BACKWARD ERROR ANALYSIS OF SPECIFIED EIGENPAIRS AND ITS APPLICATION IN SOLVING INVERSE EIGENVALUE PROBLEMS

Ph.D. Thesis

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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **BACKWARD ERROR ANALYSIS OF SPECIFIED EIGENPAIRS AND ITS APPLICATION IN SOLVING INVERSE EIGENVALUE PROBLEMS** in the partial fulfillment of the requirements for the award of the degree of **DOCTOR OF PHILOSOPHY** and submitted in the **DISCIPLINE OF MATHEMATICS, Indian Institute of Technology Indore**, is an authentic record of my own work carried out during the time period from December 2014 to September 2020 under the supervision of Dr. Sk. Safique Ahmad, Associate Professor, Indian Institute of Technology Indore.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other institute.

Pkanhya 26-09-2020

Signature of the student with date

(**PRINCE KANHYA**)

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

26/09/2021

Signature of Thesis Supervisor with date

(**DR. SK. SAFIQUE AHMAD**)

PRINCE KANHYA has successfully given his Ph.D. Oral Examination held on 8 April 2021.

Signature of Thesis Supervisor

Date: 08/04/2021

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(Prince Kanhya)

DEDICATION

Dedicated to
My beloved late Papa, late Kaku and late Pappu Bua
&
My family

LIST OF PUBLICATIONS

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ABSTRACT

KEYWORDS: Matrix pencil, matrix polynomial, two-parameter matrix system, backward error, semisimple eigenvalue, sparsity, Hankel matrix, symmetric-Toeplitz matrix, perturbation theory, generalized inverse eigenvalue problem, quadratic inverse eigenvalue problem, Frobenius norm.

This thesis deals with the structured and unstructured backward errors, and perturbation analysis of one or more specified eigenpairs for matrix pencils, matrix polynomials and two-parameter matrix systems. For a given set of specified eigenpairs, we develop a general framework on backward error analysis in such a way that various types of inverse eigenvalue problems, viz., matrix inverse eigenvalue problems, generalized inverse eigenvalue problems, and polynomial inverse eigenvalue problems are solved using our obtained backward error results.

We raise the following two questions throughout the thesis with respect to matrix pencils, matrix polynomials, and two-parameter matrix systems. The first question is, what is the cumulative backward error of one or more approximate eigenpairs ? And the second question is, what is the nearest matrix pencil for which given approximate eigenpairs become the exact eigenpairs ?

To answer the above-raised questions, first, we develop a general framework of the structured and unstructured backward error analysis of two specified eigenpairs of a double-semisimple eigenvalue for matrix pencils. We establish relationships between the unstructured backward error of a single eigenpair, structured backward error of two eigenpairs of a double semisimple eigenvalue, and the structured backward error of a single eigenpair. Next, we move towards the answers of the above-raised questions in a more general sense, i.e., the number of specified eigenpairs can be more than two and eigenvalues can be distinct. We further use the developed backward error results for solving the

different inverse eigenvalue problems; for example, we solve real symmetric quadratic inverse eigenvalue problem and the symmetric generalized inverse eigenvalue problem with submatrix constraints.

After then, we discuss the backward error analysis of symmetric-Toeplitz and Hankel matrix pencils. These two structured matrix pencils are particular types of a symmetric matrix pencil. We present the backward error analysis of these matrix pencils in such a way that the solutions of the symmetric-Toeplitz inverse eigenvalue problem and Hankel inverse eigenvalue problem are a consequence of it.

Next, we discuss the backward error analysis for structured and unstructured matrix polynomials and answer the above-raised questions. An n-by-n matrix polynomial of degree l have ln eigenvalues (finite or infinite) and the corresponding ln eigenvectors. Hence for each structured matrix polynomial, we provide the upper bound on the maximum number of approximate eigenpairs whose backward error analysis can be done simultaneously. This challenge has not arisen during the backward error analysis of a single eigenpair. Further, we use the developed backward error results in solving different kinds of *quadratic inverse eigenvalue problems*. In particular, we solve symmetric and palindromic quadratic inverse eigenvalue problems.

Finally, the backward error analysis has been developed for two-parameter matrix systems. We classify the two-parameter matrix systems based on the normal rank definition. For two-parameter matrix systems, we obtain the structured and unstructured backward error results of two approximate eigenpairs provided eigenvalue is semisimple. Throughout the thesis, we answer the above-raised questions with respect to *Frobenius* norm.

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NOTATION

$\mathbb{N}$	the set of natural numbers
$\mathbb{R}$	the set of real number
$\mathbb{C}$	the set of complex numbers
$C^{m\times n}$	the set of complex matrices of dimension $m \times n$
$R^{m\times n}$	the set of real matrices of dimension $m \times n$
$A \circ B$	Hadamard product of Matrices A and B
$A \otimes B$	Kronecker product of Matrices A and B
A^T	the transpose of $A \in \mathbb{C}^{m\times n}$
A^H	the conjugate transpose of $A \in \mathbb{C}^{m\times n}$
$\overline{A}$	the conjugate of $A \in \mathbb{C}^{m\times n}$
A^{-1}	the inverse of $A \in \mathbb{C}^{n\times n}$
A^+	the pseudoinverse of $A \in \mathbb{C}^{m\times n}$
$det(A)$	determinant of $A \in \mathbb{C}^{n\times n}$
$rank(A)$	rank of matrix A
$det(A)$	the determinant of $A \in \mathbb{C}^{n\times n}$
$tr(A)$	the trace of $A \in \mathbb{C}^{n\times n}$
I_n	the Identity matrix of order n
$\|x\|_2 = \sqrt{\sum_{i=1}^{n} \|x_i\|^2}$	the 2-norm on $\mathbb{C}^n$
$\sigma_{min}(A)$	the smallest singular value of matrix $A \in \mathbb{C}^{m\times n}$
$\sigma_{max}(A)$	the largest singular value of matrix $A \in \mathbb{C}^{m\times n}$
$\|A\|_F = \sqrt{tr(A^H A)}$	the Frobenius norm of $A \in \mathbb{C}^{m\times n}$
$\|A\|_2 = \max_{\|x\|_2=1} \|Ax\|_2$	the spectral norm of $A \in \mathbb{C}^{m\times n}$
$\Lambda(A)$	the spectrum of a matrix A
e_i	the vector in $\mathbb{C}^n$ having 1 at ith position and 0 elsewhere
i	the imaginary number
$\Re(z)$	Real part of $z \in \mathbb{C}^n$
$\Im(z)$	Imaginary part of $z \in \mathbb{C}^n$
$A([1,r])$	$r \times r$ leading principal submatrix of $A \in \mathbb{C}^{n\times n}$

CHAPTER 1

INTRODUCTION

1.1. Introduction

Matrix pencils and matrix polynomials are well-known terms in the field of numerical linear algebra. They arise in numerous applications in engineering, mechanics, control theory, linear systems theory, computer-aided graphic design, and vibration analysis, see [1, 17, 37, 38, 43, 44, 47, 51, 67, 70, 83, 84]. Eigenvalue problems of matrix pencils are known as generalized eigenvalue problems, and eigenvalue problems of matrix polynomials are known as polynomial eigenvalue problems. A two-parameter matrix system is another well discussed and most widely used form of the multi-parameter matrix system. It arises in different types of applications [11, 12, 40, 41]. In particular, it arises in mathematical physics when the separation of variables is used to solve the boundary value problems [73], in model updating [22], in three-point boundary value problems [39], and in the quadratic two-parameter eigenvalue problem [55]. The eigenvalue problem of two-parameter matrix systems is known as two-parameter eigenvalue problems. Finding solutions to a linear system of equations, finding the eigenvalues and eigenvectors of matrices, matrix pencils, matrix polynomials, and two-parameter matrix systems are always very challenging tasks from a long back. Different authors have developed many numerical algorithms to obtain the desired solutions. But due to the roundoff errors and truncation errors of the available iterative methods, one can get only approximate solutions. Due to the approximate nature of the obtained solutions, some major questions come into the picture: are these computed solutions reliable to use ? Numerical algorithms that we are using to get these solutions are stable or not? For which problem the obtained approximate solution is exact ? Answers of these questions are very much of importance as ignorance of these answers may lead to insignificant results to our original problems. For answering these questions, a term backward error is developed in numerical linear algebra. The backward error of a computed solution tells us how far a solution stands from the original problem. In other words, for a given problem and its given approximate solution, backward error tells the minimum perturbation (in some appropriate norm) to the problem for which the given approximate solution becomes exact. The term backward error is used by different authors in different aspects. Backward error analysis is one of the important and continuously developing areas in numerical linear algebra. If we recall the history of backward error analysis, we find that Wilkinson was the first to use the term backward error analysis [76, 77]. Wilkinson has developed the backward error bounds for the computed triangular factorization of a matrix and further discusses the backward error analysis of an approximate solution of the linear system using this factorization. Boor and Pinkus [16] have studied the backward error analysis for totally positive linear systems (see, [26, 58] and the references therein). Higham and Higham [35] have discussed the backward error analysis of an approximate solution to a linear system for structured as well as unstructured matrices. If we move further, we found that different authors have developed the backward error analysis of eigenvalues. For example, Malyshev [53] has discussed the minimal perturbation of a given n-by-n matrix to the nearest matrices that have $\lambda \in \mathbb{C}$ as a multiple eigenvalue with respect to 2-norm, see [31, 43, 44, 48, 49, 57] for information on the backward error analysis of one or more eigenvalues. Similar to backward error analysis of eigenvalues, different authors have developed the backward error analysis for a single eigenpair. For the matrix case, Dief [24] has discussed the backward error analysis for a single approximate eigenpair. In the series of developments of the backward error of a single eigenpair, the authors in [1] have developed the backward error analysis of a single approximate eigenpair for various structured matrix pencils. They have also provided a comparison between unstructured and structured backward errors. Many other authors have also contributed to the development of the backward error analysis of a single eigenpair for structured and unstructured matrix pencils and matrix polynomials, see [1, 2, 8, 9, 47, 50, 70]. Moving further, we find that the theory of backward error analysis of a single eigenpair for structured and unstructured multi-parameter matrix systems is also well studied in the literature, see [27, 42, 50]. At this point, a natural question is arising: What is the backward error of more than one approximate eigenpairs for structured and unstructured matrix pencils, matrix polynomials, and two-parameter matrix systems? Some work has been done in the field of backward error analysis of one or more approximate eigenpairs. For example, Tisseur [71] has obtained the backward error formulas of one or more eigenpairs for structured matrices, and for unstructured non-square matrix pencils, Chu and Golub [18] have studied the backward error analysis for one or more eigenpairs. Still, these works are not enough to answer the above-raised question. Hence finding the answer to the above-raised question in every possible aspect is one of the main aims of the thesis.

The inverse eigenvalue problem is another major discussed topic in numerical linear algebra. The term inverse eigenvalue problem refers to reconstructing the required matrix or matrix pencil or matrix polynomial from the given eigeninformation, see [21]. In this thesis, we are interested in solving the inverse eigenvalue problems from the given set of eigenpairs. For example, let us consider the following inverse eigenvalue problem: The second main aim of the thesis is to solve the different kinds of inverse eigenvalue problems.

- Let (λ_i, x_i) for $i = 1 : k$ be specified eigenpairs, where $\lambda_i \in \mathbb{R}$ and $x_i \in \mathbb{C}^n$. Construct a matrix $G \in \mathbb{C}^{n\times n}$ such that $G = G^H$ from the given set of eigenpairs.

In the above inverse eigenvalue problem, we need to construct the required structured matrix from the given eigenpairs set. This is called the matrix inverse eigenvalue problem (MIEP). Similarly one can solve different kind of MIEP for different structured matrices. See, [21].

Next, we discuss the generalized inverse eigenvalue problems. A generalized inverse eigenvalue problem is to reconstruct the required matrix pencil from the given set of eigenpairs. For example, consider the following inverse eigenvalue problem from [84].

- From a given set of eigenpairs (λ_i, x_i) for $i = 1 : p$, construct the real symmetric matrices $A_0, A_1 \in \mathbb{C}^{n\times n}$ with the (2r+1) diagonal, where $\lambda_i \in \mathbb{C}, x_i \in \mathbb{C}^n$, and $p \leq n, r \leq n$.

Next, consider a quadratic matrix polynomial which is defined as follows:

$$P(\lambda) = A_0 + \lambda A_1 + \lambda^2 A_2.$$

Matrices A_0, A_1, and A_2 can have different structures. For example $P(\lambda)$ is called a monic gyroscopic quadratic matrix polynomial if $A_2 = I_n$, A_1 is a skew-symmetric matrix, and A_0 is symmetric matrix. The inverse quadratic eigenvalue problem (IQEP) is to construct matrices A_2, A_1 and A_0 from the measured eigenpairs. Some IQEP are summarized as follows:

1. From a given set of eigenpairs (λ_i, x_i) for $i = 1 : n+1$, construct real symmetric matrices A_1 and A_0 such that $(A_0 + \lambda_i A_1 + \lambda_i^2 I_n)x_i = 0$, where $\lambda_i \in \mathbb{C}$, and $x_i \in \mathbb{C}^n$, see [85].
2. From a given set of eigenpairs (λ_i, x_i) for $i = 1 : k$, construct a T-palindromic matrix polynomial, i.e., construct A_2, A_1 and A_0 such that $A_2 = A_0^T$, and $A_1 = A^T$ such that $(A_0 + \lambda_i A_1 + \lambda_i^2 A_2)x_i = 0$, where $\lambda_i \in \mathbb{C}$, and $x_i \in \mathbb{C}^n$, and $k \leq \frac{3n+1}{2}$, see [88].
3. From a given set of eigenpairs (λ_i, x_i) for $i = 1 : k$, construct Hermitian matrices A_2, A_1 and A_0 such that $(A_0 + \lambda_i A_1 + \lambda_i^2 A_2)x_i = 0$, where $\lambda_i \in \mathbb{C}$, and $x_i \in \mathbb{C}^n$, and $k \leq n$.

We establish the backward error theory of one or more specified eigenpairs in such a way that the solutions of different kind of inverse eigenvalue problems can be obtained from the developed backward error theory.

For simplicity of presentation, first we obtain the results for matrix pencils, then for matrix polynomials and finally for two-parameter matrix systems. The first chapter is introductory in nature and provides the history of backward error analysis, basic definitions, background ideas and pre-requisites for the remaining chapters. Chapter 2 dedicates for the backward error analysis of two approximate eigenpairs of a semisimple eigenvalue for structured and unstructured matrix pencils. This chapter also deals with the relationships of the backward error of a single approximate eigenpair and the backward error of two approximate eigenpairs. Chapter 3 discusses the backward error analysis of one or more approximate eigenpairs for several structured matrix pencils. In this chapter we also solve the quadratic symmetric inverse eigenvalue problem by linearizing it into a large matrix pencil and applying the backward error results. Further, we also solve the symmetric inverse eigenvalue problem with submatrix constraints. Chapter 4 provides the backward error analysis of approximate eigenpairs for the special class of symmetric matrix pencils, i.e., symmetric-Toeplitz and Hankel matrix pencils. This chapter also deals with the inverse eigenvalue problem for matrices as well as the generalized inverse eigenvalue problems. In Chapter 5, we generalize the backward error results from matrix pencils to matrix polynomials. We show that different kind of structured quadratic inverse eigenvalue problems are also solvable from our developed backward error results. In Chapter 6, we classify the two-parameter matrix systems on the basis of normal rank definition. Further, we obtain the backward error formulas for structured and unstructured two-parameter matrix systems. Finally, Chapter 7 concludes with important remarks and some open problems.

1.2. Preliminaries

This section deals with some basic definitions and results which will be used throughout the thesis. Throughout this thesis, $\mathbb{C}^{m\times n}$ denotes the vector space of m-by-n matrices with entries from $\mathbb{C}$, and $\mathbb{C}^n$ denotes the vector space of column vectors $[x_1, x_2, \ldots, x_n]^T$, where $x_i \in \mathbb{C}$. We denote the $n \times n$ identity matrix by I_n.

Kernel and range of a matrix: Let $B \in C^{m\times n}$. Then the kernel of B is defined by $ker(B) := \{x \in \mathbb{C}^n : Bx = 0\}$. We denote the dimension of kernel of B by $\dim ker(B)$. Kernel is also known as Null space and $\dim ker(B)$ is also known as $nullity(B)$. The range space of B is defined by $range(B) := \{Bx : x \in \mathbb{C}^n\}$. The dimension of range of B is called the rank of B and it is denoted by $rank(B)$.

Spectrum of a matrix. Let $B \in \mathbb{C}^{n\times n}$. Let $\lambda \in \mathbb{C}$ is said to be an eigenvalue of B if $\det(B - \lambda I_n) = 0$. The set of all eigenvalue is said to be spectrum of B and it is denoted by $\Lambda(B)$.

Let $\lambda \in \Lambda(B)$. Then algebraic multiplicity of λ is defined as its multiplicity as a zero of the characteristic polynomial $\det(B - \lambda I_n)$. Geometric multiplicity of λ is defined as the dimension of the $ker(B - \lambda I_n)$. An eigenvalue is said to be semisimple if its algebraic multiplicity is equal to its geometric multiplicity. When both the multiplicities are equal to one then the eigenvalue is said to be simple.

Inner product: Let V be a vector space over a field $\mathbb{F}$. Then define $\langle .,. \rangle$: $V \times V \to \mathbb{F}$ is said to be a inner product if the following conditions hold:

1. $\langle v\,, v\rangle$ for all $v \in V$.
2. $\langle v\,, v\rangle = 0$ if and only if $v = 0$.
3. $\langle v + u\,, w\rangle = \langle v\,, w\rangle + \langle u\,, w\rangle$ for all $u, v, w \in V$.
4. $\langle cv\,, w\rangle = c\langle v\,, w\rangle$ for all $c \in \mathbb{F}$ and $v, w \in V$.
5. $\langle v\,, w\rangle = \overline{\langle v\,, w\rangle}$.

Note that $\langle v\,, w\rangle = w^H v$ defines the inner product on $\mathbb{C}^n$.

Norm: We recall the definition of norm and its basic properties. Next we define the norm for the space of two-parameter.

Definition 1.2.1. *For a given vector space V over a field $\mathbb{F}$, a function $\|.\| : V(\mathbb{F}) \to \mathbb{R}$ is said to be norm if it satisfies the following conditions:*

- $\|v\| \geq 0$ *for* $v \in V$.
- $\|v\| = 0 \iff v = 0$.
- $\|\alpha v\| = |\alpha|\|v\|$ *for* $v \in V$ *and* $\alpha \in \mathbb{F}$.
- $\|v + u\| \leq \|v\| + \|u\|$ *for* $v, u \in V$.

Clearly, For $x \in \mathbb{C}^n$, $\|x\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2}$ is a norm on $\mathbb{C}^n$. It is called 2-norm on $\mathbb{C}^n$. For $x = [x_1, \dots, x_n] \in \mathbb{C}^n$ and $w := [w_1, \dots, w_n] \in \mathbb{R}^n$, we define

$$\|x\|_w = \sqrt{\sum_{i=1}^{n} |w_i x_i|^2}.$$

$\|x\|_w$ is called the norm if only of each component of w is a positive real number. Otherwise, $\|x\|_w$ is called the seminorm. Similarly, for $A \in \mathbb{C}^{m \times n}$, $\|A\|_F = \sqrt{tr(A^H A)}$ and $\|A\|_2 = \max_{\|x\|_2=1}(\|Ax\|_2)$ define the Frobenius norm and spectral norm on $\mathbb{C}^{m \times n}$, respectively.

Unitary matrix: A matrix $U \in \mathbb{C}^{n \times n}$ is said to be unitary if $U^H U = I_n = U U^H$. The Frobenius and spectral norms satisfy the following properties:

- $\|Ux\| = \|U\|$ for any unitary matrix U and $x \in \mathbb{C}^n$.
- $\|\overline{U} A U^H\| = \|A\|$ for any unitary matrix U and $A \in \mathbb{C}^{n \times n}$.

Kronecker and Hadamard product: Next, we will discuss the definitions and basic properties of kronecker and Hadamard product.

Definition 1.2.2. [12] *Let $A = (a_{ij}) \in \mathbb{C}^{m \times n}$ and $B = (b_{ij}) \in \mathbb{C}^{p \times q}$. Then the kronecker product (tensor product) of A and B is defined by*

$$A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{bmatrix} \in \mathbb{C}^{mp \times nq}.$$

Example 1.2.3. *Let* $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 7 & 8 \end{bmatrix}$ *and* $B = \begin{bmatrix} 2 & 10 & 11 \\ 12 & 20 & 30 \end{bmatrix}$. *Then*

$$A \otimes B = \begin{bmatrix} 4 & 20 & 22 & 6 & 30 & 33 \\ 24 & 40 & 60 & 36 & 60 & 90 \\ 8 & 40 & 44 & 10 & 50 & 55 \\ 48 & 80 & 120 & 60 & 100 & 150 \\ 14 & 70 & 77 & 16 & 160 & 240 \\ 84 & 140 & 210 & 96 & 160 & 240 \end{bmatrix}.$$

Some properties of kronecker product are as follows:

- For all A and B, $(A \otimes B)^T = A^T \otimes B^T$ and $(A \otimes B)^H = A^H \otimes B^H$.
- Let $A \in \mathbb{C}^{m \times n}, B \in \mathbb{C}^{r \times s}, C \in \mathbb{C}^{n \times t}$ and $D \in \mathbb{C}^{s \times q}$. Then

$$(A \otimes B)(C \otimes D) = (AC \otimes BD) \in \mathbb{C}^{mr \times sq}.$$

Definition 1.2.4. [12] *Let* $A = (a_{ij}) \in \mathbb{C}^{m \times n}$ *and* $B = (b_{ij}) \in \mathbb{C}^{m \times n}$. *Then the Hadamard product of* A *and* B *is defined by*

$$A \circ B = [a_{ij} b_{ij}].$$

Example 1.2.5. *Let* $A = \begin{bmatrix} 2 & -3 & -2\mathrm{i} \\ 7 & 8 - \mathrm{i} & 3 \end{bmatrix}$ *and* $B = \begin{bmatrix} \mathrm{i} & 2 & 3 \\ -8 & 9 & 10 \end{bmatrix}$. *Then*

$$A \circ B = \begin{bmatrix} 2\mathrm{i} & -6 & -6\mathrm{i} \\ -56 & 72 - 9\mathrm{i} & 30 \end{bmatrix}.$$

Some properties of Hadamard product are as follows:

Suppose $A, B, C \in \mathbb{C}^{m \times n}$ and $c \in \mathbb{C}$

- For all $A, B \in \mathbb{C}^{m \times n}$, $A \circ B = B \circ A$.
- For all $A, B, C \in \mathbb{C}^{m \times n}$, $A \circ (B + C) = A \circ B + A \circ C$.
- For all $A, B, C \in \mathbb{C}^{m \times n}$ and $\gamma \in \mathbb{C}$, $A \circ (\gamma B) = \gamma (A \circ B)$.

Pseudoinverse: A pseudoinverse $A^+ \in \mathbb{C}^{n \times m}$ of a matrix $A \in \mathbb{C}^{m \times n}$ is satisfied the following four properties, known as the Moore-Penrose conditions:

- $AA^+A = A$.
- $A^+AA^+ = A^+$.
- $(AA^+)^H = (AA^+)$.
- $(A^+A)^H = (A^+A)$.

Theorem 1.2.6. [75, Theorem 4.3.7] *Let $A \in \mathbb{C}^{m\times n}$ and $b \in \mathbb{C}^m$, and let $x \in \mathbb{C}^n$ be the minimum-norm solution of $\|b - Ax\|_2 = \min_{w\in\mathbb{C}^m} \|b - Aw\|_2$. Then $x = A^+b$, where A^+ is the pseudo inverse of A.*

Remark 1.2.7. *A^+ exists for every matrix A, but, when A has full column rank then A^HA is invertible and*

$$A^+ = (A^HA)^{-1}A^H.$$

On the other hand, if A has full row rank then AA^H is invertible and

$$A^+ = A^H(AA^H)^{-1}.$$

Singular value decomposition [75]**:** Let $A \in \mathbb{C}^{n\times m}$ be a nonzero matrix with rank r. Then A can be expressed as a product

$$A = U\Sigma V^H, \tag{1.1}$$

where $U \in \mathbb{C}^{n\times n}$ and $V \in \mathbb{C}^{m\times m}$ are orthonormal matrices, and $\Sigma = \mathrm{diag}([\sigma_1, \ldots, \sigma_r, 0, \ldots, 0]^T)$ $\in \mathbb{C}^{n\times m}$ is a diagonal matrix, where $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > 0$. The decomposition (1.1) is called the singular value decomposition of A. We usually use the abbreviation SVD.

Remark 1.2.8. *Let $A \in \mathbb{C}^{n\times m}$ be a nonzero matrix with rank r. Then using SVD, we get $A^+ = V\Sigma^+U^H$, where $\Sigma^+ = \mathrm{diag}([1/\sigma_1, \ldots, 1/\sigma_r, 0, \ldots, 0]^T) \in \mathbb{C}^{m\times n}$.*

Orthonormal vectors and Gram-Schmidt process:

Definition 1.2.9. [75] *Let $u_1, u_2 \in \mathbb{C}^n$ be called orthonormal if $u_1^Hu_2 = 0$, and $u_i^Hu_i = 1$ for $i = 1:2$.*

Definition 1.2.10. [75] *Let $S = \{u_1, u_2, \ldots, u_k\}$ be the set of linearly independent vectors then S is said to be orthonormal set if $u_i^Hu_j = 0$ for $i \neq j$, and $u_i^Hu_i = 1$ for $i = 1:k$.*

Gram-Schmidt: Let $S = \{u_1, u_2, \ldots, u_k\}$ be the set of linearly independent vectors, where $u_i \in \mathbb{C}^n$, and $k \leq n$. Define

$$v_1 = u_1,$$

$$v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1,$$

$$v_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 - \frac{\langle u_3, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2,$$

$$\vdots$$

$$v_k = u_k - \sum_{i=1}^{k-1} \frac{\langle u_k, v_i \rangle}{\langle v_i, v_i \rangle} v_i.$$

The Gram-Schmidt process is an algorithm that produces a orthonormal set of vectors $\{q_1, q_2, \ldots, q_k\}$, where $q_i = \frac{v_i}{\|v_i\|}$ or $i = 1 : k$.

Structured matrices:

Definition 1.2.11. *A matrix $B \in \mathbb{R}^{n\times n}$ is called a symmetric matrix if $B = B^T$.*

Definition 1.2.12. *A matrix $B \in \mathbb{R}^{n\times n}$ is called a skew-symmetric matrix if $B = -B^T$.*

Definition 1.2.13. *A matrix $B \in \mathbb{C}^{n\times n}$ is called a complex-symmetric matrix if $B = B^T$.*

Definition 1.2.14. *A matrix $B \in \mathbb{C}^{n\times n}$ is called a complex-skew-symmetric matrix if $B = -B^T$.*

Definition 1.2.15. *A matrix $B \in \mathbb{C}^{n\times n}$ is called a Hermitian matrix if $B = B^H$.*

Definition 1.2.16. *A matrix $B \in \mathbb{C}^{n\times n}$ is called a skew-Hermitian matrix if $B = -B^H$.*

Definition 1.2.17. *A matrix $A \in \mathbb{C}^{n\times n}$ is said to be Hankel matrix if for any vector $[a_{11}, \ldots, a_{1n}, a_{2n}, \ldots, a_{nn}]^T \in \mathbb{C}^{2n-1}$, the matrix A is of the following form:*

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1(n-1)} & a_{1n} \\ a_{12} & a_{13} & \cdots & a_{1n} & a_{2n} \\ \vdots & & & & \vdots \\ a_{1(n-2)} & & & & a_{(n-2)n} \\ a_{1(n-1)} & \cdots & \cdots & a_{(n-2)n} & a_{(n-1)n} \\ a_{1n} & \cdots & a_{(n-2)n} & a_{(n-1)n} & a_{nn} \end{bmatrix}.$$

For Hankel matrix A, we define the generator vector of A in the following form:

$$\mathrm{vec}(A, \mathrm{Hank}) = [a_{11}, \ldots, a_{1n}, a_{2n}, \ldots, a_{nn}]^T.$$

The Hankel matrix is a particular type of the complex-symmetric matrix.

Definition 1.2.18. *A matrix $A \in \mathbb{C}^{n\times n}$ is said to be symmetric-Toeplitz matrix if for any vector $[a_1, a_2, \dots, a_n]^T \in \mathbb{C}^n$, the matrix A is of the following form:*

$$A = \begin{bmatrix} a_1 & a_2 & \dots & a_{n-1} & a_n \\ a_2 & a_1 & \ddots & & a_{n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{n-1} & & \ddots & \ddots & a_2 \\ a_n & a_{n-1} & a_3 & a_2 & a_1 \end{bmatrix}.$$

For symmetric-Toeplitz matrix A, we define the generator vector of A in the following form:

$$\mathrm{vec}(A, \mathrm{symToep}) = [a_1, a_2, \dots, a_n]^T.$$

The symmetric-Toeplitz matrix is a particular type of the complex-symmetric matrix.

Proposition 1.2.19. *Let Symm $:= \{A \in \mathbb{F}^{n\times n} : A = A^T\}$. Then Symm is a vector space over a field $\mathbb{F}$ and the dimension of Symm is $(n^2+n)/2$, where $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$.*

Proposition 1.2.20. *Let Ssymm $:= \{A \in \mathbb{F}^{n\times n} : A = -A^T\}$. Then Ssymm is vector space over a field $\mathbb{F}$ and the dimension of Symm is $(n^2-n)/2$, where $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$.*

Proposition 1.2.21. *Let Herm $:= \{A \in \mathbb{C}^{n\times n} : A = A^H\}$. Then Herm is a vector space over the field $\mathbb{R}$ and the dimension of Herm is n^2.*

Proof. Let A be a *Hermitian* matrix of the form $A := E + \mathrm{i}F$, where $E, F \in \mathbb{R}^{n\times n}$. Since $A^H = A$, we get $E = E^T$ and $F = -F^T$. Then by Proposition 1.2.19 and Proposition 1.2.20, we get that the dimension of *Herm* is $(n^2+n)/2 + (n^2-n)/2 = n^2$. ∎

Proposition 1.2.22. *Let Sherm $:= \{A \in \mathbb{C}^{n\times n} : A = -A^H\}$. Then Sherm is a vector space over the field $\mathbb{R}$ and the dimension of Sherm is n^2.*

Proof. Let A be a *skew-Hermitian* matrix of the form $A = E + \mathrm{i}F$ where $E, F \in \mathbb{R}^{n\times n}$. Since $A^H = -A$, we get $E = -E^T$ and $F = F^T$. Then by Proposition 1.2.19 and Proposition 1.2.20, we conclude that the dimension of *Sherm* is $(n^2-n)/2+(n^2+n)/2 = n^2$.

Proposition 1.2.23. [62] *Let* Hank $= \{A \in \mathbb{C}^{n\times n} : A$ *is Hankel matrix*$\}$. *Then* Hank *is a vector space over the field $\mathbb{C}$ and dimension of* Hank *is* $2n-1$.

Proof. Consider $e_i \in \mathbb{C}^{2n-1}$, for $i = 1 : (2n-1)$. Then the *Hankel* matrices generated by these $2n-1$ vectors form a basis for Hank.

Proposition 1.2.24. [62] *Let* $\mathrm{symToep} := \{A \in \mathbb{C}^{n\times n} : A$ *is symmetric-Toeplitz matrix*$\}$. *Then* symToep *is a vector space over the field* $\mathbb{C}$ *and dimension of* symToep *is* n.

Proof. Consider $e_i \in \mathbb{C}^n$, for $i = 1 : n$. Then the *symmetric-Toeplitz* matrices generated by these n vectors form a basis for symToep.

Definition 1.2.25. *Let* $x \in \mathbb{C}^n$, *and* $x = [x_1, x_2, \ldots, x_n]^T$. *Then*

$$\mathrm{diag}(x) = \begin{bmatrix} x_1 & 0 & \cdots & \cdots & 0 \\ 0 & x_2 & 0 & \ldots & 0 \\ \vdots & 0 & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & \cdots & x_n \end{bmatrix} \in \mathbb{C}^{n\times n}.$$

$$\mathrm{diag}(x) = \begin{bmatrix} x_1 & 0 & \cdots & \cdots & 0 & 0 & \ldots & 0 \\ 0 & x_2 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & 0 & \ddots & & \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \ddots & 0 & 0 & \ldots & 0 \\ 0 & 0 & \cdots & \cdots & x_n & 0 & \ldots & 0 \end{bmatrix} \in \mathbb{C}^{n\times m} \ \textit{for } (n < m).$$

$$\mathrm{diag}(x) = \begin{bmatrix} x_1 & 0 & \cdots & \cdots & 0 \\ 0 & x_2 & 0 & \ldots & 0 \\ \vdots & 0 & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & \cdots & x_n \\ 0 & 0 & \ldots & \ldots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \ldots & \ldots & 0 \end{bmatrix} \in \mathbb{C}^{n\times n} \ \textit{for } (n > m).$$

Definition 1.2.26. *Let* $A = (a_{ij}) \in \mathbb{C}^{m\times n}$. *Then* $\mathrm{vec}(A) \in \mathbb{C}^{mn}$ *is defined as follows:*

$$\mathrm{vec}(A) = [a_{11}, \ldots, a_{1n}, \ldots, \ldots, a_{n1}, \ldots, a_{mn}]^T.$$

Definition 1.2.27. *Let* $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ *be a symmetric matrix. Then* $\mathrm{vec}(A) \in \mathbb{C}^{\frac{n^2+n}{2}}$ *is defined as follows:*

$$\mathrm{vec}(A) := [a_{11}, \ldots, a_{1n}, a_{22}, \ldots, a_{2n}, \ldots, a_{(n-1)(n-1)}, a_{(n-1)n}, a_{nn}]^T.$$

Definition 1.2.28. *Let* $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ *be a skew-symmetric matrix. Then* $\mathrm{vec}(A) \in \mathbb{C}^{\frac{n^2-n}{2}}$ *is defined as follows:*

$$\mathrm{vec}(A) := [a_{12}, \ldots, a_{1n}, a_{23}, \ldots, a_{2n}, \ldots, a_{(n-1)n}]^T.$$

Definition 1.2.29. *Let* $a \in \mathbb{C}$. *Then define* $\mathrm{sgn}(a) = 1$, *when* $a \neq 0$ *and* $\mathrm{sgn}(a) = 0$ *when* $a = 0$.

