ENTANGLEMENT PERCOLATION IN QUANTUM NETWORKS

M.Sc. Thesis

By

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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **"ENTANGLEMENT PERCOLATION IN QUANTUM NETWORKS"** in the partial fulfillment of the requirements for the award of the degree of **MASTER OF SCIENCE** and submitted in the **DISCIPLINE OF PHYSICS, Indian Institute of Technology Indore**, is an authentic record of my own work carried out during the time period from July 2019 to June 2020, under the supervision of Prof. Ujjwal Sen, Professor H, Harish Chandra Research Institute, Homi Bhabha National Institute, Allahabad and Dr. Sarika Jalan, Professor, Indian Institute of Technology, Indore.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other institute.

14th June 2020
Signature of the student with date
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This is to certify that the above statement made by the candidate is correct to the best of my/our knowledge.

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ABSTRACT and MOTIVATION

Quantum entanglement is a phenomenon in which two or more systems interact in such a way that the quantum state of the entire system cannot be expressed as a probabilistic mixture of pure product quantum states of the individual systems even when the systems are far away. The correlations in an entangled state are of a quite different nature and there is no classical analogy to them [24,26].

Entanglement is a very valuable resource for a variety of quantum information applications and has been applied to develop very useful protocols like teleportation [12], dense coding [11], quantum key distribution [9, 10] and entanglement swapping [4, 13]. Quantum Entanglement also has widespread applications in large-scale communication. It can be applied for effective long distance communication.

It is thus essential to understand the non-trivial task of creating and distributing entanglement between distant parties. The subject is greatly complicated, largely due to the fact that entanglement is a very fragile resource, in the sense that it inevitably deteriorates while being manipulated or being stored [24].

This report introduces the strategies for establishing long distance entanglement distribution in quantum networks, building upon from the basics of entanglement manipulation and then presenting the details of entanglement distribution through entanglement swapping and entanglement percolation. The last chapter of the thesis deals with the devising of another measurement strategy in a quantum network which may help entanglement percolation succeed with the usage of a lower amount of resources.

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Nomenclature

$\mid \psi\rangle$	A general pure quantum state
ρ	A general mixed quantum state
$\mid 0\rangle, \mid 1\rangle$	Two possible states of a qubit which can exist in their superposition
$\otimes$	Tensor Product
$\lvert\uparrow\rangle$	Same as $\mid 0\rangle$
$\lvert\downarrow\rangle$	Same as $\mid 1\rangle$
$\mid \psi\rangle\langle\phi \mid$	Outer product of ψ and ϕ
$\langle\psi \mid \phi\rangle$	Inner product of ψ and ϕ
$A^\dagger$	Hermitian Conjugate of A
δ	Kronecker Delta
$x \prec y$	x is majorized by y
$\{A, B, \ldots\}$	Set of $A, B \ldots$
Σ	Summation
Π	Product
$A^\star$	Complex Conjugate of A
A^T	Transpose of A

Acronyms

LOCC	Local Operations and Classical Communication
CEP	Classical Entanglement Percolation
QEP	Quantum Entanglement Percolation
Cat States	GHZ states of higher number of qubits

Chapter 1

Entanglement Manipulation

1.1 Introduction

Quantum theory, contains elements that are very different from the classical description of Nature. An important aspect in these fundamental differences is the existence of quantum correlations in the quantum formalism. There are correlations in classical systems and in statistical mechanics as well but the nature of quantum correlations is very different.

In the classical description of Nature, if a system is formed by different subsystems, complete knowledge of the whole system implies that the sum of the information of the subsystems makes up the complete information for the whole system [22, 31]. This is no longer true in the quantum mechanics. In quantum mechanics, there exist states of composite systems for which we might have the complete knowledge, while our knowledge about the subsystems might be completely random. In technical terms, one can have pure quantum states of a two-party system, whose local states are completely mixed, that is one's knowledge about the subsystems might be completely random (for maximally entangled states).

One of the most important theorems in all of physics, **Bell's Inequalities** [15, 16] mark the significant difference between classical correlations and the nature of quantum correlations. Indeed the certain no-go theorems in quantum information and quantum foundations reflect the peculiarity of quantum mechanics and in particular of all entangled states over all the domain of classical physics. For a subtle introduction to quantum mechanics in general and quantum foundations in particular refer to the engrossing text by Asher Peres [22] and to [24] for an understanding of the various applications of entanglement and entangled states.

During the last two decades, it has been realized that these fundamentally "entangled states", can provide us with something other than paradoxes. They may be used to perform tasks that cannot be achieved with the classical states. The study of these entangled states in the context of using them to process information and using them to perform tasks which are not possible with classical states forms the core of the field of Quantum Information and Computation [23].

1.2 Entanglement vs Separability

1.2.1 Pure States

By the postulates of Quantum Mechanics any *composite pure quantum system* $|\psi\rangle \in \mathscr{H}_{\mathscr{A}} \otimes \mathscr{H}_{\mathscr{B}}$ can be written as

$$|\psi\rangle = \sum_i a_i |A_i\rangle \otimes b_i |B_i\rangle \tag{1.1}$$

where $\{A_i, B_i\}$ is basis for the composite Hilbert Space $\mathscr{H}_{\mathscr{A}} \otimes \mathscr{H}_{\mathscr{B}}$ of the two qubits.

If further $|\psi\rangle \in \mathscr{H}_{\mathscr{A}} \otimes \mathscr{H}_{\mathscr{B}}$ can be written as,

$$|\psi\rangle = |A\rangle \otimes |B\rangle \tag{1.2}$$

where $|A\rangle \in \mathscr{H}_{\mathscr{A}}$ and $|B\rangle \in \mathscr{H}_{\mathscr{B}}$, i.e, if it can be written as a *tensor product* of the two individual states, then $|\psi\rangle$ is called a *separable state*, otherwise the composite state $|\psi\rangle$ is said to be an *entangled state*. Thus $|\psi\rangle$ is entangled if

$$|\psi\rangle \neq |A\rangle \otimes |B\rangle \tag{1.3}$$

Thus a composite state is a product state if the states of local subsystems are also pure states. Operationally, product states correspond to those states, that can be locally prepared by Alice and Bob at two separate locations. Entangled states can, however, be prepared only after the particles of Alice and Bob have interacted either directly or by means of an ancillary system.

This is because just the application of local unitary operations coupled with classical communication cannot create entanglement (though local unitary operations do not affect the amount of entanglement [30]). To create entanglement a global unitary operation needs to be performed on both the particles (eg: application of CNOT gate on the 2 particles) collectively to project them onto an entangled state.

An interaction between particles is also global in its effect and can be used to create a global unitary operation acting on both the particles collectively and thus creating entanglement between them. Naturally, in decay processes there are interactions and due to conservation laws (eg: angular momentum conservation or linear momentum conservation), the output particles might be in an entangled state.

The second option of creating entanglement is necessary due to the existence of the phenomenon of entanglement swapping (in which measurement in the basis of entangled states forms a non-unitary operation) which will be presented in the second chapter.

1.2.2 Mixed States

Similarly a bipartite mixed state ρ^{AB} is said to be *product* or *uncorrelated* if

$$\rho^{AB} \;=\; \rho^{A} \otimes \rho^{B} \tag{1.4}$$

where ρ^A and ρ^B are the mixed states for the two subsystems.
A mixed state which can be written as

$$\rho^{AB} \;=\; \sum_r p_r\, \rho_r^A \otimes \rho_r^B \;=\; \sum_r p_r \mid A_r\rangle\langle A_r \mid \;\otimes\; \mid B_r\rangle\langle A_r \mid \tag{1.5}$$

with $\sum_r p_r \;=\; 1$

(where $\mid A_r\rangle$ and $\mid B_r\rangle$ are state vectors on the spaces $\mathscr{H}_{\mathscr{A}}$ and $\mathscr{H}_{\mathscr{B}}$ of the subsystems A and B and $1 \geq p_r \geq 0$)

is said to *separable* and *correlated* but is not *entangled*. If ρ_{AB} cannot be written as tensor product as in Eq.(1.5) then it depicts a *mixed entangled state*. Note that a composite mixed state can be separable and *still can* be correlated as in Eq.(1.5), but this correlation does not imply that the composite mixed state is entangled. Since these are not pure states but mixed states so we need to go ahead and include classical correlations into the separable mixed states as well. Thus a composite mixed state is separable if it can be written as a *convex combination of pure product states*.

Also note that in Eq.(1.5), in general $\langle A_i \mid A_j \rangle \neq \delta_{ij}$ and $\langle B_i \mid B_j \rangle \neq \delta_{ij}$

Only the states which cannot be written in the form of Eq.(1.5) are said to entangled mixed states. Thus a mixed state is entangled if,

$$\rho^{AB} \neq \sum_r p_r \, \rho_r^A \otimes \rho_r^B \tag{1.6}$$

An example of separable correlated state containing classical correlation but no quantum correlations is

$$\rho = \frac{1}{2} \left(\; |\uparrow\uparrow\rangle\langle\uparrow\uparrow| \; + \; |\downarrow\downarrow\rangle\langle\downarrow\downarrow| \; \right) \tag{1.7}$$

It is worth mentioning that to find whether a mixed state is entangled or not is a very difficult problem in Quantum Information and there is no general protocol to establish whether a given mixed state is entangled or not.

1.2.3 Partial Trace

Suppose the joint bipartite state of two subsystems, A and B is denoted as ρ_{AB}, then the density matrix for subsystems A and B is given by (by tracing out the other system);

$$\begin{aligned} \rho_A &= tr_B(\rho_{AB}) \\ \rho_B &= tr_A(\rho_{AB}) \end{aligned} \tag{1.8}$$

Now consider that the joint state of the 2 qubits, i.e. the joint system AB is a pure state.

Now if ρ_A (or ρ_B) represents a mixed state, i.e if $tr(\rho_A^2) < 1$ (or $\rho_B^2 < 1$), then the bipartite state ρ_{AB} is an entangled state. If, however ρ_A (or ρ_B) represents a pure state, i.e if $tr(\rho_A^2) = 1$ (or

$\rho_B^2 = 1$), then the bipartite state ρ_{AB} is a separable state.

1.2.4 Partial Transposition

Let ρ_{AB} be a bipartite density matrix, and let us express it as

$$\rho_{AB} = \sum_{\substack{1\le i,j\le d_A \\ 1\le \mu,\nu\le d_B}} a_{ij}^{\mu\nu} (|i\rangle\langle j|)_A \otimes (|\mu\rangle\langle\nu|)_B, \tag{1.9}$$

where ($|i\rangle$) ($|\mu\rangle$) is a set of real orthonormal vectors in ($\mathscr{H}_A$) (($\mathscr{H}_B$)), with $d_A = \dim\mathscr{H}_A$ and $d_B = \dim\mathscr{H}_B$. The partial transposition, ($\rho_{AB}^{T_A}$), of (ρ_{AB}) with respect to subsystem A, is defined as

$$\rho_{AB}^{T_A} = \sum_{\substack{1\le i,j\le d_A \\ 1\le \mu,\nu\le d_B}} a_{ij}^{\mu\nu} (|j\rangle\langle i|)_A \otimes (|\mu\rangle\langle\nu|)_B. \tag{1.10}$$

(where the Roman subscripts i, j are for Alice's subsystem and the Greek subscripts μ,ν are for Bob's subsystem)

A similar definition exists for the partial transposition of (ρ_{AB}) with respect to Bob's subsystem. Notice that $\rho_{AB}^{T_B} = (\rho_{AB}^{T_A})^T$. Although the partial transposition depends upon the choice of the basis in which (ρ_{AB}) is written, its eigenvalues are basis independent (since eigenvalues of an operator are independent of the basis which we choose to represent the operator as a matrix). We say that a state has positive partial transposition (PPT), whenever ($\rho_{AB}^{T_A} \ge 0$), i.e. the eigenvalues of ($\rho_{AB}^{T_A}$) are non-negative. Otherwise, the state is said to be non-positive under partial transposition (NPT) [31].

If a state ρ_{AB} is separable, then $\rho_{AB}^{T_A} \ge 0$ and $\rho_{AB}^{T_B} = \left(\rho_{AB}^{T_A}\right)^T \ge 0$.

Proof:

Since ρ_{AB} is separable, it can be written as

$$\rho_{AB} = \textstyle\sum_{i=1}^{K} p_i |e_i\rangle\langle e_i| \otimes |f_i\rangle\langle f_i| \ge 0. \tag{1.11}$$

Now performing the partial transposition w.r.t. A, we have

$$\begin{aligned} \rho_{AB}^{T_A} &= \sum_{i=1}^{K} p_i \left(|e_i\rangle\langle e_i|\right)^{T_A} \otimes |f_i\rangle\langle f_i| \\ &= \sum_{i=1}^{K} p_i |e_i^*\rangle\langle e_i^*| \otimes |f_i\rangle\langle f_i| \ge 0. \end{aligned} \tag{1.12}$$

Note that in the second line, we have used the fact that $A^\dagger = (A^*)^T$.

The *partial transposition criterion*, for detecting entanglement is simple: Given a bipartite state ρ_{AB}, find the eigenvalues of any of its partial transpositions. A negative eigenvalue immediately implies that the state is entangled. Examples of states for which the partial transposition has negative eigenvalues include the singlet state.

1.2.5 Local Quantum Operations and Classical Communication

In LOCC protocols [31], two parties Alice and Bob can perform local quantum operations separately in their respective Hilbert spaces, i.e on their separate particles and they are also allowed to communicate classical information about the results of their local operations.

