point modes of the field. The classical electromagnetic field get modified in the presence of material boundaries. In any space, even it is unbounded, the quantum field modes may modified in presence of conducting surfaces. To observe these possibilities we look into a simple example of an infinite plane ($x_3 = 0$) in ubounded four dimensional Minkowski space. The field modes for a massless scalar field vanish at the plane' surface and is given by

$$\sin |k_3| x_3 e^{\iota(k_1 x_1 + k_2 x_2 - \omega t)}$$

The vacuum state will be different in this case from that of a simple Minkowski space without boundary. The Green function can be calculated by using the method of images[2]

$$D_B^{(1)}(x, x') = \frac{1}{2\pi^2}\Big(\frac{1}{(x_1 - x_1')^2 + (x_2 - x_2')^2 + (x_3 - x_3')^2 - (t - t')^2}$$

$$-\frac{1}{(x_1 - x_1')^2 + (x_2 - x_2')^2 + (x_3 + x_3')^2 - (t - t')^2} \Big)$$

The first part of this expression corresponds to the unbounded Minkowski space, which diverges when $x \to x'$. The second term corresponds to the Minkowski space with boundary. To analize the boundary effects we calculate the vacuum expectation value of Stress - Energy- Momentum tensor in latter case

$$\langle 0 | T_{tt} | 0 \rangle_B = -\frac{1}{16\pi^2 x^4}$$

similarly

$$\langle 0 | T_{ii} | 0 \rangle_B = \frac{1}{16\pi^2 x^4}$$

We can see that the vacuum energy diverges as one approaches to surface $x_3 \to 0$, even though we have already subtracted the infinite vacuum energy of unbounded Minkowski space. Hence we observe that these boundary surfaces alter the topology of field configuration.

We can generalize the problem, to the case where more than one boundaries are present, easily for plane boundaries and conformally invariant fields. **Casmire (1948)** considered the vacuum energy associated with the electromagnetic field between two parallel reflecting planes. Conservation and tracelessness of stress tensor require that

$$\langle T^{\mu\nu} \rangle = A(\frac{1}{4}\eta^{\mu\nu} + \hat{x}_3^\mu \hat{x}_3^\nu)$$

the two parallel reflecting planes are orthogonal to X_3, and A is a constant

$$A = -\frac{\pi^2}{180a^4}$$

Here a is the separation between the two parallel plates.

Present Work :

3.1 Casimir effect for a spherical shell in De Sitter spacetime

The Casimir force depends on the nature of the quantum field, the specific boundary conditions imposed on the field, the type of spacetime manifold and its dimensionality. Here we analyse the Casimir effect for a spherical shell in De Sitter background for a scalar field satisfying Dirichlet boundary condition. Using the lagrangian density (1.1) we obtain the scalar field equation

$$\left[\Box_x + m^2 + \xi R(x)\right] \phi(x) = 0 \tag{3.1}$$

De Sitter space is a maximally symmetric Lorentzian manifold, represented as the hyperboloid

$${z_0}^2 - {z_1}^2 - {z_2}^2 - {z_3}^2 - {z_4}^2 = -\alpha^2$$

embedded in (n+1) dimensional Minkowski space with metric

$$ds^2 = d{z_0}^2 - d{z_1}^2 - d{z_2}^2 - d{z_3}^2 - d{z_4}^2$$

Using the coordinates (t,x) described as[2]

$$z_0 = \alpha \sinh(t/\alpha) + \frac{1}{2}\alpha^{-1} e^{t/\alpha} |x|^2$$

$$z_4 = \alpha \cosh(t/\alpha) - \frac{1}{2}\alpha^{-1} e^{t/\alpha} |x|^2$$

$$z_i = e^{t/\alpha} x_i, \qquad i = 1, 2, 3, \qquad -\infty < t, x < \infty$$

it covers the half of the de Sitter manifold with $z_0 + z_4 > 0$, the line element becomes

$$ds^2 = dt^2 - e^{2t/\alpha} \sum_i (dx^i)^2$$

In terms of conformal time parameter $\eta = -\alpha e^{-t/\alpha}$ the line element becomes

$$ds^2 = \frac{\alpha^2}{\eta^2}[d\eta^2 - \sum_i (dx^i)^2], \qquad -\infty < \eta < 0 \tag{3.2}$$

which is the same as Robertson-Walker sapcetime with $C(\eta) = \frac{\alpha}{\eta}$. This metric is conformal to the metric for Minkowski space. Also the metric is symmetric under $\eta \to -\eta$ so we can take range $0 < \eta < \infty$ instead of $-\infty < \eta < 0$ for studying cosmology forward in time. The mode decomposition for the field ϕ (3.1)

$$\phi(x) = \int d^{n-1}k[a_k \tilde{u_k}(x) + {a_k}^{\dagger} \tilde{u_k}\star(x)]$$

The modes can be expressed as[2]

$$\tilde{u_k}(x) = (2\phi)^{\frac{1-n}{2}} e^{\iota \mathbf{k.x}} C^{\frac{2-n}{4}(\eta)} \chi_k(\eta)$$

Where $a_k \left|\tilde{0}\right\rangle = 0$, $\left|\tilde{0}\right\rangle$ is the vacuum associated with the mode $\tilde{u_k}(x)$, known as conformal vaccum.