Definition 1.2.30. *Let* $A = (a_{ij}) \in \mathbb{C}^{m\times n}$. *Then*

$$\mathrm{sgn}\, A = (\mathrm{sgn}\, a_{ij}) \in \mathbb{C}^{m\times n}.$$

Example 1.2.31. *Let* $A = \begin{bmatrix} -4+\mathrm{i} & 9 & 0 & 3-\mathrm{i} \\ 10 & 0 & -5-\mathrm{i} & 89 \\ 19 & 2i & -77 & 0 \end{bmatrix}$. *Then*

$$\mathrm{sgn}\, A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

1.2.1. Generalized eigenvalue problems

A matrix pencil is a pair of two matrices defined in the following manner:

$$\mathbf{L}(\alpha, \beta) := \alpha A_0 + \beta A_1, \quad A_0, A_1 \in \mathbb{C}^{n\times n},\ \alpha = (\alpha, \beta) \in \mathbb{C}^2. \tag{1.2}$$

Finding $\lambda = (\lambda_0, \lambda_1) \in \mathbb{C}^2 \setminus \{(0,0)\}$, $0 \neq x \in \mathbb{C}^n$ such that $\mathbf{L}(\lambda)x = 0$, is called *generalized eigenvalue problem* (GEP). We denote the matrix pencil defined in (1.2) by $\mathbf{L}$, then λ is called the eigenvalue and x is the corresponding right eigenvector of matrix pencil $\mathbf{L}$. Further, (λ, x) is called the *eigenpair* of $\mathbf{L}$. If $0 \neq y \in \mathbb{C}^n$ such that $y^H\mathbf{L}(\lambda) = 0$, then y is called the left eigenvector corresponding to λ. We denote the space of matrix pencils by $\mathbf{L}(\mathbb{C}^{n\times n})$. A matrix pencil of the form (1.2) is called the homogenous matrix pencil. When we substitute $\alpha = 1$ in (1.2), then matrix pencil $\mathbf{L}$ is called the non-homogeneous matrix pencil.

Definition 1.2.32. *A matrix pencil* $\mathbf{L} \in \mathbf{L}(\mathbb{C}^{n\times n})$ *of the form (1.2) is said to be regular if* $\det(\mathbf{L}(\lambda)) \neq 0$ *for some* $\lambda = (\lambda_0, \lambda_1) \in \mathbb{C}^2 \setminus \{(0,0)\}$, *otherwise it is called a singular matrix pencil.*

At this point, let us consider the following homogeneous matrix pencil $\mathbf{L}$, where

$$A_0 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, A_1 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then

$$\mathbf{L}(\alpha, \beta) = \begin{bmatrix} \alpha + 2\beta & \alpha \\ \alpha & \alpha \end{bmatrix}.$$

Cleary, we get $\det(\mathbf{L}(1,1)) = 2 \neq 0$. Hence the given homogeneous matrix pencil is regular. Next, $\det(\mathbf{L}(\alpha, \beta)) = 2\alpha\beta$. Then $(1,0)$ and $(0,1)$ are the eigenvalues of homogeneous matrix pencil $\mathbf{L}$. One can also see that $(a, 0)$ and $(0, b)$ are also eigenvalues of $\mathbf{L}$, where a, b are arbitrary nonzero complex numbers. Hence at this point, it is important to differentiate between two eigenvalues. For a homogeneous matrix pencil, two eigenvalues (λ_0, λ_1) and (μ_0, μ_1) are called distinct if $\lambda_0\mu_1 - \lambda_1\mu_0 \neq 0$. If $\lambda = (\lambda_0, \lambda_1)$ is an eigenvalue of a homogeneous matrix pencil, then for nonzero a, $(a\lambda_0, a\lambda_1)$ is just an another representation of eigenvalue λ. Hence for simplicity, one can also choose the normalized (λ_0, λ_1), i.e., (λ_0, λ_1) is an eigenvalue of $\mathbf{L}$ if $|\lambda_0|^2 + |\lambda_1|^2 = 1$. On the other hand, let us consider the non-homogeneous version of the above matrix pencil as follows:

$$\mathbf{L}(1, \beta) := \mathbf{L}(\beta) = \begin{bmatrix} 1 + 2\beta & 1 \\ 1 & 1 \end{bmatrix}.$$

We get $\det(\mathbf{L}(\beta)) = 2\beta$, which gives only one eigenvalue and it is equal to zero. Clearly, for non-homogenous matrix pencil version, one eigenvalue is missing. This missing eigenvalue is called the infinite eigenvalue. For a homogeneous matrix pencil an infinite eigenvalue is denoted by $(0,1)$, and an eigenvalue (λ_0, λ_1) with $\lambda_0 \neq 0$ corresponds to finite eigenvalue $\frac{\lambda_1}{\lambda_0}$ of the non-homogenous matrix pencil, see [**3, 5**]. So while dealing with homogeneous matrix pencil instead of a non-homogeneous matrix pencil, we can handle both finite and infinite eigenvalue together. From the overall discussion and the eigenvalues of the homogeneous matrix pencil, we can see that $(1,0)$ corresponds to the finite eigenvalue $\frac{0}{1} = 0$, of the non-homogenous matrix pencil.

Spectrum of a matrix pencil. Let $\mathbf{L}$ be a matrix pencil of the form (1.2). Then spectrum of $\mathbf{L}$ is given as follows:

$$\Lambda(\mathbf{L}) := \{\lambda = (\lambda_0, \lambda_1) \in \mathbb{C}^2 \setminus \{(0,0)\} : \det(\mathbf{L}(\lambda)) = 0\}.$$

For a matrix pencil $\mathbf{L}$, the algebraic multiplicity of an eigenvalue $\lambda = (\lambda_0, \lambda_1) \in \Lambda(\mathbf{L})$ is its multiplicity as a zero of the characteristic polynomial $\det(\mathbf{L}(\lambda))$. The geometric multiplicity of an eigenvalue $(\lambda_0, \lambda_1) \in \Lambda(\mathbf{L})$ is defined as the dimension of the subspace $\ker(\mathbf{L}(\lambda))$. Finally, an eigenvalue is said to be semisimple if its algebraic multiplicity is equal to its geometric multiplicity.

Structured matrix pencil: Let $\mathbf{L}$ be a matrix pencil of the form (1.2). We define the different kind of structured matrix pencils based on the properties of matrices A_0 and A_1 by the following tables.

S	**Matrix structure**
T-symmetric	$A_0 = A_0^T, A_1 = A_1^T$
T-skew-symmetric	$A_0 = -A_0^T, A_1 = -A_1^T$
Hermitian	$A_0 = A_0^H, A_1 = A_1^H$
skew-Hermitian	$A_0 = -A_0^H, A_1 = -A_1^H$
T-even	$A_0 = A_0^T, A_1 = -A_1^T$
T-odd	$A_0 = -A_0^T, A_1 = A_1^T$
H-even	$A_0 = A_0^H, A_1 = -A_1^H$
H-odd	$A_0 = -A_0^H, A_1 = A_1^H$

S	**Matrix structure**
T-palindromic	$A_0 = A_1^T$
T-anti-palindromic	$A_0 = -A_1^T$
H-palindromic	$A_0 = A_1^H$
H-anti-palindromic	$A_0 = -A_1^H$

Definition 1.2.33. *A matrix pencil $\mathbf{L}$ of the form (1.2) is said to be Hankel matrix pencil if both A_0 and A_1 are Hankel matrices.*

Definition 1.2.34. *A matrix pencil $\mathbf{L}$ of the form (1.2) is said to be symmetric-Toeplitz matrix pencil if both A_0 and A_1 are symmetric-Toeplitz matrices.*

1.2.2. Polynomial eigenvalue problems

Similar to a matrix pencil, a matrix polynomial is defined as follows:

$$\mathbf{P}(\alpha,\beta) := \alpha^l A_0 + \alpha^{l-1}\beta A_1 + \cdots + \beta^l A_l, \;\; A_i \in \mathbb{C}^{n\times n} \quad \text{for } i = 0,\ldots,l. \tag{1.3}$$

$\mathbf{P}(\alpha,\beta)$ defined in (1.3) is called the homogeneous matrix polynomial in $(\alpha,\beta) \in \mathbb{C}^2$. We denote a matrix polynomial defined in (1.3) by $\mathbf{P}$, and l is called its degree. Finding $(c,d) \in \mathbb{C}^2 \setminus \{(0,0)\}$, $0 \neq x \in \mathbb{C}^n$ such that $\mathbf{P}(c,d)x = 0$ is called the polynomial eigenvalue problem (PEP). Together (c,d) is called the eigenvalue and x is called the right eigenvector of the matrix polynomial $\mathbf{P}$. $((c,d),x)$ is called the right eigenpair of matrix polynomial $\mathbf{P}$. Similarly if $y^H\mathbf{P}(c,d) = 0$ for some nonzero y, then y is called the left eigenvector corresponding to (c,d). We denote $\mathbf{P}_l(\mathbb{C}^{n\times n})$ be the space of matrix polynomials up to degree l. Similar to a matrix pencil by substituting $\alpha = 1$ in matrix polynomial $\mathbf{P}$ of the form (1.3), we can get the non-homogeneous matrix polynomial. If (c, d) is an eigenvalue of a homogeneous matrix polynomial $\mathbf{P}$, then $a(c, d)$ is also an eigenvalue of $\mathbf{P}$ for each nonzero $a \in \mathbb{C}$. Hence, to differentiate between the eigenvalues, we consider the normalized eigenvalue $(c, d) \in \mathbb{C}^2 \setminus \{(0, 0)\}$, i.e, $|c|^2 + |s|^2 = 1$.

Spectrum of a matrix pencil. Let $\mathbf{P}$ be a matrix polynomial of the form (1.3). Then spectrum of $\mathbf{P}$ is given as follows:

$$\Lambda(\mathbf{P}) := \{(c, d) \in \mathbb{C}^2 \setminus \{(0, 0)\} : \det(\mathbf{P}(c, d)) = 0\}.$$

For a matrix polynomial $\mathbf{P}$, the algebraic multiplicity of an eigenvalue $(c, d) \in \Lambda(\mathbf{P})$ is its multiplicity as a zero of the characteristic polynomial $\det(\mathbf{P}(c, d))$. The geometric multiplicity (G.M.) of an eigenvalue $(c, d) \in \Lambda(\mathbf{P})$ is defined as the dimension of the subspace $\ker(\mathbf{P}(c, d))$. Finally, an eigenvalue is said to be semisimple if its algebraic multiplicity is equal to its geometric multiplicity.

Definition 1.2.35. *A matrix polynomial $\mathbf{P} \in \mathbf{P}(\mathbb{C}^{n \times n})$ of the form (1.3) is said to be regular if and only if $\det(\mathbf{P}(c, d)) \neq 0$ for some $(c, d) \in \mathbb{C}^2 \setminus \{(0, 0)\}$, otherwise it is called a singular matrix polynomial.*

Structured matrix polynomial: Let $\mathbf{P}$ be a matrix polynomial of the form (1.3). We define the different kind of structured matrix polynomials based on the properties of matrices $A_j, j = 0 : l$, by the following tables.

S	**Matrix structure**
T-symmetric	$A_j = A_j^T$
T-skew-symmetric	$A_j = -A_j^T$
Hermitian	$A_j = A_j^H$
skew-Hermitian	$A_j = -A_j^H$
T-even	$A_j = A_j^T$ for j even, $A_j = -A_j^T$ for j odd
T-odd	$A_j = -A_j^T$ for j even, $A_j = A_j^T$ for j odd
H-even	$A_j = A_j^H$ for j even, $A_j = -A_j^H$ for j odd
H-odd	$A_j = -A_j^H$ for j even, $A_j = A_j^H$ for j odd

S	Matrix structure
T-palindromic	$A_j = A_{l-j}^T$
T-anti-palindromic	$A_j = -A_{l-j}^T$
H-palindromic	$A_j = A_{l-j}^H$
H-anti-palindromic	$A_j = -A_{l-j}^H$

Linearization of a matrix polynomial: Let $\mathbf{P}(\beta) = \sum_{i=0}^{l} \beta^i A_i$ be a matrix polynomial of degree l. A standard way to solve a polynomial eigenvalue problem $\mathbf{P}(\beta)x = 0$ is to convert it into a generalized eigenvalue problem $\mathbf{L}(\beta)z = 0$, where

$$\mathbf{L}(\beta) = X + \beta Y, \quad X, Y \in \mathbb{C}^{ln\times ln},$$

with

$$X = \begin{bmatrix} 0 & I_n & 0 & \dots & 0 \\ 0 & 0 & I_n & \ddots & \vdots \\ \vdots & & & \ddots & 0 \\ & & & & I_n \\ -A_0 & -A_1 & -A_2 & \dots & -A_{l-1} \end{bmatrix}, Y = \begin{bmatrix} -I_n & & & & \\ & -I_n & & & \\ & & \ddots & & \\ & & & -I_n & \\ & & & & -A_l \end{bmatrix}, \text{ and, } z = \begin{bmatrix} x \\ \beta x \\ \vdots \\ \beta^{l-1}x \end{bmatrix}.$$

Then we can use different available numerical methods to solve the generalized eigenvalue problem. QZ algorithm is used if all the eigenpairs are required or the problem is of small to medium size. An Arnoldi or nonsymmetric lanczos-type method can be used if a few eigenpairs are required or one can use Krylov method for large sparse problems.

If $\mathbf{P}(\beta)$ is a matrix polynomial, then the matrix pencils

$$C_1(\beta) := \begin{bmatrix} A_{l-1} & A_{l-2} & A_{l-3} & \dots & A_0 \\ -I_n & 0 & \dots & \dots & \vdots \\ 0 & -I_n & \ddots & & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \dots & & -I_n & 0 \end{bmatrix} + \beta \begin{bmatrix} A_l & & & & \\ & I_n & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & I_n \end{bmatrix}$$

and

$$C_2(\beta) := \begin{bmatrix} A_{l-1} & -I_n & 0 & \dots & 0 \\ A_{l-2} & 0 & -I_n & \dots & \vdots \\ A_{l-3} & 0 & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & -I_n \\ A_0 & 0 & \dots & \dots & 0 \end{bmatrix} + \beta \begin{bmatrix} A_l & & & & \\ & I_n & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & I_n \end{bmatrix}$$

are called the first and second companion forms of $\mathbf{P}(\lambda)$. The above forms are the most common linearization forms.

Let $\mathbf{P}$ be a quadratic matrix polynomial given by $\mathbf{P}(\beta) = A_0 + \beta A_1 + \beta^2 A_2$. In many applications the given quadratic matrix polynomial have special structure, for example when $\mathbf{P}(\beta)$ is Hermitian, then the following linearizations can be used to convert the quadratic polynomial eigenvalue problem $\mathbf{P}(\beta)x = 0$ into a Hermitian generalized eigenvalue problem $\mathbf{L}(\beta)z = 0$.

$$\mathbf{L}(\beta) = \begin{bmatrix} A_2 & 0 \\ 0 & -A_0 \end{bmatrix} + \beta \begin{bmatrix} 0 & -A_2 \\ -A_2 & -A_1 \end{bmatrix}, z = \begin{bmatrix} \beta x \\ x \end{bmatrix}$$

$$\mathbf{L}(\beta) = \begin{bmatrix} A_1 & A_0 \\ A_0 & 0 \end{bmatrix} + \beta \begin{bmatrix} A_2 & 0 \\ 0 & -A_0 \end{bmatrix}, z = \begin{bmatrix} \beta x \\ x \end{bmatrix}.$$

For more information on linearization, see [**1, 37, 70**].

1.2.3. Two-parameter eigenvalue problem

A two-parameter matrix system is defined in the following manner:

$$W(\alpha) := (W_1(\alpha), W_2(\alpha)), \text{ where } W_i(\alpha) := \alpha_0 V_{i0} + \alpha_1 V_{i1} + \alpha_2 V_{i2}, \; i = 1:2, \tag{1.4}$$

where $V_{ij} \in \mathbb{C}^{n_i \times n_i}$ for $i = 1:2, j = 0:2$, and $\alpha = (\alpha_0, \alpha_1, \alpha_2) \in \mathbb{C}^3$. We denote the system (1.4) by $W := (W_1, W_2) \in \mathbb{C}^{n_1 \times n_1} \times \mathbb{C}^{n_2 \times n_2}$. Finding $(\lambda_0, \lambda_1, \lambda_2) \in \mathbb{C}^3 \setminus \{(0,0,0)\}$, and non zero vectors $x_i \in \mathbb{C}^{n_i}$ such that $W_i(\lambda)x_i = 0$ for $i = 1:2$ is called a two parameter eigenvalue problem (TEP). Further, $(\lambda_0, \lambda_1, \lambda_2) = \lambda \in \mathbb{C}^3 \setminus \{(0,0,0)\}$ is called an eigenvalue of (1.4), and the pair (x_1, x_2) is called an eigenvector of W corresponding to λ. We also denote an eigenvector corresponding to λ by $x = x_1 \otimes x_2 \in \mathbb{C}^{n_1 n_2}$. (λ, x) is called the eigenpair of W. We denote $\mathbb{K}$ be the space of two-parameter matrix systems. By substituting $\alpha_0 = 1$ in (1.4), we can get the non-homogeneous form of a two-parameter matrix system.

For a two-parameter matrix system of the form (1.4), we define the following operators:

$$\Delta_0 = V_{11} \otimes V_{22} - V_{12} \otimes V_{21}, \Delta_1 = V_{12} \otimes V_{20} - V_{10} \otimes V_{22}, \Delta_2 = V_{10} \otimes V_{21} - V_{12} \otimes V_{20}$$

If for any $\alpha_0, \alpha_1, \alpha_2 \in \mathbb{C}$, we have $\alpha_0 \Delta_0 + \alpha_1 \Delta_1 + \alpha_2 \Delta_2$ is nonsingular, then we said that a two-parameter matrix system is nonsingular and then from system of generalized eigenvalue problems $\Delta_0 z = \lambda_0 \Delta z, \Delta_1 z = \lambda_1 \Delta z$, $\Delta_2 z = \lambda_2 \Delta z$, we can get the eigenvalue

$(\lambda_0, \lambda_1, \lambda_2)$, and corresponding eigenvector z, of W, where $z \in \mathbb{C}^{n_1 n_2}, z = x_1 \otimes x_2$ is a decomposable tensor. If for every $\alpha_0, \alpha_1, \alpha_2 \in \mathbb{C}$, we have $\alpha_0 \Delta_0 + \alpha_1 \Delta_1 + \alpha_2 \Delta_2$ is singular, then W is said to a singular two-parameter matrix system.

Spectrum of a two-parameter matrix system. Let W be a two-parameter matrix system of the form (1.4). Then the set of eigenvalues of W is defined as

$$\Lambda(W) = \{\lambda \in \mathbb{C}^3 \setminus \{(0,0,0)\} : \det(W_i(\lambda)) = 0 \text{ for } i = 1, 2\}.$$

If λ is an eigenvalue of a homogeneous two-parameter matrix system W, then $a\lambda$ is also an eigenvalue of W for each nonzero $a \in \mathbb{C}$. Hence, to differentiate between the eigenvalues, we consider the normalized eigenvalue $\lambda \in \mathbb{C}^3 \setminus \{(0,0,0)\}$, i.e., $|\lambda_0|^2 + |\lambda_1|^2 + |\lambda_2|^2 = 1$.

Definition 1.2.36. [42] *The geometric multiplicity (G.M.) of an eigenvalue* $\lambda = (\lambda_0, \lambda_1, \lambda_2)$ *of a two-parameter* W *is defined in the following manner:*

$$G.M. = \dim(\ker(W_1(\lambda))) \times \dim(\ker(W_2(\lambda))).$$

Definition 1.2.37. [60] *The algebraic multiplicity (A.M.) of* $\lambda = (\lambda_0, \lambda_1, \lambda_2)$ *is equal to the intersection multiplicity of two curves* $w_1 = 0$ *and* $w_2 = 0$ *at* λ*. Here* $w_i = \det(W_i(\alpha))$ *for* $i = 1, 2$.

Definition 1.2.38. *An eigenvalue* $\lambda = (\lambda_0, \lambda_1, \lambda_2)$ *of* W *is semisimple if its algebraic and geometric multiplicity coincide.*

Structured two-parameter matrix systems: Let W be a two-parameter matrix system of the form (1.4). We define the different kind of structured two-parameter matrix systems based on the properties of matrices $V_{ij}, i = 1:2, j = 0:2$, by the following tables.

S	**Matrix structure**
Complex *symmetric*	$V_{ij} = V_{ij}^T$ for $i = 1:2, j = 0:2$
Complex *skew-symmetric*	$V_{ij} = -V_{ij}^T$ for $i = 1:2, j = 0:2$
Hermitian	$V_{ij} = V_{ij}^H$ for $i = 1:2, j = 0:2$
Skew-Hermitian	$V_{ij} = -V_{ij}^H$ for $i = 1:2, j = 0:2$
T*-even alternating*	$V_{ij} = V_{ij}^T$ for $i = 1:2, j = 0, 2$ and $V_{i1} = -V_{i1}^T$ for $i = 1:2$.
T*-odd alternating*	$V_{ij} = -V_{ij}^T$ for $i = 1:2, j = 0, 2$ and $V_{i1} = V_{i1}^T$ for $i = 1:2$.
H*-even alternating*	$V_{ij} = V_{ij}^H$ for $i = 1:2, j = 0, 2$ and $V_{i1} = -V_{i1}^H$ for $i = 1:2$.
H*-odd alternating*	$V_{ij} = -V_{ij}^H$ for $i = 1:2, j = 0, 2$ and $V_{i1} = V_{i1}^H$ for $i = 1:2$.

Two-parameter norm: Let $W \in \mathbb{K}$ be a two-parameter matrix system of the form (1.4). Then $|||.||| : \mathbb{K} \to \mathbb{R}$ is defined as $|||W||| = \sqrt{\sum_{i=1}^{2}\sum_{j=0}^{2}\|V_{ij}\|_F^2}$, forms a norm over vector space $\mathbb{K}$.

- Clearly $|||W||| \geq 0$.
- For $W \in \mathbb{K}$ and $|||W||| = 0$ implies that $\sqrt{\sum_{i=1}^{2}\sum_{j=0}^{2}\|V_{ij}\|_F^2} = 0$. This gives $\|V_{ij}\|_F = 0$. We get $V_{ij} = 0$ for $i = 1:2; j = 0:2$. Hence we get $W_1 = 0 = W_2$. In particular $W = 0$. On the other hand if $W = 0$ then clearly $W_1 = 0 = W_2$ which gives $|||W||| = 0$.
- For $a \in \mathbb{C}$ we have $|||aW||| = \sqrt{\sum_{i=1}^{2}\sum_{j=0}^{2}|a|^2\|V_{ij}\|_F^2} = |a|\sqrt{\sum_{i=1}^{2}\sum_{j=0}^{2}\|V_{ij}\|_F^2} = |a||||W|||$.
- If $W, \overline{W} \in \mathbb{K}$ then $|||W + \overline{W}|||^2 = \sum_{i=1}^{2}\sum_{j=0}^{2}\|V_{ij} + \overline{V}_{ij}\|_F^2 \leq \sum_{i=1}^{2}\sum_{j=0}^{2}\|V_{ij}\|^2 + \sum_{i=1}^{2}\sum_{j=0}^{2}\|\overline{V}_{ij}\|_F^2 = |||W|||^2 + |||\overline{W}|||^2$.

$|||.|||$ is called the two-parameter norm. In the similar manner, we can define the weighted two-parameter norm as follows:

Let $W \in \mathbb{K}$ be a two-parameter matrix system of the form (1.4), and $w = \begin{bmatrix} w_1^T \\ w_2^T \end{bmatrix} \in \mathbb{R}^{2\times 3}$ be a nonnegative matrix, where $w_i = [w_{i0}, w_{i1}, w_{i2}]^T \in \mathbb{R}^3 \setminus \{(0,0,0)\}$ and $w_{ij}, i = 1:2, j = 0:2$ are nonnegative real numbers. Then $|||.|||_w : \mathbb{K} \to \mathbb{R}$ is defined as $|||W|||_w = \sqrt{\sum_{i=1}^{2}\sum_{j=0}^{2}\|w_{ij}V_{ij}\|_F^2}$, forms a norm over vector space $\mathbb{K}$ if every w_{ij} is positive and form a seminorm if otherwise.

Let $W = (W_1, W_2)$ be a two-parameter matrix system of the form (1.4). Then we define the normal rank of W_i for $i = 1:2$ by

$$\mathrm{Nrank}\,(W_i) = \max_{\lambda \in \mathbb{C}^3 \setminus \{(0,0,0)\}} \mathrm{rank}(W_i(\lambda)).$$

Let us discuss the above definition with the following example.

Example 1.2.39. *Let W be a two-parameter matrix system of the form (1.4), where*

$$W_1(\alpha) = \begin{bmatrix} \alpha_0 + 2\alpha_1 + 3\alpha_2 & 0 \\ 0 & \alpha_1 - 2\alpha_2 \end{bmatrix}, W_2(\alpha) = \begin{bmatrix} 0 & 0 \\ 0 & \alpha_1 + 2\alpha_2 \end{bmatrix}.$$

Clearly for $\lambda = (1,1,1)$, we have $\mathrm{rank}(W_1(\lambda)) = 2$. *Hence* $\mathrm{Nrank}\,(W_1) = 2$. *But on the other hand, for all $\lambda \in \mathbb{C}^3 \setminus \{(0,0,0)\}$, we get* $\mathrm{rank}(W_2(\lambda)) = 1$. *Hence* $\mathrm{Nrank}\,(W_2) = 1$.

CHAPTER 2

BACKWARD ERROR ANALYSIS OF TWO APPROXIMATE EIGENPAIRS OF A DOUBLE-SEMISIMPLE EIGENVALUE

Abstract: This chapter deals with the backward error analysis of two approximate eigenpairs of a double-semisimple eigenvalue for structured and unstructured matrix pencils. We develop the backward error results in such a way that one can theoretically compare the structured and unstructured backward errors of a single eigenpair and backward error of two eigenpairs of a double semisimple eigenvalue.

2.1. Introduction

Matrix pencils arise in many applications, see [**1**, **17**, **67**, **51**]. Backward erorr analysis of eigenvalues has been developed by different authors in the literature. Malyshev [**53**] has discussed the minimal perturbation of a given n-by-n matrix to the nearest matrices that have $\lambda \in \mathbb{C}$ as a multiple eigenvalue with respect to 2-norm. Further, this work has been extended for two distinct prescribed numbers, and the nearest matrix has been obtained that contains these prescribed numbers in its spectrum, see [**31**, **48**, **57**]. For a given n-by-n matrix, the above work has been extended for $k\,(k \leq n)$ prescribed eigenvalues by Lippert [**49**] and Kokabifar et al. [**43**]. For the matrix polynomial setup, E. Kokabifara et al. [**44**] have extended the above idea for k specified distinct eigenvalues and provided the backward error and the minimum perturbed matrix polynomial for the unstructured case. Similar to the backward error of eigenvalues, different authors have developed the backward error analysis of a single approximate eigenpair for unstructured as well as structured matrix pencils and matrix polynomials (see, [**1**, **8**, **9**, **45**, **67**]). For the matrix case, Tisseur [**71**] has extended the backward error results from one specified eigenpair to more specified eigenpairs. The author has obtained the backward error formula for *Hermitian, skew-Hermitian, complex symmetric, complex skew-symmetric* and doubly structured matrices using [**68**, Lemma 1.4], [**71**, Lemma 2.4] along with "W-trick". Tisseur has investigated the structured backward error analysis by imposing the appropriate conditions on approximate eigenpairs, for example, while computing the backward error result for *Hermitian* matrices, the author has assumed that the columns of X_k, the approximate eigenvectors matrix, are orthonormal. This condition seems to be natural as we always get a set of orthonormal vectors for a given *Hermitian* matrix. Similar to the *Hermitian* case, the author has imposed two natural conditions during the backward error analysis of *Hermitian unitary* matrices, first is the orthonormality condition on X_k, and second is the approximate eigenvalues matrix, $\Lambda_k = \text{diag}(\pm 1)$. In the same manner, in this Chapter we discuss the natural conditions on the given approximate eigenpairs to perform the backward error analysis. Next, in [**18**] Chu and Golub have studied the backward error analysis of one or more approximate eigenpairs for unstructured nonsquare matrix pencils when approximate eigenvalues are distinct, and eigenvectors are linearly independent. Though they worked on one or more eigenpairs and obtained the unstructured backward error, results of backward error analysis of more than one approximate eigenpairs for structured matrix pencils are still unanswered.

The above discussion on the backward error analysis of approximate eigenvalues or eigenpairs for unstructured/structured matrices, matrix pencils and matrix polynomials leads to a natural question that what will be the cumulative backward error of two approximate eigenpairs of a given matrix pencil ? Before finding the answer to this question, we want to emphasize on the point that whenever the author in [**71**] has imposed a condition on X_k or Λ_k to obtain the structured backward error formula, that condition seems to be a natural one for that particular structure. In a similar manner, to answer the question raised above, we shall propose certain conditions, which we believe are natural, in order to approximate eigenpairs under which we can obtain the backward error results for a large class of matrix pencils. To understand the natural condition, we recall one important result: if an eigenvalue of a matrix pencil is repeating but semisimple, we always get a set of orthonormal eigenvectors corresponding to that eigenvalue (see, Lemma 2.2.6 for more information). Using this result, we obtain the backward error formula for two approximate eigenpairs of a semisimple eigenvalue with multiplicity two. Here we add that a generic situation for a multiple eigenvalue is a double eigenvalue (see, for example, [**56**]). For obtaining backward error results, we adopt and extend the technique of [**1, 8, 9**]. This technique works on the orthonormal properties of approximate eigenvectors. In general, we can not get the orthonormal vectors corresponding to distinct eigenvalues; hence the question of finding the structured backward error of two approximate eigenpairs is still open when eigenvalues are distinct or defective. We answer the above-raised question for structured as well as unstructured matrix pencils. We work with *T-symmetric/T-skew-symmetric, Hermitian/skew-Hermitian, T-even/T-odd,* and *H-even/H-odd* matrix pencils (see, [**17, 32, 62, 87**] for more on structured matrix pencils and matrix polynomials).

Let $\mathbf{L}(\mathbb{C}^{n\times n})$ be the space of matrix pencils, and let $\mathbf{L} \in \mathbf{L}(\mathbb{C}^{n\times n})$ be of the form $\mathbf{L}(\alpha,\beta) := \alpha A_0 + \beta A_1$, where $A_0, A_1 \in \mathbb{C}^{n\times n}, \alpha = (\alpha,\beta) \in \mathbb{C}^2$. Suppose (λ, x_i) for $i = 1,2$ are two approximate eigenpairs of $\mathbf{L}$, where $\lambda \in \mathbb{C}^2 \setminus \{(0,0)\}$ and $0 \neq x_i \in \mathbb{C}^n$. In this chapter, we find the nearest $\delta\mathbf{L} \in \mathbf{L}(\mathbb{C}^{n\times n})$ of the form $\delta\mathbf{L}(\alpha,\beta) := \alpha\delta A_0 + \beta\delta A_1$, $\delta A_0, \delta A_1 \in \mathbb{C}^{n\times n}$ such that two approximate eigenpairs (λ, x_1) and (λ, x_2) become the exact eigenpairs of $\mathbf{L} + \delta\mathbf{L}$. We use the *Frobenius norm to investigate the structured backward error analysis.* Results are developed in such a way that *T-symmetric* & *T-skew-symmetric* cases are presented in a single platform. Similarly, *Hermitian* & *skew-Hermitian, T-even* & *T-odd,* and *H-even* & *H-odd cases are also presented* in a single platform. Further, we find relationships between the backward error of a single approximate eigenpair, the backward error of two approximate eigenpairs for a semisimple eigenvalue with multiplicity two, and the structured backward error of two approximate eigenpairs for a semisimple eigenvalue with multiplicity two.

2.2. Structured matrix pencils and preliminaries

Let $\mathbf{L}(\mathbb{C}^{n\times n})$ be the space of matrix pencils and homogeneous matrix pencil $\mathbf{L} \in \mathbf{L}(\mathbb{C}^{n\times n})$ is defined as follows :

$$\mathbf{L}(\alpha,\beta) := \alpha A_0 + \beta A_1, \quad A_0, A_1 \in \mathbb{C}^{n\times n},\ (\alpha,\beta) \in \mathbb{C}^2. \tag{2.1}$$

Finding $\lambda = (\lambda_0, \lambda_1) \in \mathbb{C}^2 \setminus \{(0,0)\}$, $0 \neq x \in \mathbb{C}^n$ such that $\mathbf{L}(\lambda)x = 0$, is called the *generalized eigenvalue problem* (GEP). λ is called an eigenvalue of (2.1) and x is the corresponding right eigenvector. If $0 \neq y \in \mathbb{C}^n$ such that $y^H\mathbf{L}(\lambda) = 0$, then y is called the left eigenvector corresponding to λ. We denote (2.1) by $\mathbf{L}$, and (λ, x) is an eigenpair of $\mathbf{L}$. We define $|||\mathbf{L}|||_F := \|(\|A_0\|_F, \|A_1\|_F)\|_2 = (\|A_0\|_F^2 + \|A_1\|_F^2)^{1/2}$, where $\|.\|_F$ denotes the *Frobenius norm* on $\mathbb{C}^{n\times n}$, and $\|.\|_2$ denotes the 2-*norm* on $\mathbb{C}^n$. Non-homogeneous matrix pencils can be obtain by fixing $\alpha = 1$ in (2.1). We denote the spectrum of $\mathbf{L}$ by $\Lambda(\mathbf{L})$, and it is given by

$$\Lambda(\mathbf{L}) := \{\lambda = (\lambda_0, \lambda_1) \in \mathbb{C}^2 \setminus \{(0,0)\} : \det(\mathbf{L}(\lambda)) = 0\}.$$

When $(\lambda_0, \lambda_1) \in \mathbb{C}^2 \setminus \{(0,0)\}$ is an eigenvalue of $\mathbf{L}$, then $(a\lambda_0, a\lambda_1)$ is another representation of the eigenvalue (λ_0, λ_1) for any $0 \neq a \in \mathbb{C}$. Hence for a given homogeneous $\mathbf{L}$, we normalize $(\lambda_0, \lambda_1) \in \Lambda(\mathbf{L})$ as $|\lambda_0|^2 + |\lambda_1|^2$ and consider $\Lambda(\mathbf{L})$ is a subset of unit sphere $\mathbf{S}^1 := \{(\lambda_0, \lambda_1) \in \mathbb{C}^2 : |\lambda_0|^2 + |\lambda_1|^2 = 1\}$. By working in the homogeneous setup, one can handle the infinity eigenvalue together with the finite eigenvalue (see, [**4**] for more detail on homogeneous eigenvalue problems). Throughout this chapter, we consider regular matrix pencils for the establishment of our results.

Definition 2.2.1. *The algebraic multiplicity (A.M.) of an eigenvalue* $\lambda = (\lambda_0, \lambda_1) \in \Lambda(\mathbf{L})$ *is its multiplicity as a zero of the characteristic polynomial* $\det(\mathbf{L}(\lambda))$.

Definition 2.2.2. *The geometric multiplicity (G.M.) of an eigenvalue* $(\lambda_0, \lambda_1) \in \Lambda(\mathbf{L})$ *is defined as the dimension of the subspace* $\ker(\mathbf{L}(\lambda))$.

Definition 2.2.3. *An eigenvalue is said to be semisimple if its algebraic multiplicity is equal to its geometric multiplicity (see,* [**81**] *for more detail on semisimple eigenvalues).*

Let $\mathbf{L}$ be a matrix pencil of the form (2.1), and let $\lambda = (\lambda_0, \lambda_1) \in \mathbb{C}^2 \setminus \{(0,0)\}$ be its eigenvalue. Then λ is said to be a *double eigenvalue* if its algebraic multiplicity is two. We will consider a *double-semisimple eigenvalue* for the backward error analysis, since a generic situation for a multiple eigenvalue is a double eigenvalue (see, [**56, 80**] for more information on *double-semisimple eigenvalues*). We work with structured matrix pencils of the form (2.1). These structured matrix pencils are defined by Table 2.1 based on the properties of matrices $A_0, A_1 \in \mathbb{C}^{n\times n}$. After defining the different structured matrix pencils, we extend the backward error definition from a single approximate eigenpair to two approximate eigenpairs. Backward error analysis for a single approximate eigenpair has been discussed in [**1**].