Suppose Alice and Bob share a quantum state ρ_{AB} defined on the Hilbert space $\mathscr{H}_A \otimes \mathscr{H}_B$. Alice performs a quantum operation on her local Hilbert space $\mathscr{H}_A$, using a complete set of complete general quantum operations $\{A_i^{(1)}\}$, satisfying $\sum_i (A_i^{(1)})^\dagger A_i^{(1)} = I_A$, and sends her measurement result i to Bob via a classical channel. Depending on the measurement result of Alice, Bob operates a complete set of general quantum operations $\{B_{ij}^{(1)}\}$, satisfying $\sum_j (B_{ij}^{(1)})^\dagger B_{ij}^{(1)} = I_B$ on his part belonging to the Hilbert space $\mathscr{H}_B$. This joint operation along with the classical communication is called one-way LOCC.

Furthermore, Bob can send his result j to Alice, and she can choose another set of local operations $\{A_{ijk}^{(2)}\}$, satisfying $\sum_k (A_{ijk}^{(2)})^\dagger A_{ijk}^{(2)} = I_A$, according to Bob's outcome. They can continue this process as long as required, and the entire operation is termed as LOCC, or two-way LOCC. The operators I_A and I_B are the identity operators on $\mathscr{H}_A$ and $\mathscr{H}_B$ respectively.

Entangled states cannot be prepared by two parties if only LOCC is allowed between them. To prepare such states, the physical systems must be brought together to interact. Because of the phenomenon of entanglement swapping (presented in chapter 2) one must suitably enlarge the notion of preparation of entangled states.

So, an entangled state between two particles can be prepared if and only if, either the two particles (call them A and B) themselves came together to interact at a time in the past, or two particles (call them C and D) do the same, with C (D) having interacted beforehand with A (B).

1.3 Schmidt Decomposition

If $|\psi\rangle$ is a pure state of composite system AB, there exist orthonormal bases $\{|i_A\rangle\}$ and $\{|i_B\rangle\}$ for system A and B respectively, such that,

$$|\psi\rangle = \sum_i \alpha_i |i_A\rangle |i_B\rangle \tag{1.13}$$

where α_i are real non-negative numbers satisfying $\sum_i \alpha^2 = 1$, known as *Schmidt Coefficients* [5]

The number of Schmidt Coefficients counted in multiplicity form the Schmidt Rank or Schmidt Number of a state. Any bi-partite pure state having Schmidt Number greater than 1 is entangled. This is because then the number of non zero (and equal) eigenvalues of the reduced density matrices of the subsystems will be greater than 1 and thus the reduced density matrices of the subsystems will represent mixed states. If a state has Schmidt Number equal to one then the state is separable, because then the reduced density matrix for the subsystems will represent pure states.

Notice that the squares of the Schmidt coefficients of a pure bipartite state $|\psi_{AB}\rangle$ are the non-zero eigenvalues of both the reduced density matrices $\rho_A = (\text{tr}_B |\psi_{AB}\rangle\langle\psi_{AB}|)$ and $\rho_B = (\text{tr}_A |\psi_{AB}\rangle\langle\psi_{AB}|)$ of $|\psi_{AB}\rangle$. This fact gives us an easy method to find the Schmidt coefficients and the Schmidt vectors.

Any bipartite pure state can be written in Schmidt Decomposition form but Schmidt Decomposition for more than 2 subsystems is generally not possible. The best way to obtain the Schmidt Decomposition form is to exploit the Singular Value Decomposition or calculate the eigenvalues and eigenvectors of the reduced density matrices of the subsystems and then directly obtain the Schmidt Decomposition form where the basis used in writing the Schmidt Decomposition form will be formed out of the eigenvectors of the reduced density matrices.

It is important to note that for pure states, if the combined system of the 2 particles is entangled only then will the Von Neumann entropy of both the reduced states be non-zero. If the combined system of the 2 particles is is separable then the Von Neumann entropy of both the reduced states will be zero.

Hence Schmidt Decomposition form not only helps us write the Von Neumann entropy of both the reduced states as $-\sum_i \alpha_i|^2 \log(|\alpha_i|^2)$ very easily but also lets us know if the bipartite state is entangled or not [5].

1.4 Majorization

Majorization is a large and active area of research in linear algebra [27]. Suppose $x = (x_1, ..., x_d)$ and $y = (y_1, ..., y_d)$ are real and d dimensional vectors. Then x is said to be majorized by y (equivalently y majorizes x), written as $x \prec y$, if for each k in the range $1, ..., d$,

$$\sum_{i=1}^{k} x_i \downarrow \leq \sum_{i=1}^{k} y_i \downarrow \quad \forall \quad k = 1, 2 \ldots n \tag{1.14}$$

with equality holding when $k = d$, and where the $\downarrow$ indicates that elements are to be taken in descending order, so for example, $x_1 \downarrow$ is the largest element in $(x_1, ..., x_d)$. The majorization relation is a partial order on real vectors, with $x \prec y$ and $y \prec x$ if and only if $x \downarrow = y \downarrow$.

1.5 Nielsen's Majorization Criterion

Nielsen showed [8] that any pure state $|\psi\rangle$ can be converted deterministically under LOCC into another pure state $|\psi\rangle$ under certain conditions. Consider the pure state $|\psi\rangle \in C^n \otimes C^n$.

$$|\psi\rangle = \sum_{i=0}^{n} \sqrt{\alpha_i} \, |i_A i_B\rangle \;\; ; \; where \quad \sum_{i=0}^{n} \alpha_i = 1 \;\; and \;\; \alpha_i \geq \alpha_{i+1} \geq 0 \tag{1.15}$$

It can be converted to another pure state $|\phi\rangle \in C^n \otimes C^n$,

$$|\phi\rangle = \sum_{i=0}^{n} \sqrt{\beta_i} \, |i_A i_B\rangle \quad where \quad \sum_{i=0}^{n} \beta_i = 1 \;\; and \;\; \beta_i \geq \beta_{i+1} \geq 0 \tag{1.16}$$

Denoting, the Schmidt Vectors as,

$$\lambda_\psi = (\alpha_1, \alpha_2 \ldots \alpha_n) \quad and \quad \lambda_\phi = (\beta_1, \beta_2 \ldots \beta_n) \tag{1.17}$$

Nielsen's criteria then says that $|\psi\rangle \rightarrow |\phi\rangle$, only if λ_ψ is majorized by λ_ϕ, denoted by $\lambda_\psi \prec \lambda_\phi$ and described as,

$$\sum_{i=1}^{k} \alpha_i \leq \sum_{i=1}^{k} \beta_i \quad \forall \quad k = 1, 2 \ldots n \tag{1.18}$$

As a specific example consider Alice and Bob each possessing a three dimensional quantum system, with respective orthonormal bases denoted as $|1\rangle$, $|2\rangle$, $|3\rangle$. Their joint states are $|\psi\rangle$ and $|\phi\rangle$, where,

$$\begin{aligned} |\psi\rangle &= \sqrt{\frac{1}{2}}\,|11\rangle + \sqrt{\frac{2}{5}}\,|22\rangle + \sqrt{\frac{1}{10}}\,|33\rangle \\ |\phi\rangle &= \sqrt{\frac{3}{5}}\,|11\rangle + \sqrt{\frac{1}{5}}\,|22\rangle + \sqrt{\frac{1}{5}}\,|33\rangle \end{aligned} \tag{1.19}$$

From, Nielsen's Majorization criteria it is evident that neither $|\phi\rangle \rightarrow |\psi\rangle$ is possible nor is $|\psi\rangle \rightarrow |\phi\rangle$ this possible.

1.6 Vidal's Monotones

1.6.1 Probability of Local Conversion between Bi-partite Pure States

Using the majorization conditions that Nielsen derived [8], Vidal showed [3] that any pure state $|\psi\rangle \in C^n \otimes C^n$, written in its Schmidt Decomposition form,

$$|\psi\rangle = \sum_{i=0}^{n} \sqrt{\alpha_i}\,|i_A i_B\rangle \quad where \quad \sum_{i=0}^{n} \alpha_i = 1 \quad and \quad \alpha_i \geq \alpha_{i+1} \geq 0 \tag{1.20}$$

(where α_i are the Schmidt Coefficients and $\{|i_A\rangle\}$ and $\{|i_B\rangle\}$ are local orthonormal basis for the 2 qubits respectively)

can be converted to another pure state $|\phi\rangle \in C^n \otimes C^n$,

$$|\phi\rangle = \sum_{i=0}^{n} \sqrt{\beta_i}\,|i_A i_B\rangle \quad where \quad \sum_{i=0}^{n} \beta_i = 1 \quad and \quad \beta_i \geq \beta_{i+1} \geq 0 \tag{1.21}$$

with a maximum probability given by,

$$P(\psi \to \phi) = \min_{l \in [1,n]} \frac{\sum_{i=l}^{n} \alpha_i}{\sum_{i=l}^{n} \beta_i} \tag{1.22}$$

For conversion of a partially entangled state with Schmidt Coefficients ϕ_0, ϕ_1 such that $\phi_0 > \phi_1$, to a maximally entangled Bell Pair the above formula yields a singlet conversion probability (SCP) = $2\phi_1$.

Eq.(1.22) shows an important fact that the optimal local conversion between any two states with non-identical Schmidt Coefficients is always an irreversible process. Here irreversible means that the parties can not, with certainty, convert locally one state into another and then get the initial state back. Also it shows that the non-local resources of entangled states are not additive in general.

1.7 Procrustean Method of Entanglement Concentration

1.7.1 Probability of Conversion to a Maximally Entangled Bi-partite Pure State

Consider that we have to obtain

$$\mid \psi_+\rangle \;=\; \frac{\mid 00\rangle_{AB} \;+\; \mid 11\rangle_{AB}}{\sqrt{2}} \quad from \quad \mid \psi\rangle_{AB} \;=\; \sqrt{\phi_0} \mid 00\rangle_{AB} \;+\; \sqrt{\phi_1} \mid 11\rangle_{AB} \tag{1.23}$$

Now we act upon one of the qubits of the pair with the operators

$$M_1 = \begin{bmatrix} \sqrt{\frac{\phi_1}{\phi_0}} & 0 \\ 0 & 1 \end{bmatrix} \qquad M_2 = \begin{bmatrix} \sqrt{1-\frac{\phi_1}{\phi_0}} & 0 \\ 0 & 0 \end{bmatrix} \tag{1.24}$$

The above operation results into either a maximally entangled Bell state or a separable state.

It is easy to see that the corresponding maximally entangled Bell state is obtained with a probability of $2\phi_1$.

1.7.2 Generalization to the Probability of Conversion between certain Multi-party States

1.7.2.1 Strategy - 1

The Procrustean Method is generalized to show that an ‘n’qubit partially entangled GHZ state can be converted to the corresponding ‘n’qubit maximally entangled GHZ state with essentially the same probability as the one we obtain for the conversion of a 2 qubit partially entangled state to a Bell state. We begin with the state,

$$\mid \psi\rangle_{AB\ldots n} \;=\; \cos(\theta)\mid 00\ldots 0\rangle_{AB\ldots n} \;+\; \sin(\theta)\mid 11\ldots 1\rangle_{AB\ldots n} \tag{1.25}$$

(where $0\leq \theta \leq \frac{\pi}{4}$ and $\cos(\theta) = \sqrt{\phi_0} \;\geq\; \sin(\theta) = \sqrt{\phi_1}$, where the equality holds at $\theta = \frac{\pi}{4}$)

and apply LOCC operations to convert it to the corresponding maximally entangled state

$$\mid \psi_+\rangle \;=\; \frac{\mid 00\ldots 0\rangle_{AB\ldots n} \;+\; \mid 11\ldots 1\rangle_{AB\ldots n}}{\sqrt{2}} \tag{1.26}$$

An auxiliary qubit A' in the state $\mid 0\rangle_{A'}$ is appended to the qubit “A” so that the total state becomes

$$\mid \psi\rangle_{AB\ldots n}\otimes \mid 0\rangle_{A'} \;=\; \cos(\theta)\mid 00\rangle_{AA'}\otimes \mid 00\ldots 0\rangle_{B\ldots n} \;+\; \sin(\theta)\mid 10\rangle_{AA'}\otimes \mid 11\ldots 1\rangle_{B\ldots n} \tag{1.27}$$

Next a Unitary operation, denoted as $U_{AA'}$ is applied, whose action is defined as follows

$$U_{AA'}\mid 10\rangle_{AA'} \;=\; \mid 10\rangle_{AA'} \tag{1.28}$$

$$U_{AA'}\mid 00\rangle_{AA'} \;=\; \tan(\theta)\mid 00\rangle_{AA'} \;+\; (1-\tan^2(\theta))^{\frac{1}{2}}\mid 01\rangle_{AA'} \tag{1.29}$$

so that the total state becomes,

$$\begin{aligned}(U_{AA'}\otimes I_B)(\mid \psi\rangle_{AB\ldots n}\otimes \mid 0\rangle_{A'}) \;=\; & [\sin(\theta)\mid 00\rangle_{AA'} \;+\; (1-2\sin^2(\theta))^{\frac{1}{2}}\mid 01\rangle_{AA'}]\otimes \mid 00\ldots 0\rangle_{B\ldots n} \\ & +\; \sin(\theta)\mid 10\rangle_{AA'}\otimes \mid 11\ldots n\rangle_{B\ldots n}\end{aligned} \tag{1.30}$$

$$= \sqrt{2}\sin(\theta)\mid 0\rangle_{A'}\otimes[\frac{\mid 00\ldots 0\rangle_{AB\ldots n} \;+\; \mid 11\ldots 1\rangle_{AB\ldots n}}{\sqrt{2}}] \;+\; (1-2\sin^2(\theta))^{\frac{1}{2}}\mid 1\rangle_{A'}\otimes \mid 10\ldots 0\rangle_{AB\ldots n} \tag{1.31}$$

Thus we see that the corresponding maximally entangled state is obtained with a probability of $2\sin^2(\theta) = 2\phi_1$

1.7.2.2 Strategy - 2

Another way to get the state

$$|\,\psi_+\rangle \;=\; \frac{|\,00\ldots0\rangle_{AB\ldots n} \;+\; |\,11\ldots1\rangle_{AB\ldots n}}{\sqrt{2}} \quad from \quad |\,\psi\rangle_{AB\ldots n} \;=\; \sqrt{\phi_0}\,|\,00\ldots0\rangle_{AB\ldots n} \;+\; \sqrt{\phi_1}\,|\,11\ldots1\rangle_{AB\ldots n} \tag{1.32}$$

is to act upon one of the qubits of the pair with the operators

$$M_1 = \begin{bmatrix} \sqrt{\frac{\phi_1}{\phi_0}} & 0 \\ 0 & 1 \end{bmatrix} \qquad M_2 = \begin{bmatrix} \sqrt{1-\frac{\phi_1}{\phi_0}} & 0 \\ 0 & 0 \end{bmatrix} \tag{1.33}$$

The above operation results into either a “maximally” entangled cat state (a generalizations of Bell states and GHZ states to higher number of particles) or a separable state.