Substituting in the field equation, $\chi_k(\eta)$ satisies

$$\frac{d^2}{d\eta^2}\chi_k(\eta) + [k^2 + C(\eta)(m^2 + (\xi - \xi(\eta))R(\eta))]\chi_k(\eta) = 0 \tag{3.3}$$

Now, the metric (3.2) is conformal to flat space, if we take $\xi = \frac{n-2}{4(n-1)}$, then for massless field ($m = 0$)the field equation (3.1) is invariant under conformal transformation. Therefore considering the massless scalar field i.e.

$$\left[\Box_x + \frac{n-2}{4(n-1)}R(x)\right]\phi(x) = 0 \tag{3.4}$$

With Dirichlet boundary condition

$$\phi(x)|_{x=a} = 0$$

Where a is the radius of the spherical shell.

Now, using the properties of conformal symmetry $\tilde{g_{\mu\nu}} = \omega^2(x)g_{\mu\nu}$

$$\tilde{D^+}(x,x^{'}) = \omega^{(2-n)/2}(x)D^+(x,x^{'})\omega^{(2-n)/2}(x^{'}) \tag{3.5}$$

where $D^+(x,x^{'})$ is the Wightman function for the Minkowski space, which is given by equation (1.6).

Wightman function for the four dimensional ($n=4$) De Sitter vacuum is obained by using (1.6) and (3.2) in (3.5)

$$\tilde{D^+}(x,x^{'}) = \frac{\eta\eta^{'}}{\alpha^2}D^+(x,x^{'}) = \frac{-\eta\eta^{'}}{4\pi^2\alpha^2[(\eta-\eta^{'}-\iota\epsilon)^2-|x-x^{'}|^2]} \tag{3.6}$$

The Casimir stress (radial casimir force per unit area) on the spherical shell is obtained as

$$\frac{F}{A} = \langle 0|\,[T_{(in)r}{}^{r} - T_{(out)r}{}^{r}]\,|0\rangle\,|_{r=a} \tag{3.7}$$

The corresponding energy momentum tensor for two conformally related metrics $g_{\mu\nu}$ and $\tilde{g_{\mu\nu}}$ is given as [10]

$$\langle 0|\,T_{\mu}{}^{\nu}[\tilde{g_{\mu\nu}}]\,|0\rangle\,|_{ren} = (g/\tilde{g})^{1/2}\,\langle 0|\,T_{\mu}{}^{\nu}[g_{\mu\nu}]\,|0\rangle\,|_{ren} - \frac{1}{2880}[\frac{1}{6}{}^{(1)}\tilde{H}_{\mu}{}^{\nu} - {}^{(3)}\tilde{H}_{\mu}{}^{\nu}] \tag{3.8}$$

Where $\tilde{H}^{\nu}_{\mu}$ are some combination of curvature components, the second term denotes the vacuum polarization due to the gravitational field (due to time dependent curved background). The Casimir stress due to pure gravitational field (without using boundary condition) can be calculated by substituting [10]

$$\langle 0|\,T_{\mu}{}^{\nu}[g_{\mu\nu}]\,|0\rangle\,|_{ren} = 0$$

$${}^{(1)}\tilde{H}_{\mu}{}^{\nu} = 0$$

$$^{(1)}\tilde{H}_{\mu}{}^{\nu} = \frac{3}{\alpha^4}\delta_{\mu}{}^{\nu}$$

in (3.8), we have

$$\langle 0|\, T_{\mu}{}^{\nu}[\tilde{g}_{\mu\nu}]\, |0\rangle\, |_{ren} = \frac{1}{960\pi^2\alpha^4}\delta_{\mu}{}^{\nu}$$

Therefore the pressure on the spherical shell due to gravitaional field

$$-\,\langle 0|\, T_{r}{}^{r}[\tilde{g}_{\mu\nu}]\, |0\rangle\, |_{ren} = -\frac{1}{960\pi^2\alpha^4}$$