Definition 2.2.4. *Let* $\mathbf{L}$ *be a matrix pencil of the form (2.1). Let* (λ, x_1) *and* (λ, x_2) *be two approximate eigenpairs of* $\mathbf{L}$ *where* $\lambda \in \mathbb{C}^2 \setminus \{(0,0)\}$, *and* $0 \neq x_1, x_2 \in \mathbb{C}^n$. *Then unstructured and structured backward errors of two approximate eigenpairs* (λ, x_1) *and* (λ, x_2) *are defined by*

$$\eta_F(\lambda, x_1, x_2, \mathbf{L}) := \inf\{\|\delta\mathbf{L}\|_F, \quad (\mathbf{L}(\lambda) + \delta\mathbf{L}(\lambda))x_i = 0; \ \textit{for } i = 1, 2\}, \ \textit{and}$$

$$\eta_F^{\mathbf{S}}(\lambda, x_1, x_2, \mathbf{L}) := \inf\{\|\delta\mathbf{L}\|_F,\ \delta\mathbf{L} \in \mathbf{S},\ \ (\mathbf{L}(\lambda) + \delta\mathbf{L}(\lambda))x_i = 0;\ \textit{for } i = 1, 2\},$$

respectively. Here $\delta\mathbf{L}$ is a matrix pencil of the form (2.1) such that $\delta\mathbf{L}(\alpha, \beta) = \alpha\delta A_0 + \beta\delta A_1$ with $\delta A_0, \delta A_1 \in \mathbb{C}^{n\times n}$, $\|\delta\mathbf{L}\|_F := \sqrt{\|\delta A_0\|_F^2 + \|\delta A_1\|_F^2}$, and

$$\mathbf{S} := \{\ T\text{-symmetric, } T\text{-skew-symmetric, Hermitian, skew-Hermitian, } T\text{-even, } T\text{-odd,}\\ H\text{-even, } H\text{-odd}\}.$$

After defining the backward error formulas, now we recall some useful results.

Remark 2.2.5. *Eigenvectors corresponding to a double-semisimple eigenvalue of a matrix pencil* $\mathbf{L}$*, are not uniquely determined. Using this information, we will establish the following lemma.*

Lemma 2.2.6. *Suppose* $p = (p_0, p_1) \in \mathbb{C}^2 \setminus \{(0, 0)\}$ *is a double-semisimple eigenvalue of a matrix pencil* $\mathbf{L}$*. Then there exists two orthonormal vectors* $y_1, y_2 \in \mathbb{C}^n$*, such that* $\mathbf{L}(p)y_i = 0$ *for* $i = 1, 2$*. In particular, every double-semisimple eigenvalue* p *of* $\mathbf{L}$ *has two orthonormal eigenvectors.*

Proof. Let (p_0, p_1) be a *double-semisimple eigenvalue* of $\mathbf{L}$. It implies that its algebraic and geometric multiplicity will be two. Then there exists two linearly independent eigenvectors $z_1, z_2 \in \mathbb{C}^n$ such that $\mathbf{L}(p)z_i = 0$ for $i = 1, 2$. By *Gram-Schmidt* process, we can set $y_1 = \frac{z_1}{\|z_1\|}$ and $y_2 = \frac{z_2 - \gamma z_1}{\|z_2 - \gamma z_1\|}$, where $\gamma = \frac{z_1^H z_2}{z_1^H z_1} \in \mathbb{C}$. We can easily see that $\mathbf{L}(p)y_i = 0$, and y_1, y_2 are orthonormal. ■

S	**Matrix structure**
T-*symmetric*	$A_0 = A_0^T, A_1 = A_1^T$
T-*skew-symmetric*	$A_0 = -A_0^T, A_1 = -A_1^T$
Hermitian	$A_0 = A_0^H, A_1 = A_1^H$
skew-Hermitian	$A_0 = -A_0^H, A_1 = -A_1^H$
T-*even*	$A_0 = A_0^T, A_1 = -A_1^T$
T-*odd*	$A_0 = -A_0^T, A_1 = A_1^T$
H-*even*	$A_0 = A_0^H, A_1 = -A_1^H$
H-*odd*	$A_0 = -A_0^H, A_1 = A_1^H$

TABLE 2.1. An overview for structured matrix pencils

The above lemma gives a guarantee that for *a double-semisimple eigenvalue*, we always get two orthonormal eigenvectors.

Remark 2.2.7. *Using the Gram-Schmidt process, we can extend the above lemma for a semisimple eigenvalue with algebraic multiplicity more than two.*

After recalling the preliminary results, we will establish backward error results for two eigenpairs of a *double-semisimple eigenvalue.*

Remark 2.2.8. *Since we are interested in finding the backward error of two approximate eigenpairs of a double-semisimple eigenvalue, hence in light of Lemma 2.2.6, from now onwards, we will take the orthonormal eigenvectors corresponding to a double-semisimple eigenvalue.*

Lemma 2.2.9. *Let $x_1, x_2 \in \mathbb{C}^n$ be orthonormal vectors. Define $P_{x_1:x_2} := (I - x_1x_1^H - x_2x_2^H)$, $\quad P^c_{x_1} := (I - x_2x_2^H)$, and $P^c_{x_2} := (I - x_1x_1^H)$. Then*

1. $P_{x_1:x_2} = P^H_{x_1:x_2}$,
2. $P_{x_1:x_2}x_1 = P_{x_1:x_2}x_2 = 0$,
3. $P^c_{x_1}x_2 = 0 = P^c_{x_2}x_1$.

Proof. Proof is computational and is easy to check.

Next, we discuss the backward error analysis of T-symmetric and T-skew-symmetric matrix pencils.

2.3. Backward error for T-symmetric and T-skew-symmetric matrix pencils

In this section, we present the structured backward error analysis of two approximate eigenpairs of a *double-semisimple eigenvalue* for T*-symmetric* and T*-skew-symmetric* matrix pencils. We start this section with the following existence theorem for T*-symmetric/*T*-skew-symmetric* matrix pencils. Throughout this section, $\epsilon = 1$ represents a T-symmetric matrix pencil and $\epsilon = -1$ represents a T*-skew-symmetric* matrix pencil.

Theorem 2.3.1. *Let $\mathbf{L} \in \mathbf{L}(\mathbb{C}^{n\times n})$ be a T-symmetric/ T-skew-symmetric homogeneous matrix pencil of the form (2.1). Let (λ, x_1) and (λ, x_2) be two approximate eigenpairs of*

$\mathbf{L}$, *where* $\lambda = (\lambda_0, \lambda_1) \in \mathbb{C}^2 \setminus \{(0,0)\}$ *is a double-semisimple eigenvalue, and* $x_1, x_2 \in \mathbb{C}^n$ *are orthonormal vectors. Set* $k_i := -\mathbf{L}(\lambda)x_i$ *for* $i = 1:2$, *and define*

$$\delta A_0 = \sum_{i=1}^{2} \overline{\lambda}_0 \frac{k_i x_i^H + \epsilon \overline{x_i} k_i^T P_{x_1:x_2}}{H_2^2(\lambda)} \text{ and } \delta A_1 = \sum_{i=1}^{2} \overline{\lambda}_1 \frac{k_i x_i^H + \epsilon \overline{x_i} k_i^T P_{x_1:x_2}}{H_2^2(\lambda)},$$

where $H_2(\lambda) = (|\lambda_0|^2 + |\lambda_1|^2)^{1/2}$. *Then there exists a* T*-symmetric/*T*-skew-symmetric matrix pencil* $\delta\mathbf{L} \in \mathbf{L}(\mathbb{C}^{n\times n})$ *of the form* $\delta\mathbf{L}(\alpha,\beta) = \alpha\delta A_0 + \beta\delta A_1$, *such that* $(\mathbf{L}(\lambda) + \delta\mathbf{L}(\lambda))x_i = 0$ *for* $i = 1:2$.

Proof. The proof is computational and is easy to check. ■

Lemma 2.3.2. *Let* $\mathbf{L} \in \mathbf{L}(\mathbb{C}^{n\times n})$ *be a* T*-symmetric/* T*-skew-symmetric homogeneous matrix pencil of the form (2.1). Let* (λ, x_1) *and* (λ, x_2) *be two approximate eigenpairs of* $\mathbf{L}$, *where* $\lambda = (\lambda_0, \lambda_1) \in \mathbb{C}^2 \setminus \{(0,0)\}$ *is a double-semisimple eigenvalue, and* $x_1, x_2 \in \mathbb{C}^n$ *are orthonormal vectors. Set* $k_i := -\mathbf{L}(\lambda)x_i$ *for* $i = 1:2$. *Then the following equality holds for* $t = 1,2$

$$(x_1^T k_2)(\epsilon \overline{x}_2 x_1^H + \overline{x}_1 x_2^H)x_t = \sum_{j=1, j\neq t}^{2} \overline{x_j} x_j^T k_t.$$

Proof. The proof is computational and obtained by using the fact that $\epsilon x_1^T k_2 = x_2^T k_1$. ■

Next, we establish the main result of this section.

Theorem 2.3.3. *Let* $\mathbf{L} \in \mathbf{L}(\mathbb{C}^{n\times n})$ *be a* T*-symmetric/* T*-skew-symmetric homogeneous matrix pencil of the form (2.1). Let* (λ, x_1) *and* (λ, x_2) *be two approximate eigenpairs of* $\mathbf{L}$, *where* $\lambda = (\lambda_0, \lambda_1) \in \mathbb{C}^2 \setminus \{(0,0)\}$ *is a double-semisimple eigenvalue, and* $x_1, x_2 \in \mathbb{C}^n$ *are orthonormal vectors. Set* $k_i := -\mathbf{L}(\lambda)x_i$ *for* $i = 1:2$. *Then there exists a* T*-symmetric/* T*-skew-symmetric* $\delta\mathbf{L}$ *of the form* $\delta\mathbf{L}(\alpha,\beta) = \alpha\delta A_0 + \beta\delta A_1$ *such that* $(\mathbf{L}(\lambda) + \delta\mathbf{L}(\lambda))x_i = 0$. *The perturbation matrices are given by*

$$\delta A_0 = \sum_{i=1}^{2} (\overline{\lambda}_0 \frac{\overline{P}_{x_i}^c k_i x_i^H + \epsilon \overline{x_i} k_i^T P_{x_1:x_2}}{H_2^2(\lambda)}) + \frac{\overline{\lambda}_0 (x_1^T k_2)(\epsilon \overline{x}_2 x_1^H + \overline{x}_1 x_2^H)}{H_2^2(\lambda)},$$

$$\delta A_1 = \sum_{i=1}^{2} (\overline{\lambda}_1 \frac{\overline{P}_{x_i}^c k_i x_i^H + \epsilon \overline{x_i} k_i^T P_{x_1:x_2}}{H_2^2(\lambda)}) + \frac{\overline{\lambda}_1 (x_1^T k_2)(\epsilon \overline{x}_2 x_1^H + \overline{x}_1 x_2^H)}{H_2^2(\lambda)}.$$

The backward error is given by

$$(\eta_F^{\mathbf{S}}(\lambda, x_1, x_2, \mathbf{L}))^2 = \sum_{i=1}^{2} (\frac{2\|k_i\|_2^2 - ((1+\epsilon)/2))|x_i^T k_i|^2}{H_2^2(\lambda)}) - 2\frac{|x_2^T k_1|^2}{H_2^2(\lambda)}.$$

Proof. From Theorem 2.3.1, there exists a T-symmetric/T-skew-symmetric $\delta\mathbf{L}$ of the form $\delta\mathbf{L}(\alpha,\beta) = \alpha\delta A_0 + \beta\delta A_1$ such that $\mathbf{L}(\lambda)x_i + \delta\mathbf{L}(\lambda)x_i = 0$ for $i = 1,2$. For constructing δA_j for $j = 0,1$ such that $\delta A_j = \epsilon\delta A_j^T$, we consider

$$\widetilde{\delta A_j} = U^T \delta A_j U = \begin{array}{c} \\ 2 \\ n-2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} \quad 2 \quad & \quad n-2 \quad \end{array} \\ \left[\begin{array}{c|c} \widehat{\delta A_j} & \epsilon{\delta B_j}^T \\ \hline \delta B_j & \delta D_j \end{array}\right] \end{array}, \tag{2.2}$$

where $\widehat{\delta A_j} = \begin{bmatrix} \frac{(1+\epsilon)}{2}\delta a_{j,11} & \epsilon\delta a_{j,12} \\ \delta a_{j,12} & \frac{(1+\epsilon)}{2}\delta a_{j,22} \end{bmatrix}$, $\delta B_j = \begin{bmatrix} b_{j1} & b_{j2} \end{bmatrix}$, $\delta D_j = \epsilon{\delta D_j}^T$ for $j = 0,1$, and $U \in \mathbb{C}^{n\times n}$ is a unitary matrix such that $U = \begin{bmatrix} V_1 & V_2 \end{bmatrix}$ with $V_1 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \in \mathbb{C}^{n\times 2}$.

We need to construct $\delta\mathbf{L}$ such that $(\mathbf{L}(\lambda) + \delta\mathbf{L}(\lambda))x_i = 0$ for $i = 1:2$. Since it is given that $k_i = -\mathbf{L}(\lambda)x_i$, we get $k_i = \delta\mathbf{L}(\lambda)x_i$. From (2.2), we get $\widetilde{\delta\mathbf{L}}(\lambda) = U^T\delta\mathbf{L}(\lambda)U$. Using the properties of U, we get $\widetilde{\delta\mathbf{L}}(\lambda)U^H x_i = U^T\delta\mathbf{L}(\lambda)x_i = U^T k_i$. This implies $\lambda_0 \begin{bmatrix} \widehat{\delta A_0} & \epsilon\delta B_0^T \\ \delta B_0 & \delta D_0 \end{bmatrix} \begin{bmatrix} e_i \\ 0 \end{bmatrix} + \lambda_1 \begin{bmatrix} \widehat{\delta A_1} & \epsilon\delta B_1^T \\ \delta B_1 & \delta D_1 \end{bmatrix} \begin{bmatrix} e_i \\ 0 \end{bmatrix} = \begin{bmatrix} V_1^T k_i \\ V_2^T k_i \end{bmatrix}$. Further simplification gives

$$\begin{bmatrix} (\lambda_0\widehat{\delta A_0} + \lambda_1\widehat{\delta A_1})e_i \\ (\lambda_0\delta B_0 + \lambda_1\delta B_1)e_i \end{bmatrix} = \begin{bmatrix} V_1^T k_i \\ V_2^T k_i \end{bmatrix}, \tag{2.3}$$

where $e_i \in \mathbb{C}^2$ is a vector having 1 at i^{th} position and 0 elsewhere. From (2.3), we get the following equations

$$((1+\epsilon)/2)\lambda_0\delta a_{0,ii} + ((1+\epsilon)/2)\lambda_1\delta a_{1,ii} = x_i^T k_i, \quad i = 1,2, \tag{2.4}$$

$$\lambda_0 b_{0i} + \lambda_1 b_{1i} = V_2^T k_i,\ i = 1,2. \tag{2.5}$$

The minimum norm solutions of (2.4) and (2.5) are given by $\delta a_{0,ii} = \frac{(1+\epsilon)}{2}\frac{\bar{\lambda}_0}{H_2^2(\lambda)}x_i^T k_i$, $\delta a_{1,ii} = \frac{(1+\epsilon)}{2}\frac{\bar{\lambda}_1}{H_2^2(\lambda)}x_i^T k_i$, $b_{0i} = \frac{\bar{\lambda}_0}{H_2^2(\lambda)}V_2^T k_i, b_{1i} = \frac{\bar{\lambda}_1}{H_2^2(\lambda)}V_2^T k_i$. By Equation 2.3, we get two more equations

$$\lambda_0\delta a_{0,12} + \lambda_1\delta a_{1,12} = x_2^T k_1, \tag{2.6}$$

$$\lambda_0\delta a_{0,12} + \lambda_1\delta a_{1,12} = \epsilon x_1^T k_2. \tag{2.7}$$

Since $A_j = \epsilon A_j^T$ for $j = 0,1$, we get $\epsilon x_1^T k_2 = x_2^T k_1$. Hence Equations 2.6 and 2.7 are the same. The minimum norm solution of (2.7) is given by $\delta a_{0,12} = \frac{\epsilon\bar{\lambda}_0}{H_2^2(\lambda)}x_1^T k_2, \delta a_{1,12} =$

$\frac{\epsilon\overline{\lambda}_1}{H_2^2(\lambda)}x_1^Tk_2$. Substituting back all these obtained entries in (2.2), we get

$$\delta A_j = \overline{U}\begin{bmatrix} \frac{(1+\epsilon)}{2}\frac{\overline{\lambda}_j}{H_2^2(\lambda)}x_1^Tk_1 & \frac{\overline{\lambda}_j}{H_2^2(\lambda)}x_1^Tk_2 & \epsilon\frac{\overline{\lambda}_j}{H_2^2(\lambda)}(V_2^Tk_1)^T \\ \frac{\epsilon\overline{\lambda}_j}{H_2^2(\lambda)}x_1^Tk_2 & \frac{(1+\epsilon)}{2}\frac{\overline{\lambda}_j}{H_2^2(\lambda)}x_2^Tk_2 & \epsilon\frac{\overline{\lambda}_j}{H_2^2(\lambda)}(V_2^Tk_2)^T \\ \frac{\overline{\lambda}_j}{H_2^2(\lambda)}V_2^Tk_1 & \frac{\overline{\lambda}_j}{H_2^2(\lambda)}V_2^Tk_2 & \delta D_j \end{bmatrix}U^H. \tag{2.8}$$

Further simplifying (2.8) and setting $\delta D_j = 0$, we get the desired structured perturbation matrices δA_0 and δA_1 whose *Frobenius* norms are minimum. For $i = 1, 2$, we need to show $((\mathbf{L}(\lambda) + \delta\mathbf{L}(\lambda))x_i = 0$. Consider $(\mathbf{L}(\lambda) + \delta\mathbf{L}(\lambda))x_i = \mathbf{L}(\lambda)x_i + \delta\mathbf{L}(\lambda)x_i = -k_i + \lambda_0\delta A_0x_i + \lambda_1\delta A_1x_i = -k_i + \overline{P}_{x_i}^ck_i + (x_1^Tk_2)(\epsilon\overline{x}_2x_1^H + \overline{x}_1x_2^H)x_i$, using Lemma 3.3.3, we get $(\mathbf{L}(\lambda) + \delta\mathbf{L}(\lambda))x_i = -k_i + \overline{P}_{x_i}^ck_i + \sum_{j=1,j\neq i}^{2}\overline{x}_jx_j^Tk_i = 0$.

Since the *Frobenius* norms of δA_0 and δA_1 are minimum, hence $(\eta_F^{\mathbf{S}}(\lambda, x_1, x_2, \mathbf{L}))^2 = \|\delta A_0\|_F^2 + \|\delta A_1\|_F^2$ where $\|\delta A_0\|_F^2 + \|\delta A_1\|_F^2 = \sum_{j=0}^{1}\|\widehat{\delta A_j}\|_F^2 + (1+\epsilon^2)\|\delta B_j\|_F^2 = \sum_{j=0}^{1}\|\widehat{\delta A_j}\|_F^2 + 2\|\delta B_j\|_F^2 = \sum_{i=1}^{2}((1+\epsilon)/2)\frac{|x_i^Tk_i|^2}{H_2^2(\lambda)} + 2\frac{|x_1^Tk_2|^2}{H_2^2(\lambda)} + 2\frac{\|V_2^Tk_i\|^2}{H_2^2(\lambda_i)}$. Since $\|V_2^Tk_i\|^2 = \|k_i\|^2 - |x_1^Tk_i|^2 - |x_2^Tk_i|^2$, and using Remark 2.3.6, we get

$$(\eta_F^{\mathbf{S}}(\lambda, x_1, x_2, \mathbf{L}))^2 = \sum_{i=1}^{2}(\frac{2\|k_i\|_2^2 - ((1+\epsilon)/2)|x_i^Tk_i|^2}{H_2^2(\lambda)}) - 2\frac{|x_2^Tk_1|^2}{H_2^2(\lambda)}. \blacksquare$$

Remark 2.3.4. *Results for non-homogeneous matrix pencils can be obtained by fixing $\lambda_0 = 1$ in Theorem 2.3.3.*

Remark 2.3.5. *By extending $\{x_1, x_2\}$ to the basis of $\mathbb{C}^n$, we get another $(n-2)$ linearly independent vectors $\{x_3, \ldots, x_n\}$. Then using the Gram-Schmidt process on $\{x_1, \ldots, x_n\}$, we get the desired $V_2 \in \mathbb{C}^{n\times(n-2)}$.*

Remark 2.3.6. *For $\epsilon = 1, -1$ we have $\epsilon x_1^Tk_2 = x_2^Tk_1$, and $|\epsilon x_1^Tk_2|^2 = |x_1^Tk_2|^2 = |x_2^Tk_1|^2$.*

Corollary 2.3.7. *Let $\mathbf{L}$ be a non-homogeneous T-symmetric/T-skew-symmetric non-homogeneous matrix pencil $\mathbf{L}$ of the form $\mathbf{L}(\gamma) = A_0 + \gamma A_1$. Let (μ, x_1) and (μ, x_2) be two approximate eigenpairs such that $\mu \in \mathbb{C}$ is a double-semisimple eigenvalue and $x_1, x_2 \in \mathbb{C}^n$ are orthonormal vectors. Set $k_i := -\mathbf{L}(\mu)x_i$ for $i = 1 : 2$. Then the following holds:*

$$(\eta_F^{\mathbf{S}}(\mu, x_1, x_2, \mathbf{L})) \leq \sqrt{\eta_F^{\mathbf{S}}(\mu, x_1, \mathbf{L})^2 + \eta_F^{\mathbf{S}}(\mu, x_2, \mathbf{L})^2}.$$

Proof. For the T-*symmetric* case by substituting $\lambda_0 = 1, \lambda_1 = \mu$, and $\epsilon = 1$ in Theorem 2.3.3, we get the following relation

$$(\eta_F^{\mathbf{S}}(\mu, x_1, x_2, \mathbf{L}))^2 = \sum_{i=1}^{2}(\frac{2\|k_i\|_2^2 - |x_i^Tk_i|^2}{(1+|\mu|^2)}) - 2\frac{|x_2^Tk_1|^2}{(1+|\mu|^2)}. \tag{2.9}$$

From [1, Theorem 3.1], we have

$$(\eta_F^{\mathbf{S}}(\mu, x_i, \mathbf{L}))^2 = \frac{2\|k_i\|_2^2 - |x_i^T k_i|^2}{(1+|\mu|^2)}, \quad i = 1:2. \tag{2.10}$$

By substituting Equation 2.10 in Equation 2.9, we get

$$(\eta_F^{\mathbf{S}}(\mu, x_1, x_2, \mathbf{L}))^2 = \sum_{i=1}^{2} \left(\eta_F^{\mathbf{S}}(\mu, x_i, \mathbf{L})\right)^2 - 2\frac{|x_2^T k_1|^2}{(1+|\mu|^2)}. \tag{2.11}$$

Since $\dfrac{|x_2^T k_1|^2}{(1+|\mu|^2)} \geq 0$, we get the desired result.

Remark 2.3.8. *The result for the T-skew-symmetric case can be obtained in a similar manner by using $\epsilon = -1$ and* [1, Theorem 3.2].

Next, we present the backward error analysis for unstructured matrix pencils, and by that analysis, we will establish a relationship between structured and unstructured backward errors.

2.4. Backward error analysis for unstructured matrix pencils

In this section, we derive the backward error formula for two approximate eigenpairs of a *double-semisimple* eigenvalue without imposing any structure on matrix pencils. We start this section with the following theorem, which gives a guarantee that there always exists a matrix pencil for two approximate eigenpairs of a *double-semisimple eigenvalue.*

Theorem 2.4.1. *Let $\mathbf{L} \in \mathbf{L}(\mathbb{C}^{n\times n})$ be a matrix pencil of the form (2.1). Let (λ, x_1) and (λ, x_2) be two approximate eigenpairs of $\mathbf{L}$, where $\lambda = (\lambda_0, \lambda_1) \in \mathbb{C}^2 \setminus \{(0,0)\}$ is a double-semisimple eigenvalue, and $x_1, x_2 \in \mathbb{C}^n$ are orthonormal vectors. Set $k_i := -\mathbf{L}(\lambda)x_i$ for $i = 1:2$, and define*

$$\delta A_0 = \sum_{i=1}^{2} \overline{\lambda}_0 \frac{k_i x_i^H + \overline{x}_i x_i^T P_{x_1:x_2}}{H_2^2(\lambda)} \text{ and } \delta A_1 = \sum_{i=1}^{2} \overline{\lambda}_1 \frac{k_i x_i^H + \overline{x}_i x_i^T P_{x_1:x_2}}{H_2^2(\lambda)}, \tag{2.12}$$

where $H_2(\lambda) = (|\lambda_0|^2 + |\lambda_1|^2)^{1/2}$. Then there exists a matrix pencil $\delta\mathbf{L} \in \mathbf{L}(\mathbb{C}^{n\times n})$ of the form $\delta\mathbf{L}(\alpha, \beta) = \alpha\delta A_0 + \beta\delta A_1$, such that $(\mathbf{L}(\lambda) + \delta\mathbf{L}(\lambda))x_i = 0$ for $i = 1:2$.

Proof. The proof is computational and is easy to check. ■

Now we present the main result of this section.

Theorem 2.4.2. *Let $\mathbf{L} \in \mathbf{L}(\mathbb{C}^{n\times n})$ be a homogeneous matrix pencil of the form (2.1). Let (λ, x_1) and (λ, x_2) be two approximate eigenpairs of $\mathbf{L}$, where $\lambda = (\lambda_0, \lambda_1) \in \mathbb{C}^2 \setminus \{(0,0)\}$ is a double-semisimple eigenvalue, and $x_1, x_2 \in \mathbb{C}^n$ are orthonormal vectors. Set $k_i := -\mathbf{L}(\lambda)x_i$ for $i = 1, 2$. Then there exists a matrix pencil $\delta\mathbf{L}$ of the form $\delta\mathbf{L}(\alpha, \beta) = \alpha\delta A_0 + \beta\delta A_1$ such that $(\mathbf{L}(\lambda) + \delta\mathbf{L}(\lambda))x_i = 0$. The perturbation matrices are given by*

$$\delta A_0 = \sum_{i=1}^{2} \frac{\overline{\lambda}_0 k_i x_i^H}{H_2^2(\lambda)},\ \delta A_1 = \sum_{i=1}^{2} \frac{\overline{\lambda}_1 k_i x_i^H}{H_2^2(\lambda)}.$$

The unstructured backward error is given by

$$(\eta_F(\lambda, x_1, x_2, \mathbf{L}))^2 = \sum_{i=1}^{2} \frac{\|k_i\|_2^2}{H_2^2(\lambda)}.$$

Proof. From Theorem 2.4.1, there always exists a matrix pencil $\delta\mathbf{L}$ of the form $\delta\mathbf{L}(\alpha, \beta) = \alpha\delta A_0 + \beta\delta A_1$ such that $\mathbf{L}(\lambda)x_i + \delta\mathbf{L}(\lambda)x_i = 0$ for $i = 1, 2$. To construct δA_j for $j = 0, 1$, we consider

$$\widetilde{\delta A_j} = U^T \delta A_j U = \begin{array}{c} \\ 2 \\ n-2 \end{array} \!\!\begin{array}{c} \begin{array}{cc} 2 & n-2 \end{array} \\ \left[\begin{array}{c|c} \widehat{\delta A_j} & \delta C_j^{\ T} \\ \hline \delta B_j & \delta D_j \end{array}\right] \end{array}, \tag{2.13}$$

where $\widehat{\delta A_j} = \begin{bmatrix} \delta a_{j,11} & \delta a_{j,12} \\ \delta a_{j,21} & \delta a_{j,22} \end{bmatrix}, \delta B_j = \begin{bmatrix} b_{j1} & b_{j2} \end{bmatrix}, \delta C_j = \begin{bmatrix} c_{j1} & c_{j2} \end{bmatrix}$ for $j = 0, 1$, and $U \in \mathbb{C}^{n\times n}$ is a unitary matrix such that $U = \begin{bmatrix} V_1 & V_2 \end{bmatrix}$ with $V_1 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \in \mathbb{C}^{n\times 2}$. Since we need to construct $\delta\mathbf{L}$ such that $(\mathbf{L}(\lambda) + \delta\mathbf{L}(\lambda))x_i = 0$, we get $k_i = \delta\mathbf{L}(\lambda)x_i$ for $i = 1 : 2$. From $\widetilde{\delta\mathbf{L}}(\lambda) = U^T\delta\mathbf{L}(\lambda)U$, we have $\widetilde{\delta\mathbf{L}}(\lambda)U^H x_i = U^T\delta\mathbf{L}(\lambda)x_i = U^T k_i$. This implies

$$\lambda_0 \begin{bmatrix} \widehat{\delta A_0} & \delta C_0^T \\ \delta B_0 & \delta D_0 \end{bmatrix} \begin{bmatrix} e_i \\ \mathbf{0} \end{bmatrix} + \lambda_1 \begin{bmatrix} \widehat{\delta A_1} & \delta C_1^T \\ \delta B_1 & \delta D_1 \end{bmatrix} \begin{bmatrix} e_i \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} V_1^T k_i \\ V_2^T k_i \end{bmatrix},$$ further simplification gives

$$\begin{bmatrix} (\lambda_0\widehat{\delta A_0} + \lambda_1\widehat{\delta A_1})e_i \\ (\lambda_0\delta B_0 + \lambda_1\delta B_1)e_i \end{bmatrix} = \begin{bmatrix} V_1^T k_i \\ V_2^T k_i \end{bmatrix}, \tag{2.14}$$

where $e_i \in \mathbb{C}^2$ is a vector having 1 at i^{th} position and 0 elsewhere. From (2.14), we get the following equations

$$\lambda_0\delta a_{0,ii} + \lambda_1\delta a_{1,ii} = x_i^T k_i, \quad i = 1, 2, \tag{2.15}$$

$$\lambda_0 b_{0i} + \lambda_1 b_{1i} = V_2^T k_i,\ i = 1, 2 \tag{2.16}$$

The minimum norm solutions of (2.15) and (2.16) are given by $\delta a_{0,ii} = \frac{\overline{\lambda}_0}{H_2^2(\lambda)} x_i^T k_i, \delta a_{1,ii} = \frac{\overline{\lambda}_1}{H_2^2(\lambda)} x_i^T k_i$ and $b_{0i} = \frac{\overline{\lambda}_0}{H_2^2(\lambda)} V_2^T k_i, b_{1i} = \frac{\overline{\lambda}_1}{H_2^2(\lambda)} V_2^T k_i$. Further from (2.14), we get the following two equations:

$$\lambda_0 \delta a_{0,21} + \lambda_1 \delta a_{1,21} = x_2^T k_1, \tag{2.17}$$

$$\lambda_0 \delta a_{0,12} + \lambda_1 \delta a_{1,12} = x_1^T k_2. \tag{2.18}$$

The minimum norm solutions of (2.17) and (2.18) are given by $\delta a_{0,21} = \frac{\overline{\lambda}_0}{H_2^2(\lambda)} x_2^T k_1$, $\delta a_{1,21} = \frac{\overline{\lambda}_1}{H_2^2(\lambda)} x_2^T k_1$; $\delta a_{0,12} = \frac{\overline{\lambda}_0}{H_2^2(\lambda)} x_1^T k_2, \delta a_{1,12} = \frac{\overline{\lambda}_1}{H_2^2(\lambda)} x_1^T k_2$.

Similar to the T-*symmetric/*T-*skew-symmetric* case, substituting back all these obtained entries in (2.13) along with $\delta D_1 = \delta D_2 = 0$, and $\delta C_1 = \delta C_2 = 0$, we get the desired perturbation matrices with the minimum *Frobenius* norms. Similar to Theorem 2.3.3, we can obtain the backward error for the unstructured case, which is given by

$$\eta_F(\lambda, x_1, x_2, \mathbf{L}) = \sqrt{\sum_{i=1}^{2} \frac{\|k_i\|_2^2}{H_2^2(\lambda)}}.$$

After establishing the unstructured backward error formula for two approximate eigenpairs, we now establish a relationship between unstructured and T-*symmetric/*T-*skew-symmetric* backward errors.

Corollary 2.4.3. *Let* $\mathbf{L} \in \mathbf{L}(\mathbb{C}^{n\times n})$ *be a* T*-symmetric/*T*-skew-symmetric matrix pencil of the form (2.1). Let* (λ, x_1) *and* (λ, x_2) *be two approximate eigenpairs of* $\mathbf{L}$, *where* $x_1, x_2 \in \mathbb{C}^n$ *are orthonormal vectors, and* $\lambda = (\lambda_0, \lambda_1) \in \mathbb{C}^2 \setminus \{(0,0)\}$ *is a double-semisimple eigenvalue. Then the following holds:*

$$(\eta_F^{\mathbf{S}}(\lambda, x_1, x_2, \mathbf{L})) \leq \sqrt{2}\,(\eta_F(\lambda, x_1, x_2, \mathbf{L})).$$

Proof. From Theorem 2.3.3, we get $(\eta_F^{\mathbf{S}}(\lambda, x_1, x_2, \mathbf{L}))^2 \leq \sum_{i=1}^{2} \frac{2\|k_i\|_2^2}{H_2^2(\lambda)}$. Also using Theorem 2.4.2, we have $(\eta_F(\lambda, x_1, x_2, \mathbf{L}))^2 = \displaystyle\sum_{i=1}^{2} \frac{\|k_i\|_2^2}{H_2^2(\lambda)}$. Hence we get

$$(\eta_F^{\mathbf{S}}(\lambda, x_1, x_2, \mathbf{L})) \leq \sqrt{2}\,(\eta_F(\lambda, x_1, x_2, \mathbf{L})).\blacksquare$$

Now we present a relationship between the backward error of a single eigenpair and the backward error of two approximate eigenpairs of a *double-semisimple eigenvalue.*

Corollary 2.4.4. *Let* (μ, x_1) *and* (μ, x_2) *be two approximate eigenpairs such that* $\mu \in \mathbb{C}$ *is a double-semisimple eigenvalue of a non-homogeneous matrix pencil* $\mathbf{L}$ *of the form*

$\mathbf{L}(\gamma) = A_0 + \gamma A_1$. *Set* $k_i := -\mathbf{L}(\mu)x_i$, *where* $x_i \in \mathbb{C}^n$ *for* $i = 1, 2$. *Then the following holds:*

$$\eta_F(\mu, x_1, x_2, \mathbf{L}) = \sqrt{\eta_F^2(\mu, x_1, \mathbf{L}) + \eta_F^2(\mu, x_2, \mathbf{L})}.$$

Proof. By substituting $\lambda_0 = 1$, and $\lambda_1 = \mu$ in Theorem 2.4.2, we get $\eta_F(\mu, x_1, x_2, \mathbf{L}) = \sqrt{\sum_{i=1}^{2} \frac{\|k_i\|_2^2}{(1+|\mu|^2)}}$. On the other hand by [1], we have $\eta_F(\mu, x_i, \mathbf{L}) = \frac{\|k_i\|}{(1+|\mu|^2)^{1/2}}$ for $i = 1, 2$. On combining these two results, we get $\eta_F(\mu, x_1, x_2, \mathbf{L}) = \sqrt{\eta_F^2(\mu, x_1, \mathbf{L}) + \eta_F^2(\mu, x_2, \mathbf{L})}$. ■

Remark 2.4.5. *For T-symmetric/ T-skew-symmetric matrix pencils, a relation between the unstructured backward error of a single approximate eigenpair and the structured backward error of two approximate eigenpairs of a double-semisimple eigenvalue can be established by using Corollary 2.4.4 and Corollary 2.4.3.*

Remark 2.4.6. *From now onwards, we will not invoke the existence theorem separately as we did for T-symmetric/T-skew-symmetric and unstructured cases by Theorem 2.3.1 and Theorem 2.4.1, respectively, because the construction of δA_0 and δA_1 in each case itself gives a guarantee of the existence of the required structured matrix pencil.*

2.5. Backward error analysis for Hermitian/skew-Hermitian matrix pencils

This section deals with the backward error analysis of *Hermitian* and *skew-Hermitian* matrix pencils. First, we state and prove the main result of this section. Later, we establish a relationship between the backward error of a single approximate eigenpair and the backward error of two approximate eigenpairs of a *double-semisimple eigenvalue*. Before moving to the main result of this section, we now present an important lemma as follows.