It is easy to see that the corresponding maximally entangled state is again obtained with a probability of $2\phi_1$.

Chapter 2

Entanglement Distribution

2.1 Introduction

One of the major problems in Quantum Information is of distributing entangled states over large distances. Entanglement is a fragile resource and decoherence tends to make the problem of distributing entanglement a difficult one. But distributing entanglement over large distances can have vast uses ranging from secure communication through quantum key distribution [9,18] to dense coding [11] . Quantum Networks (Figure 1) are employed in solutions regarding this problem. Quantum Networks consist of nodes where each node can station many qubits. Different qubits at a particular node can be entangled with other qubits at different nodes. A class of operations called local operations and classical communication (LOCC) is employed to manipulate the structure and connectivity of the Quantum Network.

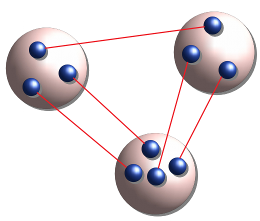


Figure 1: Different Nodes in a Quantum Network

An associated problem is that of establishing maximally entangled states between two far away places. Using maximally entangled states for different protocols like Teleportation [12], Quantum

Cryptography[10, 25] is extremely important, since maximally entangled states usually provide the highest quantum advantage over the corresponding classical ones. Also in quantum communication over large distances,due to the presence of noise in quantum communication channels, the quality of entangled states decreases exponentially, thus making the use of maximally entangled states necessary. It is thus evident that the usage of maximally entangled states is significant in the study of different protocols and hence several studies have been done addressing the problem of distributing maximally entangled states.

2.2 Quantum Teleportation

The protocol for quantum teleportation, introduced by Bennett, *et all* [12]. opened up new doors for exciting research in quantum information. Teleportation, presents itself as one of the most finest uses of quantum entanglement. Contrary to what the name "Teleportation" suggests, it doesn't violate any known physical law and particularly pays strict obeisance to the principles of the theory of relativity (Figure 2).

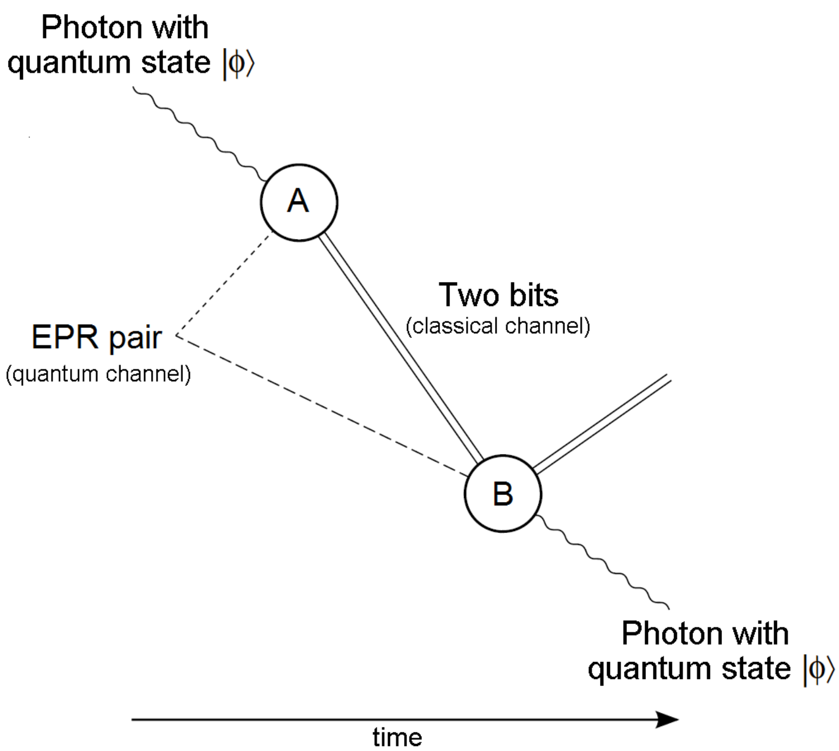


Figure 2: Space-Time approach to Quantum Teleportation

2.2.1 The Protocol

2.2.1.1 Algorithm

1) Take a pair of maximally entangled particles (Bell states), separate them by giving each to 1 party, Alice and Bob, respectively.
2) Alice wants to "Teleport" another quantum state to Bob over a distance. So Alice has effectively 2 states with her.
3) Alice makes a Von Neumann type Measurement in Bell basis on the 2 particles she possess.
4) This projects the particles she has got into one of the 4 Bell states and meanwhile destroys the entanglement which was shared between the maximally entangled (Bell pairs) states shared by Alice and Bob initially.
5) Now Bob's state changes depending on the measurement outcome of Alice's projective measurement in the Bell basis.
6) Alice sends classical information to Bob regarding the result she gets.
7) Using that information Bob does a unitary transformation on his state.
8) Bob then gets the exact state Alice wanted to send.

2.2.1.2 Formal Presentation

The protocol begins with Alice and Bob sharing one of the 4 Bell states i.e their collective state is either $\mid \Phi^{\pm}\rangle_{AB} = \frac{1}{\sqrt{2}}(\mid i_A i_B\rangle \pm \mid \bar{i_A}\bar{i_B}\rangle)$ or $\mid \Psi^{\pm}\rangle_{AB} = \frac{1}{\sqrt{2}}(\mid i_A \bar{i_B}\rangle \pm \mid \bar{i_A} i_B\rangle)$ where i is a binary variable and $i \in \{0,1\}$ and $\bar{i}$ is the respective complement. Thus it is understood that $\mid i\rangle$ and $\mid \bar{i}\rangle$ are the 2 orthogonal states of a 2 level quantum system. Further the subscripts used denote the owner of the respective qubits (A- Alice and B- Bob). Alice possess another pure state $\mid \psi\rangle_C = \alpha \mid 0\rangle_C + \beta \mid 1\rangle_C$ which she wants to "Teleport" to Bob.
To demonstrate teleportation let us assume that Alice and Bob share the state $\mid \Phi^{+}\rangle_{AB}$ With a bit of algebraic manipulation we can write the total 3 particle state as

$$\begin{aligned} \mid \psi\rangle_C \otimes \mid \Phi^{+}\rangle_{AB} &= \frac{1}{2}[\ \mid \Phi^{+}\rangle_{CA} \otimes (\alpha \mid 0\rangle_B + \beta \mid 1\rangle_B) \ + \ \mid \Phi^{-}\rangle_{CA} \otimes (\alpha \mid 0\rangle_B - \beta \mid 1\rangle_B) \\ &\quad + \ \mid \Psi^{+}\rangle_{CA} \otimes (\alpha \mid 1\rangle_B + \beta \mid 0\rangle_B) \ + \ \mid \Psi^{-}\rangle_{CA} \otimes (\alpha \mid 1\rangle_C - \beta \mid 0\rangle_C)\ ] \end{aligned} \tag{2.1}$$

Now, the actual teleportation begins with Alice measuring her qubits A and C in the Bell basis. It is evident form (1) that on doing so, the state of Bob's particle is one of the following

1) $|\psi\rangle_B = \alpha \mid 0\rangle_B \pm \beta \mid 1\rangle_B$
2) $|\psi\rangle_C = \alpha \mid 1\rangle_B \pm \beta \mid 0\rangle_B$

depending on what outcome Alice gets. So, Alice needs to send 2 bits of classical information through a classical channel (and hence the speed at which teleportation occurs will be quite less than the speed of light) to Bob telling him the outcome of her measurement. After receiving this information, Bob performs a unitary operation (depending upon the information he gets) out of I, Z, X, $\imath Y$ quantum gates respectively on his qubit to get back the original state $|\psi\rangle = \alpha \mid 0\rangle + \beta \mid 1\rangle$ and thus completing teleportation.

2.3 Entanglement Swapping Scheme of Zukowoski, *et all.*

The Entanglement Swapping Scheme of Zukowoski, *et all.* [13] is presented in this section. Following the convention from the above section, we can write down any Bell state (not normalized) of two particles in terms of a binary variable $i \in \{0,1\}$ and its complement i^c (defined as $1-u_i$),as

$$|\Psi(1,2)\rangle_{\pm} = |i_1,i_2\rangle \pm |i_1^c, j_2^c\rangle \ . \tag{2.2}$$

In the above it is again understood that $|i\rangle$ and $|i^c\rangle$ are two orthogonal states of a two state system. Consider the initial state of four particles 1,2,3 and 4 to be

$$\begin{aligned} |\Psi(1,2)\rangle_+ \otimes |\Psi(3,4)\rangle_+ &= |i_1,i_2,i_3,i_4\rangle \\ &+ |i_1^c,i_2^c,i_3,i_4\rangle \\ &+ |i_1,i_2,i_3^c,i_4^c\rangle \\ &+ |i_1^c,i_2^c,i_3^c,i_4^c\rangle \end{aligned} \tag{2.3}$$

That is, particles 1 and 2 are mutually entangled (in a Bell state), and particles 3 and 4 are mutually entangled (also in a Bell state). Measurement in the Bell basis is conducted on the particles 2 and 3 (which projects particles 2 and 3 onto a Bell state)and the joint state of the four particles becomes

$$|\Phi_1\rangle = (|i_2,i_3\rangle + |i_2^c,i_3^c\rangle) \otimes (|i_1,i_4\rangle + |i_1^c,i_4^c\rangle) \ , \tag{2.4}$$

$$|\Phi_2\rangle = (|i_2, i_3\rangle - |i_2^c, i_3^c\rangle) \otimes (|i_1, i_4\rangle - |i_1^c, i_4^c\rangle) \ , \tag{2.5}$$

$$|\Phi_3\rangle = (|i_2, i_3^c\rangle + |i_2^c, i_3\rangle) \otimes (|i_1, i_4^c\rangle + |i_1^c, i_4\rangle) \ , \tag{2.6}$$

$$|\Phi_4\rangle = (|i_2, i_3^c\rangle - |i_2^c, i_3\rangle) \otimes (|i_1, i_4^c\rangle - |i_1^c, i_4\rangle) \ . \tag{2.7}$$

To derive the above, only the orthogonality of $|i\rangle$ and $|i^c\rangle$ is required. Also, no matter what the outcome is, the particles 1 and 4 always end up in one of the four Bell states. Prior to the measurement, the Bell pairs were (1,2) and (3,4), after the measurement the Bell pairs are (2,3) and (1,4). Following the same procedure, it can be shown that the same fact would hold true even if (1,2) and (3,4) started in some other Bell states than those in Eq. (1.3). Figure 3, shows a pictorial description of entanglement swapping. A very important feature to note is that particles 1 and 4 which shared no common past are now entangled and the use of a global unitary has been avoided.

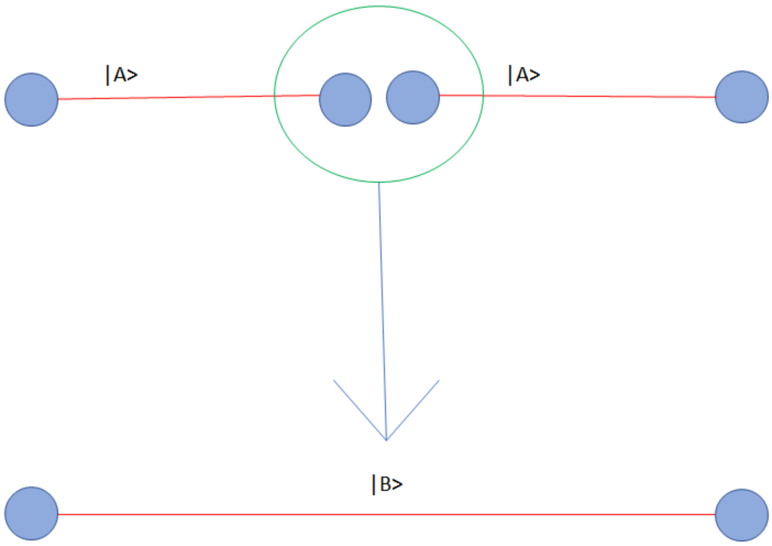


Figure 3: Entanglement Swapping

Thus we see that entanglement swapping scheme forms a very efficient way of entangling particles which shared no common past. It is also easy to see that entanglement swapping can be seen as a teleportation scheme where the particle which is to be teleported is itself entangled.