Using this equation in (3.7) the effective pressure on spherical shell due to pure gravitational vacuum polarisation

$$P_G = P_{in} - P_{out} = 0 \tag{3.9}$$

i.e. pressure is same on both side of shell and cancel each other.
Now we observe the Casimir stress on spherical shell due to Dirichlet boundary condition. The stress on spherical shell in flat space can be obtain by [16]

$$\frac{F}{A} = \langle 0|\, [T_{(in)r}{}^{r} - T_{(out)r}{}^{r}]\, |0\rangle\, |_{r=a} = -\frac{1}{4\pi a^2}\frac{\partial E}{\partial a} \tag{3.10}$$

where E is the total Casimir energy due to boundary condition. this relation can be generalized for De Sitter space using conformal properties (3.5) i.e.

$$\frac{\tilde{F}}{A} = -\frac{1}{4\pi a^2}\frac{\partial \tilde{E}}{\partial a} = \frac{\eta^2}{\alpha^2}\frac{F}{A} \tag{3.11}$$

In case of spherical shell Casimir energies are individually divergent for inside and outside of the shell. For a massless scalar field in Minkowski spacetime with Dirichlet boundary condition the Casimir energy is given as

$$E_{in} = \frac{1}{2a}[c_1 + c_1^{'}/\epsilon], \qquad E_{out} = \frac{1}{2a}[c_2 - c_1^{'}/\epsilon]$$

Where

$$c_1 = 0.008873, \qquad c_2 = -0.003234.$$

After renormalization and using (3.11) the total Casimir energy in De Sitter space due to boundary condition is obtained as

$$\tilde{E} = \frac{\eta^2}{2a}\left(\frac{c_1}{\alpha^2} + \frac{c_2}{\alpha^2}\right) \tag{3.12}$$

Substituting this equation in (3.11) the Casimir stress on spherical shell due to boundary condition is obtained

$$P_B = \frac{\tilde{F}}{A} = \frac{\eta^2}{8\pi a^4}\left(\frac{c_1}{\alpha^2} + \frac{c_2}{\alpha^2}\right) \tag{3.13}$$

Calculating the non-vanishing component of Ricci tensor for metric (3.2), we obtain the Ricci scalar for the Di Sitter space

$$R_{\mu\nu} = \frac{n-1}{\alpha^2} g_{\mu\nu}, \qquad R = 12/\alpha^2 \tag{3.14}$$

Where the adiabatic parameter α, which gives the measure of curvature, is related to cosmological constant.

$$\Lambda = 3/\alpha^2 \tag{3.15}$$

i.e. De Sitter space is a vacuum solution of Einstein's equation with cosmological constant given by (3.15). And the nature of Casimir stress (3.13) depends on the cosmological constant.

3.1.1 Spherical shell in De Sitter space with different background (vacua) outside and inside

To study the model for expansion of universe let us consider that vacua in the inside and outside of the spherical shell (bubble) are different. The different vaccua correspond to the different adiabatic parameter α_{in} and α_{out} for the metric (3.2). The total Casimir stress due to boundary condition and due to gravitational field for this case can be evaluated by substituting the different α's in the equations (3.9) and (3.93) and adding them

$$P = P_G + P_B = -\frac{1}{960\pi^2}\left(\frac{1}{{\alpha_{in}}^4} - \frac{1}{{\alpha_{out}}^4}\right) + \frac{\eta^2}{8\pi a^4}\left(\frac{c_1}{{\alpha_{in}}^2} + \frac{c_2}{{\alpha_{out}}^2}\right)$$

Using the relation (3.15), we have

$$P = P_G + P_B = -\frac{1}{8640\pi^2}({\Lambda_{in}}^2 - {\Lambda_{out}}^2) + \frac{\eta^2}{24\pi a^4}(c_1\Lambda_{in} + c_2\Lambda_{out}) \quad (3.16)$$

where P_G is the stress due to pure gravitational effect (i.e. due to non-trivial geometry of spacetime) and P_B is the stress on the shell due to boundary condition. This pressure is due to quantum effect only. Total stress on spherical cell may be positive or negative, to analyse this result, let us first assume (in 3.16)

$$c_1\Lambda_{in} + c_2\Lambda_{out} > o$$

Now a true vacuum is a global minimum of energy, and is commonly assumed to coincide with a physical vacuum state, if we consider the true vacuum outside and false vacuum inside i.e. $\Lambda_{in} > \Lambda_{out}$ then the first term will be always negative. So total pressure can be negative or positive. However, if we assume the true vacuum inside i.e. $\Lambda_{in} < \Lambda_{out}$, it is important for cosmological consideration. Initially, when $\eta \to 0$ the first term dominates, ($P_G > 0$), which leads to the expansion of universe (i.e. spherical bubble). At latter times, when $\eta >> 1$, the second term which is η dependent contributes in (3.16) and leads to further acceleration in the expansiom of universe. Thus in this case Casimir force is always repulsive. On the other hand if we assume

$$c_1\Lambda_{in} + c_2\Lambda_{out} < o$$

In this case inside will be the true vaccum i.e. $\Lambda_{in} < \Lambda_{out}$ and for $P < 0$ initially, the pressure will remain negative and bubble will collapse.