Lemma 2.5.1. *Let* $\mathbf{L}$ *be a Hermitian/skew-Hermitian matrix pencil of the form (2.1). Let* $\lambda = (\lambda_0, \lambda_1) \in \mathbb{C}^2 \setminus \{(0,0)\}$ *be a double-semisimple eigenvalue of* $\mathbf{L}$ *satisfying* $\Im(\overline{\lambda}_0\lambda_1) \neq 0$, *i.e.,* $\mathbf{L}(\lambda_0, \lambda_1)y_i = 0$ *for* $i = 1, 2$ *where* $y_1, y_2 \in \mathbb{C}^n$ *are the eigenvectors corresponding to* λ. *Then* $y_1^H A_j y_1 = 0, y_2^H A_j y_2 = 0$ *for* $j = 0, 1$.

Proof. Given that λ is a *double-semisimple eigenvalue* of $\mathbf{L}$, i.e., $(\lambda_0 A_0 + \lambda_1 A_1)y_i = 0$ for $i = 1, 2$. This gives $y_i^H(\lambda_0 A_0 + \lambda_1 A_1)y_i = 0$. Using the fact that $A_j = \epsilon A_j^H$ for $j = 0, 1$, we get $y_i^H(\lambda_0 A_0 + \lambda_1 A_1)y_i = 0$ and $y_i^H(\overline{\lambda}_0 A_0 + \overline{\lambda}_1 A_1)y_i = 0$. Solving these two equations along with $\Im(\overline{\lambda}_0\lambda_1) \neq 0$, we get the desired result. ■

Throughout this section, $\epsilon = 1$ represents the *Hermitian* case and $\epsilon = -1$ represents the *skew-Hermitian* case.

Remark 2.5.2. *For $\epsilon = -1$ we have $\sqrt{\epsilon} = \sqrt{-1} = \mathrm{i}$, an imaginary number.*

Remark 2.5.3. *Let $\mathbf{L} \in \mathbf{L}(\mathbb{C}^{n\times n})$ be a Hermitian/ skew-Hermitian homogeneous matrix pencil of the form (2.1). Suppose (λ, x_1) and (λ, x_2) are two approximate eigenpairs of $\mathbf{L}$ with $\lambda = (\lambda_0, \lambda_1) \in \mathbb{C}^2 \setminus \{(0,0)\}$ is a double-semisimple eigenvalue such that $\Im(\overline{\lambda}_0\lambda_1) = 0$, and $x_1, x_2 \in \mathbb{C}^n$ are orthonormal vectors. Set $k_i := -\mathbf{L}(\lambda)x_i$ for $i = 1:2$. Then $\lambda_j\epsilon\overline{x_2^H k_1} = \overline{\lambda}_j x_1^H k_2$ for $j = 0:1$, and $|\epsilon x_2^H k_1|^2 = |x_1^H k_2|^2$.*

Now, we state and prove the main result of this section.

Theorem 2.5.4. *Let $\mathbf{L} \in \mathbf{L}(\mathbb{C}^{n\times n})$ be a Hermitian/ skew-Hermitian homogeneous matrix pencil of the form (2.1). Let (λ, x_1) and (λ, x_2) be two approximate eigenpairs of $\mathbf{L}$, where $\lambda = (\lambda_0, \lambda_1) \in \mathbb{C}^2 \setminus \{(0,0)\}$ is a double-semisimple eigenvalue, and $x_1, x_2 \in \mathbb{C}^n$ are orthonormal vectors. Set $k_i := -\mathbf{L}(\lambda)x_i$ for $i = 1:2$. Then there exists a Hermitian/ skew-Hermitian matrix pencil $\delta\mathbf{L}$ of the form $\delta\mathbf{L}(\alpha, \beta) = \alpha\delta A_0 + \beta\delta A_1$ such that $(\mathbf{L}(\lambda) + \delta\mathbf{L}(\lambda))x_i = 0$. The perturbation matrices, for $\Im(\overline{\lambda}_0\lambda_1) = 0$, are given by*

$$\delta A_0 = \sum_{i=1}^{2} \frac{\overline{\lambda}_0 k_i x_i^H + \epsilon\lambda_0 x_i k_i^H P_{x_1:x_2}}{H_2^2(\lambda)} \text{ and } \delta A_1 = \sum_{i=1}^{2} \frac{\overline{\lambda}_1 k_i x_i^H + \epsilon\lambda_1 x_i k_i^H P_{x_1:x_2}}{H_2^2(\lambda)},$$

and the backward error is given by

$$(\eta_F^{\mathbf{S}}(\lambda, x_1, x_2, \mathbf{L}))^2 = \sum_{i=1}^{2}\left(\frac{2\|k_i\|_2^2 - |x_i^H k_i|^2}{H_2^2(\lambda)}\right) - 2\frac{|x_2^H k_1|^2}{H_2^2(\lambda)}.$$

The perturbation matrices, for $\Im(\overline{\lambda}_0\lambda_1) \neq 0$, are given by

$$\begin{aligned}
\delta A_0 &= \sum_{i=1}^{2}\left(-x_i x_i^H A_0 x_i x_i^H + \overline{\lambda}_0 \frac{P_{x_1:x_2} k_i x_i^H}{H_2^2(\lambda)} + \epsilon\lambda_0 \frac{x_i k_i^H P_{x_1:x_2}}{H_2^2(\lambda)}\right) \\
&+ \frac{x_2(\overline{\lambda}_1 x_2^H k_1 - \epsilon\lambda_1 \overline{x_1^H k_2})x_1^H}{\lambda_0\overline{\lambda}_1 - \overline{\lambda}_0\lambda_1} + \frac{x_1(\epsilon\lambda_1 \overline{x_2^H k_1} - \overline{\lambda}_1 x_1^H k_2)x_2^H}{\overline{\lambda}_0\lambda_1 - \lambda_0\overline{\lambda}_1} \quad \text{and} \\
\delta A_1 &= \sum_{i=1}^{2}\left(-x_i x_i^H A_1 x_i x_i^H + \overline{\lambda}_1 \frac{P_{x_1:x_2} k_i x_i^H}{H_2^2(\lambda)} + \epsilon\lambda_1 \frac{x_i k_i^H P_{x_1:x_2}}{H_2^2(\lambda)}\right) \\
&+ \frac{x_2(\epsilon\lambda_0 \overline{x_1^H k_2} - \overline{\lambda}_0 x_2^H k_1)x_1^H}{\lambda_0\overline{\lambda}_1 - \overline{\lambda}_0\lambda_1} + \frac{x_1(\overline{\lambda}_0 x_1^H k_2 - \epsilon\lambda_0 \overline{x_2^H k_1})x_2^H}{\overline{\lambda}_0\lambda_1 - \lambda_0\overline{\lambda}_1}.
\end{aligned}$$

In this case, the backward error is given by

$$(\eta_F^{\mathbf{S}}(\lambda,x_1,x_2,\mathbf{L}))^2 = \sum_{i=1}^{2}\sum_{j=0}^{1}(|x_i^HA_jx_i|^2+2\frac{\|k_i\|_2^2-|x_i^Hk_i|^2}{H_2^2(\lambda)})-\epsilon\frac{\Re((\overline{\lambda}_0^2+\overline{\lambda}_1^2)(x_1^Hk_2)(x_2^Hk_1))}{|\Im(\overline{\lambda}_0\lambda_1)|^2}$$
$$+\frac{|x_2^Hk_1|^2+|x_1^Hk_2|^2}{2|\Im(\overline{\lambda}_0\lambda_1)|^2H_2^2(\lambda)}(H_2^4(\lambda)-4|\Im(\overline{\lambda}_0\lambda_1)|^2).$$

Proof. For constructing δA_j for $j=0,1$ such that $\delta A_j=\epsilon\delta A_j^H$, we consider

$$\widetilde{\delta A_j}=U^H\delta A_jU=\begin{array}{c c} & \begin{array}{cc} 2 & n-2 \end{array}\\ \begin{array}{c} 2\\ n-2\end{array} & \left[\begin{array}{c|c}\widehat{\delta A_j} & \epsilon\delta B_j{}^H\\ \hline \delta B_j & \delta D_j\end{array}\right]\end{array}, \tag{2.19}$$

where $\widehat{\delta A_j}=\begin{bmatrix}\sqrt{\epsilon}\delta a_{j,11} & \delta a_{j,12}\\ \epsilon\overline{\delta a_{j,12}} & \sqrt{\epsilon}\delta a_{j,22}\end{bmatrix}$ with $\delta a_{j,tt}\in\mathbb{R}$ for $t=1:2$, $\delta B_j=\begin{bmatrix}b_{j1} & b_{j2}\end{bmatrix}$, $\delta D_j=\epsilon\delta D_j{}^H$ for $j=0,1$, and $U\in\mathbb{C}^{n\times n}$ is a unitary matrix such that $U=\begin{bmatrix}V_1 & V_2\end{bmatrix}$ with $V_1=\begin{bmatrix}x_1 & x_2\end{bmatrix}\in\mathbb{C}^{n\times 2}$. Since we need to construct $\delta\mathbf{L}$ such that $(\mathbf{L}(\lambda)+\delta\mathbf{L}(\lambda))x_i=0$, we get $k_i=\delta\mathbf{L}(\lambda)x_i$ for $i=1:2$. From $\widetilde{\delta\mathbf{L}}(\lambda)=U^H\delta\mathbf{L}(\lambda)U$ we have $\widetilde{\delta\mathbf{L}}(\lambda)U^Hx_i=U^H\delta\mathbf{L}(\lambda)x_i=U^Hk_i$. This implies

$\lambda_0\begin{bmatrix}\widehat{\delta A_0} & \epsilon\delta B_0^H\\ \delta B_0 & \delta D_1\end{bmatrix}\begin{bmatrix}e_i\\ 0\end{bmatrix}+\lambda_1\begin{bmatrix}\widehat{\delta A_1} & \epsilon\delta B_1^H\\ \delta B_1 & \delta D_1\end{bmatrix}\begin{bmatrix}e_i\\ 0\end{bmatrix}=\begin{bmatrix}V_1^Hk_i\\ V_2^Hk_i\end{bmatrix}$. Further simplification gives

$$\begin{bmatrix}(\lambda_0\widehat{\delta A_0}+\lambda_1\widehat{\delta A_1})e_i\\ (\lambda_0\delta B_0+\lambda_1\delta B_1)e_i\end{bmatrix}=\begin{bmatrix}V_1^Hk_i\\ V_2^Hk_i\end{bmatrix}, \tag{2.20}$$

where $e_i\in\mathbb{C}^2$ is a vector having 1 at i^{th} position and 0 elsewhere. From (2.20), we get the following four equations and one system of equation:

$$\sqrt{\epsilon}\lambda_0\delta a_{0,ii}+\sqrt{\epsilon}\lambda_1\delta a_{1,ii}=x_i^Hk_i,\quad i=1,2, \tag{2.21}$$

$$\lambda_0b_{0i}+\lambda_1b_{1i}=V_2^Hk_i,\ i=1,2, \tag{2.22}$$

$$\begin{bmatrix}\overline{\lambda}_0 & \overline{\lambda}_1\\ \lambda_0 & \lambda_1\end{bmatrix}\begin{bmatrix}\delta a_{0,12}\\ \delta a_{1,12}\end{bmatrix}=\begin{bmatrix}\epsilon\overline{x_2^Hk_1}\\ x_1^Hk_2\end{bmatrix}. \tag{2.23}$$

The minimum norm solution of (2.22) is given by $b_{0i}=\frac{\overline{\lambda}_0}{H_2^2(\lambda)}V_2^Hk_i$ and $b_{1i}=\frac{\overline{\lambda}_1}{H_2^2(\lambda)}V_2^Hk_i$.

Case-1: If $\Im(\overline{\lambda}_0\lambda_1)=0$, then the minimum norm solution of (2.21) is given by $\delta a_{0,ii}=\frac{\sqrt{\epsilon}\overline{\lambda}_0}{H_2^2(\lambda)}x_i^Hk_i, \delta a_{1,ii}=\frac{\sqrt{\epsilon}\overline{\lambda}_1}{H_2^2(\lambda)}x_i^Hk_i$. Since $A_j=\epsilon A_j^H$ for $j=0,1$, and $\Im(\overline{\lambda}_0\lambda_1)=0$, we get system (2.23) is consistent by using Remark 2.5.3. The minimum norm solution of (2.23) is given by $\delta a_{0,12} = \frac{\epsilon\lambda_0}{H_2^2(\lambda)}\overline{x_2^H k_1}$ and $\delta a_{1,12} = \frac{\epsilon\lambda_1}{H_2^2(\lambda)}\overline{x_2^H k_1}$. Substituting back all these obtained entries in (2.19), we get

$$\delta A_j = U \begin{bmatrix} \frac{\overline{\lambda}_j}{H_2^2(\lambda)} x_1^H k_1 & \epsilon \frac{\lambda_j}{H_2^2(\lambda)} \overline{x_2^H k_1} & \epsilon \frac{\lambda_j}{H_2^2(\lambda)} (V_2^H k_1)^H \\ \frac{\overline{\lambda}_j}{H_2^2(\lambda)} x_2^H k_1 & \frac{\overline{\lambda}_j}{H_2^2(\lambda)} x_2^H k_2 & \epsilon \frac{\lambda_j}{H_2^2(\lambda)} (V_2^H k_2)^H \\ \frac{\overline{\lambda}_j}{H_2^2(\lambda)} V_2^H k_1 & \frac{\overline{\lambda}_j}{H_2^2(\lambda)} V_2^H k_2 & D_j \end{bmatrix} U^H. \tag{2.24}$$

Further, simplifying (2.24) and setting $\delta D_j = 0$ along with Remark 2.5.3, we get the desired perturbation matrices $\delta A_0, \delta A_1$ whose *Frobenius* norms are minimum.

Next, we need to show $(\mathbf{L}(\lambda) + \delta\mathbf{L}(\lambda))x_i = 0$ for $i = 1, 2$. Consider $(\mathbf{L}(\lambda) + \delta\mathbf{L}(\lambda))x_i = \mathbf{L}(\lambda)x_i + \delta\mathbf{L}(\lambda)x_i = -k_i + \lambda_0\delta A_0 x_i + \lambda_1 \delta A_1 x_i = -k_i + k_i = 0$. Since the *Frobenius* norms of δA_0 and δA_1 are minimum, hence $(\eta_F^{\mathbf{S}}(\lambda, x_1, x_2, \mathbf{L}))^2 = \|\delta A_0\|_F^2 + \|\delta A_1\|_F^2$, where $\|\delta A_0\|_F^2 + \|\delta A_1\|_F^2 = \sum_{j=0}^1 \|\widehat{\delta A_j}\|_F^2 + 2\|\delta B_j\|_F^2 = \sum_{i=1}^2 \frac{|x_i^H k_i|^2}{H_2^2(\lambda)} + 2\frac{|x_2^H k_1|^2}{H_2^2(\lambda)} + 2\frac{\|V_2^H k_i\|^2}{H_2^2(\lambda)})$. Since $\|V_2^H k_i\|_2^2 = \|k_i\|_2^2 - |x_1^H k_i|^2 - |x_2^H k_i|^2$, and using Remark 2.5.3, we get

$$(\eta_F^{\mathbf{S}}(\lambda, x_1, x_2, \mathbf{L}))^2 = \sum_{i=1}^2 (\frac{2\|k_i\|_2^2 - |x_i^T k_i|^2}{H_2^2(\lambda)}) - 2\frac{|x_2^H k_1|^2}{H_2^2(\lambda)}.$$

Case-2: If $\Im(\overline{\lambda}_0\lambda_1) \neq 0$, then using Lemma 2.5.1, we get $\delta a_{0,ii} = -\sqrt{\epsilon}x_i^H A_0 x_i, \delta a_{1,ii} = -\sqrt{\epsilon}x_i^H A_1 x_i$. When $\Im(\overline{\lambda}_0\lambda_1) \neq 0$, i.e., $\overline{\lambda}_0\lambda_1 - \lambda_0\overline{\lambda}_1 \neq 0$, then the unique solution of system (2.23) is given by $\delta a_{0,12} = \frac{\epsilon\lambda_1\overline{x_2^H k_1} - \overline{\lambda}_1 x_1^H k_2}{\overline{\lambda}_0\lambda_1 - \lambda_0\overline{\lambda}_1}$ and $\delta a_{1,12} = \frac{-\epsilon\lambda_0\overline{x_2^H k_1} + \overline{\lambda}_0 x_1^H k_2}{\overline{\lambda}_0\lambda_1 - \lambda_0\overline{\lambda}_1}$, which is the minimum norm solution. Similar to Case-1, we get the desired perturbed matrices by substituting back the obtained entries in (2.19). In this case $(\eta_F^{\mathbf{S}}(\lambda, x_1, x_2, \mathbf{L}))^2 = \|\delta A_0\|_F^2 + \|\delta A_1\|_F^2$, where

$$\begin{aligned} \|\delta A_0\|_F^2 + \|\delta A_1\|_F^2 &= \sum_{j=0}^1 \sum_{i=1}^2 |x_i^H A_j x_i|^2 + \sum_{i=1}^2 \frac{2\|k_i\|^2 - 2|x_i^H k_i|^2}{H_2^2(\lambda)} \\ &+ 2\frac{|\epsilon\lambda_1\overline{x_2^H k_1} - \overline{\lambda}_1 x_1^H k_2|^2 + |\overline{\lambda}_0 x_1^H k_2 - \epsilon\lambda_0\overline{x_2^H k_1}|^2}{|\overline{\lambda}_0\lambda_1 - \lambda_0\overline{\lambda}_1|^2} - 2\frac{|x_1^H r_2|^2 + |x_2^H r_1|^2}{H_2^2(\lambda)}. \end{aligned}$$

Since

$$\begin{aligned} \frac{|\epsilon\lambda_1\overline{x_2^H k_1} - \overline{\lambda}_1 x_1^H k_2|^2 + |\overline{\lambda}_0 x_1^H k_2 - \epsilon\lambda_0\overline{x_2^H k_1}|^2}{|\overline{\lambda}_0\lambda_1 - \lambda_0\overline{\lambda}_1|^2} &= H_2^2(\lambda)\frac{|x_2^H k_1|^2 + |x_1^H k_2|^2}{4|\Im(\overline{\lambda}_0\lambda_1)|^2} \\ &- \epsilon\frac{\Re((\overline{\lambda}_0^2 + \overline{\lambda}_1^2)(x_1^H k_2)(x_2^H k_1))}{2|\Im(\overline{\lambda}_0\lambda_1)|^2}, \end{aligned}$$

we get

$$(\eta_F^{\mathbf{S}}(\lambda, x_1, x_2, \mathbf{L}))^2 = \sum_{i=1}^{2}\sum_{j=0}^{1}\left(|x_i^H A_j x_i|^2 + 2\frac{\|k_i\|_2^2 - |x_i^H k_i|^2}{H_2^2(\lambda)}\right) - \epsilon\frac{\Re((\overline{\lambda}_0^2 + \overline{\lambda}_1^2)(x_1^H k_2)(x_2^H k_1))}{|\Im(\overline{\lambda}_0\lambda_1)|^2}$$
$$+\frac{|x_2^H k_1|^2 + |x_1^H k_2|^2}{2|\Im(\overline{\lambda}_0\lambda_1)|^2 H_2^2(\lambda)}\left(H_2^4(\lambda) - 4|\Im(\overline{\lambda}_0\lambda_1)|^2\right).$$

Corollary 2.5.5. *Let* $\mathbf{L}$ *be a non-homogeneous Hermitian/skew-Hermitian matrix pencil of the form (2.1). Let* (μ, x_1) *and* (μ, x_2) *be two approximate eigenpairs where* $\mu \in \mathbb{R}$ *is a double-semisimple eigenvalue and* $x_1, x_2 \in \mathbb{C}^n$ *are orthonormal vectors. Set* $k_i := -\mathbf{L}(\mu)x_i$*. Then the following inequality holds:*

$$(\eta_F^{\mathbf{S}}(\mu, x_1, x_2, \mathbf{L})) \leq \sqrt{\eta_F^{\mathbf{S}}(\mu, x_1, \mathbf{L})^2 + \eta_F^{\mathbf{S}}(\mu, x_2, \mathbf{L})^2}.$$

Proof. Substituting $\lambda_0 = 1, \lambda_1 = \mu$ in Theorem 2.5.4 and using [**1**, Theorem 3.6], we get the desired backward error relation. ■

Remark 2.5.6. *Let* (μ, x_1) *and* (μ, x_2) *be two approximate eigenpairs of a non-homogeneous Hermitian/skew-Hermitian matrix pencil* $\mathbf{L}$ *where* $\mu \in \mathbb{C}$ *is a double-semisimple eigenvalue, and* $x_1, x_2 \in \mathbb{C}^n$ *are orthonormal eigenvectors. Then similar to Corollary 2.3.7, substituting* $\lambda_0 = 1, \lambda_1 = \mu$ *in Theorem 2.5.4, and using Theorem 3.6 of* [**1**]*, we get* $\eta_F^{\mathbf{S}}(\mu, x_1, x_2, \mathbf{L}) = \sqrt{(\eta_F^{\mathbf{S}}(\mu, x_1, \mathbf{L}))^2 + (\eta_F^{\mathbf{S}}(\mu, x_2, \mathbf{L}))^2} = \sqrt{2}\sqrt{(\eta_F(\mu, x_1, \mathbf{L}))^2 + (\eta_F(\mu, x_2, \mathbf{L}))^2}$ *when* $\mu^2 = -1$*. Further, using Corollary 2.4.4 in the above relation, we get* $\eta_F^{\mathbf{S}}(\mu, x_1, x_2, \mathbf{L}) = \sqrt{2}(\eta_F(\mu, x_1, x_2, \mathbf{L}))$ *when* $\mu^2 = -1$ *and for* $\mu^2 = -1$*, we get* $H_2^4(\lambda) - 4|\Im(\overline{\lambda}_0\lambda_1)|^2 = 0$.

Similar to *Hermitian/skew-Hermitian* matrix pencils next, we present the backward error analysis for *H-even/H-odd* matrix pencils.

2.6. Backward error analysis for H-even/H-odd matrix pencils

In this section, we will discuss the backward error analysis for *H-even* and *H-odd* matrix pencils. We start this section with the following important lemma.

Throughout this section, $\epsilon = 1$ represents the *H-even* case and $\epsilon = -1$ represents the *H-odd* case.

Lemma 2.6.1. *Let* $\mathbf{L}$ *be a H-even/H-odd matrix pencil of the form (2.1). Suppose* $\lambda = (\lambda_0, \lambda_1) \in \mathbb{C}^2 \setminus \{(0,0)\}$ *is a double-semisimple eigenvalue of* $\mathbf{L}$ *satisfying* $\Re(\overline{\lambda}_0\lambda_1) \neq 0$, *i.e.*, $\mathbf{L}(\lambda_0,\lambda_1)y_i = 0$, *for* $i = 1, 2$ *where* $y_1, y_2 \in \mathbb{C}^n$ *are the eigenvectors corresponding to* λ. *Then* $y_1^H A_j y_1 = 0, y_2^H A_j y_2 = 0$ *for* $j = 0, 1$.

Proof. Proof follows similar to Lemma 2.5.1 by using the fact that $A_0 = \epsilon A_0^H, A_1 = -\epsilon A_1^H$. ■

Remark 2.6.2. *Let* $\mathbf{L} \in \mathbf{L}(\mathbb{C}^{n\times n})$ *be a H-even/ H-odd homogeneous matrix pencil of the form (2.1). Suppose* (λ, x_1) *and* (λ, x_2) *are two approximate eigenpairs of* $\mathbf{L}$, *where* $\lambda = (\lambda_0, \lambda_1) \in \mathbb{C}^2 \setminus \{(0, 0)\}$ *is a double-semisimple eigenvalue such that* $\Re(\overline{\lambda}_0\lambda_1) = 0$, *and* $x_1, x_2 \in \mathbb{C}^n$ *are orthonormal vectors. Let* $k_i := -\mathbf{L}(\lambda)x_i$ *for* $i = 1 : 2$. *Then* $\lambda_0\epsilon\overline{x_2^H k_1} = \overline{\lambda}_0 x_1^H k_2$, $\lambda_1\epsilon\overline{x_2^H k_1} = -\overline{\lambda}_1 x_1^H k_2$, *and* $|\epsilon x_2^H k_1|^2 = |x_1^H k_2|^2$.

Now we state and prove the main result of this section.

Theorem 2.6.3. *Let* $\mathbf{L} \in \mathbf{L}(\mathbb{C}^{n\times n})$ *be a H-even/H-odd matrix pencil of the form (2.1). Let* (λ, x_1) *and* (λ, x_2) *be two approximate eigenpairs of* $\mathbf{L}$, *where* $x_1, x_2 \in \mathbb{C}^n$ *are orthonormal vectors and* $\lambda = (\lambda_0, \lambda_1) \in \mathbb{C}^2 \setminus \{(0, 0)\}$ *is a double-semisimple eigenvalue. Set* $k_i = -\mathbf{L}(\lambda)x_i$ *for* $i = 1 : 2$. *Then there exists a H-even/H-odd matrix pencil* $\delta\mathbf{L}$ *of the form* $\delta\mathbf{L}(\alpha, \beta) = \alpha\delta A_0 + \beta\delta A_1$ *such that* $(\mathbf{L}(\lambda) + \delta\mathbf{L}(\lambda))x_i = 0$. *The perturbation matrices, for* $\Re(\overline{\lambda}_0\lambda_1) = 0$, *are given by*

$$\delta A_0 = \sum_{i=1}^{2} \frac{\overline{\lambda}_0 k_i x_i^H + \epsilon\lambda_0 x_i k_i^H P_{x_1:x_2}}{H_2^2(\lambda)} \text{ and } \delta A_1 = \sum_{i=1}^{2} \frac{\overline{\lambda}_1 k_i x_i^H - \epsilon\lambda_1 x_i k_i^H P_{x_1:x_2}}{H_2^2(\lambda)}.$$

In this case, the backward error is given by

$$(\eta_F^{\mathbf{S}}(\lambda, x_1, x_2, \mathbf{L}))^2 = \sum_{i=1}^{2}\left(\frac{2\|k_i\|_2^2 - |x_i^T k_i|^2}{H_2^2(\lambda)}\right) - 2\frac{|x_1^H k_2|^2}{H_2^2(\lambda)}.$$

The perturbation matrices, for $\Re(\overline{\lambda}_0\lambda_1) \neq 0$, *are given by*

$$\begin{aligned}
\delta A_0 &= \sum_{i=1}^{2}\left(-x_i x_i^H A_0 x_i x_i^H + \overline{\lambda}_0 \frac{P_{x_1:x_2} k_i x_i^H}{H_2^2(\lambda)} + \epsilon\lambda_0 \frac{x_i k_i^H P_{x_1:x_2}}{H_2^2(\lambda)}\right) \\
&+ \frac{x_2(\overline{\lambda}_1 x_2^H k_1 + \epsilon\lambda_1 \overline{x_1^H k_2})x_1^H}{\overline{\lambda}_0\lambda_1 + \lambda_0\overline{\lambda}_1} + \frac{x_1(\epsilon\lambda_1 \overline{x_2^H k_1} + \overline{\lambda}_1 x_1^H k_2)x_2^H}{\overline{\lambda}_0\lambda_1 + \lambda_0\overline{\lambda}_1}, \text{ and} \\
\delta A_1 &= \sum_{i=1}^{2}\left(-x_i x_i^H A_1 x_i x_i^H + \overline{\lambda}_1 \frac{P_{x_1:x_2} k_i x_i^H}{H_2^2(\lambda)} - \epsilon\lambda_1 \frac{x_i k_i^H P_{x_1:x_2}}{H_2^2(\lambda)}\right) \\
&+ \frac{x_2(\overline{\lambda}_0 x_2^H k_1 - \epsilon\lambda_0 \overline{x_1^H k_2})x_1^H}{\overline{\lambda}_0\lambda_1 + \lambda_0\overline{\lambda}_1} + \frac{x_1(\overline{\lambda}_0 x_1^H k_2 - \epsilon\lambda_0 \overline{x_2^H k_1})x_2^H}{\overline{\lambda}_0\lambda_1 + \lambda_0\overline{\lambda}_1}.
\end{aligned}$$

In this case, the backward error is given by

$$(\eta_F^{\mathbf{S}}(\lambda, x_1, x_2, \mathbf{L}))^2 = \sum_{i=1}^{2}\sum_{j=0}^{1}(|x_i^H A_j x_i|^2 + \frac{2\|k_i\|_2^2 - 2|x_i^H k_i|^2}{H_2^2(\lambda)}) + \epsilon \frac{\Re((\overline{\lambda}_1^2 - \overline{\lambda}_0^2)(x_1^H k_2)(x_2^H k_1))}{|\Re(\overline{\lambda}_0\lambda_1)|^2}$$
$$+ \frac{|x_2^H k_1|^2 + |x_1^H k_2|^2}{2|\Re(\overline{\lambda}_0\lambda_1)|^2 H_2^2(\lambda)}\left(H_2^4(\lambda) - 4|\Re(\overline{\lambda}_0\lambda_1)|^2\right).$$

Proof. To construct δA_j for $j = 0, 1$ such that $\delta A_0 = \epsilon \delta A_0^H$ and $\delta A_1 = -\epsilon \delta A_1^H$, we consider

$$\widetilde{\delta A_0} = U^H \delta A_0 U = \begin{array}{c c} & \begin{array}{cc} 2 & n-2 \end{array} \\ \begin{array}{c} 2 \\ n-2 \end{array} & \left[\begin{array}{c|c} \widehat{\delta A_0} & \epsilon \delta {B_0}^H \\ \hline \delta B_0 & \delta D_0 \end{array}\right] \end{array}, \tag{2.25}$$

$$\widetilde{\delta A_1} = U^H \delta A_1 U = \begin{array}{c c} & \begin{array}{cc} 2 & n-2 \end{array} \\ \begin{array}{c} 2 \\ n-2 \end{array} & \left[\begin{array}{c|c} \widehat{\delta A_1} & -\epsilon \delta {B_1}^H \\ \hline \delta B_1 & \delta D_1 \end{array}\right] \end{array}, \tag{2.26}$$

where $\widehat{\delta A_0} = \begin{bmatrix} \sqrt{\epsilon}\delta a_{0,11} & \delta a_{0,12} \\ \epsilon\overline{\delta a_{0,12}} & \sqrt{\epsilon}\delta a_{0,22} \end{bmatrix}$, $\widehat{\delta A_1} = \begin{bmatrix} \sqrt{-\epsilon}\delta a_{1,11} & \delta a_{1,12} \\ -\epsilon\overline{\delta a_{1,12}} & \sqrt{-\epsilon}\delta a_{1,22} \end{bmatrix}$, $\delta D_0 = \epsilon \delta {D_0}^H$, $\delta D_1 = -\epsilon \delta {D_1}^H$, $\delta B_j = \begin{bmatrix} b_{j1} & b_{j2} \end{bmatrix}$ for $j = 0, 1$, and $U \in \mathbb{C}^{n\times n}$ is a unitary matrix such that $U = \begin{bmatrix} V_1 & V_2 \end{bmatrix}$ with $V_1 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \in \mathbb{C}^{n\times 2}$. Similar to Theorem 2.5.4, we get the following equations:

$$\sqrt{\epsilon}\lambda_0 \delta a_{0,ii} + \sqrt{-\epsilon}\lambda_1 \delta a_{1,ii} = x_i^H k_i, \quad i = 1, 2, \tag{2.27}$$

$$\lambda_0 b_{0i} + \lambda_1 b_{1i} = V_2^H k_i,\ i = 1, 2, \tag{2.28}$$

$$\begin{bmatrix} \overline{\lambda}_0 & -\overline{\lambda}_1 \\ \lambda_0 & \lambda_1 \end{bmatrix} \begin{bmatrix} \delta a_{0,12} \\ \delta a_{1,12} \end{bmatrix} = \begin{bmatrix} \epsilon\overline{x_2^H k_1} \\ x_1^H k_2 \end{bmatrix}. \tag{2.29}$$

The minimum norm solution of (2.28) is given by $b_{0i} = \frac{\overline{\lambda}_0}{H_2^2(\lambda)} V_2^H k_i, b_{1i} = \frac{\overline{\lambda}_1}{H_2^2(\lambda)} V_2^H k_i$. Next, we have the following two cases.