2.4 Generalized Entanglement Swapping Scheme of Bose, Vedral and Knight

Definition- Generalization of Bell states and GHZ states to a higher number of particles is referred to as a cat state.

The method of entanglement swapping described in the previous section can be generalized to cases where a greater number of particles are involved [4].
To explicitly see the swapping scheme, consider an initial state in which there are N different sets of entangled particles in cat states. Let each of these sets be labelled by m (where $m = 1, 2, .., N$), the ith particle of the mth set be labelled by $i(m)$ and the total number of particles in the mth set be n_m. Then the initial state can be represented by

$$|\Psi\rangle = \prod_{m=1}^{N} |\Psi\rangle_m, \tag{2.8}$$

in which each of the cat states $|\Psi\rangle_m$ is given by

$$|\Psi\rangle_m = \prod_{i=1}^{n_m} |u_{i(m)}\rangle \pm \prod_{i=1}^{n_m} |u^c_{i(m)}\rangle \tag{2.9}$$

Here I have reserved the symbol i to denote the number of particles and used the convention that $u_{i(m)}$ stand for binary variables $\in \{0, 1\}$ with $u^c_{i(m)} = 1 - u_{i(m)}$. Now imagine that the first p_m particles from all the entangled sets are brought together (i.e a total of $p = \sum_{m=1}^{N} p_m$ particles) and a joint measurement is performed on all of them. Note that the set of all cat states of p particles forms a complete orthonormal basis. Let the nature of the measurement on the selected particles be such that it projects them to this basis. Such a basis will be composed of states of the type

$$|\Psi(p)\rangle = \prod_{m=1}^{N} \prod_{i=1}^{p_m} |u_{i(m)}\rangle \pm \prod_{m=1}^{N} \prod_{i=1}^{p_m} |u^c_{i(m)}\rangle. \tag{2.10}$$

By simply operating with $|\Psi(p)\rangle\langle\Psi(p)|$ on $|\Psi\rangle$, we find that the rest of the particles (i.e those not being measured) are projected to states of the type

$$|\Psi(\sum_{m=1}^{N} n_m - p)\rangle = \prod_{m=1}^{N} \prod_{i=p_m+1}^{n_m} |u_{i(m)}\rangle \pm \prod_{m=1}^{N} \prod_{i=p_m+1}^{n_m} |u^c_{i(m)}\rangle, \tag{2.11}$$

which represents a cat state of the rest of the particles. In a schematic way the above process can be represented as

$$\prod_{m=1}^{N} |E(n_m)\rangle \rightarrow |E(p)\rangle \otimes |E(\sum_{m=1}^{N} n_m - p)\rangle \tag{2.12}$$

where $|E(n)\rangle$ denotes a n particle cat state. As a specific application of the above generalized entanglement swapping scheme consider Figure 4, which shows how generalized entanglement swapping can be applied to convert a single GHZ state and two 2 Bell states to a four party cat state and another GHZ state. It is thus simple to understand that the outcome can be obtained by connecting the particles being measured to form a polygon and also connecting the particles not being measured to form another complementary polygon. These polygons form the cat states that one gets after employing the generalized entanglement swapping scheme.

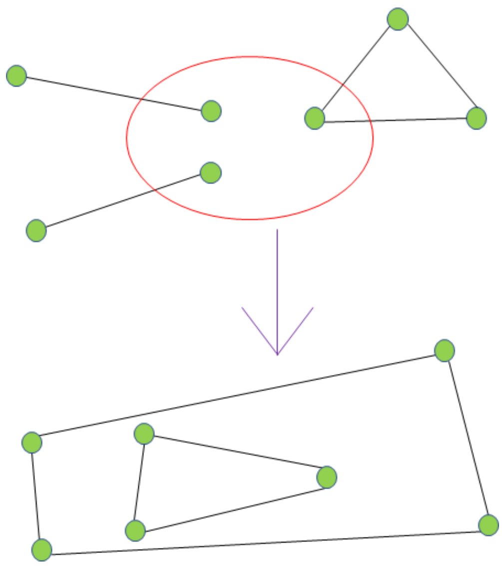


Figure 4: Generalized Entanglement Swapping

Figure 5 shows 2 pairs of qubits being brought together to perform entanglement swapping on qubits C and D.

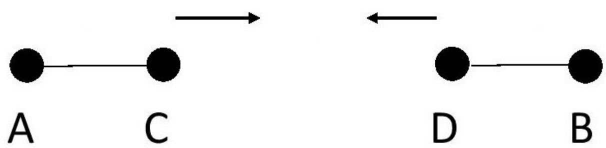


Figure 5: Two pairs of qubits being brought together to perform entanglement swapping on qubits C and D to create entanglement between qubits A and B

2.5 Application of Generalized Entanglement Swapping to a Communication Network

To begin with, each user initially needs to share entangled states with some central exchange. In Figure 6, users A, B, C and D share entangled qubit pairs (1,2), (3,4), (5,6), (7,8) with a central exchange O. Now suppose A, B, D wish to share a GHZ triplet between them then a measurement in the GHZ basis on the qubits (2,3,7) needs to performed at the central exchange to accomplish the task [4]. Establishing GHZ states in this way can be very efficient in other protocols like Quantum Entanglement Percolation (see Chapter 3). Also, utilizing such a measurement strategy turns out to very fruitful, as I show in Chapter 4, in transforming the geometry of a lattice, which can thus consequently help us in succeeding with Entanglement Percolation.

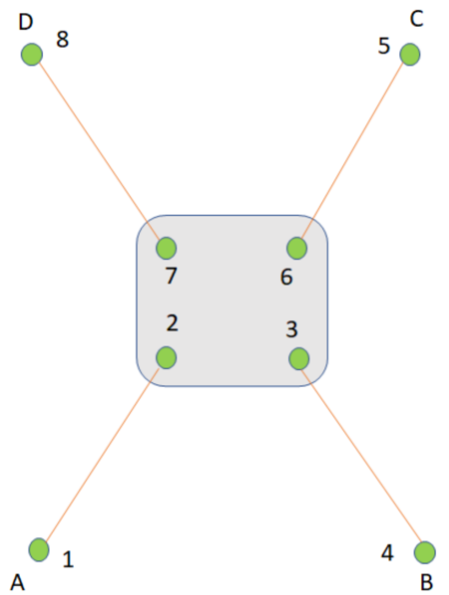


Figure 6: A Communication Network

2.6 Average Singlet Conversion Probability at One Repeater

Consider a one repeater configuration (see Figure 7). The probability to establish a singlet between nodes A and C by converting each of the 2 states $|\,\phi\rangle = \sqrt{\phi_0}\,|\,00\rangle + \sqrt{\phi_1}\,|\,11\rangle$ to singlets happens with a probability equal to $(2\phi_1^2)^2$. But if entanglement swapping is employed at the single repeater, then the average probability to establish a singlet between nodes A and B is seen to be just $2\phi_1$ [1]. This can be seen as follows:

After entanglement swapping at the one repeater the state between (which is a mixed state) nodes A and B is either of the 4 states:

1) $\frac{\phi_0|00\rangle \pm \phi_1|11\rangle}{\sqrt{\phi_0^2+\phi_1^2}}$, each occurring with probability $P = \frac{\phi_0^2+\phi_1^2}{2}$
2) $\frac{|01\rangle \pm |10\rangle}{\sqrt{2}}$ with corresponding probability for each state being $P = \phi_1\phi_2$

Thus the average probability to establish a singlet between nodes A and B is seen to be $P_{av} = (2\phi_0\phi_1 + 2\phi_1^2) = 2\phi_1$ (where Vidal's theorem has been used).

2.7 Concurrence

Concurrence is an entanglement monotone and is a non-negative function whose value does not increase under local operations and classical communication. Hence it serves as a measure of entanglement [24]. For a two-qubit pure state $\mid\phi\rangle = \sum_{i,j} t_{ij} \mid ij\rangle$, its concurrence is given by $C(\phi) = 2|\det(T)|$, where T is the 2×2 matrix such that $(T)_{ij} = t_{ij}$.

When considering the repeater configuration, the maximization of the averaged concurrence [1] turns out to be equal to

$$C_N = \sup_{\mathscr{M}} \sum_r 2|\det(\phi_1 M_{r_1} \phi_2 \ldots M_{r_N} \phi_{N+1})|. \tag{2.13}$$

Here $\mathscr{M}$ briefly denotes the choice of measurements, while ϕ_k represent the 2×2 diagonal matrices given by the Schmidt coefficients of the states $\mid\phi_k\rangle$. M_{r_k} are also 2×2 matrices, corresponding to the pure state $\mid\mu_k\rangle$ associated to the measurement result r_k of the k-th repeater, that is $\mid r_k\rangle = \sum_{i,j}(M_{r_k})_{ij} \mid ij\rangle$. Note that the computational basis i and j in the previous expressions are the Schmidt bases for the states $\mid\phi_k\rangle$ and $\mid\phi_{k+1}\rangle$ entering the repeater k. Using the fact that $\det(AB) = \det(A)\det(B)$, the previous maximization gives

$$C_N = \prod_{k=1}^{N} 2|\det(\phi_k)|. \tag{2.14}$$

Note that $|2\det(\varphi_k)| = 1$ if and only if $\mid\phi_k\rangle$ is maximally entangled. Hence Concurrence decreases exponentially if the initial states are not maximally entangled. Consequently the exponential decay of singlet conversion probability (for many repeaters) is apparent when the initial states are not maximally entangled and entanglement swapping at the repeaters is employed as a method to generate a singlet between distant nodes [1].

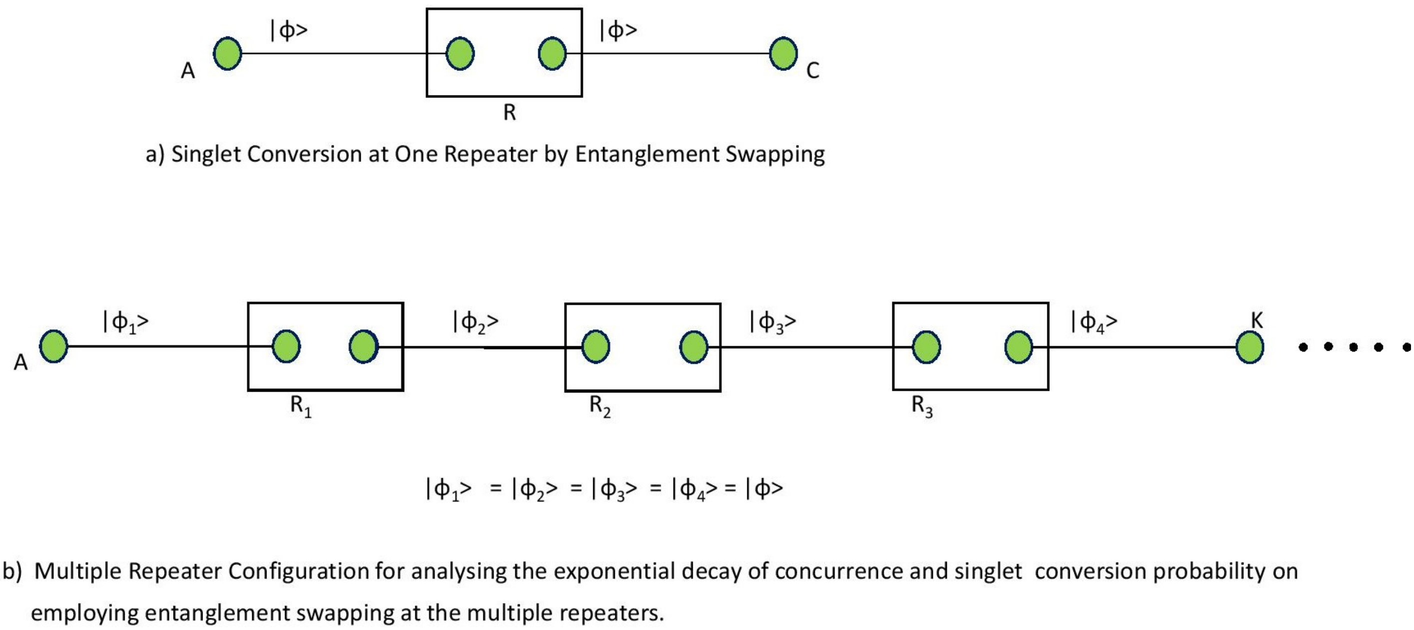


Figure 7:

a) Average Singlet Conversion Probability at one repeater

b) Exponential Decay of Concurrence at Multiple Repeater configuration

Chapter 3

Entanglement Percolation

3.1 Percolation

As a model for a disordered medium, percolation is one of the simplest, incorporating as it does a minimum of statistical dependence [17, 20]. Its attractions are manyfold. It is easy to formulate but not unrealistic in its qualitative predictions for random media [17, 20].

It has been claimed that percolation theory is a cornerstone of the theory of disordered media. As evidence to support this claim, I will briefly give a few examples, explaining how percolation occurs in them and emphasizing the role of percolation in them.

3.1.1 Site Percolation

The concept of site percolation can be best understood by using a simple example. Suppose that there is a wire mesh between 2 nodes (see Figure 8) which are connected to respective terminals of a battery. The problem under study is determination of the conductivity of the wire mesh. Suppose that some sites on the wire mesh are blocked by putting some rubber (insulator) on the sites (or cutting all the 4 wires meeting at a site).

It is seen that the current is still able to flow from one node to the other, but its magnitude has decreased (since by blocking certain sites, the resistance of the mesh increases or its conductivity decreases).