3.2 Non-zero response of a comoving detector to a conformal vacuum

The comoving worldline is defined as $\mathbf{x} = constant$ in Robertson-Walker spacetime. The response function of a comoving detector in De Sitter spacetime can be calculated by substituting (3.6) in (1.4) with $\Delta x = 0$ and performing the contour integral in complex plane with $E > 0$,

$$\frac{\mathcal{F}(E)}{unit \quad area} = \frac{E}{2\pi} \frac{1}{e^{2\alpha\pi E} - 1} \tag{3.17}$$

which is equivalent to the thermal spectrum with temperature $T = 1/(2\pi\alpha k_B)$. Thus it seems that the comoving observer will detect the particle in conformal vacuum of de Sitter space. Also metric (3.2) is time dependent in curved background, so one may expect some kind of particle production in analogy with dynamical Casimir effect.

However the metric (3.2) is conformal to Minkowski space, therefore to analize the problem we calculate the Bogolubov coefficient for the *in* and *out* region modes, where the *in* and *out* region correspond to $\eta \to -\infty$ and $\eta \to T < 0$ respectively. The equation of motion for a massless scalar field in Minkowski space is

$$(\partial^2/\partial\eta^2 - \Delta^2)\phi(x, \eta) = 0$$

Which can be solved by separation of variables in polar coordinates

$$\phi(r, \theta, \phi, \eta) = A(r, \theta, \phi)W(\eta)$$

then

$$W(\eta) = e^{-\iota\omega\eta}$$

Now using the conformal property $\tilde{\phi} = \Omega^{(2-n)/2}\phi$, the corresponding Solution for massless scalar field equation in De Sitter space (3.4) is

$$\tilde{W(\eta)} = (\eta/\alpha)e^{-\iota\omega\eta} \tag{3.18}$$

Since particle creation is related to the time dependence of space (3.2), the time dependent part of solution will satisfy the Bogolubov transformation

$$\tilde{W}^{in}(\eta) = \alpha_n \tilde{W}^{out}(\eta) + \beta_n \tilde{W}^{*out}(\eta) \tag{3.19}$$

The mode function (3.19) and its derivative should be continuous at $\eta = T$. using (3.18) and (3.19)

$$\eta e^{-\iota\omega\eta} = \alpha_n \eta e^{-\iota\omega\eta} + \beta_n \eta e^{\iota\omega\eta}|_{\eta=T}$$

$$-\iota\omega\eta e^{-\iota\omega\eta} = -\iota\omega\alpha_n \eta e^{-\iota\omega\eta} + \iota\omega\beta_n \eta e^{\iota\omega\eta}|_{\eta=T}$$

Comparing the both sides we obtain

$$\alpha_n = 1, \qquad \beta_n = 0 \tag{3.20}$$

The expectation value of number operator in conformal vacuum

$$\langle\tilde{0}|\, \hat{N}_{ni}\, |\tilde{0}\rangle = \sum_m |\beta_{mn}|^2$$

Therefore, there will be no particle creation and the conformal vacuum will remain so for all the time. Particle production takes place only when the conformal symmetry is broken by the presence of mass, which provides a length scale for the theory. If the cosmological expansion is allowed to cease (smoothly), then an inertial particle detector adiabatically switched on after the expansion has ceased will register no quanta [2].

3.3 Summary and Conclusion

In curved spacetime, we quantize the matter field in usual way but the quantum field theory looses its privilege of natural vacuum, particle number etc. As we can see in sec. [3.2] that a comoving observer, which is the most natural generalization of the inertial observer in De Sitter space, has nonzero response (i.e. it observes a thermal spectrum) to the conformal vacuum for de Sitter space, even there is no particle creation for metric (3.2). It is an illustration that in curved spacetime (with nonzero curvature) the concept of particle looses much of its intuitive meaning. The field quanta (particle) is a global property of the space and depends upon the mode solutions of the field. Two different observer, in general, will disagree on the number of particles in curved space which leads to the effect like particle creation in curved geometry. Unruh effect is one such prediction of the quantum field theory in curved spacetime.
In second part of the project We have studied the effect of boundaries and non-trivial geometry on the quantum fields, the field modes get modified in this case and we observe the stress on the bounding surface known as Casimir effect. It is the surprising manifestation of the zero-point energy which reveals the nature of vacuum : Vacuum is not empty at all.
However, the vacuum contains infinite ammount of energy and we have to use different kind of regularization and renormalization techniques to obtain a finite result. Further, in future we will use this quantum field theory in curved space to study the quantum blackholes in view of blackhole thermodynamics which will give us some insight into the subject quantum gravity.

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