Case-1: If $\Re(\overline{\lambda}_0\lambda_1) = 0$, then the minimum norm solution of (2.27) is given by $\delta a_{0,ii} = \frac{\overline{\sqrt{\epsilon}\lambda_0}}{H_2^2(\lambda)} x_i^H k_i$, $\delta a_{1,ii} = \frac{\overline{\sqrt{-\epsilon}\lambda_1}}{H_2^2(\lambda)} x_i^H k_i$. Since $A_0 = \epsilon A_0^H, A_0 = -\epsilon A_0^H$ and $\Re(\overline{\lambda}_0\lambda_1) = 0$, we get system (2.29) is consistent by using Remark 2.6.2. The minimum norm solution of (2.29) is given by $\delta a_{0,12} = \frac{\epsilon\lambda_0}{H_2^2(\lambda)}\overline{x_2^H k_1}$ and $\delta a_{1,12} = \frac{-\epsilon\lambda_1}{H_2^2(\lambda)}\overline{x_2^H k_1}$. Substituting these obtained values in (2.25) and (2.26), we get the desired perturbed matrices and *backward error.*

Case-2: If $\Re(\overline{\lambda}_0\lambda_1) \neq 0$, then using Lemma 2.6.1, we get $\delta a_{0,ii} = -\sqrt{\epsilon}x_i^H A_0 x_i$, and $\delta a_{1,ii} = -\sqrt{-\epsilon}x_i^H A_1 x_i$. Since $\Re(\overline{\lambda}_0\lambda_1) \neq 0$, i.e., $\overline{\lambda}_0\lambda_1 + \lambda_0\overline{\lambda}_1 \neq 0$, the unique solution of system (2.29) is given by $\delta a_{0,12} = \dfrac{\epsilon\lambda_1\overline{x_2^H k_1} + \overline{\lambda}_1 x_1^H k_2}{\overline{\lambda}_0\lambda_1 + \lambda_0\overline{\lambda}_1}$ and $\delta a_{1,12} = \dfrac{-\epsilon\lambda_0\overline{x_2^H k_1} + \overline{\lambda}_0 x_1^H k_2}{\overline{\lambda}_0\lambda_1 + \lambda_0\overline{\lambda}_1}$, which is the minimum norm solution. Using these obtained values, we can get the desired $\delta A_0, \delta A_1$ and backward error in this case. ■

Remark 2.6.4. *Suppose (μ, x_1) and (μ, x_2) are two approximate eigenpairs of a non-homogeneous H-even/H-odd matrix pencil $\mathbf{L}$ such that $\mu \in \mathbb{C}$ is a double-semisimple eigenvalue and $x_1, x_2 \in \mathbb{C}^n$ are orthonormal vectors. Then similar to Corollary 2.3.7, substituting $\lambda_0 = 1, \lambda_1 = \mu$ in Theorem 2.6.3, and using* [1, Theorem 3.7], *we get $(\eta_F^{\mathbf{S}}(\mu, x_1, x_2, \mathbf{L}))^2 = (\eta_F^{\mathbf{S}}(\mu, x_1, \mathbf{L}))^2 + (\eta_F^{\mathbf{S}}(\mu, x_2, \mathbf{L}))^2 = 2((\eta_F(\mu, x_1, \mathbf{L}))^2 + (\eta_F(\mu, x_2, \mathbf{L}))^2)$ for $\mu^2 = 1$. Further, using Corollary 2.4.4 in this obtained relation, we get $\eta_F^{\mathbf{S}}(\mu, x_1, x_2, \mathbf{L}) = \sqrt{2}(\eta_F(\mu, x_1, x_2, \mathbf{L}))$ when $\mu^2 = 1$. Note that for $\mu^2 = 1$, we have $H_2^4(\lambda) - 4|\Re(\overline{\lambda}_0\lambda_1)|^2 = 0$.*

Next section deals with the backward error analysis of T-even and T-odd matrix pencils.

2.7. Backward error analysis for T-even/T-odd matrix pencils

In this section, we state and prove the structured backward error theorem for T-*even/*T-*odd* matrix pencils. The derivation of the theorem is similar to the previous section. Hence we discuss only those steps which are unique for this section. We start this section with two important lemmas as follows:

Lemma 2.7.1. *Suppose $\lambda = (\lambda_0, \lambda_1) \in \mathbb{C}^2$ such that $\lambda_0 \neq 0$, $\lambda_1 \neq 0$, and $\epsilon = 1, -1$. Then the following equality holds:*

$$\frac{1}{G_\epsilon^2(\lambda)} - \frac{1}{H^2(\lambda)} = \frac{\epsilon(|\lambda_1|^2 - |\lambda_0|^2)}{G_\epsilon^2(\lambda)H^2(\lambda)},$$

where $G_\epsilon(\lambda) = \sqrt{\frac{|\lambda_0|^2(1+\epsilon)+|\lambda_1|^2(1-\epsilon)}{2}}$ and $H_2(\lambda) = (|\lambda_0|^2 + |\lambda_1|^2)^{1/2}$.

Proof. The proof follows by using the definitions of $G_\epsilon(\lambda)$ and $H_2(\lambda)$. ■

Throughout this section, $\epsilon = 1$ and $\epsilon = -1$ represent the T-even and T-odd cases, respectively.

Lemma 2.7.2. *Let* $\mathbf{L} \in \mathbf{L}(\mathbb{C}^{n\times n})$ *be a* T*-even/* T*-odd homogeneous matrix pencil of the form (2.1). Suppose* (λ, x_1) *and* (λ, x_2) *are two approximate eigenpairs of* $\mathbf{L}$*, where* $\lambda = (\lambda_0, \lambda_1) \in \mathbb{C}^2 \setminus \{(0,0)\}$ *is a double-semisimple eigenvalue and* $x_1, x_2 \in \mathbb{C}^n$ *are orthonormal vectors. Set* $k_i = -\mathbf{L}(\lambda)x_i$ *for* $i = 1:2$*. Then the following equality holds for* $t = 1, 2$ *:*

$$(\sum_{i=1}^{2} \sum_{j=1, j\neq i}^{2} \overline{x}_j x_j^T k_i x_i^H) x_t = \sum_{j=1, j\neq t}^{2} \overline{x_j} x_j^T k_t.$$

Proof. The proof is computational and obtained by using the fact that x_1 and x_2 are orthonormal vectors.■

Remark 2.7.3. *For* $\lambda_0 = 0$*, we have* $x_2^T k_1 = -\epsilon x_1^T k_2$*, and for* $\lambda_1 = 0$ *we have* $x_2^T k_1 = \epsilon x_1^T k_2$*.*

Remark 2.7.4. *We have* $(1+\epsilon)^2/4 = (1+\epsilon)/2$ *for* $\epsilon = 1, -1$*.*

Now we present the main theorem of this section.

Theorem 2.7.5. *Let* $\mathbf{L} \in \mathbf{L}(\mathbb{C}^{n\times n})$ *be a* T*-even/* T*-odd matrix pencil of the form (2.1). Let* (λ, x_1) *and* (λ, x_2) *be two approximate eigenpairs of* $\mathbf{L}$*, where* $x_1, x_2 \in \mathbb{C}^n$ *are orthonormal vectors, and* $\lambda = (\lambda_0, \lambda_1) \in \mathbb{C}^2 \setminus \{(0,0)\}$ *is a double-semisimple eigenvalue. Set* $k_i := -\mathbf{L}(\lambda)x_i$ *for* $i = 1, 2$*. Then there exists a* T*-even/* T*-odd matrix pencil* $\delta\mathbf{L}$ *of the form* $\delta\mathbf{L}(\alpha,\beta) = \alpha\delta A_0 + \beta\delta A_1$ *such that* $(\mathbf{L}(\lambda) + \delta\mathbf{L}(\lambda))x_i = 0$*. Then we have*

Case-1: *If* $\lambda_0 \neq 0$ *and* $\lambda_1 \neq 0$*, then the perturbation matrices are given by*

$$\delta A_0 = \sum_{i=1}^{2} (\overline{\lambda}_0 \frac{(1+\epsilon)}{2} \frac{\overline{x}_i x_i^T k_i x_i^H}{G_\epsilon^2(\lambda)} + \overline{\lambda}_0 \frac{\overline{P}_{x_1:x_2} k_i x_i^H + \epsilon \overline{x_i} k_i^T P_{x_1:x_2}}{H_2^2(\lambda)}) + \sum_{i=1}^{2} \sum_{j=1, j\neq i}^{2} \frac{\overline{x}_j x_j^T k_i x_i^H + \epsilon \overline{x}_j x_i^T k_j x_i^H}{2\lambda_0},$$

$$\delta A_1 = \sum_{i=1}^{2} (\overline{\lambda}_1 \frac{(1-\epsilon)}{2} \frac{\overline{x}_i x_i^T k_i x_i^H}{G_\epsilon^2(\lambda)} + \overline{\lambda}_1 \frac{\overline{P}_{x_1:x_2} k_i x_i^H - \epsilon \overline{x_i} k_i^T P_{x_1:x_2}}{H_2^2(\lambda)}) + \sum_{i=1}^{2} \sum_{j=1, j\neq i}^{2} \frac{\overline{x}_j x_j^T k_i x_i^H - \epsilon \overline{x}_j x_i^T k_j x_i^H}{2\lambda_1},$$

where $G_\epsilon(\lambda) = \sqrt{\dfrac{|\lambda_0|^2(1+\epsilon)+|\lambda_1|^2(1-\epsilon)}{2}}$. *In this case, the backward error is given by*

$$\begin{aligned}(\eta_F^{\mathbf{S}}(\lambda, x_1, x_2, \mathbf{L}))^2 &= \sum_{i=1}^{2}\Big(\frac{2\|k_i\|_2^2}{H_2^2(\lambda)} + \frac{\epsilon(|\lambda_1|^2-|\lambda_0|^2)|x_i^T k_i|^2}{G_\epsilon^2(\lambda)H_2^2(\lambda)}\Big) + \frac{|x_2^T k_1|^2+|x_1^T k_2|^2}{2H_2^2(\lambda)}\Big(\frac{|\lambda_0|}{|\lambda_1|}-\frac{|\lambda_1|}{|\lambda_0|}\Big)^2 \\ &- \Big(\frac{\epsilon}{|\lambda_1|^2}-\frac{\epsilon}{|\lambda_0|^2}\Big)\Re((x_1^T k_2)(\overline{x_2^T k_1})).\end{aligned}$$

Case-2: *If* $\lambda_0 = 0$ *or* $\lambda_1 = 0$, *then we have the following two cases:*

(*i*) *If* $\lambda_0 = 0$ *and* $\lambda_1 \neq 0$, *the perturbation matrices are given by* $\delta A_0 = 0$ *and*

$$\delta A_1 = \sum_{i=1}^{2}(-((1-\epsilon)/2)\overline{x}_i x_i^T A_1 x_i x_i^H - \overline{P}_{x_1:x_2} A_1 x_i x_i^H + \overline{x_i} x_i^T A_1 P_{x_1:x_2}) + \frac{x_2^T k_1}{\lambda_1}(\overline{x}_2 x_1^H - \epsilon \overline{x}_1 x_2^H).$$

In this case the backward error is given by

$$(\eta_F^{\mathbf{S}}(\lambda, x_1, x_2, \mathbf{L}))^2 = \sum_{i=1}^{2} 2\|A_1 x_i\|_2^2 - \frac{(1-\epsilon)}{2}|x_i^T A_0 x_i|^2 - 2|x_1^T A_1 x_2|^2.$$

(*ii*) *If* $\lambda_0 \neq 0$, *and* $\lambda_1 = 0$, *the perturbation matrices are given by* $\delta A_1 = 0$ *and*

$$\delta A_0 = \sum_{i=1}^{2}(-((1+\epsilon)/2)\overline{x}_i x_i^T A_0 x_i x_i^H - \overline{P}_{x_1:x_2} A_0 x_i x_i^H - \overline{x_i} x_i^T A_0 P_{x_1:x_2}) + \frac{x_2^T k_1}{\lambda_0}(\overline{x}_2 x_1^H + \epsilon \overline{x}_1 x_2^H).$$

In this case, the backward error is given by

$$(\eta_F^{\mathbf{S}}(\lambda, x_1, x_2, \mathbf{L}))^2 = \sum_{i=1}^{2} 2\|A_0 x_i\|_2^2 - ((1+\epsilon)/2)|x_i^T A_0 x_i|^2 - 2|x_1^T A_0 x_2|^2.$$

Proof. For constructing δA_j for $j = 0, 1$ such that $\delta A_0 = \epsilon \delta A_0^T, \delta A_1 = -\epsilon \delta A_1^T$, we consider

$$\widetilde{\delta A_0} = U^T \delta A_0 U = \begin{array}{c c} & \begin{array}{cc} 2 & n-2 \end{array} \\ \begin{array}{c} 2 \\ n-2 \end{array} & \left[\begin{array}{c|c} \widehat{\delta A_0} & \epsilon {\delta B_0}^T \\ \hline \delta B_0 & \delta D_0 \end{array}\right] \end{array}, \tag{2.30}$$

$$\widetilde{\delta A_1} = U^T \delta A_1 U = \begin{array}{c c} & \begin{array}{cc} 2 & n-2 \end{array} \\ \begin{array}{c} 2 \\ n-2 \end{array} & \left[\begin{array}{c|c} \widehat{\delta A_1} & -\epsilon {\delta B_1}^T \\ \hline \delta B_1 & \delta D_1 \end{array}\right] \end{array}, \tag{2.31}$$

where $\widehat{\delta A_0} = \begin{bmatrix} \frac{(1+\epsilon)}{2}\delta a_{0,11} & \epsilon \delta a_{0,12} \\ \delta a_{0,12} & \frac{(1+\epsilon)}{2}\delta a_{0,22} \end{bmatrix}$, $\widehat{\delta A_1} = \begin{bmatrix} \frac{(1-\epsilon)}{2}\delta a_{1,11} & -\epsilon \delta a_{1,12} \\ \delta a_{1,12} & \frac{(1-\epsilon)}{2}\delta a_{1,22} \end{bmatrix}$, $\delta D_0 = \epsilon {\delta D_0}^T$, $\delta D_1 = -\epsilon {\delta D_1}^T$, $\delta B_j = \begin{bmatrix} b_{j1} & b_{j2} \end{bmatrix}$ for $j = 0, 1$, and $U \in \mathbb{C}^{n \times n}$ is a unitary matrix such that

$U = \begin{bmatrix} V_1 & V_2 \end{bmatrix}$ with $V_1 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \in \mathbb{C}^{n\times 2}$. Similar to Theorem 2.5.4, we get the following equations

$$((1+\epsilon)/2)\lambda_0\delta a_{0,ii} + ((1-\epsilon)/2)\lambda_1\delta a_{1,ii} = x_i^T k_i, \quad i = 1,2 \tag{2.32}$$

$$\lambda_0 b_{0i} + \lambda_1 b_{1i} = V_2^T k_i,\ i = 1,2 \tag{2.33}$$

$$\begin{bmatrix} \lambda_0 & -\lambda_1 \\ \lambda_0 & \lambda_1 \end{bmatrix} \begin{bmatrix} \delta a_{0,12} \\ \delta a_{1,12} \end{bmatrix} = \begin{bmatrix} x_2^T k_1 \\ \epsilon x_1^T k_2 \end{bmatrix}. \tag{2.34}$$

The minimum norm solution of (2.33) is given by $b_{0i} = \frac{\overline{\lambda}_0}{H_2^2(\lambda)} V_2^T k_i$ and $b_{1i} = \frac{\overline{\lambda}_1}{H_2^2(\lambda)} V_2^T k_i$.
Case-1: When $\lambda_0 \neq 0$ and $\lambda_1 \neq 0$, then the minimum norm solution of (2.32) is given by $\delta a_{0,ii} = ((1+\epsilon)/2)\frac{\overline{\lambda}_0}{G_\epsilon^2(\lambda)} x_i^T k_i$ and $\delta a_{1,ii} = ((1-\epsilon)/2)\frac{\overline{\lambda}_1}{G_\epsilon^2(\lambda)} x_i^T k_i$. In this case $\delta a_{0,12} = \frac{x_2^T k_1 + \epsilon x_1^T k_2}{2\lambda_0}$, $\delta a_{1,12} = \frac{x_2^T k_1 - \epsilon x_1^T k_2}{2\lambda_1}$.

Case-2: When $\lambda_0 = 0$ but $\lambda_1 \neq 0$, we get system (2.34) is consistent by using Remark 2.7.3. The minimum norm solution of (2.34) is given by $\delta a_{0,12} = 0$ and $\delta a_{1,12} = \frac{x_2^T k_1}{\lambda_1}$. In this case $\delta a_{0,ii} = 0$ and $\delta a_{1,ii} = -\frac{(1-\epsilon)}{2} x_i^T A_1 x_i$. When $\lambda_0 \neq 0$ but $\lambda_1 = 0$, we get system (2.34) is consistent by using Remark 2.7.3. The minimum norm solution of (2.34) is given by $\delta a_{0,12} = \frac{x_2^T k_1}{\lambda_0}$, $\delta a_{1,12} = 0$. In this case $\delta a_{0,ii} = -\frac{(1+\epsilon)}{2} x_i^T A_0 x_i$ and $\delta a_{1,ii} = 0$.

Similar to earlier sections, we can get the backward error expression and perturbation matrices for Case-1 and Case-2 each case. ■

Remark 2.7.6. *Let (μ, x_1) and (μ, x_2) be two approximate eigenpairs of a non-homogeneous T-even/T-odd matrix pencil $\mathbf{L}$, where $\mu \in \mathbb{C}$ is a double-semisimple eigenvalue and $x_1, x_2 \in \mathbb{C}^n$ are orthonormal vectors. Then similar to Corollary 2.3.7, substituting $\lambda_0 = 1, \lambda_1 = \mu$ in Theorem 2.7.5, and using* [1, Theorem 3.4], *we get $(\eta_F^{\mathbf{S}}(\mu, x_1, x_2, \mathbf{L}))^2 = ((\eta_F^{\mathbf{S}}(\mu, x_1, \mathbf{L}))^2 + (\eta_F^{\mathbf{S}}(\mu, x_2, \mathbf{L}))^2) = 2((\eta_F(\mu, x_1, \mathbf{L}))^2 + (\eta_F(\mu, x_2, \mathbf{L}))^2)$ when $|\mu| = 1$. Further, using Corollary 2.4.4, we get $(\eta_F^{\mathbf{S}}(\mu, x_1, x_2, \mathbf{L})) = \sqrt{2}(\eta_F(\mu, x_1, x_2, \mathbf{L}))$ when $|\mu| = 1$.*

Finally, we summarize the relation between unstructured and structured backward errors of a single approximate eigenpair and two approximate eigenpairs. Let (μ, x_1) and (μ, x_2) be two approximate eigenpairs such that $\mu \in \mathbb{C}$ is a *double-semisimple eigenvalue* of a non-homogeneous matrix pencil $\mathbf{L}$. Then by Table 2.2, we present relationships between the structured backward error of two approximate eigenpairs of a *double-semisimple eigenvalue* $(\eta_F^{\mathbf{S}}(\mu, x_1, x_2, \mathbf{L}))$, the unstructured backward error of two approximate eigenpairs of a *double-semisimple eigenvalue* $(\eta_F(\mu, x_1, x_2, \mathbf{L}))$, and the structured backward error of a single approximate eigenpair $(\eta_F^{\mathbf{S}}(\mu, x_i, \mathbf{L}))$ for $i = 1, 2$.

TABLE 2.2. Relation between unstructured and structured backward errors of a single approximate eigenpair and two approximate eigenpairs for non-homogeneous matrix pencils.

Structure $(\mathbf{S})$	**Relation between** $\eta_F^{\mathbf{S}}(\mu, x_1, x_2, \mathbf{L})$ & $\eta_F^{\mathbf{S}}(\mu, x_i, \mathbf{L}), i = 1, 2$	**Relation between** $\eta_F^{\mathbf{S}}(\mu, x_1, x_2, \mathbf{L})$ & $\eta_F(\mu, x_1, x_2, \mathbf{L})$	**Relation between** $\eta_F^{\mathbf{S}}(\mu, x_1, x_2, \mathbf{L})$ & $\eta_F(\mu, x_i, \mathbf{L}), i = 1, 2$
T-sym./ T-skew-sym.	$\eta_F^{\mathbf{S}}(\mu, x_1, x_2, \mathbf{L}) \leq \sqrt{\eta_F^{\mathbf{S}}(\mu, x_1, \mathbf{L})^2 + \eta_F^{\mathbf{S}}(\mu, x_2, \mathbf{L})^2}$	$\eta_F^{\mathbf{S}}(\mu, x_1, x_2, \mathbf{L}) \leq \sqrt{2}\,\eta_F(\mu, x_1, x_2, \mathbf{L})$	$\eta_F^{\mathbf{S}}(\mu, x_1, x_2, \mathbf{L}) \leq \sqrt{2}\sqrt{\eta_F^2(\mu, x_1, \mathbf{L}) + \eta_F^2(\mu, x_2, \mathbf{L})}$
Herm./ skew-Herm.	$\eta_F^{\mathbf{S}}(\mu, x_1, x_2, \mathbf{L}) = \sqrt{\eta_F^{\mathbf{S}}(\mu, x_1, \mathbf{L})^2 + \eta_F^{\mathbf{S}}(\mu, x_2, \mathbf{L})^2}$ for $\mu^2 = -1$	$\eta_F^{\mathbf{S}}(\mu, x_1, x_2, \mathbf{L}) = \sqrt{2}\,\eta_F(\mu, x_1, x_2, \mathbf{L})$ for $\mu^2 = -1$	$\eta_F^{\mathbf{S}}(\mu, x_1, x_2, \mathbf{L}) = \sqrt{2}\sqrt{\eta_F^2(\mu, x_1, \mathbf{L}) + \eta_F^2(\mu, x_2, \mathbf{L})}$ for $\mu^2 = -1$
T-even/ T-odd	$\eta_F^{\mathbf{S}}(\mu, x_1, x_2, \mathbf{L}) = \sqrt{\eta_F^{\mathbf{S}}(\mu, x_1, \mathbf{L})^2 + \eta_F^{\mathbf{S}}(\mu, x_2, \mathbf{L})^2}$ for $\lvert\mu\rvert = 1$	$\eta_F^{\mathbf{S}}(\mu, x_1, x_2, \mathbf{L}) = \sqrt{2}\eta_F(\mu, x_1, x_2, \mathbf{L})$ for $\lvert\mu\rvert = 1$	$\eta_F^{\mathbf{S}}(\mu, x_1, x_2, \mathbf{L}) = \sqrt{2}\sqrt{\eta_F^2(\mu, x_1, \mathbf{L}) + \eta_F^2(\mu, x_2, \mathbf{L})}$ for $\lvert\mu\rvert = 1$
H-even/ H-odd	$\eta_F^{\mathbf{S}}(\mu, x_1, x_2, \mathbf{L}) = \sqrt{\eta_F^{\mathbf{S}}(\mu, x_1, \mathbf{L})^2 + \eta_F^{\mathbf{S}}(\mu, x_2, \mathbf{L})^2}$ for $\mu^2 = 1$	$\eta_F^{\mathbf{S}}(\mu, x_1, x_2, \mathbf{L}) = \sqrt{2}\eta_F(\mu, x_1, x_2, \mathbf{L})$ for $\mu^2 = 1$	$\eta_F^{\mathbf{S}}(\mu, x_1, x_2, \mathbf{L}) = \sqrt{2}\sqrt{\eta_F^2(\mu, x_1, \mathbf{L}) + \eta_F^2(\mu, x_2, \mathbf{L})}$ for $\mu^2 = 1$

2.8. Numerical example

In this section, we illustrate our developed theory with a numerical example using Matlab 7.11.0. Let $\mathbf{L}$ be a T-*skew-symmetric* non-homogeneous ($\alpha = 1$) matrix pencil of the form (2.1). Let A_0, A_1 be defined by

$$A_0 = \begin{bmatrix} 0 & -0.2600 + 0.6487i & -0.1135 + 0.3416i & -0.3040 - 0.6366i \\ 0.2600 - 0.6487i & 0 & -0.0914 - 0.5687i & -0.7628 + 0.4553i \\ 0.1135 - 0.3416i & 0.0914 + 0.5687i & 0 & 0.3138 - 0.3496i \\ 0.3040 + 0.6366i & 0.7628 - 0.4553i & -0.3138 + 0.3496i & 0 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 0 & -0.0996 - 0.8100i & 0.6837 + 0.2671i & 0.0716 + 0.0580i \\ 0.0996 + 0.8100i & 0 & 0.2214 - 0.5972i & -0.2433 - 0.0032i \\ -0.6837 - 0.2671i & -0.2214 + 0.5972i & 0 & 0.2821 + 0.2661i \\ -0.0716 - 0.0580i & 0.2433 + 0.0032i & -0.2821 - 0.2661i & 0 \end{bmatrix}.$$

These are random matrices such that $A_0 = -A_0^T$ and $A_1 = -A_1^T$. Clearly $\mathbf{L}$ is a regular matrix pencil. The approximate eigenpairs of $\mathbf{L}$ are obtained by using Matlab formula $[V, D] = eig(A_0, A_1)$. Let $\mu = -D(2,2) = -D(3,3)$ be an approximate multiple eigenvalue of $\mathbf{L}$, and its corresponding eigenvectors are $V(:,2)$ and $V(:,3)$. Orthonormal eigenvectors x_1, x_2 corresponding to μ, are obtained by $x_1 := V(:,2)/\|V(:,2)\|$ and $x_2 := (V(:,3) - \gamma * V(:,2))/\|V(:,3) - \gamma * V(:,2)\|$, where $\gamma = \frac{V(:,2)^H V(:,3)}{V(:,2)^H V(:,2)}$. Using Theorem 2.3.3 for $\epsilon = -1$, we get the following perturbation matrices

$$\delta A_0 = \begin{bmatrix} 0 & -0.0170 - 0.4873i & 0.3412 - 0.1463i & 0.1048 + 0.2769i \\ 0.0170 + 0.4873i & 0 & -0.0294 - 0.0085i & 0.2473 - 0.0822i \\ -0.3412 + 0.1463i & 0.0294 + 0.0085i & 0 & -0.0096 + 0.1557i \\ -0.1048 - 0.2769i & -0.2473 + 0.0822i & 0.0096 - 0.1557i & 0 \end{bmatrix},$$

$$\delta A_1 = \begin{bmatrix} 0 & -0.0891 + 0.7143i & -0.5315 + 0.1335i & -0.0880 - 0.4281i \\ 0.0891 - 0.7143i & -0 & 0.0409 + 0.0193i & -0.3797 + 0.0621i \\ 0.5315 - 0.1335i & -0.0409 - 0.0193i & 0 & 0.0503 - 0.2247i \\ 0.0880 + 0.4281i & 0.3797 - 0.0621i & -0.0503 + 0.2247i & 0 \end{bmatrix}.$$

$\eta_F^{\mathbf{S}}(\mu, x_1, x_2, \mathbf{L}) = 1.8809$. Clearly, $\mathbf{L}(\mu)x_i + \delta\mathbf{L}(\mu)x_i = 0$ for $i = 1, 2$.

Remark 2.8.1. *When we encounter with two approximate eigenpairs* (λ, x_1) *and* (λ, x_2), *where* λ *is a double-semisimple eigenvalue, the existing backward error theory of a single eigenpair fails to provide the minimum norm* $\delta\mathbf{L} \in \mathbf{L}(\mathbb{C}^{n\times n})$ *which satisfies* $(\mathbf{L}(\lambda) + \delta\mathbf{L}(\lambda))x_i = 0$ *for* $i = 1, 2$. *On the other hand, by using our theory, one can easily construct the required perturbed matrix pencil, and backward error corresponding to two approximate eigenpairs of a double-semisimple eigenvalue.* ■

CHAPTER 3

STRUCTURED PERTURBATION ANALYSIS OF SPECIFIED EIGENPAIRS FOR MATRIX PENCILS WITH SPARSITY

Abstract: This chapter is devoted for the backward error analysis of one or more approximate eigenpairs of structured matrix pencils. We worked with different structured matrix pencils which includes structures such as T-symmetric, T-skew-symmetric, Hermitian, skew-Hermitian, T-even, T-odd, H-even, H-odd, T-palindromic, T-anti-palindromic, H-palindromic, and H-anti-palindromic. Further, our backward error results are developed in such a way that one can solve the different kind of *inverse eigenvalue problems*. This shows that the backward error of one or more eigenpairs and *inverse eigenvalue problems* are interconnected.

3.1. Introduction

Backward error and *perturbation* analysis of a single *eigenpair* are well developed in case of structured and unstructured generalized eigenvalue problems (see, [**1**, **8**, **9**]). Since many applications require an only specific set of eigenpairs, for example, consider a problem of vibration in engineering applications which leads to an n-by-n *generalized eigenvalue problem* (GEP), required smallest $m \in \mathbb{N}\,(m \leq n)$ *eigenpairs* (see, [**65**]). Hence it is necessary to compute one or more specified *eigenpairs*. For computing these *eigenpairs*, several numerical methods are available in the literature, for more details see, [**46**, **54**, **64**, **66**, **69**, **83**] and references therein. Development of the *backward error analysis* for more than one approximate *eigenpairs* will provide a better understanding of the quality of the computed *eigenpairs* and stability of the numerical methods. Literature is very much restricted when it comes to the *backward error* and *perturbation theory* of more than one specified *eigenpairs*. Though in the last chapter we have taken a step towards the backward error analysis of two approximate eigenpairs provided the eigenvalue is double-semisimple, but the question that what will be the backward error in general, is still open for discussion. In this chapter, we discuss the backward error analysis of eigenpairs in more general way.

We have discussed in the last chapter that Tisseur [71] has obtained the backward error formulas and perturbation matrices of more than one specified *eigenpairs* for different structured matrices by generalizing the existing definition of backward error of a single approximate *eigenpair*. But in the case of a matrix pencil, the existing results of perturbation theory on a single approximate eigenpair are not sufficient to answer the following questions:

1. For a given matrix pencil and its given one or more eigenpairs, what is the nearest matrix pencil for which the given approximate eigenpairs simultaneously become the exact eigenpairs?
2. What is the value of *backward error* when one or more approximate *eigenpairs* of a given matrix pencil become exact *eigenpairs* of an appropriately perturbed matrix pencil ?

Inverse eigenvalue problem (IEP) deals with the construction of the perturbed matrices from the given spectral data which consists one or more *eigenpairs*. Development of the backward error analysis of GEP for more than one *eigenpairs* play an important role to provide the solution of a different kind of *inverse eigenvalue problems* (see, [19]). For example, consider the Problem 5.4 of [21, Chapter 5], which requires the construction of a quadratic matrix polynomial with prescribed *eigenpairs*. By linearization, *quadratic eigenvalue problems* (QEP) can be transformed into large GEP which has the same eigenstructure. Hence the Problem 5.4 is equivalent to solve the real symmetric GEP for more than one *eigenpairs*. For more information on the conversion of QEP to GEP, see [37, 70]. Another *inverse eigenvalue problem* which we discuss in this chapter is [89, Problem 1.1]. This problem requires to construct a symmetric matrix pencil from the given specified eigenpairs under the submatrix constraint. We explain both the IEPs by examples in Section 3.8. Further, the matrices in eigenvalue problems reflect the properties of underlying physical models; their structured backward error has a special attraction. If the coefficient matrices of GEP are structured, arbitrary perturbation to

GEP will not respect the structure and lead to insignificant results. Hence necessary care should be adopted while analyzing the structured perturbation, so the property of physical model will not be destroyed during *backward error analysis* of different structures such as *symmetric, skew-symmetric, T-even/T-odd, Hermitian/skew-Hermitian, palindromic* (see, [**1**, **10**, **17**, **52**, **51**]).

Several eigenvalue problems arise with matrices having a large number of zeroes, this is called the *zero structured* or *sparsity structure* (see, [**59**, **87**]). For this kind of matrices, it is necessary to work with those perturbed matrices which respect the sparsity structure. Hence for maintaining sparsity, we need to construct *sparse perturbation matrices*.

The main purpose of this chapter is to present a detailed structured *backward error analysis* of s-specified *eigenpairs* ($s \geq 1$) of structured matrix pencils which also preserve the sparsity. For the matrix case [**71**] provides the *backward error* formula and *perturbation matrices* for different structures without sparsity. In [**87**] the authors have adopted the methodology from [**71**, Section 3] for analyzing the structured backward error formula of one approximate *eigenpair* for *symmetric, skew-symmetric, Hermitian,* and *skew-Hermitian* polynomial eigenvalue problems. However, the results in [**71**, **87**] are unable to answers the above raised questions. Hence for given s-approximate *eigenpairs* $((\lambda_{i0}, \lambda_{i1}), x_i)$ of an n-by-n matrix pencil, where $(\lambda_{i0}, \lambda_{i1}) \in \mathbb{C}^2 \setminus \{(0,0)\}$ and $0 \neq x_i \in \mathbb{C}^n$ for $i = 1 : s$, we extend the methodology of [**35**, Section 4] and [**71**, **87**], so that these s-approximate *eigenpairs* become the exact *eigenpairs* of an appropriately perturbed matrix pencil. We discuss the perturbation analysis with respect to *Frobenius norm.*

3.2. Structured matrix pencils and preliminaries

Let us recall the definition of a matrix pencil. A matrix pencil $\mathbf{L}$ is a pair of two matrices defined as follows:

$$\mathbf{L}(\alpha, \beta) := \alpha A_0 + \beta A_1, \;\; A_0, A_1 \in \mathbb{C}^{n \times n}, \;\text{ and } (\alpha, \beta) \in \mathbb{C}^2. \tag{3.1}$$

Now, we define different structured matrix pencils of the form (3.1) by Table 3.2 based on the properties of matrices $A_0, A_1 \in \mathbb{C}^{n \times n}$.

S	**Matrix structure**
T-symmetric	$A_0 = A_0^T, A_1 = A_1^T$
T-skew-symmetric	$A_0 = -A_0^T, A_1 = -A_1^T$
Hermitian	$A_0 = A_0^H, A_1 = A_1^H$
skew-Hermitian	$A_0 = -A_0^H, A_1 = -A_1^H$
T-even	$A_0 = A_0^T, A_1 = -A_1^T$
T-odd	$A_0 = -A_0^T, A_1 = A_1^T$
H-even	$A_0 = A_0^H, A_1 = -A_1^H$
H-odd	$A_0 = -A_0^H, A_1 = A_1^H$

S	**Matrix structure**
T-palindromic	$A_0 = A_1^T$
T-anti-palindromic	$A_0 = -A_1^T$
H-palindromic	$A_0 = A_1^H$
H-anti-palindromic	$A_0 = -A_1^H$

TABLE 3.1. Types of structured matrix pencils

Throughout this chapter, $w := (w_0, w_1)^T \in \mathbb{R}^2$ be a nonnegative vector such that w_0, w_1 are nonnegative real numbers. Define $w^{-1} := (w_0^{-1}, w_1^{-1})^T$ and $w_i^{-1} = 0$ for $w_i = 0$. Next, for a given nonnegative weight vector $w = (w_0, w_1)^T$, we define the pencil norm as follows

$$\|\mathbf{L}\|_{w,2} := \|(w_0\|A_0\|_F, w_1\|A_1\|_F)\|_2 = (w_0^2\|A_0\|_F^2 + w_1^2\|A_1\|_F^2)^{1/2}. \tag{3.2}$$

Further, we generalize the definition of backward error from one approximate eigenpair to s-approximate eigenpairs for unstructured and structured matrix pencils. Further results are developed for describing the relation between structured and unstructured backward errors.

Definition 3.2.1. *Consider* $\lambda_{1:s} := \{\lambda_1, \lambda_2, \ldots, \lambda_s\}$ *and* $x_{1:s} := \{x_1, x_2, \ldots, x_s\}$*, where* $\lambda_i \in \mathbb{C}^2\backslash\{(0,0)\}$ *and* $0 \neq x_i \in \mathbb{C}^n$*, for* $i = 1 : s$*. Let* (λ_i, x_i) *be the* s*-approximate eigenpairs of the matrix pencil* $\mathbf{L} \in \mathbf{L}(\mathbb{C}^{n\times n})$ *of the form (3.1), for* $i = 1 : s$*. Then unstructured and structured backward errors of* s*-approximate eigenpairs* (λ_i, x_i) *for matrix pencil* $\mathbf{L}$ *are defined by*

$$\eta_{w,F}(\lambda_{1:s}, x_{1:s}, \mathbf{L}) := \inf\{\|\delta\mathbf{L}\|_{w,2}, \quad (\mathbf{L}(\lambda_i) + \delta\mathbf{L}(\lambda_i))x_i = 0; \textit{for } 1 \leq i \leq s\}, \textit{ and}$$

$$\eta_{w,F}^{\mathbf{S}}(\lambda_{1:s}, x_{1:s}, \mathbf{L}) := \inf\{\|\delta\mathbf{L}\|_{w,2}, \ \delta\mathbf{L} \in \mathbf{S}, \ (\mathbf{L}(\lambda_i) + \delta\mathbf{L}(\lambda_i))x_i = 0; \ \textit{for} \ \ 1 \leq i \leq s\},$$

respectively, where $\delta\mathbf{L} \in \mathbf{L}(\mathbb{C}^{n\times n})$ *is of the form (3.1),* $\|\delta\mathbf{L}\|_{w,2}$ *is given by (3.2), and*

$\mathbf{S} :=$ { *T-symmetric, T-skew-symmetric, T-even, T-odd, Hermitian, skew-Hermitian, H-even, H-odd, T-palindromic, T-anti-palindromic, H-palindromic and H-anti-palindromic* }.