Now, some more sites are blocked but still it is found that the current can still 'search' its path between the 2 nodes though its magnitude has decreased significantly. In each step the site to be blocked is decided using a random number generator and associating the outcome of the random generator to a site in the wire mesh.

A quantity of significant importance is x which is the ratio of non-blocked sites to the total number of sites in the wire mesh. If we continue blocking the sites, a certain x_c is reached at which the current drops and the conductivity of the wire mesh drop to zero. This value of x_c is termed as the critical threshold of the given wire mesh [20].

If we repeat the experiment with a similar wire mesh, then we will get get a significantly different value of x_c. So the x_c we get is a random variable. If we perform the experiment Q number of times and calculate the mean value of x_c in these Q experiments, we will still get fluctuations in x_c. If the same experiment is repeated again for a Q number of times, we will get a different value for the mean value of x_c.

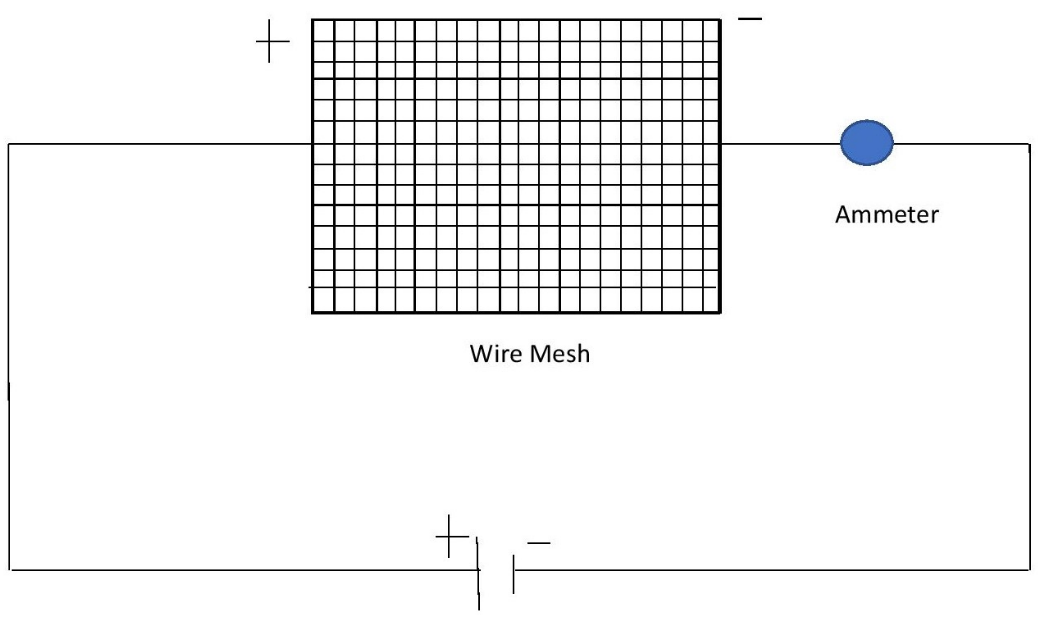


Figure 8: Site Percolation

However, the larger the number of experiments, Q, within the series, the smaller the differences between the averages representing different series. The point is that random fluctuations of x_c cancel each other in the long run, so that as Q increases, the arithmetic mean of x_c tends to a quite definite value independent; of Q but dependent on the conditions under which the experiments were run. This limiting value which is the mean value of x_c is called the percolation threshold for the wire mesh. This is an example of site percolation.

It is evident that such type of probabilistic problems can be modelled on to a graph or a lattice too with the same mathematics being carried but with just a little change in terminology or notations. For the knowledge of the rigorous mathematics of site percolation theory refer [17, 20].

3.1.2 Bond Percolation

A bond percolation problem can be easily mapped onto a graph. A bond or a connection between two nodes of a lattice exists with a probability, p. Now if this occupation probability,p is greater than the percolation threshold for the lattice, then percolation will succeed in the lattice else it will fail. The below example should make things a bit clearer.

To understand the problem of Bond percolation consider the following simple example. Consider a square lattice (whose threshold is known), the vertices of which denote people. Consider a person infected with Covid-19 on one vertex (marked A in Figure 9). Let the virulence of Covid-19 be independent of the number of infections it causes.

Then, there is a probability which depends on the distance between two successive vertices of the lattice $p(x)$ (where x is the distance between 2 successive vertices), that this infected person can infect his nearest neighbours and his nearest neighbours can then infect their nearest neighbours with essentially the same probability $p(x)$ (so we can use a single specific value of probability $p(x)$ for all lattice points and consequently the lattice will remain square, though its size may increase or decrease with increase or decrease in x and since the lattice remains square we can use the percolation threshold already known for it).

To prevent a single infected person from endangering a significant population of people, it is necessary to choose the lattice spacing to be large enough that $p(x)$ is smaller than the critical probability of bond percolation on a square lattice. Thus a minimum value for x can be obtained.

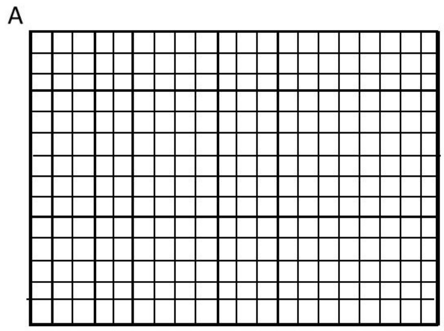


Figure 9: Bond Percolation

Now, consider a case where a single person has infected a very a large population of people. So it is known that $p(x)$ is atleast greater than the bond percolation threshold for the square lattice which is 0.5 . Similar analysis can be done for a square lattice (for a different value of x but still the value of x being such that $p(x) > 0.5$, i.e a single person still being able infecting a large number of people) composed of vertices denoting people in some other place (like say another city).

Doing this many times and gathering all the data, one can generate a good estimation of the probability distribution of spread of the disease from one person to another, i.e one can model how successfully the disease spreads depending upon the distance between two people.

Though this model is very crude in estimating the spread of Covid-19 since people will not be always at the same distance from each other, but still it can help atleast to estimate the maximum distance between two people till which the disease can spread from one to another.

This was just introduced to explain the widespread applications that Percolation theory can have and albeit one would depend on other resources to calculate the maximum distance between two people till which the disease can spread, because the collection of such a data would result particularly in a huge loss of life amongst other effects. But how wide can the applications of percolation theory be, is clearly evident through this example.

For a knowledge of percolation based models for forest fires and study of properties of ferromagnetic materials refer to [17] and [20] respectively. Even the formation of curd can be modelled using percolation.

Also the above problem can be mapped onto a site percolation problem. Suppose with probability $p'(x)$ (such that x is small enough that an infected person can infect his nearest neighbour with certainty i.e with probability = 1) people are present at the nodes or vertices of the square lattice (with probability $1 - p'(x)$ a node is devoid of a person).

Now if the the single infected person has to infect a very big population (infinite cluster), then $p'(x)$ should be greater than the percolation threshold (0.5) for the square lattice. If $p'(x)$ is less than the percolation threshold then the infection cannot be propagated to vast population (breaking the chain !). It is evident that this problem is easier to handle mathematically when mapped onto a site percolation problem.

It can also be see that the problem given in the example of site percolation can be mapped on to a bond percolation problem. Suppose instead of blocking the sites by putting insulator over there we decide to cut the wires between 2 sites. With probability p a connection between 2 sites exists (with probability ($1 - p$, it doesn't).

Now if p is greater than the percolation threshold for the wire mesh then current can find its way from one node to the other. If p is less than the percolation threshold current will not be able to find its way from one node to the other.

3.1.3 Infinite Cluster Formation

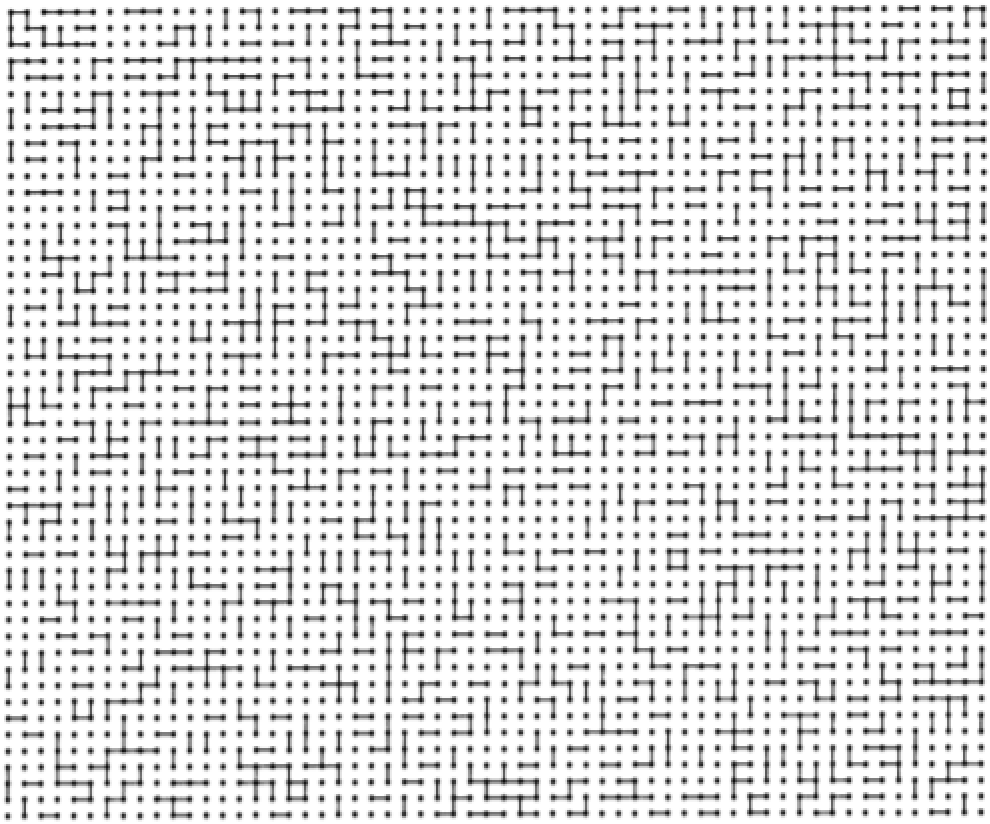
When the occupation probability is greater than the percolation threshold of the lattice, then an infinite cluster forms inside the lattice. All nodes are connected in this cluster. In other words, if the problem is mapped on to a graph then an infinite cluster just denotes a region of infinite connectivity [17, 20].

The figures below, which have been taken from [17], will show as to what an infinite cluster means and how small clusters grow in size and combine together to form an infinite cluster as the occupation probability increase and finally becomes greater than the percolation threshold of the lattice.

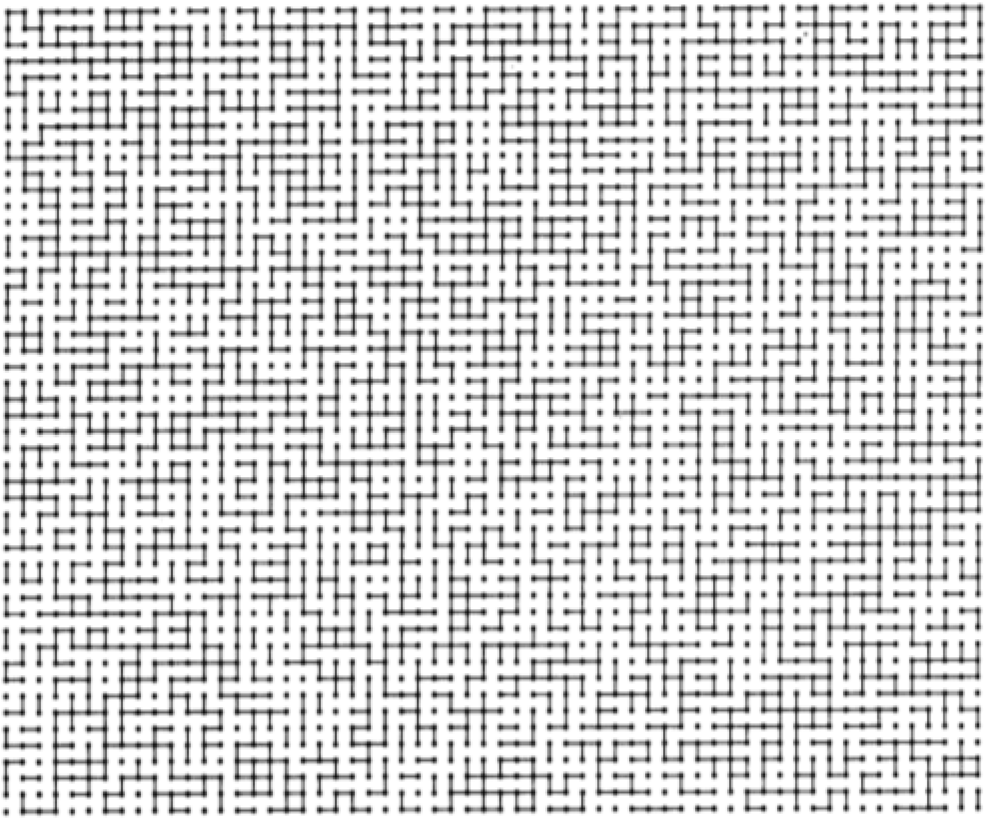
Percolation Threshold for several lattices is given below in Table 3.1 .