Remark 3.2.2. *Substituting $s = 1$ in Definition 3.2.1 corresponds to the unstructured and the structured* backward errors *for a single approximate* eigenpair *(see, [1]).*

Remark 3.2.3. *For obtaining the backward error and the perturbation matrices of different structured matrix pencils, we take $s \leq n$.*

Remark 3.2.4. *The following relations are the immediate consequences of the definitions of backward error:*

$$\eta_{w,F}(\lambda_{1:s}, x_{1:s}, \mathbf{L}) \leq \eta^{\mathbf{S}}_{w,F}(\lambda_{1:s}, x_{1:s}, \mathbf{L}), \max_{i=1:s} \eta_{w,F}(\lambda_i, x_i, \mathbf{L}) \leq \eta_{w,F}(\lambda_{1:s}, x_{1:s}, \mathbf{L}), \text{ and}$$

$$\max_{i=1:s} \eta^{\mathbf{S}}_{w,F}(\lambda_i, x_i, \mathbf{L}) \leq \eta^{\mathbf{S}}_{w,F}(\lambda_{1:s}, x_{1:s}, \mathbf{L}).$$

3.2.1. Construction

For the establishment of the backward error analysis of specified *eigenpairs*, we need the following constructions.

1. Let $y_p \in \mathbb{C}^n$. Then $N^\epsilon(y_p) \in \mathbb{C}^{n\times(n^2+\epsilon n)/2}$ is defined by

$$N^\epsilon(y_p) := \begin{bmatrix} N^\epsilon_1(y_p) & \dots & N^\epsilon_{n-(1-\epsilon)/2}(y_p) \end{bmatrix}, \quad \text{for } p = 1:s. \tag{3.3}$$

For $\epsilon = 1$, define $N^1_1(y_p) \in \mathbb{C}^{n\times n}$, $N^1_2(y_p) \in \mathbb{C}^{n\times(n-1)}$, and $N^1_n(y_p) \in \mathbb{C}^n$ as follows:

$$N^1_1(y_p) := \begin{bmatrix} y_p^1 & y_p^2 & y_p^3 & \dots & y_p^n \\ 0 & y_p^1 & 0 & \dots & 0 \\ 0 & 0 & y_p^1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & y_p^1 \end{bmatrix}, N^1_2(y_p) := \begin{bmatrix} 0 & 0 & \dots & \dots & 0 \\ y_p^2 & y_p^3 & y_p^4 & \dots & y_p^n \\ 0 & y_p^2 & 0 & \dots & 0 \\ 0 & 0 & y_p^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & y_p^2 \end{bmatrix}, \text{ and } N^1_n(y_p) := \begin{bmatrix} 0 \\ 0 \\ \vdots \\ y_p^n \end{bmatrix}.$$

Similarly for $\epsilon = -1$, define $N^{-1}_1(y_p) \in \mathbb{C}^{n\times(n-1)}$, $N^{-1}_2(y_p) \in \mathbb{C}^{n\times(n-2)}$ and $N^{-1}_{n-1}(y_p) \in \mathbb{C}^n$ as follows:

$$N^{-1}_1(y_p) := \begin{bmatrix} y_p^2 & y_p^3 & \dots & \dots & y_p^n \\ -y_p^1 & 0 & \dots & \dots & 0 \\ 0 & -y_p^1 & \dots & \dots & 0 \\ \vdots & 0 & -y_p^1 & \dots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \dots & -y_p^1 \end{bmatrix}, N^{-1}_2(y_p) := \begin{bmatrix} 0 & 0 & \dots & 0 \\ y_p^3 & y_p^4 & \dots & y_p^n \\ -y_p^2 & 0 & \dots & 0 \\ 0 & -y_p^2 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & -y_p^2 \end{bmatrix},$$

and

$$N_{n-1}^{-1}(y_p) := \begin{bmatrix} 0 & 0 & \dots & y_p^n & -y_p^{n-1} \end{bmatrix}^T.$$

Remark 3.2.5. *Superscript "-1" in $N^{-1}(y_p)$ is only for notational point of view. It should not mismatch with the inverse of $N(y_p)$.*

2. Matrices $C := (c_{ij})$ and $D := (d_{ij})$ are defined in the following manner:

$$C = \begin{cases} \frac{1}{\sqrt{2}} & \text{when} \quad i \neq j, \\ 1 & \text{when} \quad i = j, \end{cases} \quad D = \begin{cases} \sqrt{2} & \text{when} \quad i \neq j, \\ 1 & \text{when} \quad i = j. \end{cases} \tag{3.4}$$

3. Let $A_j = (a_{j,tk}) \in \mathbb{C}^{n\times n}, \delta A_j = (\delta a_{j,tk}) \in \mathbb{C}^{n\times n}$, and w_j be nonnegative real number for $j = 0, 1$. Then for $\epsilon = \pm 1$, we define $\Delta_j^{\epsilon} = w_j \mathrm{vec}(\delta A_j \circ \mathrm{sgn}\, A_j \circ D, \epsilon)$ as follows:

$$\Delta_j^1 = \begin{bmatrix} w_j\, \delta a_{j,11}\, \mathrm{sgn}\, a_{j,11} \\ \sqrt{2}\, w_j\, \delta a_{j,12}\, \mathrm{sgn}\, a_{j,12} \\ \vdots \\ \sqrt{2}\, w_j \delta a_{j,1n}\, \mathrm{sgn}\, a_{j,1n} \\ w_j\, \delta a_{j,22}\, \mathrm{sgn}\, a_{j,22} \\ \sqrt{2}\, w_j \delta a_{j,23}\, \mathrm{sgn}\, a_{j,23} \\ \vdots \\ \sqrt{2}\, w_j \delta a_{j,2n} \mathrm{sgn}\, a_{j,2n} \\ \vdots \\ w_j\, \delta a_{j,nn} \mathrm{sgn}\, a_{j,nn} \end{bmatrix} \text{and } \Delta_j^{-1} = \begin{bmatrix} \sqrt{2}\, w_j \delta a_{j,12}\, \mathrm{sgn}\, a_{j,12} \\ \vdots \\ \sqrt{2}\, w_j \delta a_{j,1n}\, \mathrm{sgn}\, a_{j,1n} \\ \sqrt{2}\, w_j \delta a_{j,23}\, \mathrm{sgn}\, a_{j,23} \\ \vdots \\ \sqrt{2}\, w_j \delta a_{j,2n}\, \mathrm{sgn}\, a_{j,2n} \\ \vdots \\ \sqrt{2}\, w_j \delta a_{j,(n-1)n}\, \mathrm{sgn}\, a_{j,(n-1)n} \end{bmatrix}.$$

4. For $\epsilon = \pm 1$, define $\mathrm{vec}(\mathrm{sgn}\, A_j \circ C, \epsilon) \in \mathbb{C}^{(n^2+\epsilon n)/2}$ as follows:

$$\mathrm{vec}(\mathrm{sgn}\, A_j \circ C, 1) = [\mathrm{sgn}\, a_{j,11}, \dots, \frac{\sqrt{2}}{2} \mathrm{sgn}\, a_{j,1n}, \dots, \mathrm{sgn}\, a_{j,(n-1)(n-1)}, \frac{\sqrt{2}}{2} \mathrm{sgn}\, a_{j,(n-1)n}, \mathrm{sgn}\, a_{j,nn}]^T,$$

$$\mathrm{vec}(\mathrm{sgn}\, A_j \circ C, -1) = \frac{\sqrt{2}}{2} [\mathrm{sgn}\, a_{j,12}, \dots, \mathrm{sgn}\, a_{j,1n}, \dots, \mathrm{sgn}\, a_{j,(n-2)(n-1)}, \mathrm{sgn}\, a_{j,(n-2)n}, \mathrm{sgn}\, a_{j,(n-1)n}]^T.$$

We use the above terminologies in the subsequent sections for the development of perturbation theory of different structured matrix pencils.

3.3. Perturbation on T-symmetric and T-skew-symmetric matrix pencils with s-specified eigenpair(s)

This section deals with the perturbation theory and the backward error analysis of matrix pencils in which both the matrices are *T-symmetric/T-skew-symmetric.* Further, backward error results are obtained for a single approximate *eigenpair* and two-approximate *eigenpairs.* Before stating the theorem, let $0 \neq x_p \in \mathbb{C}^n$ and $0 \neq \lambda_p = (\lambda_{p0}, \lambda_{p1}) \in \mathbb{C}^2 \setminus \{(0,0)\}$ for $p = 1 : s$. Then we define

$$N^\epsilon = \begin{bmatrix} N^\epsilon_{10} & N^\epsilon_{11} \\ N^\epsilon_{20} & N^\epsilon_{21} \\ \vdots & \vdots \\ N^\epsilon_{s0} & N^\epsilon_{s1} \end{bmatrix} \in \mathbb{C}^{sn \times (n^2 + \epsilon n)}, \tag{3.5}$$

with $N^\epsilon_{pj} = w_j^{-1} \lambda_{pj} N^\epsilon(x_p) \text{diag}(\text{vec}(\text{sgn}\, A_j \circ C, \epsilon))$, where $N^\epsilon(x_p)$ is defined by (3.3).

Now, we introduce the following two important lemmas, which are useful for deriving the main result of this section.

Lemma 3.3.1. *Let $\delta A = (\delta\, a_{ij}) \in \mathbb{C}^{n\times n}$ be a symmetric matrix, $x = [x^1, x^2, \ldots, x^n]^T \in \mathbb{C}^n$, and $b = [b^1, b^2, \ldots, b^n]^T \in \mathbb{C}^n$. Then $\delta A x = b$ is equivalent to $N^1(x)\text{vec}(\delta A) = b$, where $\text{vec}(\delta A) := [\delta a_{11}, \ldots, \delta a_{1n}, \delta a_{22}, \ldots, \delta a_{2n}, \ldots,\ \delta a_{(n-1)(n-1)}, \delta a_{(n-1)n}, \delta a_{nn}]^T$, and $N^1(x)$ is defined by (3.3).*

Proof. Expanding $\delta A x = b$, we get the following n equations:

$$\begin{aligned}
&\delta a_{11} x^1 + \delta a_{12} x^2 + \cdots + \delta a_{1(n-1)} x^{(n-1)} + \delta a_{1n} x^n = b^1, \\
&\delta a_{12} x^1 + \delta a_{22} x^2 + \cdots + \delta a_{2(n-1)} x^{(n-1)} + \delta a_{2n} x^n = b^2, \\
&\ldots \qquad\qquad \ldots \qquad\qquad \ldots \qquad\qquad \ldots \\
&\ldots \qquad\qquad \ldots \qquad\qquad \ldots \qquad\qquad \ldots \\
&\delta a_{1n} x^1 + \delta a_{2n} x^2 + \cdots + \delta a_{(n-1)n} x^{(n-1)} + \delta a_{nn} x^n = b^n.
\end{aligned}$$

Further rearranging these equations by writing δA in vector form, we get $N^1(x)\text{vec}(\delta A) = b$, which is required. ■

Remark 3.3.2. *Similar to the symmetric case, we can obtain the result for skew-symmetric matrices. For skew-symmetric matrix δA, $N^{-1}(x)$ is defined by (3.3), and* $\mathrm{vec}(\delta A) := [\delta a_{12}, \dots, \delta a_{1n}, \delta a_{23}, \dots, \delta a_{2n}, \dots, \delta a_{(n-2)(n-1)}, \delta a_{(n-2)n}, \delta a_{(n-1)n}]^T$.

Lemma 3.3.3. *Let $A, \delta A \in \mathbb{C}^{n\times n}$ be symmetric matrices and $x, b \in \mathbb{C}^n$. Then $(\delta A \circ \mathrm{sgn} A \circ C \circ D)x = b$ is equivalent to $N^1(x)\mathrm{diag}(\mathrm{vec}(\mathrm{sgn}\, A \circ C, 1))\phi = b$, where $N^1(x)$ is defined by (3.3), $\phi = \mathrm{vec}(\delta A \circ \mathrm{sgn}\, A \circ D)$ and C, D are defined in Subsection 3.2.1.*

Proof. Since $\delta A \circ \mathrm{sgn} A = (\delta a_{ij}\, \mathrm{sgn}\, a_{ij})$, on considering $(\delta A \circ \mathrm{sgn} A \circ C \circ D)x = b$, we get the following n equations similar to Lemma 3.3.1

$$
\begin{array}{l}
\delta a_{11}\mathrm{sgn}\, a_{11}x^1 + (\sqrt{2}\delta a_{12})(\frac{\mathrm{sgn}\, a_{12}}{\sqrt{2}})x^2 + \cdots + (\sqrt{2}\delta a_{1n})(\frac{\mathrm{sgn}\, a_{1n}}{\sqrt{2}})x^n = b^1, \\
(\sqrt{2}\delta a_{12})(\frac{\mathrm{sgn}\, a_{12}}{\sqrt{2}})x^1 + \delta a_{22}\, \mathrm{sgn}\, a_{22}x^2 + \cdots + (\sqrt{2}\delta a_{2n})(\frac{\mathrm{sgn}\, a_{2n}}{\sqrt{2}})x^n = b^2, \\
\dots \qquad \dots \qquad \dots \qquad \dots \qquad \dots \\
\dots \qquad \dots \qquad \dots \qquad \dots \qquad \dots \\
(\sqrt{2}\delta a_{1n})(\frac{\mathrm{sgn}\, a_{1n}}{\sqrt{2}})x^1 + (\sqrt{2}\delta a_{2n})(\frac{\mathrm{sgn}\, a_{2n}}{\sqrt{2}})x^2 + \cdots + \delta a_{nn}\mathrm{sgn}\, a_{nn}x^n = b^n.
\end{array}
$$

Further rearrangement gives $N^1(x)\mathrm{diag}(\mathrm{vec}(\mathrm{sgn}\, A \circ C, 1))\phi = b$, which is required. ∎

Remark 3.3.4. *For more clarity, we present the proof of Lemma 3.3.3 for* $n = 3$.

Consider $(\delta A \circ \mathrm{sgn}\, A \circ C \circ D)x = b$, we get

$$
\left(\begin{bmatrix} \delta a_{11} & \delta a_{12} & \delta a_{13} \\ \delta a_{12} & \delta a_{22} & \delta a_{23} \\ \delta a_{13} & \delta a_{23} & \delta a_{33} \end{bmatrix} \circ \begin{bmatrix} \mathrm{sgn}\, a_{11} & \mathrm{sgn}\, a_{12} & \mathrm{sgn}\, a_{13} \\ \mathrm{sgn}\, a_{12} & \mathrm{sgn}\, a_{22} & \mathrm{sgn}\, a_{23} \\ \mathrm{sgn}\, a_{13} & \mathrm{sgn}\, a_{23} & \mathrm{sgn}\, a_{33} \end{bmatrix} \circ \begin{bmatrix} 1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 1 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 \end{bmatrix} \circ \begin{bmatrix} 1 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & 1 & \sqrt{2} \\ \sqrt{2} & \sqrt{2} & 1 \end{bmatrix}\right) \begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix} = \begin{bmatrix} b^1 \\ b^2 \\ b^3 \end{bmatrix}.
$$

Expanding the above expression, we get the following three equations:

$$
\begin{array}{r}
\delta a_{11}\, \mathrm{sgn}\, a_{11}x^1 + (\sqrt{2}\, \delta a_{12})(\frac{\mathrm{sgn}\, a_{12}}{\sqrt{2}})x^2 + (\sqrt{2}\, \delta a_{13})(\frac{\mathrm{sgn}\, a_{13}}{\sqrt{2}})x^3 = b^1, \\
(\sqrt{2}\, \delta a_{12})(\frac{\mathrm{sgn}\, a_{12}}{\sqrt{2}})x^1 + \delta a_{22}\, \mathrm{sgn}\, a_{22}\, x^2 + (\sqrt{2}\, \delta a_{23})(\frac{\mathrm{sgn}\, a_{23}}{\sqrt{2}})x^3 = b^2, \\
(\sqrt{2}\, \delta a_{13})(\frac{\mathrm{sgn}\, a_{13}}{\sqrt{2}})x^1 + (\sqrt{2}\, \delta a_{23})(\frac{\mathrm{sgn}\, a_{23}}{\sqrt{2}})x^2 + \delta a_{33}\, \mathrm{sgn}\, a_{33}x^3 = b^3.
\end{array}
$$

By rearranging the above equations, we get

$$(3.6)\qquad \begin{bmatrix} x^1 & x^2 & x^3 & 0 & 0 & 0 \\ 0 & x^1 & 0 & x^2 & x^3 & 0 \\ 0 & 0 & x^1 & 0 & x^2 & x^3 \end{bmatrix} Y \begin{bmatrix} \delta a_{11}\mathrm{sgn}\, a_{11} \\ \sqrt{2}\,\delta a_{12}\mathrm{sgn}\, a_{12} \\ \sqrt{2}\,\delta a_{13}\mathrm{sgn}\, a_{13} \\ \delta a_{22}\mathrm{sgn}\, a_{22} \\ \sqrt{2}\,\delta a_{23}\mathrm{sgn}\, a_{23} \\ \delta a_{33}\mathrm{sgn}\, a_{33} \end{bmatrix} = \begin{bmatrix} b^1 \\ b^2 \\ b^3 \end{bmatrix},$$

where

$$Y = \begin{bmatrix} \mathrm{sgn}\, a_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\mathrm{sgn}\, a_{12}}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\mathrm{sgn}\, a_{13}}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{sgn}\, a_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\mathrm{sgn}\, a_{23}}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathrm{sgn}\, a_{33} \end{bmatrix}.$$

System (3.6) is the same as $N^1(x)\mathrm{diag}(\mathrm{vec}(\mathrm{sgn}\, A \circ C, 1))\phi = b$.

Remark 3.3.5. *For the skew-symmetric case* $\mathrm{diag}(\mathrm{vec}(\mathrm{sgn}\, A \circ C, 1))$, *and* $N^1(x)$ *have to be replaced by* $\mathrm{diag}(\mathrm{vec}(\mathrm{sgn}\, A \circ C, -1))$ *and* $N^{-1}(x)$, *respectively.*

Now we state and prove the main result of this section in the light of the above lemmas. Throughout this section, $\epsilon = 1$ represents the T-*symmetric* case, and $\epsilon = -1$ represents the T-*skew-symmetric* case.

Theorem 3.3.6. *Let* $\mathbf{L} \in \mathbf{L}(\mathbb{C}^{n\times n})$ *be a* T*-symmetric/*T*-skew symmetric homogeneous matrix pencil of the form (3.1). Let* (λ_p, x_p) *be the* $s\,(s \leq n)$ *approximate eigenpairs of* $\mathbf{L}$ *with* $\lambda_p = (\lambda_{p0}, \lambda_{p1}) \in \mathbb{C}^2 \setminus \{(0,0)\}$, *and* $0 \neq x_p \in \mathbb{C}^n$ *for* $p = 1 : s$. *Set* $r := \begin{bmatrix} r_1^T & r_2^T & \dots & r_s^T \end{bmatrix}^T$, *where* $r_p := -\mathbf{L}(\lambda_p)x_p$ *for* $p = 1 : s$. *If* N^ϵ *(defined in Equation 3.5) has full row rank, then the backward error is given by*

$$\eta^{\mathbf{S}}_{w,F}(\lambda_{1:s}, x_{1:s}, \mathbf{L}) = \|N^{\epsilon H}(N^\epsilon N^{\epsilon H})^{-1} r\|_F.$$

A minimizing T*-symmetric/skew-symmetric matrix pencil* $\delta\mathbf{L} \in \mathbf{L}(\mathbb{C}^{n\times n})$, *such that* $(\mathbf{L}(\lambda_p) + \delta\mathbf{L}(\lambda_p))x_p = 0$, *is of the form* $\delta\mathbf{L}(\alpha, \beta) := \alpha\delta A_0 + \beta\delta A_1$, *where* $\delta A_j = (\delta a_{j,tk})$, *for* $j = 0, 1$, *are given by*

$$\delta a_{j,tk} = \begin{cases} \sum_{p=1}^{s} \frac{1}{2} w_j^{-2} \overline{\lambda}_{pj} (\operatorname{sgn} a_{j,tk}) (\overline{x}_p^k e_{t+(p-1)n}^T + \epsilon \overline{x}_p^t e_{k+(p-1)n}^T) (N^\epsilon N^{\epsilon H})^{-1} r, & \textit{for} \quad t \neq k, \\ \sum_{p=1}^{s} \frac{1+\epsilon}{2} w_j^{-2} \overline{\lambda}_{pj} (\operatorname{sgn} a_{j,tk}) \overline{x}_p^k e_{t+(p-1)n}^T (N^\epsilon N^{\epsilon H})^{-1} r, & \textit{for} \quad t = k. \end{cases}$$

Here $e_{k+(p-1)n}, e_{t+(p-1)n} \in \mathbb{C}^{sn}$.

Proof. For the given s-approximate eigenpairs (λ_p, x_p) of matrix pencil $\mathbf{L}$, we need to construct $\delta\mathbf{L} \in \mathbf{L}(\mathbb{C}^{n\times n})$ which preserves sparsity such that $(\mathbf{L}(\lambda_p) + \delta\mathbf{L}(\lambda_p))x_p = 0$. By assumption $\mathbf{L}(\lambda_p)x_p + r_p = 0$, for $p = 1 : s$. Then $r_p = \delta\mathbf{L}(\lambda_p)x_p = \sum_{j=0}^{1} \lambda_{pj}\delta A_j x_p = (\lambda_{p0}\,\delta A_0 + \lambda_{p1}\delta A_1)x_p = (\lambda_{p0}\delta A_0 \circ \operatorname{sgn}A_0 + \lambda_{p1}\delta A_1 \circ \operatorname{sgn}A_1)x_p$, where δA_j are replaced by $(\delta A_j \circ \operatorname{sgn} A_j)$ to maintain the sparsity in the perturbed matrices. Now, we have $r_p = \sum_{j=0}^{1} \lambda_{pj}(\delta A_j \circ \operatorname{sgn}A_j)x_p = \sum_{j=0}^{1} \lambda_{pj}(\delta A_j \circ \operatorname{sgn}A_j \circ D \circ C)x_p$. Further, we get $r_p = \sum_{j=0}^{1} w_j w_j^{-1}\lambda_{pj}(\delta A_j \circ \operatorname{sgn}A_j \circ D \circ C)x_p$. Finally rearranging r_p by using Lemma 3.3.3, we get $r_p = \sum_{j=0}^{1} w_j^{-1}\lambda_{pj}N^\epsilon(x_p)\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}A_j \circ C, \epsilon))\Delta_j^\epsilon = \sum_{j=0}^{1} N_{pj}^\epsilon \Delta_j^\epsilon$, where $\Delta_j^\epsilon = w_j \operatorname{vec}(\delta A_j \circ \operatorname{sgn} A_j \circ D, \epsilon)$ is a column vector defined in Section 3.2.1, and $N_{pj}^\epsilon = w_j^{-1}\lambda_{pj}N^\epsilon(x_p)\operatorname{diag}(\operatorname{vec}(\operatorname{sgn} A_j \circ C, \epsilon))$. Here $N^\epsilon(x_p) \in \mathbb{C}^{n\times(n^2+\epsilon n)/2}$, $\operatorname{diag}(\operatorname{vec}(\operatorname{sgn} A_j \circ C, \epsilon)) \in \mathbb{C}^{(n^2+\epsilon n)/2\times(n^2+\epsilon n)/2}$, and $\Delta_j^\epsilon \in \mathbb{C}^{(n^2+\epsilon n)/2}$.

Using $r_p = \sum_{j=0}^{1} N_{pj}^\epsilon \Delta_j^\epsilon$, $N_p^\epsilon = \begin{bmatrix} N_{p0}^\epsilon & N_{p1}^\epsilon \end{bmatrix}$ and $\Delta^\epsilon = \begin{bmatrix} \Delta_0^{\epsilon T} & \Delta_1^{\epsilon T} \end{bmatrix}^T$, we get $r_p = N_p^\epsilon \Delta^\epsilon$. For $p = 1 : s$, we get the following system of equations:

$r_1 = N_1^\epsilon \Delta^\epsilon = \begin{bmatrix} N_{10}^\epsilon & N_{11}^\epsilon \end{bmatrix} \begin{bmatrix} \Delta_0^\epsilon \\ \Delta_1^\epsilon \end{bmatrix}$, $r_2 = N_2^\epsilon \Delta^\epsilon = \begin{bmatrix} N_{20}^\epsilon & N_{21}^\epsilon \end{bmatrix} \begin{bmatrix} \Delta_0^\epsilon \\ \Delta_1^\epsilon \end{bmatrix}, \ldots,$ and $r_s = N_s^\epsilon \Delta^\epsilon = \begin{bmatrix} N_{s0}^\epsilon & N_{s1}^\epsilon \end{bmatrix} \begin{bmatrix} \Delta_0^\epsilon \\ \Delta_1^\epsilon \end{bmatrix}$. On writing these s equations in the combined form, we get

$$\begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_s \end{bmatrix} = \begin{bmatrix} N_{10}^\epsilon & N_{11}^\epsilon \\ N_{20}^\epsilon & N_{21}^\epsilon \\ \vdots & \vdots \\ N_{s0}^\epsilon & N_{s1}^\epsilon \end{bmatrix} \begin{bmatrix} \Delta_0^\epsilon \\ \Delta_1^\epsilon \end{bmatrix}. \tag{3.7}$$

By Equation 3.7, we have $r = N^\epsilon \Delta^\epsilon$, and under the assumption that N^ϵ is a full row rank matrix, the minimum norm solution of $r = N^\epsilon\Delta^\epsilon$ is given by $\Delta^\epsilon = N^{\epsilon H}(N^\epsilon N^{\epsilon H})^{-1}r$. Expanding the first $N^{\epsilon H}$ in the minimum norm solution, we get the desired entrywise perturbations of matrices δA_j, and the backward error in *Frobenius* norm case is given by $\eta_{w,F}^{\mathbf{S}}(\lambda_{1:s}, x_{1:s}, \mathbf{L}) = |||\delta\mathbf{L}|||_{w,F}$, where

$$|||\delta\mathbf{L}|||_{w,F}^2 = w_0^2\|\delta A_0\|_F^2 + w_1^2\|\delta A_1\|_F^2 = \|\Delta^\epsilon\|_{w,F}^2 = \|N^{\epsilon H}(N^\epsilon N^{\epsilon H})^{-1}r\|_F^2.$$

In particular, the *backward error*, when N^ϵ has full row rank, is given by

$$\eta^{\mathbf{S}}_{w,F}(\lambda_{1:s}, x_{1:s}, \mathbf{L}) = \|\Delta^\epsilon\|_F = \|N^{\epsilon H}(N^\epsilon N^{\epsilon H})^{-1} r\|_F.$$

Now we need to show that $(\mathbf{L}(\lambda_p) + \delta\mathbf{L}(\lambda_p))x_p = 0$ for $p = 1 : s$, for this consider

$$\begin{bmatrix} (\mathbf{L}(\lambda_1) + \delta\mathbf{L}(\lambda_1))x_1 \\ (\mathbf{L}(\lambda_2) + \delta\mathbf{L}(\lambda_2))x_2 \\ \vdots \\ (\mathbf{L}(\lambda_s) + \delta\mathbf{L}(\lambda_s))x_s \end{bmatrix} = \begin{bmatrix} -r_1 + \delta\mathbf{L}(\lambda_1)x_1 \\ -r_2 + \delta\mathbf{L}(\lambda_2)x_2 \\ \vdots \\ -r_s + \delta\mathbf{L}(\lambda_s)x_s \end{bmatrix} = \begin{bmatrix} -r_1 \\ -r_2 \\ \vdots \\ -r_s \end{bmatrix} + \begin{bmatrix} N^\epsilon_1 \\ N^\epsilon_2 \\ \vdots \\ N^\epsilon_s \end{bmatrix} \Delta^\epsilon = \begin{bmatrix} -r_1 \\ -r_2 \\ \vdots \\ -r_s \end{bmatrix} + \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_s \end{bmatrix} = 0,$$

where $\delta\mathbf{L}(\lambda_p))x_p = N^\epsilon_p\Delta^\epsilon$, for $p = 1 : s, N^\epsilon = \begin{bmatrix} N^{\epsilon T}_1 & N^{\epsilon T}_2 & \ldots & N^{\epsilon T}_s \end{bmatrix}^T$, and $\Delta^\epsilon = N^{\epsilon H}(N^\epsilon N^{\epsilon H})^{-1} r$.$\blacksquare$

Remark 3.3.7. *When N^ϵ is not a full row rank matrix but Equation 3.7 is consistent, then using Theorem 1.2.6 and singular value decomposition, we get $N^\epsilon = U^\epsilon \Sigma^\epsilon V^{\epsilon H}$ and the minimum norm solution $\Delta^\epsilon = V^\epsilon \Sigma^{\epsilon +} U^{\epsilon H} r$. The backward error is given by*

$$\eta^{\mathbf{S}}_{w,F}(\lambda_{1:s}, x_{1:s}, \mathbf{L}) = \|V^\epsilon \Sigma^{\epsilon +} U^{\epsilon H} r\|_F.$$

Here U^ϵ, V^ϵ are unitary matrices and Σ^ϵ contains the singular values of N^ϵ. In this case, we can not get the general formula for the perturbed matrix entries because singular value decomposition of N^ϵ is not known explicitly.

Remark 3.3.8. *If Equation 3.7 is inconsistent, i.e. $rank(N^\epsilon) \neq rank([N^\epsilon, r])$, then a minimal perturbation matrix pencil does not exist. In this case the backward error is given by $\eta^{\mathbf{S}}_{w,F}(\lambda_{1:s}, x_{1:s}, \mathbf{L}) = \infty$.*

Now we present the backward error results for two approximate eigenpairs and a single approximate eigenpair by the following corollaries. The obtained result for a single approximate eigenpair will be the same as the existing result of [87, Theorem 2].

Corollary 3.3.9. *Let $\mathbf{L} \in \mathbf{L}(\mathbb{C}^{n\times n})$ be a T-symmetric/T-skew-symmetric homogeneous matrix pencil of the form (3.1). Suppose (λ_1, x_1), (λ_2, x_2) are the approximate eigenpairs of $\mathbf{L}$ with $0 \neq x_p \in \mathbb{C}^n$ and $\lambda_p = (\lambda_{p0}, \lambda_{p1}) \in \mathbb{C}^2 \setminus \{(0,0)\}$ for $p = 1, 2$. Set $r = \begin{bmatrix} r_1^T & r_2^T \end{bmatrix}^T$, where $r_p := -\mathbf{L}(\lambda_p)x_p$ for $p = 1, 2$. If N^ϵ (defined as below) has full row rank, then the backward error is given by*

$$\eta^{\mathbf{S}}_{w,F}(\lambda_{1:2}, x_{1:2}, \mathbf{L}) = \|N^{\epsilon H}(N^\epsilon N^{\epsilon H})^{-1} r\|_F.$$

A minimizing T-symmetric/skew-symmetric $\delta\mathbf{L} \in \mathbf{L}(\mathbb{C}^{n\times n})$, such that $(\mathbf{L}(\lambda_p)+\delta\mathbf{L}(\lambda_p))x_p = 0$, is of the form $\delta\mathbf{L}(\alpha,\beta) := \alpha\delta A_0 + \beta\delta A_1$, where $\delta A_j = (\delta a_{j,tk})$ for $j = 0,1$, are given by

$$\delta a_{j,tk} := \begin{cases} \frac{1}{2}w_j^{-2}(\operatorname{sgn} a_{j,tk})[\overline{\lambda}_{1j}(\overline{x}_1^k e_t^T + \epsilon\overline{x}_1^t e_k^T) + \overline{\lambda}_{2j}(\overline{x}_2^k e_{t+n}^T + \\ \qquad\qquad \epsilon\overline{x}_2^t e_{k+n}^T)](N^\epsilon N^{\epsilon H})^{-1}r\,, & \text{for} \quad t \neq k, \\ \frac{1+\epsilon}{2}w_j^{-2}(\operatorname{sgn} a_{j,tk})[\overline{\lambda}_{1j}\overline{x}_1^k e_t^T + \overline{\lambda}_{2j}\overline{x}_2^k e_{t+n}^T](N^\epsilon N^{\epsilon H})^{-1}r, & \text{for} \quad t = k, \end{cases}$$

where $N^\epsilon = \begin{bmatrix} N_{10}^\epsilon & N_{11}^\epsilon \\ N_{20}^\epsilon & N_{21}^\epsilon \end{bmatrix} \in \mathbb{C}^{2n\times(n^2+\epsilon n)}$ is defined in Theorem 3.3.6, and $e_t, e_k, e_{k+n}, e_{t+n} \in \mathbb{C}^{2n}$.

Proof. Substituting $s = 2$ in Theorem 3.3.6, we get the desired result for two specified *eigenpairs*.■

After obtaining the backward error result for 2-specified *eigenpairs*, now we establish the result for a single specified eigenpair which coincides with the existing result.

Corollary 3.3.10. *Let $\mathbf{L} \in \mathbf{L}(\mathbb{C}^{n\times n})$ be a T-symmetric/T-skew-symmetric homogeneous matrix pencil of the form (3.1). Let (λ_1, x_1) be an approximate eigenpair of $\mathbf{L}$ with $0 \neq x_1 \in \mathbb{C}^n$ and $\lambda_1 = (\lambda_{10}, \lambda_{11}) \in \mathbb{C}^2 \setminus \{(0,0)\}$. Set $r_1 := -\mathbf{L}(\lambda_1)x_1$. If N^ϵ (defined as below) has full row rank, then the backward error is given by*

$$\eta_{w,F}^{\mathbf{S}}(\lambda_1, x_1, \mathbf{L}) = \|N^{\epsilon H}(N^\epsilon N^{\epsilon H})^{-1}r\|_F.$$

A minimizing T-symmetric/skew-symmetric $\delta\mathbf{L} \in \mathbf{L}(\mathbb{C}^{n\times n})$, such that $(\mathbf{L}(\lambda_1)+\delta\mathbf{L}(\lambda_1))x_1 = 0$, is of the form $\delta\mathbf{L}(\alpha,\beta) := \alpha\delta A_0 + \beta\delta A_1$, where $\delta A_j = (\delta a_{j,tk})$, for $j = 0,1$, are given by

$$\delta a_{j,tk} := \begin{cases} \frac{1}{2}w_j^{-2}\overline{\lambda}_{1j}(\overline{x}_1^k e_t^T + \epsilon\overline{x}_1^t e_k^T)(N^\epsilon N^{\epsilon H})^{-1}r\,(\operatorname{sgn} a_{j,tk}), & \text{for} \quad t \neq k, \\ \frac{1+\epsilon}{2}w_j^{-2}\overline{\lambda}_{1j}(\operatorname{sgn} a_{j,tk})\overline{x}_1^k e_t^T(N^\epsilon N^{\epsilon H})^{-1}r, & \text{for} \quad t = k, \end{cases}$$

here $N^\epsilon = \begin{bmatrix} N_{10}^\epsilon & N_{11}^\epsilon \end{bmatrix} \in \mathbb{C}^{n\times(n^2+\epsilon n)}$ is defined in Theorem 3.3.6 and $e_t, e_k \in \mathbb{C}^n$.