Lattice	Percolation Threshold Probability
Square	$\frac{1}{2}$
Triangular	$2\sin\left(\frac{\pi}{18}\right) \approx 0.3473$
Honeycomb	$1-2\sin\left(\frac{\pi}{18}\right) \approx 0.6527$
Bowtie	0.4045
Kagome	0.5244053 MC estimate

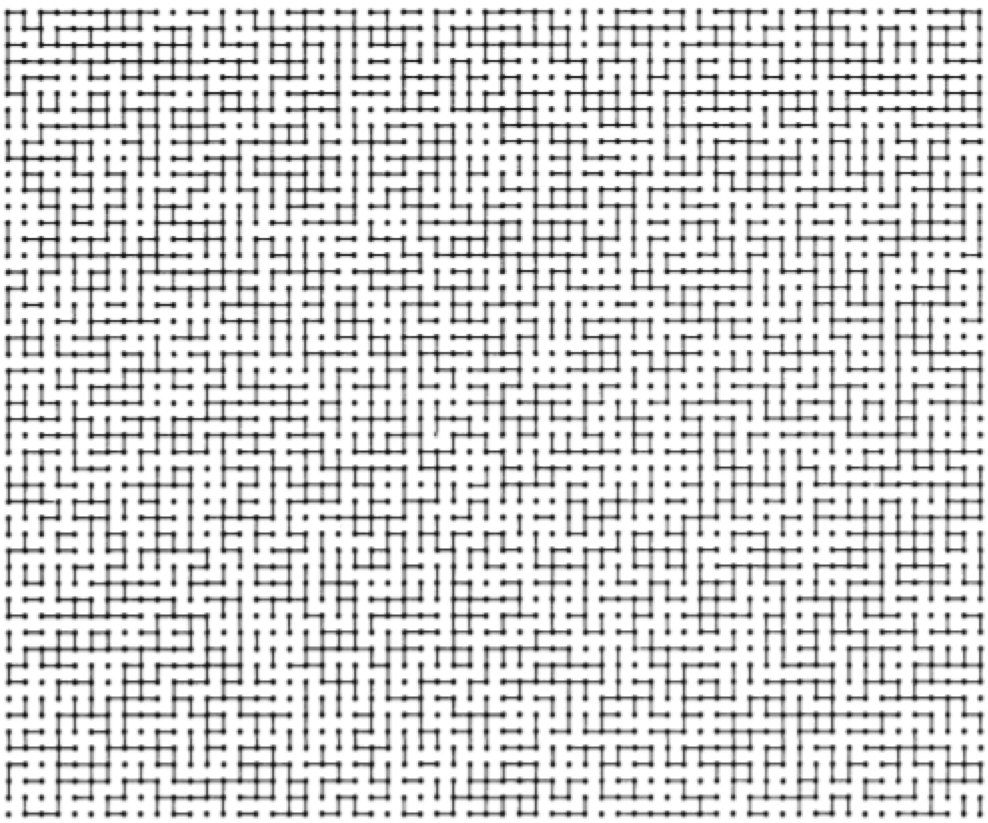
Table 3.1: Bond Percolation Threshold Probabilities for some examples of 2D lattices.



(a) $p = 0.25$



(b) $p = 0.49$



(c) $p = 0.51$

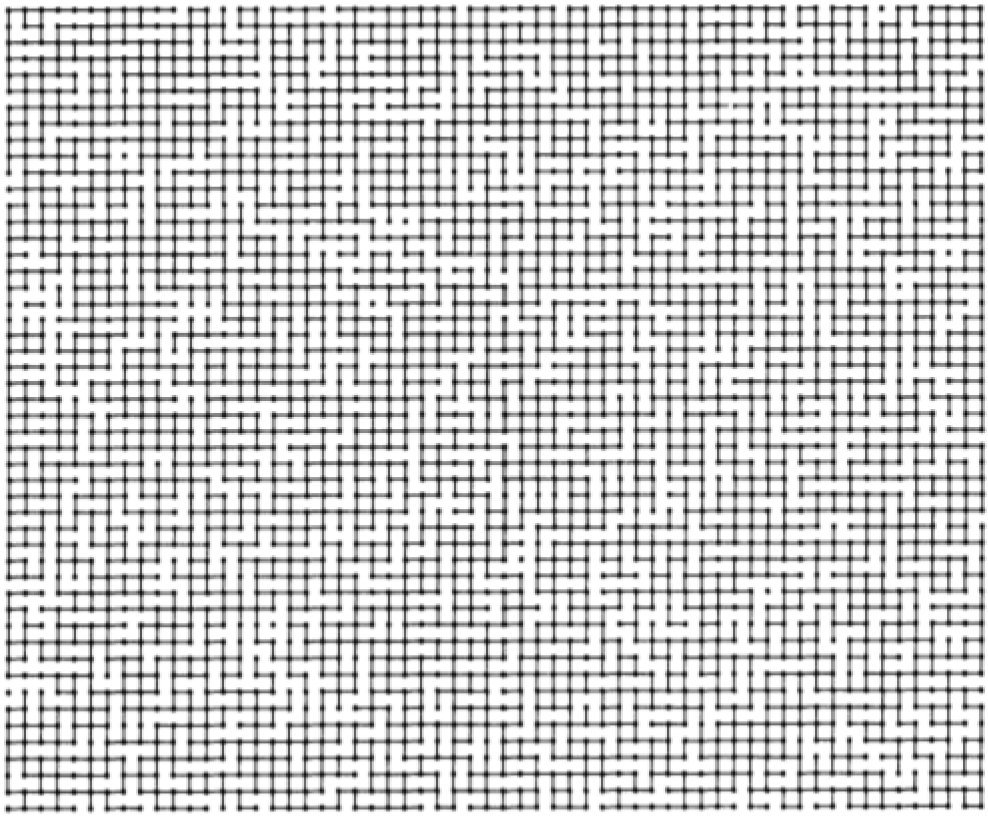

(d) $p = 0.75$

Figure 10: Infinite Cluster Formation

3.2 Classical Entanglement Percolation (CEP)

In [1] Acin *et al.* developed a protocol which they termed as "Classical Entanglement Percolation" (CEP) and used it to show how the concept of percolation can be drawn from statistical mechanics and can be applied in the context of quantum entanglement. This is to achieve the task of distributing entanglement to large distances and to establish a maximally entangled Bell state between two distant nodes of an asymptotically large lattice.

They further developed a protocol which they termed as "Quantum Entanglement Percolation" (QEP) and showed how QEP can succeed where CEP couldn't, thus, helping to accomplish the task of entanglement distribution. Later, QEP was also considered in [1, 6].

The protocol for CEP is quite simple. Firstly, any node of a lattice can contain any number of qubits and secondly, the qubits of two different nodes can be connected via partial entanglement. See

Figure 10, where the nodes within a quantum network, which can be imagined as a part of a lattice, are shown. The partially entangled state is given by $|\phi\rangle = \sqrt{\phi_0}|00\rangle + \sqrt{\phi_1}|11\rangle$, $\phi_0 + \phi_1 = 1$, where $\sqrt{\phi_0}$ and $\sqrt{\phi_1}$ are the Schmidt coefficients with $\phi_0 > \phi_1$. The geometry, created due to these partially entangled qubits at different nodes, forms the structure of the lattice.

The protocol begins by applying LOCC operations at the nodes to convert the partially entangled states to maximally entangled states. After this, some of the previous links are broken and the probability that an initial partially entangled state is converted to a maximally entangled one, is governed by the singlet conversion probability (SCP) of the initial states. It has been already showed in chapter 1, as to how one can calculate the singlet conversion probability for any arbitrary 2 qubit states using Vidal's results.

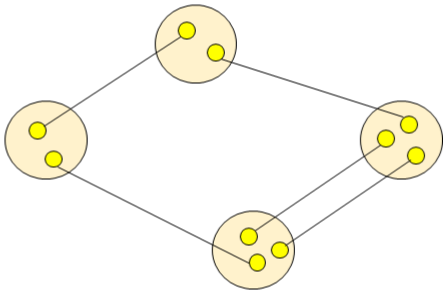


Figure 11: Nodes of a quantum network which can be imagined as a part of a lattice

Now for every lattice, there exists a percolation threshold which is the critical value of the occupation probability in the lattice, such that infinite connectivity (percolation) first occurs. In CEP if the SCP is greater than the percolation threshold for the given lattice then an infinite cluster forms in the lattice.

This infinite cluster consists of nodes which are all linked with maximally entangled states and thus, one finds many paths along which one can do entanglement swapping to create a maximally entangled state between two far away nodes of the given lattice (provided the two such nodes lie in the same cluster, the probability of which is $\theta(p)$, which is strictly greater than zero if SCP is greater than the percolation threshold of the lattice).

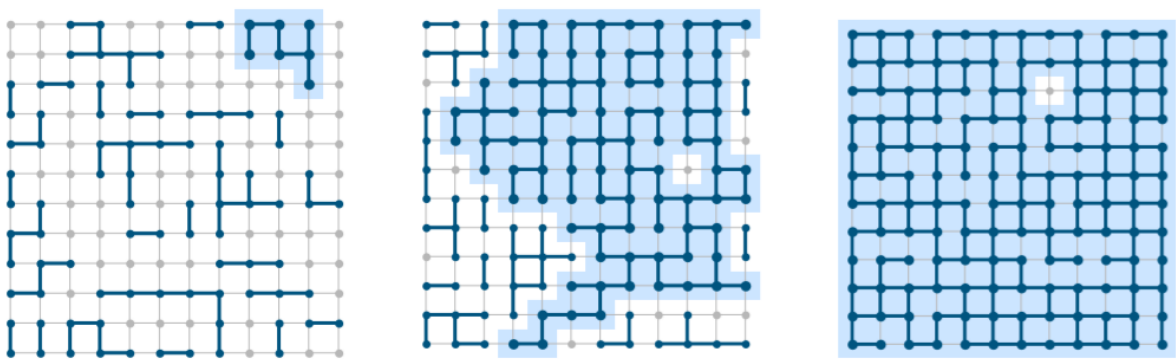
So, the task of creating a maximally entangled state between two end nodes of the lattice has been accomplished with a strictly non-vanishing probability, whereas the same would not have been possible without CEP, since it was shown in [1] that if the initial states were partially entangled then the probability of succeeding would have decayed exponentially by using entanglement swapping only.

See Figure 12, taken from [7], for the details of Classical Entanglement Percolation on a square lattice. The bonds in blue connecting the qubits (shown in blue) are the ones that could get successfully converted to a maximally entangled state and represent a maximally entangled state between the qubits which they link. Whereas, the the bonds shown in grey connecting the qubits (shown in grey) failed to get converted to a maximally entangled one and thus depict separable states (i.e there is no link between the qubits connected with grey bonds.

The percolation threshold for bond percolation is $p_c = 0.5$. Thus when SCP of the bonds is less than 0.5 as in the first case (where it is 0.25) no infinite cluster forms and entanglement percolation fails. Whereas in the last case (where SCP is 0.75), an infinite cluster (shown as a blue shade) appears and entanglement percolation succeeds.

It is also evident that more is the SCP greater than the percolation threshold for a given lattice, the larger is the probability of the success of entanglement percolation because the probability that the two nodes $\theta^2(p)$ between which one needs to establish a maximally entangled state lie in the same infinite cluster will be greater.

To understand the above fact, consider the second and the third case in Figure 11. In the second case since $SCP = 0.5 = p_c$, the infinite cluster just forms and hence $\theta^2(p)$, for nodes lying on the opposite sides of the square lattice is considerably less than $\theta^2(p)$ in the third case for the same pair of nodes. It is but obvious that if $SCP = 1$ (not shown in Figure 9) then the infinite cluster will span the whole lattice since all the states will be maximally entangled.



(a) $p = 0.25$ (b) $p = p_c = 0.5$ (c) $p = 0.75$

Figure 12: Classical Entanglement Percolation

3.3 Quantum Entanglement Percolation (QEP)

In QEP, the original lattice structure, using some particular quantum measurements is converted to some other lattice for which the percolation threshold is lesser, so that CEP which is applied next in the new lattice is successful. To show the effectiveness of their protocol, Acin *et al.* used a double layered honeycomb lattice in which percolation is not possible due to the the critical amount of entanglement (which is governed by the SCP here) being less than the percolation threshold.

Carrying out measurements in the Bell basis at the nodes, they converted their original lattice structure to a triangular lattice which has a lesser percolation threshold and thus meets the criteria of the critical amount of entanglement being greater than the percolation threshold for entanglement percolation to succeed in the new lattice. As evident this protocol of entanglement percolation uses the richness of the geometry of two-dimensional lattices.

Further, the particular lattice transformation used is one of the most important factors defining the success of QEP. But in the above case a double layered honeycomb lattice was used since using the measurement strategy described in [1], it is not possible to convert the honeycomb lattice to a triangular lattice (which has a lower percolation threshold). So a lot of quantum resource in terms of entanglement is put in for this purpose.

Refer to Figure 12 and Figure 13, to see as to how measurement strategies can transform a lattice structure where the figures are taken from [6].

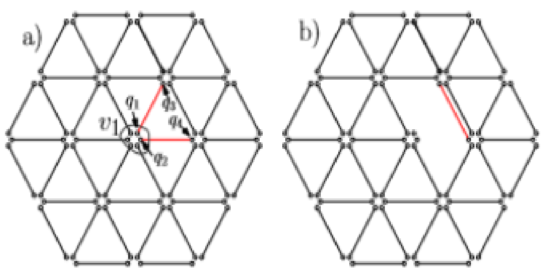


Figure 13: (Color online) a) Mapping a quantum network to a percolation problem. Qubits are represented by small circles. Each vertex (node) contains six qubits. b) using entanglement swapping to transform the lattice structure. The two red (grey) bonds in a) are replaced by the red bond in b). This process can be continued to produce a double-bond hexagonal lattice.

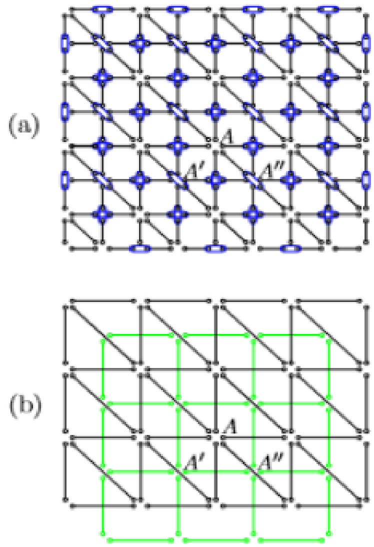


Figure 14: Transformation of the bowtie lattice to decoupled square and triangular lattices. Loops marking pairs of qubits represent swapping measurements.

It is worth mentioning here an important generalization. I have generalized the procrustean method (section 1.7) for arbitrary cat states. It is seen that even for arbitrary cat states, the probability of conversion of a "partially entangled" cat state of any number of qubits to the corresponding "maximally entangled" cat state can still be found using Vidal's formula.

This fact is quite important and would be used in Chapter 4, wherein it will play an important role in devising a measurement strategy for a single layered honeycomb lattice, that helps entanglement percolation succeed in it, with a better probability, while using much less quantum resource in term of entanglement than compared to any previous results.

Several examples of "Lattice Transformations", which effectively constitute the measurement strategies employed for the success of entanglement percolation are shown below. The pictures have been taken from [7].

In Figure 15, a measurement in the Bell basis is done on the qubits which are encircled in blue which changes the kagome lattice to a square lattice.

If the initial state is such, or the critical amount of entanglement ($2\phi_1$) is such that percolation is not possible in a kagome lattice, then this measurement strategy (and remembering the result from chapter 1, that a single measurement done in the Bell basis does not change the average singlet conversion probability), enables entanglement percolation to succeed in a square lattice (since it has a lower percolation threshold).

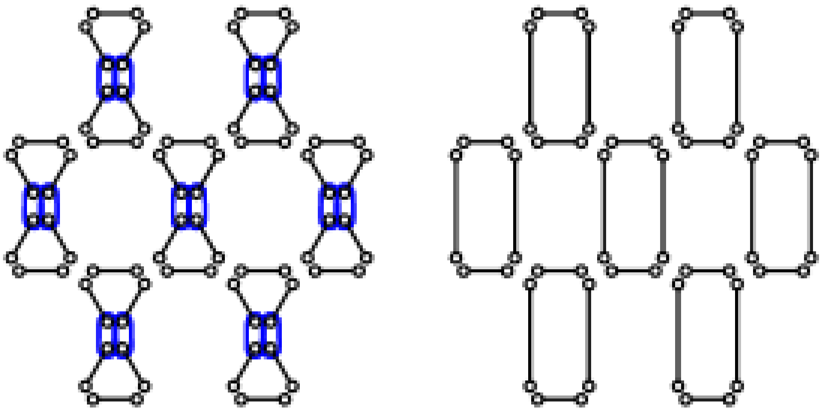


Figure 15: Transformation of Kagome lattice to square lattice (which has a lower percolation threshold)

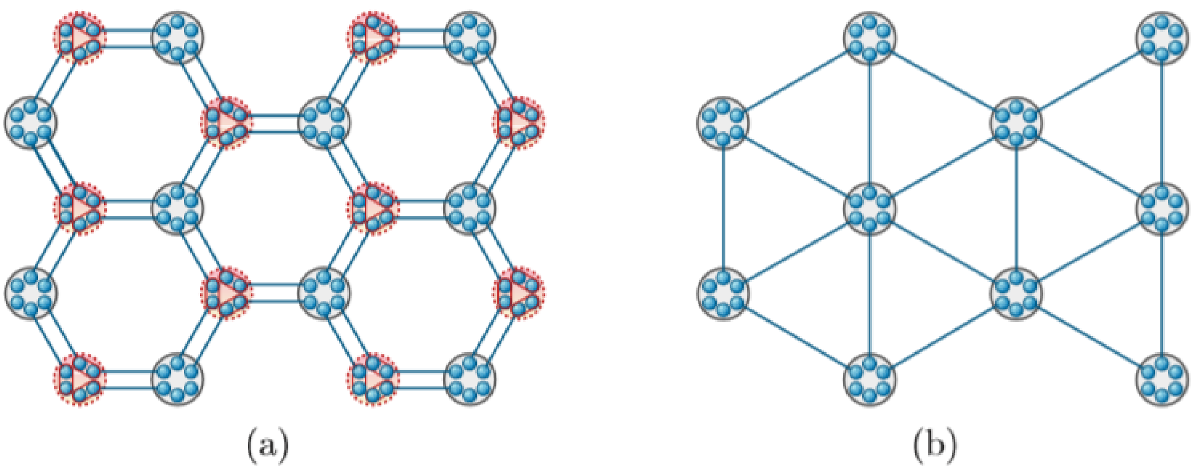


Figure 16: Transformation of Honeycomb Lattice to a Triangular Lattice (which has a lower percolation threshold)

In Figure 16, a measurement in the Bell basis is done on the qubits which are encircled in red which changes the double layered honeycomb lattice into a triangular lattice.

If the initial state is such, or the critical amount of entanglement ($2\phi_1$) is such that percolation is not possible in a double layered honeycomb lattice, then this measurement strategy (and remembering the result from chapter 1, that a single measurement done in the Bell basis does not change the average singlet conversion probability), enables entanglement percolation to succeed in a triangular lattice (since it has a lower percolation threshold.

In Chapter 4, we will see how the same can be accomplished using just a single layered honeycomb lattice, that is, by using much less quantum resource in terms of entanglement.

Chapter 4

Quantum Entanglement Percolation in a Mono-layer Honeycomb Lattice

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4.1 Main Results

The problem of establishing a maximally entangled state between two far away places is a difficult one, albeit an extremely important one. An efficient protocol which addresses this problem is that of entanglement percolation which has been introduced in the previous chapter.

In this chapter, **we construct a specific quantum measurement strategy which uses a lower amount of quantum resources, compared to the previous protocols** [1], and we will work further to show that at the same time this measurement strategy succeeds in establishing a maximally entangled state between two distant nodes of the given lattice.

The usage of much less quantum resource in terms of entanglement than compared to previous protocols [1] will call for the efficiency of the present strategy.

Our task is to establish maximally entangled states between two distant nodes of an asymptotically large lattice. As shown in [1], the average of SCP over all four possible outcomes that may result due to entanglement swapping at one repeater, is same as that of the original states.

Further the authors of [1] used this result to convert a doubly bonded (double layered) honeycomb lattice (which has a higher percolation threshold) to a triangular lattice of lower percolation threshold using one repeater Bell measurements.

If the initial state was such that the critical amount of entanglement (i.e., the SCP) was less than the amount needed to do entanglement percolation in a honeycomb lattice, the authors showed that their quantum measurement strategy enables them to do entanglement percolation in a triangular lattice. In [2], the authors used their specific measurement strategy to decrease the percolation threshold for a given lattice.

Beginning with a *single layered honeycomb lattice*, **another quantum measurement strategy is proposed** which can be showed to be more efficient than that of [1]. *The essential point is that this measurement strategy begins with just a single layered honeycomb lattice whereas in [1], the authors needed 2 layers of honeycomb lattice* for their measurement strategy to succeed.

So we are **essentially using much less quantum resource in terms of entanglement** and we further *show that this measurement strategy succeeds in entanglement percolation while using considerably less quantum resource*

We begin with the partially entangled initial states, $|\phi\rangle = \sqrt{\phi_0}|00\rangle + \sqrt{\phi_1}|11\rangle$, $\phi_0 > \phi_1$, and $\phi_0 + \phi_1 = 1$, spanning our honeycomb lattice. We carry out a measurement in the three-qubit GHZ basis which is

$$\{\frac{(|\,ijk\rangle \pm |\,\bar{i}\bar{j}\bar{k}\rangle)}{\sqrt{2}}\} \tag{4.1}$$

i.e composed of the following eight states:

$$\begin{aligned}
&1)\ (|000\rangle \pm |111\rangle)/\sqrt{2},\\
&2)\ (|001\rangle \pm |110\rangle)/\sqrt{2},\\
&3)\ (|010\rangle \pm |101\rangle)/\sqrt{2},\\
&4)\ (|011\rangle \pm |100\rangle)/\sqrt{2}.
\end{aligned} \tag{4.2}$$

The measurement is carried out on the three qubits at the nodes which are shown circled in Figure 17 and Figure 18. After this measurement, we have successfully converted our single layered honeycomb lattice made up of partially entangled pure two-qubit states to a triangular lattice spanned by three-qubit partially entangled GHZ states.

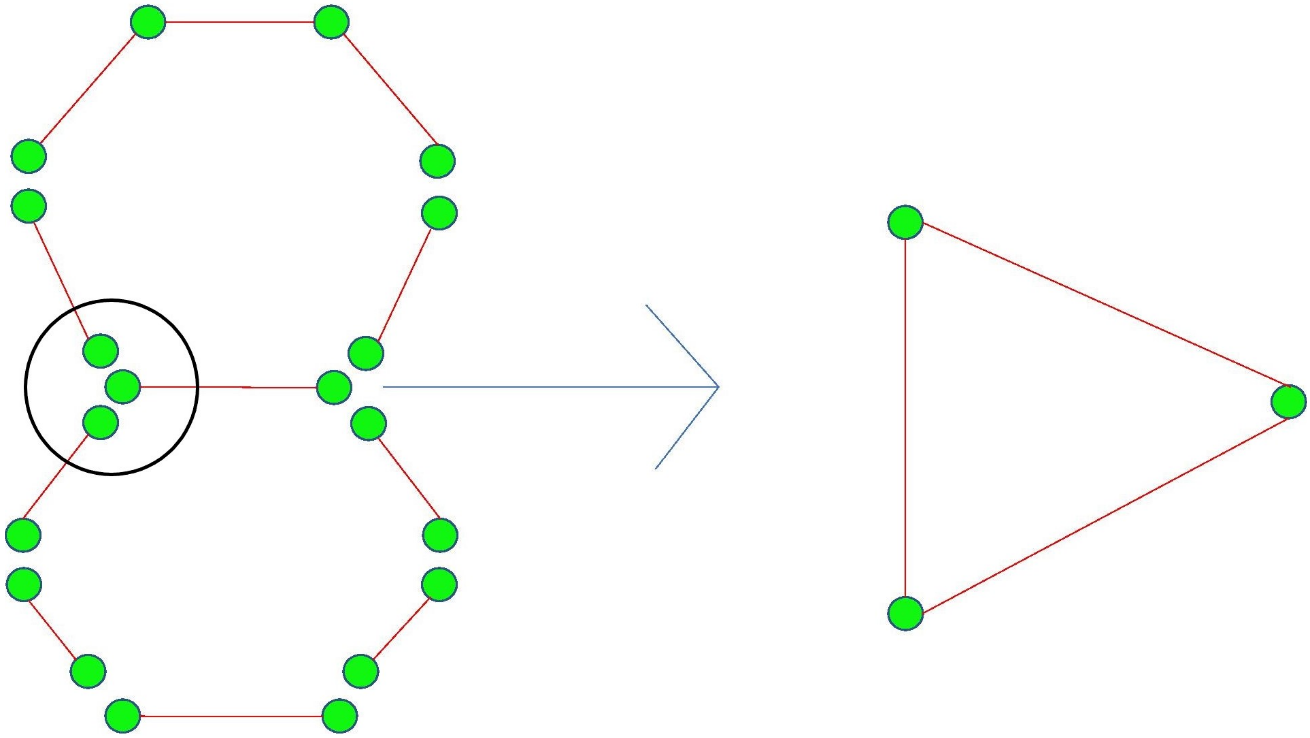

Figure 17:
Measurement in the GHZ basis carried out on the three qubits at the node, circled in black. This measurement is repeated at the consecutive nodes of the honeycomb lattice, which is made up of partially entangled two-qubit pure states and converts it to a triangular lattice which is now spanned by three-qubit partially entangled GHZ states.

The final three-qubit entangled GHZ state between the 3 respective nodes (which form a triangle in Figure 17 and Figure 18) can be calculated by computing the following expression:

$$\frac{I\otimes|A_i\rangle\langle A_i|\otimes I\otimes I}{p^{\frac{1}{2}}}\,|\,\lambda\rangle \tag{4.3}$$

where,

$$|\,\lambda\rangle \;=\; (\sqrt{\phi_0}\,|\,00\rangle \;+\; \sqrt{\phi_1}\,|\,11\rangle\,)\otimes(\sqrt{\phi_0}\,|\,00\rangle \;+\; \sqrt{\phi_1}\,|\,11\rangle\,)\otimes(\sqrt{\phi_0}\,|\,00\rangle \;+\; \sqrt{\phi_1}\,|\,11\rangle\,) \tag{4.4}$$

which can be simplified keeping in mind the correct order of the respective qubits as follows.

$$
\begin{aligned}
|\lambda\rangle = {} & \phi_0 \sqrt{\phi_0}\,|000000\rangle \\
& + \phi_0 \sqrt{\phi_1}\,|000011\rangle \\
& + \phi_0 \sqrt{\phi_1}\,|001100\rangle \\
& + \phi_1 \sqrt{\phi_0}\,|001111\rangle \\
& + \phi_0 \sqrt{\phi_1}\,|110000\rangle \\
& + \phi_1 \sqrt{\phi_0}\,|110011\rangle \\
& + \phi_1 \sqrt{\phi_0}\,|111100\rangle \\
& + \phi_1 \sqrt{\phi_1}\,|111111\rangle
\end{aligned} \tag{4.5}
$$

and where the operator $|A_i\rangle\langle A_i|$ is an element of the 3-qubit GHZ basis $\{\frac{(|ijk\rangle \pm |\bar{i}\bar{j}\bar{k}\rangle)}{\sqrt{2}}\}$ and acts on the 3 qubits present in the nodes shown circled in the figures whereas the Identity operators act on the rest of the 3 qubits and p is the corresponding probability.