Proof. Substituting $s = 1$ in Theorem 3.3.6, we get the desired result for homogeneous matrix pencil case of [87, Theorem 2]. ■

Remark 3.3.11. *To obtain zeroes at the desired places in the perturbed matrices $\delta A_j, j = 0:1$, one can replace $\operatorname{sgn} A_j$ by $\operatorname{sgn} L_j$ in Theorem 3.3.6. Here $L_j = (l_{j,tk}) \in \mathbb{R}^{n\times n}$ and $l_{j,tk} = 0$ if we require $\delta a_{j,tk} = 0$, else $l_{j,tk} = 1$.*

3.4. Perturbation for Hermitian and skew-Hermitian matrix pencils with s-specified eigenpair(s)

In this section, we discuss the perturbation matrices, and the backward error formula for one or more approximate eigenpairs for *Hermitian* and *skew-Hermitian* matrix pencils. Before we state and prove the main results of this section, let $0 \neq x_p \in \mathbb{C}^n, \lambda_p = (\lambda_{p0}, \lambda_{p1}) \in \mathbb{C}^2 \setminus \{(0,0)\}$ and $e_{t+n+2n(p-1)}, e_{t+2n(p-1)}, e_{k+n+2n(p-1)}, e_{k+2n(p-1)} \in \mathbb{C}^{2sn}$. Consider $c_{pj,tk} := (e_{t+2n(p-1)} + \mathrm{i}e_{t+n+2n(p-1)})^T(g^k_{pj} - \mathrm{i}h^k_{pj}) + (e_{k+2n(p-1)} - \mathrm{i}e_{k+n+2n(p-1)})^T(\epsilon g^t_{pj} + \mathrm{i}\epsilon h^t_{pj})$, $g_{pj} := \Re(\lambda_{pj}x_p), h_{pj} := \Im(\lambda_{pj}x_p)$, $g^t_{pj} := \Re(\lambda_{pj}x^t_p)$, $h^t_{pj} := \Im(\lambda_{pj}x^t_p)$ for $p = 1 : s$ and $t, k = 1 : n$. Define

$$N^\epsilon := \begin{bmatrix} N^\epsilon_{10} & N^\epsilon_{11} \\ N^\epsilon_{20} & N^\epsilon_{21} \\ \vdots & \vdots \\ N^\epsilon_{s0} & N^\epsilon_{s1} \end{bmatrix} \in \mathbb{C}^{2sn \times 2n^2} \tag{3.8}$$

where $N^\epsilon_{pj} := w_j^{-1} \begin{bmatrix} N^\epsilon(g_{pj}) & -N^{-\epsilon}(h_{pj}) \\ N^\epsilon(h_{pj}) & N^{-\epsilon}(g_{pj}) \end{bmatrix} \operatorname{diag}\left(\begin{bmatrix} \operatorname{vec}(\operatorname{sgn} A_j \circ C, \epsilon) \\ \operatorname{vec}(\operatorname{sgn} A_j \circ C, -\epsilon) \end{bmatrix}\right)$, for $j = 0, 1$, are constructed by (3.3). Throughout this section, $\epsilon = 1$ represents the *Hermitian* case, and $\epsilon = -1$ represents the *skew-Hermitian* case.

Theorem 3.4.1. *Let $\mathbf{L} \in \mathbf{L}(\mathbb{C}^{n\times n})$ be a Hermitian/skew-Hermitian homogeneous matrix pencil of the form (3.1). Let (λ_p, x_p) be the $s\,(s \leq n)$ approximate eigenpairs of $\mathbf{L}$ with $0 \neq x_p \in \mathbb{C}^n$ and $\lambda_p = (\lambda_{p0}, \lambda_{p1}) \in \mathbb{C}^2 \setminus \{(0,0)\}$ for $p = 1 : s$. Set $r := \begin{bmatrix} \Re(r_1)^T & \Im(r_1)^T & \ldots & \Re(r_s)^T & \Im(r_s)^T \end{bmatrix}^T$, where $r_p := -\mathbf{L}(\lambda_p)x_p$, for $p = 1 : s$. If N^ϵ (defined in Equation 3.8) has full row rank, then the backward error is given by*

$$\eta^{\mathbf{S}}_{w,F}(\lambda_{1:s}, x_{1:s}, \mathbf{L}) = \|N^{\epsilon T}(N^\epsilon N^{\epsilon T})^{-1} r\|_F.$$

A minimizing Hermitian/skew-Hermitian matrix pencil $\delta\mathbf{L} \in \mathbf{L}(\mathbb{C}^{n\times n})$, such that $(\mathbf{L}(\lambda_p) + \delta\mathbf{L}(\lambda_p))x_p = 0$, is of the form $\delta\mathbf{L}(\alpha, \beta) := \alpha\delta A_0 + \beta\delta A_1$, where $\delta A_j = (\delta a_{j,tk})$, for $j = 0, 1$, are given by

$$\delta a_{j,tk} = \begin{cases} (\operatorname{sgn} a_{j,tt}) \sum_{p=1}^{s} \sqrt{\epsilon} w_j^{-2} \big(g^t_{pj} e^T_{t+\frac{1-\epsilon}{2}n(2p-1)+\frac{1+\epsilon}{2}2n(p-1)} + \epsilon h^t_{pj} & \\ e^T_{t+\frac{1+\epsilon}{2}n(2p-1)+\frac{1-\epsilon}{2}2n(p-1)}\big)(N^\epsilon N^{\epsilon T})^{-1} r, & \text{for } t = k, \\ (\operatorname{sgn} a_{j,tk}) \sum_{p=1}^{s} \frac{1}{2} w_j^{-2} c_{pj,tk} (N^\epsilon N^{\epsilon T})^{-1} r, & \text{for } t \neq k. \end{cases}$$

Here $e_i \in \mathbb{C}^{2sn}$ for any $i \in \mathbb{N}$.

Proof. For the given s-approximate eigenpairs (λ_p, x_p) of matrix pencil $\mathbf{L}$, we need to construct minimal norm sparse $\delta\mathbf{L} \in \mathbf{L}(\mathbb{C}^{n\times n})$ such that $(\mathbf{L}(\lambda_p) + \delta\mathbf{L}(\lambda_p))x_p = 0$. By assumption $\mathbf{L}(\lambda_p)x_p + r_p = 0$ for $p = 1 : s$. Then $r_p = \delta\mathbf{L}(\lambda_p)x_p = \sum_{j=0}^{1} \lambda_{pj}\delta A_j x_p = \sum_{j=0}^{1}(\Re(\delta A_j)+\mathrm{i}\Im(\delta A_j))(\Re(\lambda_{pj}x_p)+\mathrm{i}\Im(\lambda_{pj}x_p)) = \sum_{j=0}^{1} \Re(\delta A_j)\,\Re(\lambda_{pj}x_p)-\Im(\delta A_j)\Im(\lambda_{pj}x_p) + \mathrm{i}(\Im(\delta A_j)\Re(\lambda_{pj}x_p) + \Re(\delta A_j)\Im(\lambda_{pj}x_p)) = \sum_{j=0}^{1}[\Re(\delta A_j)\Re(\lambda_{pj}x_p) - \Im(\delta A_j)\Im(\lambda_{pj}x_p) + \mathrm{i}(\Im(\delta A_j)\Re(\delta A_j) + \Re(\delta A_j)\Im(\lambda_{pj}x_p))] \circ \operatorname{sgn} A_j \circ D \circ C = \Re(r_p) + \mathrm{i}\Im(r_p)$, where

$$\Re(r_p) = \sum_{j=0}^{1}[\Re(\delta A_j)g_{pj} - \Im(\delta A_j)h_{pj}] \circ (\operatorname{sgn} A_j) \circ D \circ C, \tag{3.9}$$

$$\Im(r_p) = \sum_{j=0}^{2}[\Re(\delta A_j)h_{pj} + \Im(\delta A_j)g_{pj}] \circ (\operatorname{sgn} A_j) \circ D \circ C, \tag{3.10}$$

for $p = 1 : s$. By applying Proposition 1.2.21 and Proposition 1.2.22 for *Hermitian (skew-Hermitian)* case, we get that $\Re(\delta A_j)$, $\Im(\delta A_j)$ are real *symmetric (skew-symmetric)* and real *skew-symmetric (symmetric)* matrices, respectively. Now separating the unknown and known variables in (3.9) and (3.10) by using Lemma 3.3.3, and Remark 3.3.5, we get the following system for $p = 1 : s$

$$\begin{bmatrix}\Re(r_p)\\ \Im(r_p)\end{bmatrix} = \sum_{j=0}^{1} w_j^{-1}\begin{bmatrix}N^{\epsilon}(g_{pj}) & -N^{-\epsilon}(h_{pj})\\ N^{\epsilon}(h_{pj}) & N^{-\epsilon}(g_{pj})\end{bmatrix}\operatorname{diag}\left(\begin{bmatrix}\operatorname{vec}(\operatorname{sgn} A_j \circ C, \epsilon)\\ \operatorname{vec}(\operatorname{sgn} A_j \circ C, -\epsilon)\end{bmatrix}\right)\Delta_j^{\epsilon},$$

where $\Delta_j^{\epsilon} = w_j\begin{bmatrix}\operatorname{vec}(\Re(\delta A_j) \circ \operatorname{sgn} A_j \circ D, \epsilon)\\ \operatorname{vec}(\Im(\delta A_j) \circ \operatorname{sgn} A_j \circ D, -\epsilon)\end{bmatrix}$ for $j = 0, 1$. Writing s equations in the combined form, we get

$$\begin{bmatrix}\Re(r_1)\\ \Im(r_1)\\ \vdots\\ \Re(r_s)\\ \Im(r_s)\end{bmatrix} = \begin{bmatrix}N_{10}^{\epsilon} & N_{11}^{\epsilon}\\ N_{20}^{\epsilon} & N_{21}^{\epsilon}\\ \vdots & \vdots\\ N_{s0}^{\epsilon} & N_{s1}^{\epsilon}\end{bmatrix}\begin{bmatrix}\Delta_0^{\epsilon}\\ \Delta_1^{\epsilon}\end{bmatrix}. \tag{3.11}$$

If N^{ϵ} has full row rank, then solving the above system in the least square sense, we get

$$\Delta^{\epsilon} = N^{\epsilon T}(N^{\epsilon}N^{\epsilon T})^{-1}\begin{bmatrix}\Re(r_1)^T & \Im(r_1)^T & \dots & \Re(r_s)^T & \Im(r_s)^T\end{bmatrix}^T.$$

The backward error in *Frobenius* norm case is given by $\eta_{w,F}^{\mathbf{S}}(\lambda_{1:s}, x_{1:s}, \mathbf{L}) = \|\delta\mathbf{L}\|_{w,F}$, where

$$\|\delta\mathbf{L}\|_{w,F} = \sqrt{\sum_{i=0}^{1} w_i^2\|\delta A_i\|_F^2} = \|N^{\epsilon T}(N^{\epsilon}N^{\epsilon T})^{-1}r\|_F.$$

After obtaining the result for the general case, now we state the following corollary for a single approximate eigenpair, which is immediate from Theorem 3.4.1 for $s = 1$.

Corollary 3.4.2. *Let* $\mathbf{L} \in \mathbf{L}(\mathbb{C}^{n\times n})$ *be a Hermitian/skew-Hermitian homogeneous matrix pencil of the form (3.1). Let* (λ_1, x_1) *be an approximate eigenpair of* $\mathbf{L}$ *with* $0 \neq x_1 \in \mathbb{C}^n$ *and* $\lambda_1 = (\lambda_{10}, \lambda_{11})) \in \mathbb{C}^2 \setminus \{(0,0)\}$. *Let* $r_1 := -\mathbf{L}(\lambda_1)x_1$. *If* N^ϵ *(defined as below) has full row rank, then the backward error is given by*

$$\eta^{\mathbf{S}}_{w,F}(\lambda_1, x_1, \mathbf{L}) = \|N^{\epsilon T}(N^\epsilon N^{\epsilon T})^{-1} r\|_F.$$

A minimizing Hermitian/skew-Hermitian $\delta\mathbf{L} \in \mathbf{L}(\mathbb{C}^{n\times n})$*, such that* $(\mathbf{L}(\lambda_1) + \delta\mathbf{L}(\lambda_1))x_1 = 0$*, is of the form* $\delta\mathbf{L}(\alpha,\beta) := \alpha\delta A_0 + \beta\delta A_1$*, where* $\delta A_j = (\delta a_{j,tk})$*, for* $j = 0, 1$*, are given by*

$$\delta a_{j,tk} = \begin{cases} w_j^{-2}(\operatorname{sgn} a_{j,tt})\left(g^t_{1j}e^T_{t+\frac{1-\epsilon}{2}n} + h^t_{1j}e^T_{t+\frac{1-\epsilon}{2}n}\right)(N^\epsilon N^{\epsilon T})^{-1}r, & \textit{for} \quad t = k, \\ \frac{1}{2}w_j^{-2}(\operatorname{sgn} a_{j,tk})c_{1j,tk}(N^\epsilon N^{\epsilon T})^{-1}r, & \textit{for} \quad t \neq k, \end{cases}$$

here $c_{1j,tk} = (e_t + \mathrm{i}e_{t+n})^T(g^k_{1j} - \mathrm{i}h^k_{1j}) + (e_k - \mathrm{i}e_{k+n})^T(\epsilon g^t_{1j} + \mathrm{i}\epsilon h^t_{1j})$*, and* $N^\epsilon = \begin{bmatrix} N^\epsilon_{10} & N^\epsilon_{11} \end{bmatrix} \in \mathbb{C}^{2n\times 2n^2}$ *with* $N^\epsilon_{10}, N^\epsilon_{11}$ *are defined in Theorem 3.4.1.*

3.5. Backward error for T-even and T-odd matrix pencils with s-specified eigenpair(s)

In this section, we discuss the backward error analysis of matrix pencils of the alternative structures, i.e., A_0, $A_1 \in \mathbb{C}^{n\times n}$ are symmetric and skew-symmetric, respectively for T-*even* case and vice-versa for T-*odd* case. Before moving to the main result of this section, we define the matrix N^ϵ as follows:

$$N^\epsilon := \begin{bmatrix} N^\epsilon_{10} & N^{-\epsilon}_{11} \\ N^\epsilon_{20} & N^{-\epsilon}_{21} \\ \vdots & \vdots \\ N^\epsilon_{s0} & N^{-\epsilon}_{s1} \end{bmatrix} \in \mathbb{C}^{sn\times 2n^2} \text{ such that } N^\epsilon_{p0} = w_0^{-1}\lambda_{p0}N^\epsilon(x_p)\operatorname{diag}(\operatorname{vec}(\operatorname{sgn} A_0 \circ C, \epsilon))$$

and $N^{-\epsilon}_{p1} := w_1^{-1}\lambda_{p1}N^\epsilon(x_p)\operatorname{diag}(\operatorname{vec}(\operatorname{sgn} A_1 \circ C, -\epsilon))$ are defined in Equation 3.3. Now we state and prove the theorem for T-*even/T-odd* case.

Throughout this section, $\epsilon = 1$ and $\epsilon = -1$ exhibit the T-*even* and T-*odd* cases, respectively.

Theorem 3.5.1. *Let $\mathbf{L} \in \mathbf{L}(\mathbb{C}^{n\times n})$ be a T-even/T-odd homogeneous matrix pencil of the form (3.1). Suppose (λ_p, x_p) are the $s\,(s \leq n)$ approximate eigenpairs of $\mathbf{L}$ with $0 \neq x_p \in \mathbb{C}^n$, and $\lambda_p = (\lambda_{p0}, \lambda_{p1}) \in \mathbb{C}^2 \setminus \{(0,0)\}$. Set $r := \begin{bmatrix} r_1^T & r_2^T & \dots & r_s^T \end{bmatrix}^T$, where $r_p := -\mathbf{L}(\lambda_p)x_p$, for $p = 1:s$. If N^ϵ (defined as above) has full row rank, then the backward error is given by*

$$\eta^{\mathbf{S}}_{w,F}(\lambda_{1:s}, x_{1:s}, \mathbf{L}) = \|N^{\epsilon H}(N^\epsilon N^{\epsilon H})^{-1} r\|_F.$$

A minimizing T-even/T-odd $\delta\mathbf{L} \in \mathbf{L}(\mathbb{C}^{n\times n})$, such that $(\mathbf{L}(\lambda_p) + \delta\mathbf{L}(\lambda_p))x_p = 0$, is of the form $\delta\mathbf{L}(\alpha,\beta) := \alpha\delta A_0 + \beta\delta A_1$, where $\delta A_j = (\delta a_{j,tk})$, for $j = 0,1$, are given by

$$\delta a_{0,tk} = \begin{cases} \sum_{p=1}^{s} \frac{1}{2} w_0^{-2}\overline{\lambda}_{p0}(\operatorname{sgn} a_{0,tk})(\overline{x}_p^k e^T_{t+(p-1)n} + \epsilon \overline{x}_p^t e^T_{k+(p-1)n})(N^\epsilon N^{\epsilon H})^{-1} r, & \textit{for} \quad t \neq k, \\ \sum_{p=1}^{s} \frac{1+\epsilon}{2} w_0^{-2}\overline{\lambda}_{p0}(\operatorname{sgn} a_{0,tk})\overline{x}_p^k e^T_{t+(p-1)n}(N^\epsilon N^{\epsilon H})^{-1} r, & \textit{for} \quad t = k, \end{cases}$$

$$\delta a_{1,tk} = \begin{cases} \sum_{p=1}^{s} \frac{1}{2} w_1^{-2}\overline{\lambda}_{p1}(\operatorname{sgn} a_{1,tk})(\overline{x}_p^k e^T_{t+(p-1)n} - \epsilon \overline{x}_p^t e^T_{k+(p-1)n})(N^\epsilon N^{\epsilon H})^{-1} r, & \textit{for} \quad t \neq k, \\ \sum_{p=1}^{s} \frac{1-\epsilon}{2} w_1^{-2}\overline{\lambda}_{p1}(\operatorname{sgn} a_{1,tk})\overline{x}_p^k e^T_{t+(p-1)n}(N^\epsilon N^{\epsilon H})^{-1} r, & \textit{for} \quad t = k. \end{cases}$$

Here $e_{t+(p-1)n}, e_{k+(p-1)n} \in \mathbb{C}^{sn}$.

Proof. For the given s-approximate eigenpairs (λ_p, x_p) of the *T-even (T-odd)* matrix pencil $\mathbf{L}$, we need to construct minimal norm $\delta\mathbf{L}$ such that $(\mathbf{L}(\lambda_p)+\delta\mathbf{L}(\lambda_p))x_p = 0$. By assumption $\mathbf{L}(\lambda_p)x_p + r_p = 0$, for $p = 1:s$. For sparsity, we replace δA_j by $(\delta A_j \circ \operatorname{sgn} A_j)$, then $r_p = \delta\mathbf{L}(\lambda_p)x_p = \sum_{j=0}^{1} \lambda_{pj}\delta A_j x_p = \sum_{j=0}^{1} \lambda_{pj}(\delta A_j \circ \operatorname{sgn} A_j)x_p = \sum_{j=0}^{1} \lambda_{pj}(\delta A_j \circ \operatorname{sgn} A_j \circ D \circ C)x_p$. Let $\Delta_0^\epsilon = w_0 \operatorname{vec}(\delta A_0 \circ \operatorname{sgn} A_j \circ D, \epsilon)$, and $\Delta_1^{-\epsilon} = w_1 \operatorname{vec}(\delta A_1 \circ \operatorname{sgn} A_j \circ D, -\epsilon)$. Then similar to previous theorem, we get

$$\begin{aligned} r_p = {} & w_0^{-1}\lambda_{p0}N^\epsilon(x_p)\operatorname{diag}(\operatorname{vec}(\operatorname{sgn} A_j \circ C, \epsilon))\Delta_0^\epsilon + w_1^{-1}\lambda_{p1}N^{-\epsilon}(x_p)\operatorname{diag}(\operatorname{vec}(\operatorname{sgn} A_j \circ C, -\epsilon))\Delta_1^{-\epsilon} \\ r_p = {} & N^\epsilon_{p0}\Delta_0^\epsilon + N^{-\epsilon}_{p1}\Delta_1^{-\epsilon} = N_p^\epsilon \Delta^\epsilon. \end{aligned}$$

Combining the above s equations for $p = 1:s$, we get $r = N^\epsilon\Delta^\epsilon$. *Perturbation matrices* and *backward error* for the *T-even/T-odd* case are obtained similar to Theorem 3.3.6. ■

Remark 3.5.2. *For the T-odd case ($\epsilon = -1$), N^{-1}_{p0} is constructed according to the skew-symmetric case, and N^1_{p1} is constructed according to the symmetric case.*

Remark 3.5.3. *The difference between T-symmetric, T-skew-symmetric, Hermitian, skew-Hermitian, T-even/T-odd, and H-even/T-odd cases is the construction of matrix N^ϵ which is of different sizes for each structured matrix pencil.*

3.6. Perturbation analysis for H-even and H-odd matrix pencils with s-specified eigenpair(s)

Similar to the above section, this section deals with the *backward error* analysis of one or more approximate eigenpairs for H-even and H-odd matrix pencils. For stating the main theorem of this section, let $0 \neq x_p \in \mathbb{C}^n, \lambda_p = (\lambda_{p0}, \lambda_{p1}) \in \mathbb{C}^2 \setminus \{(0,0)\}$, and $e_{k+2n(p-1)}, e_{t+2n(p-1)}, e_{k+n+2n(p-1)}, e_{t+n+2n(p-1)} \in \mathbb{C}^{2sn}$, for $p = 1 : s$, and $t, k = 1 : n$. Next, consider the following notations as follows:

$c_{p0,tk} := (e_{t+2n(p-1)} + \mathrm{i}e_{t+n+2n(p-1)})^T(g^k_{p0} - \mathrm{i}h^k_{p0}) + (e_{k+2n(p-1)} - \mathrm{i}e_{k+n+2n(p-1)})^T(\epsilon g^t_{p0} + \mathrm{i}\epsilon h^t_{p0})$,
$c_{p1,tk} := (e_{t+2n(p-1)} + \mathrm{i}e_{t+n+2n(p-1)})^T(g^k_{p1} - \mathrm{i}h^k_{p1}) + (e_{k+2n(p-1)} - \mathrm{i}e_{k+n+2n(p-1)})^T(-\epsilon g^t_{p1} - \mathrm{i}\epsilon h^t_{p1})$,
and $g_{pj} := \Re(\lambda_{pj}x_p)$, $h_{pj} = \Im(\lambda_{pj}x_p)$, and $g^t_{pj} := \Re(\lambda_{pj}x^t_p), h^t_{pj} := \Im(\lambda_{pj}x^t_p)$. Further, we define

$$N^\epsilon := \begin{bmatrix} N^\epsilon_{10} & N^{-\epsilon}_{11} \\ N^\epsilon_{20} & N^{-\epsilon}_{21} \\ \vdots & \vdots \\ N^\epsilon_{s0} & N^{-\epsilon}_{s1} \end{bmatrix} \in \mathbb{C}^{2sn \times 2n^2}, \tag{3.12}$$

where $N^\epsilon_{pj} = w_j^{-1} \begin{bmatrix} N^\epsilon(g_{pj}) & -N^{-\epsilon}(h_{pj}) \\ N^\epsilon(h_{pj}) & N^{-\epsilon}(g_{pj}) \end{bmatrix} \operatorname{diag}\left(\begin{bmatrix} \operatorname{vec}(\operatorname{sgn}A_j \circ C, \epsilon) \\ \operatorname{vec}(\operatorname{sgn}A_j \circ C, -\epsilon) \end{bmatrix}\right)$ for $j = 0, 1$, are defined by Equation 3.3. Now we state the following theorem for *H-even/T-odd* case. Throughout this section, $\epsilon = 1$ and $\epsilon = -1$ exhibit the *H-even* and *H-odd* cases, respectively.

Theorem 3.6.1. *Let $\mathbf{L} \in \mathbf{L}(\mathbb{C}^{n\times n})$ be a H-even/H-odd homogeneous matrix pencil of the form (3.1). Suppose (λ_p, x_p) are s $(s \leq n)$ approximate eigenpairs of $\mathbf{L}$ with $0 \neq x_p \in \mathbb{C}^n$, and $\lambda_p = (\lambda_{p0}, \lambda_{p1}) \in \mathbb{C}^2 \setminus \{(0,0)\}$. Set $r := \begin{bmatrix} \Re(r_1)^T & \Im(r_1)^T & \dots & \Re(r_s)^T & \Im(r_s)^T \end{bmatrix}^T$, $r_p := -\mathbf{L}(\lambda_p)x_p$, for $p = 1 : s$. If N^ϵ (defined in Equation 3.12) has full row rank, then the backward error is given by*

$$\eta^{\mathbf{S}}_{w,F}(\lambda_{1:s}, x_{1:s}, \mathbf{L}) = \|N^{\epsilon T}(N^\epsilon N^{\epsilon T})^{-1} r\|_F.$$

A minimizing H-even/H-odd $\delta\mathbf{L} \in \mathbf{L}(\mathbb{C}^{n\times n})$, such that $(\mathbf{L}(\lambda_p) + \delta\mathbf{L}(\lambda_p))x_p = 0$, is of the form $\delta\mathbf{L}(\alpha,\beta) := \alpha\delta A_0 + \beta\delta A_1$, where $\delta A_j = (\delta a_{j,tk})$, for $j = 0, 1$, are given by

$$\delta a_{0,tk} = \begin{cases} (\operatorname{sgn} a_{0,tt}) \sum_{p=1}^{s} \sqrt{\epsilon} w_0^{-2} \big(g_{p0}^t e^T_{t+\frac{1-\epsilon}{2}n(2p-1)+\frac{1+\epsilon}{2}2n(p-1)} + \epsilon h_{p0}^t & \\ e^T_{t+\frac{1+\epsilon}{2}n(2p-1)+\frac{1-\epsilon}{2}2n(p-1)}\big)(N^\epsilon N^{\epsilon T})^{-1} r, & \text{for } t = k, \\ (\operatorname{sgn} a_{0,tk}) \sum_{p=1}^{s} \frac{1}{2} w_0^{-2} c_{p0,tk} (N^\epsilon N^{\epsilon T})^{-1} r, & \text{for } t \neq k, \end{cases}$$

$$\delta a_{1,tk} = \begin{cases} (\operatorname{sgn} a_{1,tt}) \sum_{p=1}^{s} \sqrt{-\epsilon} w_1^{-2} \big(g_{p1}^t e^T_{t+\frac{1-\epsilon}{2}n(2p-1)+\frac{1+\epsilon}{2}2n(p-1)} - \epsilon h_{p1}^t & \\ e^T_{t+\frac{1+\epsilon}{2}n(2p-1)+\frac{1-\epsilon}{2}2n(p-1)}\big)(N^\epsilon N^{\epsilon T})^{-1} r, & \text{for } t = k, \\ (\operatorname{sgn} a_{1,tk}) \sum_{p=1}^{s} \frac{1}{2} w_1^{-2} c_{p1,tk} (N^\epsilon N^{\epsilon T})^{-1} r, & \text{for } t \neq k. \end{cases}$$

Here $e_i \in \mathbb{C}^{2sn}$ for any $i \in \mathbb{N}$.

Proof. The proof follows immediately from Theorem 3.4.1 and Theorem 3.5.1. ■

After obtaining the results for *T-symmetric/skew-symmetric*, *Hermitian/skew-Hermitian*, *T-even/odd,* and *H-even/odd* matrix pencils, we present the backward error analysis of *T-palindromic/T-anti-palindromic*, and *H-palindromic/H-anti-palindromic* matrix pencils in the following section.

3.7. Perturbation analysis for palindromic matrix pencils

To understand the backward error analysis and the perturbation theory of *palindromic* matrix pencils, we define matrices $M^\epsilon(\lambda_p, y_p)$, which are obtained by the given approximate eigenpairs (λ_p, y_p), where $\lambda_p = (\lambda_{p0}, \lambda_{p1}) \in \mathbb{C}^2 \setminus \{(0,0)\}$, and $0 \neq y_p \in \mathbb{C}^n$ for $p = 1 : s$. For construction of $M^\epsilon(\lambda_p, y_p)$, we need to understand the construction of matrices $M^\epsilon(y_p)$ for $\epsilon = 1, -1$, where $M^1(y_p) \in \mathbb{C}^{n\times n^2}$ and $M^{-1}(y_p) \in \mathbb{C}^{n\times n^2}$. Superscript $'-1'$ in $M^{-1}(y_p)$ is only for notational point of view. It should not mismatch with the inverse of $M(y_p)$.

1. For deriving the *backward error* formula of specified *eigenpairs*, we define the matrices $M^1(y_p), M^{-1}(y_p)$ and $M^\epsilon(\lambda_p, y_p)$ for $p = 1 : s$ as follows:

$$M^1(y_p) = \begin{bmatrix} M_1^1(y_p) & \dots & M_n^1(y_p) \end{bmatrix} \text{ and } M^{-1}(y_p) = \begin{bmatrix} M_1^{-1}(y_p) & \dots & M_n^{-1}(y_p) \end{bmatrix},$$

$$M^\epsilon(\lambda_p, y_p) = \begin{bmatrix} \lambda_{p0} M_1^1(y_p) + \epsilon\lambda_{p1} M_1^{-1}(y_p) & \dots & \lambda_{p0} M_n^1(y_p) + \epsilon\lambda_{p1} M_n^{-1}(y_p) \end{bmatrix}.$$

For $\epsilon = 1$, define $M_1^1(y_p) \in \mathbb{C}^{n\times n}$, $M_2^1(y_p) \in \mathbb{C}^{n\times n}$, and $M_n^1(y_p) \in \mathbb{C}^{n\times n}$ as follows:

$$M_1^1(y_p) = \begin{bmatrix} y_p^1 & y_p^2 & y_p^3 & \dots & y_p^n \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}, M_2^1(y_p) = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ y_p^1 & y_p^2 & y_p^3 & \dots & y_p^n \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix},$$

$$M_n^1(y_p) = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 0 \\ y_p^1 & y_p^2 & y_p^3 & \dots & y_p^n \end{bmatrix}.$$

Similarly, for $\epsilon = -1$ define $M_i^{-1}(y_p) \in \mathbb{C}^{n \times n}$ as follows:

$$M_i^{-1}(y_p) = \mathrm{diag}([y_p^i, \dots, y_p^i]^T), i = 1, \dots, n.$$

2. Suppose $A_0 = (a_{0,tk}), \delta A_0 = (\delta a_{0,tk}) \in \mathbb{C}^{n \times n}$. Define $\Delta_0 := \begin{bmatrix} \Delta_{01} \\ \Delta_{02} \\ \vdots \\ \Delta_{0n} \end{bmatrix}$, where

$\Delta_{0i} = \begin{bmatrix} w_0\, \delta a_{0,i1} \mathrm{sgn}\, a_{0,i1} \\ \vdots \\ w_0\, \delta a_{0,in} \mathrm{sgn}\, a_{0,1n} \end{bmatrix}$, and w_0 is a nonnegative real number.

3. Define $\mathrm{vec}(\mathrm{sgn} A_0) = [\mathrm{sgn}\, a_{0,11}, \dots, \mathrm{sgn}\, a_{0,1n}, \dots, \mathrm{sgn}\, a_{0,n1}, \dots, \mathrm{sgn}\, a_{0,nn}]^T$ for $A_0 = (a_{0,ij}) \in \mathbb{C}^{n \times n}$. Here $\mathrm{vec}(\mathrm{sgn} A_0) \in \mathbb{C}^{n^2}$. Let $M_p^\epsilon = w_0^{-1} M^\epsilon(\lambda_p, y_p) \mathrm{diag}(\mathrm{vec}(\mathrm{sgn} A_0))$. Define

$$M^\epsilon := \begin{bmatrix} M_1^{\epsilon T} & M_2^{\epsilon T} & \dots & M_s^{\epsilon T} \end{bmatrix}^T, \text{where } M^\epsilon \in \mathbb{C}^{sn \times n^2}. \tag{3.13}$$

We use the above constructions in the next subsections for obtaining the backward error and perturbed matrices of *palindromic* matrix pencils.

3.7.1. Perturbation analysis for T-palindromic/T-anti-palindromic matrix pencils

This section deals with the perturbation theory and the *backward error* analysis of s-specified *eigenpairs* of T*-palindromic* and T*-anti-palindromic* matrix pencils. Throughout this section, $\epsilon = 1$ represents the T-*palindromic* case, and $\epsilon = -1$ represents the T-*anti-palindromic* case.

Theorem 3.7.1. *Let $\mathbf{L} \in \mathbf{L}(\mathbb{C}^{n\times n})$ be a T-palindromic/T-anti-palindromic homogeneous matrix pencil of the form (3.1). Let (λ_p, x_p) be the s $(s \leq n)$ approximate eigenpairs of $\mathbf{L}$ with $0 \neq x_p \in \mathbb{C}^n$, and $\lambda_p = (\lambda_{p0}, \lambda_{p1}) \in \mathbb{C}^2 \setminus \{(0,0)\}$. Set $r := \begin{bmatrix} r_1^T & r_2^T & \dots & r_s^T \end{bmatrix}^T$, where $r_p := -\mathbf{L}(\lambda_p)x_p$ for $p = 1 : s$. If M^ϵ (defined in Equation 3.13) has full row rank, then the backward error is given by*

$$\eta^{\mathbf{S}}_{w,F}(\lambda_{1:s}, x_{1:s}, \mathbf{L}) = \sqrt{2}\|M^{\epsilon H}(M^\epsilon M^{\epsilon H})^{-1} r\|_F.$$

A minimizing T-palindromic/T-anti-palindromic matrix pencil $\delta\mathbf{L} \in \mathbf{L}(\mathbb{C}^{n\times n})$, such that $(\mathbf{L}(\lambda_p) + \delta\mathbf{L}(\lambda_p))x_p = 0$, is of the form $\delta\mathbf{L}(\alpha,\beta) := \alpha\delta A_0 + \beta\delta A_1$, where $\delta A_j = (\delta a_{j,tk})$, for $j = 0, 1$, are given by

$$\delta a_{j,tk} = \begin{cases} \sum_{p=1}^s w_0^{-2}(\operatorname{sgn} a_{j,tk})(\overline{\lambda}_{pj}\overline{x}_p^k e^T_{t+(p-1)n} + \epsilon\overline{\lambda}_{p(1-j)}\overline{x}_p^t e^T_{k+(p-1)n})(M^\epsilon M^{\epsilon H})^{-1} r, & \text{for } t \neq k, \\ \sum_{p=1}^s w_0^{-2}(\operatorname{sgn} a_{j,tk})(\overline{\lambda}_{pj}\overline{x}_p^t + \epsilon\overline{\lambda}_{p(1-j)}\overline{x}_p^t) e^T_{t+(p-1)n}(M^\epsilon M^{\epsilon H})^{-1} r, & \text{for } t = k. \end{cases}$$

Here $e_{t+(p-1)n}, e_{k+(p-1)n} \in \mathbb{C}^{sn}$. If M^ϵ has not full row rank but $\operatorname{rank}(M^\epsilon) = \operatorname{rank}([M^\epsilon, r])$, then the backward error is given by

$$\eta^{\mathbf{S}}_{w,F}(\lambda_{1:s}, x_{1:s}, \mathbf{L}) = \sqrt{2}\|V^\epsilon D^{\epsilon+} U^{\epsilon H} r\|_F,$$

where U^ϵ, V^ϵ are unitary matrices and D^ϵ contains singular values of M^ϵ.