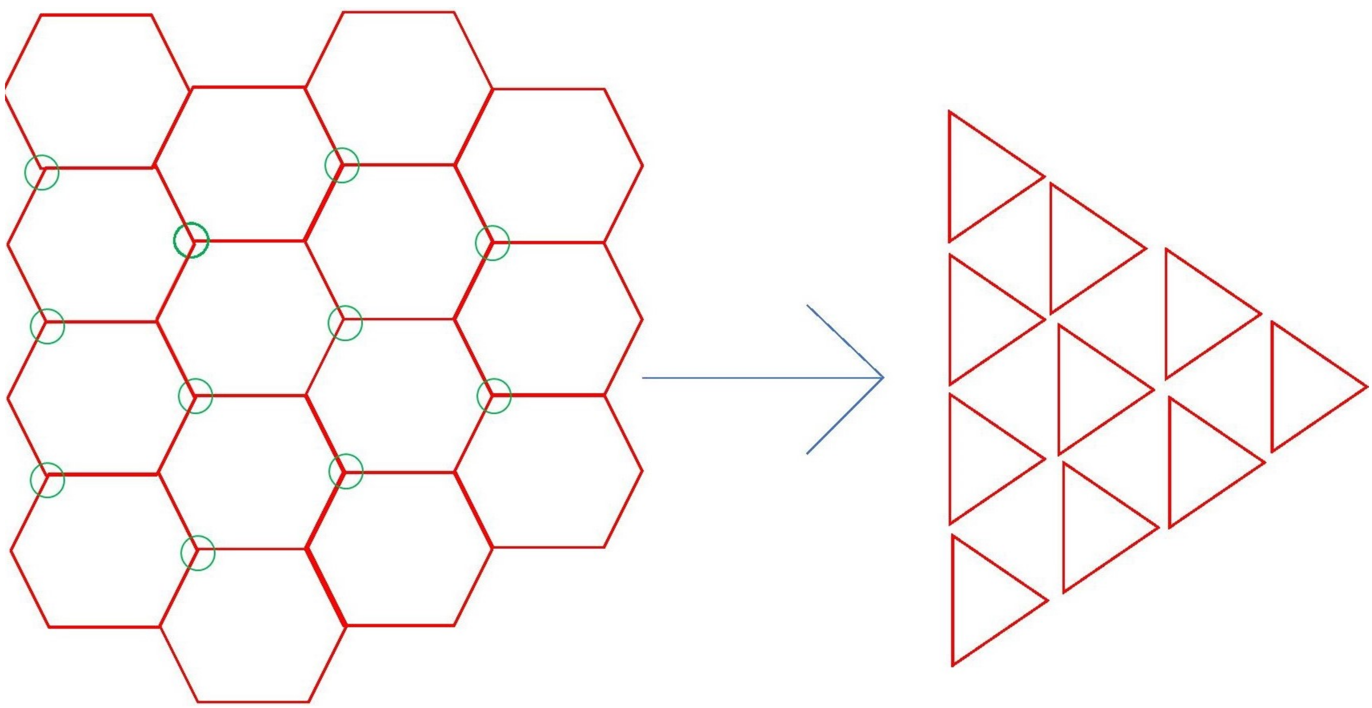

Figure 18: Conversion of the honeycomb lattice, consisting of partially entangled pure two-qubit states, to a triangular lattice, spanned by three-qubit partially entangled GHZ states

Exploiting the three-qubit measurement and orthogonality conditions, we obtain the following three-qubit partially entangled GHZ states which generate the triangles spanning the new lattice:

$$
\begin{array}{l}
1)\ \frac{\phi_0\sqrt{\phi_0}|000\rangle \pm \phi_1\sqrt{\phi_1}|111\rangle}{\sqrt{\phi_0^3+\phi_1^3}}, \\
2)\ \frac{\phi_0\sqrt{\phi_1}|001\rangle \pm \phi_1\sqrt{\phi_0}|110\rangle}{\sqrt{\phi_0^2\phi_1+\phi_1^2\phi_0}}, \\
3)\ \frac{\phi_0\sqrt{\phi_1}|010\rangle \pm \phi_1\sqrt{\phi_0}|101\rangle}{\sqrt{\phi_0^2\phi_1+\phi_1^2\phi_0}}, \\
4)\ \frac{\phi_1\sqrt{\phi_0}|011\rangle \pm \phi_0\sqrt{\phi_1}|100\rangle}{\sqrt{\phi_0^2\phi_1+\phi_1^2\phi_0}}.
\end{array}
\tag{4.6}
$$

The average SCP can be calculated by averaging the SCPs over all the eight possible outcomes (all outcomes are given above) and **and remembering the generalized result derived in section 1.7 regarding the probability of conversion of a partially entangled cat state to the corresponding maximally entangled cat state**.

The average SCP is given as

$$Avg.\,SCP \;=\; p_0 \;=\; 2\phi_1^2(\phi_1+3\phi_0) \tag{4.7}$$

4.2 Calculation of the Percolation Threshold

Now, our triangular lattice is spanned by 3- qubit partially entangled GHZ states. So we cannot apply standard bond percolation, because in bond percolation a bond exists between 2 nodes with a probability p. But here "1 triangle" plays the role of "1 bond". So we cannot apply bond percolation directly and use the bond percolation threshold of the triangular lattice. A little thought will clear the implication of the above statement.

To calculate the percolation threshold we will have to in effect "sum over the probabilities" of all the possible infinitely connected paths within the infinite cluster. Clearly the the probability over any single path will be vanishingly small but the sum of probabilities over all the possible infinite paths will be a number between 0 and 1.

To get this number we need to form a series, summing the probability over all possible paths inside the the infinite cluster. This series will particularly be intrinsic to our lattice structure and which in the limit N$\rightarrow$ inf (where N is the number of paths inside the infinite cluster) will help us evaluate the percolation threshold corresponding to our triangular lattice spanned by partially entangled GHZ states.

This approach looks a formidable task and would consequently require heavy mathematics plus a lot of numerics. Finding percolation thresholds for several different lattices remain an open problem. I present below a simple approach to attack this problem.

Denote each triangle as a site so that after our local operation of converting each partially entangled GHZ state to the corresponding maximally entangled states, an "occupied" site will denote a triangular maximally entangled GHZ state with probability $p = Average \ SCP$ and with probability $1-p$, an "open site" will denote a product state. Figure 19 and Figure 20 will clear what is actually being proposed.

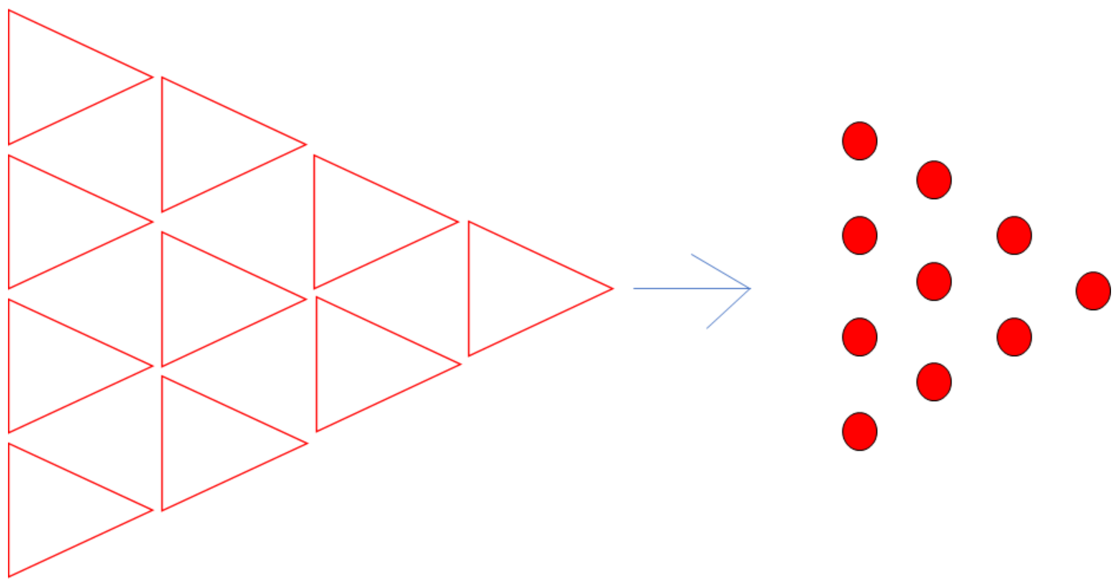

Figure 19: Calculation of Percolation Threshold of the Triangular Lattice of GHZ states

Figure 19 denotes the representation of the triangular lattice of GHZ states as a triangular lattice of sites.

In Figure 20 the sites coloured in red denote those GHZ triangles which have successfully converted to maximally entangled GHZ states whereas the uncoloured sites denote those GHZ triangles which have failed to convert to maximally entangled state and are thus in a product state. The infinite cluster is denoted too.

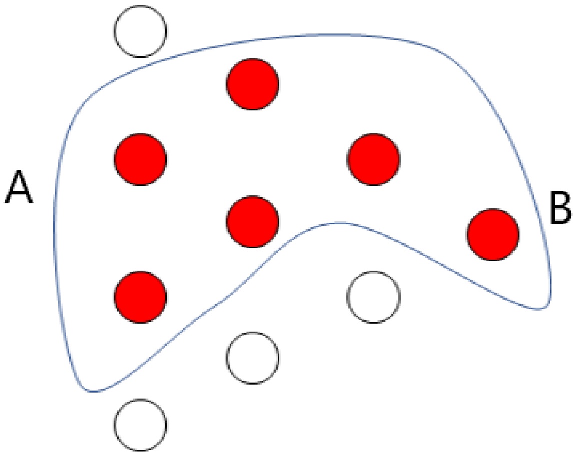

Figure 20: Infinite Cluster of Occupied Sites

4.3 Success of Entanglement Propagation

From the above discussion we arrive at the conclusion that the site percolation threshold for a triangular lattice can be used correctly used as the percolation threshold in our case too. The site percolation threshold for a triangular lattice is $p_c = 0.5$.

The site percolation threshold for several lattices is given below in Table 4.1.

Thus entanglement percolation will succeed when Average SCP is greater than the percolation threshold.

$$Avg.\,SCP = p_0 = 2\phi_1^2(\phi_1 + 3\phi_0) = 0.5. \tag{4.8}$$

$$\Rightarrow \phi_1 = 0.326352 \tag{4.9}$$

Thus we obtain a bound on our Schmidt Coefficients and above $\phi_1 = 0.326352$ entanglement percolation will succeed in our triangular lattice.

Lattice	Percolation Threshold Probability
Square	0.59274 MC estimate
Triangular	0.5
Kagome	$1 - 2\sin\left(\frac{\pi}{18}\right) \approx 0.6527$
Bowtie	0.5472 MC estimate
Honeycomb	0.6962 MC estimate

Table 4.1: Site Percolation Threshold Probabilities for some examples of 2D lattices.

4.4 Future Plan

We plan to continue the work in this direction and compare our results with [2]. Specifically we plan to compare the ratio of the number of measurements per lattice site we need to make employing our measurement strategy to the the number of measurements per lattice site made in [2] employing their measurement strategy.

If the number of measurements made in [2] is larger than what we make using our strategy, then more noise will enter into the quantum protocol of [2] than compared to our protocol based on our measurement strategy. If the number of measurements made in our protocol turns out to be less than the number made in [2], then less noise will enter when our protocol is employed and hence this fact will call for the efficiency of our protocol over [2].

4.5 Conclusion

We have shown that by using our quantum strategy of measuring in the GHZ basis, it is possible for entanglement percolation to succeed even in a single layered honeycomb lattice. A double layered honeycomb lattice would call for much more quantum resource in terms of entanglement, whereas our measurement strategy does the job with much less resource, in that we use just a single layered honeycomb lattice.

Effective lattice transformation is one of the major factors defining the success of entanglement percolation. We have shown that quantum strategies govern the effective lattice transformations, in this regard our measurement strategy succeeded with much less quantum resource in terms of entanglement.
Thus, quantum measurement strategies define the efficiency of a particular lattice transformation and hence control the success of entanglement percolation in a lattice.

A notable fact of the present measurement strategy is that we have carried out measurement on the three qubits. Though the measurement is carried out in a single location, still a joint measurement on the multiple qubits is always a difficult one to implement experimentally. Nevertheless, the amount of quantum resources used can be reduced by suggesting an improved measurement strategy.

It is obvious that that the lesser the amount of resources used, the better it is for the protocol. With our strategy we could do away with a complete layer of honeycomb lattice which used a lot of quantum resource in terms of entanglement and succeed with the use of just one layer.

We also hope to show shortly, that less noise will enter when the protocol based on our measurement strategy is employed than compared to the noise entering when using the measurement strategy developed in [2].

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