Proof. For the given s-approximate eigenpairs $(\lambda_p, x_p), p = 1 : s$, of $\mathbf{L}$, we need to construct $\delta\mathbf{L}$ such that $(\mathbf{L}(\lambda_p) + \delta\mathbf{L}(\lambda_p))x_p = 0$. By assumption $\mathbf{L}(\lambda_p)x_p + r_p = 0$ for $p = 1 : s$. Then $r_p = \delta\mathbf{L}(\lambda_p)x_p = \sum_{j=0}^1 \lambda_{pj}\delta A_j x_p = (w_0 w_0^{-1}\lambda_{p0}\delta A_0 + w_0 w_0^{-1}\lambda_1 \delta A_1)x_p$. Since $\delta A_1 = \epsilon\delta A_0^T$, we get $r_p = (w_0 w_0^{-1}\lambda_{p0}\delta A_0 \circ \operatorname{sgn} A_0 + \epsilon w_0 w_0^{-1}\lambda_{p1}\delta A_0^T \circ \operatorname{sgn} A_0^T)x_p$. Let $\Delta_0 = w_0 \operatorname{vec}(\delta A_0 \circ \operatorname{sgn} A_0)$. Then $r_p = [w_0^{-1}\lambda_{p0}M^1(x_p) + \epsilon w_0^{-1}\lambda_{p1}M^{-1}(x_p)] \operatorname{diag}(\operatorname{vec}(\operatorname{sgn} A_0))\Delta_0 = w_0^{-1}M^\epsilon(\lambda_p, x_p) \operatorname{diag}(\operatorname{vec}(\operatorname{sgn} A_0))\Delta_0 = M_p^\epsilon \Delta_0$, where

$$M_p^\epsilon = [w_0^{-1}\lambda_{p0}M^1(x_p) + \epsilon w_0^{-1}\lambda_{p1}M^{-1}(x_p)]\operatorname{diag}(\operatorname{vec}(\operatorname{sgn} A_0)).$$

On writing $r_p = M_p^\epsilon \Delta_0$ for $p = 1 : s$, in the combined form, we get

$$\begin{bmatrix} r_1^T & r_2^T & \dots & r_s^T \end{bmatrix}^T = \begin{bmatrix} M_1^{\epsilon T} & M_2^{\epsilon T} & \dots & M_s^{\epsilon T} \end{bmatrix}^T \Delta_0. \tag{3.14}$$

If M^ϵ has full row rank, then in the least square sense, we get the minimal solution $\Delta_0 = M^{\epsilon H}(M^\epsilon M^{\epsilon H})^{-1} r$. If M^ϵ has not full row rank and Equation 3.14 is consistent, then $\Delta_0 = V^\epsilon D^{\epsilon+} U^{\epsilon H} r$. Here U^ϵ, V^ϵ are unitary matrices of appropriate sizes and $D^{\epsilon+}$ contains the singular values of M^ϵ. Now using equation $\Delta_0 = M^{\epsilon H}(M^\epsilon M^{\epsilon H})^{-1} r$ and expanding the first $M^{\epsilon H}$, we get the desired entrywise perturbations. Backward error in *Frobenius* norm case is given by $\eta^{\mathbf{S}}_{w,F}(\lambda_{1:s}, x_{1:s}, \mathbf{L}) = \|\delta\mathbf{L}\|_{w,F}$, where $\|\delta\mathbf{L}\|^2_{w,F} = \sum_{i=0}^{1} w_0^2 \|\delta A_i\|_F^2 = 2w_0^2\|\delta A_0\|^2$. Then $\|\delta\mathbf{L}\|^2_{w,F} = 2\|\Delta_0\|_F^2 = 2\|M^{\epsilon H}(M^\epsilon M^{\epsilon H})^{-1}r\|_F^2$. For full row rank M^ϵ, the backward error is given by

$$\eta^{\mathbf{S}}_{w,F}(\lambda_{1:s}, x_{1:s}, \mathbf{L}) = \sqrt{2}\|M^{\epsilon H}(M^\epsilon M^{\epsilon H})^{-1}r\|_F.$$

When M^ϵ has not full row rank but Equation 3.14 is consistent, then the backward error is given by $\eta^{\mathbf{S}}_{w,2}(\lambda_{1:s}, x_{1:s}, \mathbf{L}) = \sqrt{2}\|V^\epsilon D^{\epsilon+}U^{\epsilon H}r\|$. Similar to Theorem 3.3.6, we can easily see that $(\mathbf{L}(\lambda_p) + \delta\mathbf{L}(\lambda_p))x_p = 0$ for $p = 1 : s$. ■

Remark 3.7.2. *If M^ϵ is not a full row rank matrix but Equation 3.14 is consistent, then explicit formula for the perturbed matrix pencil is not possible, though we can construct the perturbed matrices using singular value decomposition.*

Remark 3.7.3. *If Equation 3.14 is inconsistent i.e. $rank(M^\epsilon) \neq rank([M^\epsilon, r])$. Then a minimal perturbation matrix pencil does not exist, and the backward error is given by $\eta^{\mathbf{S}}_{w,F}(\lambda_{1:s}, x_{1:s}, \mathbf{L}) = \infty$.*

For simplicity, we state the following corollary for two approximate eigenpairs, which is immediate from Theorem 3.7.1 for $s = 2$.

Corollary 3.7.4. *Let $\mathbf{L} \in \mathbf{L}(\mathbb{C}^{n\times n})$ be a T-palindromic/T-anti-palindromic homogeneous matrix pencil of the form (3.1). Suppose (λ_1, x_1), (λ_2, x_2) are two approximate eigenpairs of $\mathbf{L}$ with $0 \neq x_p \in \mathbb{C}^n$, and $\lambda_p = (\lambda_{p0}, \lambda_{p1}) \in \mathbb{C}^2\backslash\{(0,0)\}$, for $p = 1, 2$. Set $r_p := -\mathbf{L}(\lambda_p)x_p$. If M^ϵ (defined as below) has full row rank, then the backward error is given by*

$$\eta^{\mathbf{S}}_{w,F}(\lambda_{1:2}, x_{1:2}, \mathbf{L}) = \sqrt{2}\|M^{\epsilon H}(M^\epsilon M^{\epsilon H})^{-1}r\|_F.$$

A minimizing T-palindromic/T-anti-palindromic $\delta\mathbf{L} \in \mathbf{L}(\mathbb{C}^{n\times n})$, such that $(\mathbf{L}(\lambda_p)+\delta\mathbf{L}(\lambda_p))x_p = 0$, is of the form $\delta\mathbf{L}(\alpha, \beta) := \alpha\delta A_0 + \beta\delta A_1$, where $\delta A_j = (\delta a_{j,tk})$ for $j = 0, 1$, are given by

$$\delta a_{j,tk} = \begin{cases} w_0^{-2}(\operatorname{sgn} a_{j,tk})[(\overline{\lambda}_{1j}\overline{x}_1^k e_t^T + \epsilon\overline{\lambda}_{1(1-j)}\overline{x}_1^t e_k^T) + (\overline{\lambda}_{2j}\overline{x}_2^k e_{t+n}^T + \\ \epsilon\overline{\lambda}_{2(1-j)}\overline{x}_2^t e_{k+n}^T)](M^\epsilon M^{\epsilon H})^{-1}r, & \text{for } t \neq k, \\ w_0^{-2}(\operatorname{sgn} a_{j,tk})[(\overline{\lambda}_{1j}\overline{x}_1^t + \epsilon\overline{\lambda}_{1(1-j)}\overline{x}_1^t)e_t^T + (\overline{\lambda}_{20}\overline{x}_2^t + \\ \epsilon\overline{\lambda}_{21}\overline{x}_2^t)e_{t+n}^T](M^\epsilon M^{\epsilon H})^{-1}r, & \text{for } t = k. \end{cases}$$

Here $M^\epsilon = \begin{bmatrix} M_1^\epsilon \\ M_2^\epsilon \end{bmatrix} \in \mathbb{C}^{2n\times n^2}$ such that $M_p^\epsilon = w_0^{-1}M^\epsilon(\lambda_p, x_p)\operatorname{diag}(\operatorname{vec}(\operatorname{sgn}A_0))$ for $p = 1, 2$.

Similar to the *T-palindromic/T-anti-palindromic* case, next we obtain the result for *H-palindromic/H-anti-palindromic* matrix pencils. ■

3.7.2. Perturbation analysis for H-palindromic/H-anti-palindromic matrix pencils

Before we prove the main result of this section, let $x_p \in \mathbb{C}^n, \lambda_p = (\lambda_{p0}, \lambda_{p1}) \in \mathbb{C}^2 \setminus \{(0,0)\}$. Define $g_{pj} := \Re(\lambda_{pj} x_p), h_{pj} := \Im(\lambda_{pj} x_p), g_{pj}^t := \Re(\lambda_{pj} x_p^t), h_{pj}^t := \Im(\lambda_{pj} x_p^t)$, for $j = 0, 1$, $p = 1 : s$, and $t, k = 1 : n$. Define

$$M^\epsilon(g_p) = \begin{bmatrix} M_1^1(g_{p0}) + \epsilon M_1^{-1}(g_{p1}) & \dots & M_n^1(g_{p0}) + \epsilon M_n^{-1}(g_{p1}) \end{bmatrix},$$

$$M^\epsilon(h_p) = \begin{bmatrix} M_1^1(h_{p0}) + \epsilon M_1^{-1}(h_{p1}) & \dots & M_n^1(h_{p0}) + \epsilon M_n^{-1}(h_{p1}) \end{bmatrix},$$

and

$$M_p^\epsilon = w_0^{-1} \begin{bmatrix} M^\epsilon(g_p) & -M^{-\epsilon}(h_p) \\ M^\epsilon(h_p) & M^{-\epsilon}(g_p) \end{bmatrix} \operatorname{diag}\left(\begin{bmatrix} \operatorname{vec}(\operatorname{sgn} A_0) \\ \operatorname{vec}(\operatorname{sgn} A_0) \end{bmatrix} \right).$$

Then $M^\epsilon := \begin{bmatrix} M_1^{\epsilon T} & M_2^{\epsilon T} & \dots & M_s^{\epsilon T} \end{bmatrix}^T \in \mathbb{C}^{2sn \times 2n^2}$. Now we are ready to provide the following theorem for H-palindromic and H-anti-palindromic matrix pencils. Throughout this section, $\epsilon = 1$ represents the *H-palindromic* case, and $\epsilon = -1$ represents the *H-anti-palindromic* case.

Theorem 3.7.5. *Let $\mathbf{L} \in \mathbf{L}(\mathbb{C}^{n\times n})$ be a H-palindromic/H-anti-palindromic homogeneous matrix pencil of the form (3.1). Let (λ_p, x_p) be the $s\,(s \le n)$ approximate eigenpairs of $\mathbf{L}$ with $0 \neq x_p \in \mathbb{C}^n$, and $\lambda_p = (\lambda_{p0}, \lambda_{p1}) \in \mathbb{C}^2 \setminus \{(0,0)\}$ for $p = 1 : s$. Set $r := \begin{bmatrix} \Re(r_1)^T & \Im(r_1)^T & \dots & \Re(r_s)^T & \Im(r_s)^T \end{bmatrix}^T$, where $r_p := -\mathbf{L}(\lambda_p) x_p$ for $p = 1 : s$. If M^ϵ (defined as above) has full row rank, then the backward error is given by*

$$\eta_{w,F}^{\mathbf{S}}(\lambda_{1:s}, x_{1:s}, \mathbf{L}) = \sqrt{2} \| M^{\epsilon T} (M^\epsilon M^{\epsilon T})^{-1} r \|_F.$$

A minimizing H-palindromic/H-anti-palindromic $\delta\mathbf{L} \in \mathbf{L}(\mathbb{C}^{n\times n})$, such that $(\mathbf{L}(\lambda_p) + \delta\mathbf{L}(\lambda_p)) x_p = 0$, is of the form $\delta\mathbf{L}(\alpha, \beta) := \alpha \delta A_0 + \beta \delta A_1$, where $\delta A_j = (\delta a_{j,tk})$ for $j = 0, 1$, are given by

$$\delta a_{j,tk} = \begin{cases} \sum_{p=1}^s w_0^{-2} (\operatorname{sgn} a_{j,tk}) [g_{pj}^k e_{t+2(p-1)n}^T + \epsilon g_{p(1-j)}^t e_{k+2(p-1)n}^T + h_{pj}^k e_{t+(2p-1)n}^T \\ +\epsilon h_{p(1-j)}^t e_{k+(2p-1)n}^T + \mathrm{i}(-h_{pj}^k e_{t+2(p-1)n}^T + \epsilon h_{p(1-j)}^t e_{k+2(p-1)n}^T) + \\ \mathrm{i}(g_{pj}^k e_{t+(2p-1)n}^T - \epsilon g_{p(1-j)}^t e_{k+(2p-1)n}^T)] (M^\epsilon M^{\epsilon T})^{-1} r, & \text{if } t \neq k, \\ \sum_{p=1}^s w_0^{-2} (\operatorname{sgn} a_{j,tk}) [(g_{pj}^t + \epsilon g_{p(1-j)}^t) e_{t+2(p-1)n}^T + (h_{pj}^t + \epsilon h_{p(1-j)}^t) e_{t+(2p-1)n}^T \\ +\mathrm{i}(-h_{pj}^t + \epsilon h_{p(1-j)}^t) e_{t+2(p-1)n}^T + \mathrm{i}(g_{pj}^t - \epsilon g_{p(1-j)}^t) e_{t+(2p-1)n}^T] (M^\epsilon M^{\epsilon T})^{-1} r, & \text{if } t = k. \end{cases}$$

Here $e_i \in \mathbb{C}^{2sn}$ for any $i \in \mathbb{N}$. If M^ϵ is not a full row rank matrix but $rank(M^\epsilon) = rank([M^\epsilon, r])$, then the backward error is given by

$$\eta^{\mathbf{S}}_{w,F}(\lambda_{1:s}, x_{1:s}, \mathbf{L}) = \sqrt{2}\|V^\epsilon D^{\epsilon +} U^{\epsilon H} r\|_F,$$

where U^ϵ, V^ϵ are unitary matrices and D^ϵ contains the singular values of M^ϵ.

Proof. For the given s-approximate eigenpairs (λ_p, x_p) of $\mathbf{L}$, we need to construct $\delta\mathbf{L}$ such that $(\mathbf{L}(\lambda_p) + \delta\mathbf{L}(\lambda_p))x_p = 0$. Since $\delta A_1 = \epsilon\delta A_0^H$, similar to the previous theorem, we get

$r_p = \delta\mathbf{L}(\lambda_p)x_p = (\lambda_{p0}\delta A_0 + \lambda_1\delta A_1)x_p = (\lambda_{p0}\delta A_0 + \lambda_{p1}\delta A_0^H)x_p = [\lambda_{p0}(\Re(\delta A_0) + \mathrm{i}\Im(\delta A_0)) + \epsilon\lambda_{p1}(\Re(\delta A_0) + \mathrm{i}\Im(\delta A_0))^H]x_p = [(\Re(\delta A_0) + \mathrm{i}\Im(\delta A_0))(\Re(\lambda_{p0}x_p) + \mathrm{i}\Im(\lambda_{p0}x_p)) + \epsilon(\Re(\delta A_0)^T - \mathrm{i}\Im(\delta A_0)^T)(\Re(\lambda_{p1}x_p) + \mathrm{i}\Im(\lambda_{p1}x_p))] = [(\Re(\delta A_0) + \mathrm{i}\Im(\delta A_0)) \circ \mathrm{sgn} A_0(\Re(\lambda_{p0}x_p) + \mathrm{i}\Im(\lambda_{p0}x_p)) + \epsilon(\Re(\delta A_0)^T - \mathrm{i}\Im(\delta A_0)^T) \circ (\mathrm{sgn}\, A_0)^T(\Re(\lambda_{p1}x_p) + \mathrm{i}\Im(\lambda_{p1}x_p))] = \Re(\delta A_0) \circ \mathrm{sgn} A_0 \Re(\lambda_{p0}x_p) + \mathrm{i}\Re(\delta A_0) \circ \mathrm{sgn}\, A_0 \Im(\lambda_{p0}x_p) + \mathrm{i}\Im(\delta A_0) \circ \mathrm{sgn}\, A_0 \Re(\lambda_{p0}x_p) - \Im(\delta A_0) \circ \mathrm{sgn}\, A_0 \Im(\lambda_{p0}x_p) + \epsilon\Re(\delta A_0)^T \circ (\mathrm{sgn}\, A_0)^T \Re(\lambda_{p1}x_p) + \mathrm{i}\epsilon\Re(\delta A_0)^T \circ (\mathrm{sgn}\, A_0)^T \Im(\lambda_{p1}x_p) - \mathrm{i}\epsilon\Im(\delta A_0)^T \circ (\mathrm{sgn}\, A_0)^T \Re(\lambda_{p1}x_p) + \epsilon\Im(\delta A_0)^T \circ (\mathrm{sgn}\, A_0)^T \Im(\lambda_{p1}x_p) = \Re(r_p) + \mathrm{i}\Im(r_p)$, where

$$\begin{aligned}
\Re(r_p) = \Re(\delta A_0) \circ (\mathrm{sgn}\, A_0)g_{p0} + \epsilon\Re(\delta A_0)^T \circ (\mathrm{sgn}\, A_0)^T g_{p1} - \Im(\delta A_0) \circ (\mathrm{sgn}\, A_0)h_{p0} + \\
\epsilon\Im(\delta A_0)^T \circ (\mathrm{sgn}\, A_0)^T h_{p1}, \\
\Im(r_p) = \Re(\delta A_0) \circ (\mathrm{sgn}\, A_0)h_{p0} + \epsilon\Re(\delta A_0)^T \circ (\mathrm{sgn}\, A_0)^T h_{p1} + \Im(\delta A_0) \circ (\mathrm{sgn}\, A_0)g_{p0} - \\
\epsilon\Im(\delta A_0)^T \circ (\mathrm{sgn}\, A_0)^T g_{p1},
\end{aligned}$$

for $p = 1 : s$. Let $\Delta_0^{\Re} = w_0(\Re(\delta A_0) \circ \mathrm{sgn} A_0)$, and $\Delta_0^{\Im} = w_0 \mathrm{vec}(\Im(\delta A_0) \circ \mathrm{sgn} A_0)$. Separating the unknown and known variables similar to the previous theorem, we get the following system for $p = 1 : s$

$$\Re(r_p) = w_0^{-1}M^\epsilon(g_p)\mathrm{diag}(\mathrm{vec}(\mathrm{sgn} A_0))\Delta_0^{\Re} - w_0^{-1}M^{-\epsilon}(h_p)\mathrm{diag}(\mathrm{vec}(\mathrm{sgn} A_0))\Delta_0^{\Im}. \tag{3.15}$$

Similarly, we get the following system for $p = 1 : s$

$$\Im(r_p) = w_0^{-1}M^\epsilon(h_p)\mathrm{diag}(\mathrm{vec}(\mathrm{sgn} A_0))\Delta_0^{\Re} + w_0^{-1}M^{-\epsilon}(g_p)\mathrm{diag}(\mathrm{vec}(\mathrm{sgn} A_0))\Delta_0^{\Im}. \tag{3.16}$$

By combining Equation 3.15 and Equation 3.16, we get

$$\begin{bmatrix} \Re(r_p) \\ \Im(r_p) \end{bmatrix} = M_p^\epsilon \begin{bmatrix} \Delta_0^{\Re} \\ \Delta_0^{\Im} \end{bmatrix}. \tag{3.17}$$

Further combining Equation 3.17, for $p = 1 : s$, we get

$$\begin{bmatrix} \Re(r_1) \\ \Im(r_1) \\ \vdots \\ \Re(r_s) \\ \Im(r_s) \end{bmatrix} = \begin{bmatrix} M_1^\epsilon \\ \vdots \\ M_s^\epsilon \end{bmatrix} \begin{bmatrix} \Delta_0^{\Re} \\ \Delta_0^{\Im} \end{bmatrix}. \tag{3.18}$$

If M^ϵ has full row rank, then in the least square sense, we get the minimal norm solution $\Delta = M^{\epsilon T}(M^\epsilon M^{\epsilon T})^{-1} r$, where $\Delta = \begin{bmatrix} \Delta_0^{\Re} \\ \Delta_0^{\Im} \end{bmatrix}$. Then the *backward error* in *Frobenius norm* case is given by $\eta_{w,F}^{\mathbf{S}}(\lambda_{1:s}, x_{1:s}, \mathbf{L}) = \|\delta\mathbf{L}\|_{w,F}$, where $\|\delta\mathbf{L}\|_{w,F}^2 = 2w_0^2\|\delta A_0\|^2 = 2w_0^2(\|\Re(\delta A_0)\|^2 + \|\Im(\delta A_0)\|^2) = 2\|\Delta_0^{\Re}\|_F^2 + 2\|\Delta_0^{\Im}\|_F^2 = 2\|M^{\epsilon T}(M^\epsilon M^{\epsilon T})^{-1} r\|_F^2$. $\blacksquare$

Now we illustrate our theory by some examples and discuss its importance in solving the *inverse eigenvalue problem*.

3.8. Numerical examples and discussion on inverse eigenvalue problem

For illustration of the theory, we present an example for the *T-palindromic generalized eigenvalue* problem.

Example 3.8.1. *Let* $\mathbf{L} \in \mathbf{L}(\mathbb{C}^{3\times 3})$ *be a T-palindromic matrix pencil of the form (3.1) with the following information:*

$$A_0 = \begin{bmatrix} 986.5689 & 1 & 1+\mathrm{i} \\ 7.2 & 3-\mathrm{i} & 0 \\ 0 & 8-\mathrm{i} & 10.236 \end{bmatrix}, A_1 = \begin{bmatrix} 986.5689 & 7.2 & 0 \\ 1 & 3-\mathrm{i} & 8-\mathrm{i} \\ 1+\mathrm{i} & 0 & 10.236 \end{bmatrix}.$$

Let (λ_1, x_1) *and* (λ_2, x_2) *be two approximate eigenpairs of* $\mathbf{L}$, *where* $\lambda_1 = (12.001 + 3\mathrm{i}, -19.66)$, $\lambda_2 = (13.96, 2 - 3\mathrm{i})$, $x_1 = [1.01125 + 0.023\mathrm{i}, 3.3, 7 - \mathrm{i}]^T$, $x_2 = [11.12, 5 + 3\mathrm{i}, 2.089]^T$, *and* $(w_0, w_1)^T = (1, 1)$. *By the given information, we get that* M^ϵ *has full row rank. Then applying Theorem 3.7.1, the perturbed matrices are given by*

$$
\delta A_0 = 10^2 \times \begin{bmatrix} -8.8567-0.0013\mathrm{i} & -1.7026+1.5291\mathrm{i} & 0.4736-0.4314\mathrm{i} \\ -0.7530+0.2216\mathrm{i} & 0.4126-1.5527\mathrm{i} & 0 \\ 0 & 0.1164+0.1446\mathrm{i} & -0.2834+0.2025\mathrm{i} \end{bmatrix},
$$

$$
\delta A_1 = 10^2 \times \begin{bmatrix} -8.8567-0.0013\mathrm{i} & -0.7530+0.2216\mathrm{i} & 0 \\ -1.7026+1.5291\mathrm{i} & 0.4126-1.5527\mathrm{i} & 0.1164+0.1446\mathrm{i} \\ 0.4736-0.4314\mathrm{i} & 0 & -0.2834+0.2025\mathrm{i} \end{bmatrix},
$$

and the backward error is given by $\eta^{\mathbf{S}}_{w,F}(\lambda_{1:2}, x_{1:2}, \mathbf{L}) = 935.1024$. *Clearly, we have* $\delta A_0 = \delta A_1^T$.

Next, we discuss the connection between *inverse eigenvalue problems* and backward error theory. First, we discuss the real symmetric inverse eigenvalue problem.

Let $Q(\lambda) = \lambda^2 I + \lambda C + K$ be a quadratic matrix polynomial. The *inverse eigenvalue problem* is to find matrices C, K such that the given approximate *s-eigenpairs* (Λ, X) satisfy the following equation

$$
X\Lambda^2 + CX\Lambda + KX = 0, \tag{3.19}
$$

where $\Lambda \in \mathbb{R}^{s\times s}$ has specified *eigenvalues* $\lambda_i \in \mathbb{R}$ on its diagonal and $X \in \mathbb{R}^{n\times s}$ has the corresponding *eigenvectors* $x_i \in \mathbb{R}^n$ as its column. We need to construct $C, K \in \mathbb{R}^{n\times n}$ with $C = C^T$ and $K = K^T$ so that (3.19) is satisfied. Solving (3.19) is the same as solving $\lambda_i^2 I x_i + \lambda_i C x_i + K x_i = 0$ for $i = 1 : s$. Further using technique of [70], we can convert the above QEP into GEP of the form

$$
(G_0 + \lambda_i G_1) y_i = 0, \tag{3.20}
$$

where $G_0 = \begin{bmatrix} C & K \\ K & 0 \end{bmatrix}$, $G_1 = \begin{bmatrix} I_n & 0 \\ 0 & -K \end{bmatrix}$, $y_i = \begin{bmatrix} \lambda_i x_i \\ x_i \end{bmatrix}$ for $i = 1 : s$. Solution of the Problem 3.20 will provide the solution for the Problem 3.19. Problem 3.20 is the generalized *T-symmetric* non-homogeneous eigenvalue problem which can be solved by Theorem 3.3.6. We will illustrate it by an example for $s = 2$.

Example 3.8.2. *Let* (λ_1, x_1) *and* (λ_2, x_2) *be two specified eigenpairs, where* $\lambda_1 = 112.001$, $\lambda_2 = -13.02$, $x_1 = [0.01125, 3.3]^T$, *and* $x_2 = [1.12, 2.25]^T$. *We construct* $C = (c_{ij}), K = (k_{ij}) \in \mathbb{C}^{2\times 2}$ *such that Equation 3.19 satisfies. Additionally, we construct* C *in such a way that* $c_{22} = 0$, *so that the desired sparsity also maintain. By the above discussion, we know* *that solving Problem 3.19 is the same as solving Problem 3.20. Suppose* $C = P_0 + \delta P_0$, $K = P_1 + \delta P_1$, *and* $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. *Choose* P_0 *and* P_1 *as follows:*

$$P_0 = \begin{bmatrix} 31.02 & 7.2 \\ 7.2 & 0 \end{bmatrix}, P_1 = \begin{bmatrix} 12.4500 & 2.0000 \\ 2.0000 & 0.0033 \end{bmatrix}.$$

For $j = 0, 1$, *we find* δP_j *using Theorem 3.3.6, where* $\delta P_0 = \begin{bmatrix} \delta p_{0,11} & \delta p_{0,12} \\ \delta p_{0,12} & 0 \end{bmatrix}$, *and* $\delta P_1 = \begin{bmatrix} \delta p_{1,11} & \delta p_{1,12} \\ \delta p_{1,12} & \delta p_{1,22} \end{bmatrix}$.

Remark 3.8.3. *Since we can choose different* P_0, P_1 *so matrices* C *and* K *are not unique. We also maintain the sparsity of matrix* C *by choosing* $\delta p_{0,22} = 0$.

By applying Theorem 3.3.6 along with Remark 3.3.11, we get

$$G_0 = 10^4 \times \begin{bmatrix} 0.3853 & -0.0211 & 0.0076 & 2.2104 \\ -0.0211 & 0 & 2.2104 & -1.2539 \\ 0.0076 & 2.2104 & 0 & 0 \\ 2.2104 & -1.2539 & 0 & 0 \end{bmatrix},$$

$$G_1 = 10^4 \times \begin{bmatrix} 0.0001 & 0 & 0 & 0 \\ 0 & 0.0001 & 0 & 0 \\ 0 & 0 & -0.0076 & -2.2104 \\ 0 & 0 & -2.2104 & 1.2539 \end{bmatrix}.$$

Clearly $G_0 = G_0^T, G_1 = G_1^T$ *and* $(G_0 + \lambda_i G_1) y_i = 0$ *for* $i = 1, 2$. *Hence our theorem provides the solution for the real symmetric inverse eigenvalue problem, which also preserves the sparsity.*

Remark 3.8.4. *Note that Remark 3.3.11 guarantees that there will be no perturbation in* G_1 *corresponding to the block matrix* I_2.

Next, we discuss an another inverse eigenvalue problem [**89**, Problem 1.1], which require to construct the symmetric matrices $K, M \in \mathbb{R}^{n \times n}$ from the given set of eigenpairs (μ_i, x_i), and from symmetric matrices $K_0, M_0 \in \mathbb{R}^{d \times d}$ such that $Kx_i - \mu_i M x_i = 0, i = 1 : s$, $K_0 = K([1, d]), M_0 = M([1, d])$, where $K([1, d])$ and $M([1, d])$ are the $d \times d$ leading principal submatrices of K and M, respectively. Here $\mu_i \in \mathbb{R}, x_i \in \mathbb{R}^n$, and $1 \leq d, s \leq n, s+d \leq n$.

To solve this problem, we set $K = Q_0 + \delta Q_0$ and $M = Q_1 + \delta Q_1$, where Q_0, Q_1 are known matrices such that $Q_0 = \begin{array}{c c} & \begin{array}{cc} d & n-d \end{array} \\ \begin{array}{c} d \\ n-d \end{array} & \left[\begin{array}{c|c} K_0 & 0 \\ \hline 0 & 0 \end{array}\right] \end{array}$, $Q_1 = \begin{array}{c c} & \begin{array}{cc} d & n-d \end{array} \\ \begin{array}{c} d \\ n-d \end{array} & \left[\begin{array}{c|c} M_0 & 0 \\ \hline 0 & 0 \end{array}\right] \end{array}$, and $\delta Q_0, \delta Q_1$ are matrices to be determine. Next, in Remark 3.3.11, we choose $L_j = (l_{j,tk}) \in \mathbb{R}^{n \times n}$ in such a way that $l_{j,tk} = 0$ for $1 \leq t, k \leq r$, else $l_{j,tk} = 1$ for $j = 0 : 1$. Now to determine matrices $\delta Q_0, \delta Q_1$, we apply Theorem 3.3.6 with $\lambda_i = (1, -\mu_i)$ along with Remark 3.3.11. We illustrate the above discussion by the following example for $n = 5, d = 2$, and $s = 3$.

Example 3.8.5. *Let* (μ_1, x_1), (μ_2, x_2), *and* (μ_3, x_3) *be three specified eigenpairs, where* $\mu_1 = -25$, $\mu_2 = 47, \mu_3 = 33.45$, $x_1 = [0.4538, 0.4324, 0.8253, 0.0835, 0.1332]^T$, $x_2 = [0.1734, 0.3909, 0.8314, 0.8034, 0.0605]^T$, *and* $x_3 = [0.3993, 0.5269, 0.4168, 0.6569, 0.6280]^T$. *Let* $K_0 = \begin{bmatrix} 1.2952 & 1.3883 \\ 1.3883 & 0.4725 \end{bmatrix}$ *and* $M_0 = \begin{bmatrix} 0.2384 & 1.4845 \\ 1.4845 & 1.2946 \end{bmatrix}$.

Now, as per the above discussion on applying the Theorem 3.3.6 along with Remark 3.3.11, we get

$$\delta Q_0 = \begin{bmatrix} 0 & 0 & -0.4031 & -0.0917 & -0.3548 \\ 0 & 0 & -0.4173 & 0.0179 & -0.2725 \\ -0.4031 & -0.4173 & 0.1402 & -0.0470 & -0.4389 \\ -0.0917 & 0.0179 & -0.0470 & 0.3998 & 0.1176 \\ -0.3548 & -0.2725 & -0.4389 & 0.1176 & -0.2185 \end{bmatrix},$$

$$\delta Q_1 = \begin{bmatrix} 0 & 0 & -0.8050 & 0.1414 & -0.9728 \\ 0 & 0 & -1.2962 & 0.5205 & -1.6919 \\ -0.8050 & -1.2962 & 0.9672 & -0.2943 & 1.2353 \\ 0.1414 & 0.5205 & -0.2943 & 0.0565 & -0.3767 \\ -0.9728 & -1.6919 & 1.2353 & -0.3767 & 1.5871 \end{bmatrix}.$$

In particular, we get

$$K = Q_0 + \delta Q_0 = \begin{bmatrix} 1.2952 & 1.3883 & -0.4031 & -0.0917 & -0.3548 \\ 1.3883 & 0.4725 & -0.4173 & 0.0179 & -0.2725 \\ -0.4031 & -0.4173 & 0.1402 & -0.0470 & -0.4389 \\ -0.0917 & 0.0179 & -0.0470 & 0.3998 & 0.1176 \\ -0.3548 & -0.2725 & -0.4389 & 0.1176 & -0.2185 \end{bmatrix},$$

$$M = Q_1 + \delta Q_1 = \begin{bmatrix} 0.2384 & 1.4845 & -0.8050 & 0.1414 & -0.9728 \\ 1.4845 & 1.2946 & -1.2962 & 0.5205 & -1.6919 \\ -0.8050 & -1.2962 & 0.9672 & -0.2943 & 1.2353 \\ 0.1414 & 0.5205 & -0.2943 & 0.0565 & -0.3767 \\ -0.9728 & -1.6919 & 1.2353 & -0.3767 & 1.5871 \end{bmatrix}.$$

Clearly, $Kx_i - \mu_i M x_i = 0$ for $i = 1:3$. Also $K = K^T$ such that $K_0 = K([1,d])$, and $M = M^T$ such that $M_0 = M([1,d])$.

Remark 3.8.6. *Similar to above inverse eigenvalue problems, one can also solve the symmetric generalized inverse eigenvalue problem of* [**84**] *which asks to construct the real symmetric matrices* $A_0, A_1 \in \mathbb{C}^{n\times n}$ *with the (2d+1) diagonal from a given set of eigenpairs* (μ_i, x_i) *for* $i = 1:s$*. Here* $\mu_i \in \mathbb{R}, x_i \in \mathbb{R}^n$*, and* $s \le n, d < n$*.*

To solve the above inverse eigenvalue problem, one can set $A_i = D_i + \delta D_i$*, where* D_i *is the known symmetric matrix with (2d+1) diagonal and* δD_i *is the unknown matrix for* $i = 1:2$*. Then applying the Theorem 3.3.6, we get the desired* A_0 *and* A_1*. Note that the sparsity property of* D_i *helps us to obtain* $A_0, A_1 \in \mathbb{C}^{n\times n}$ *with the (2d+1) diagonal.*

For further understanding of the developed backward error theory, we present an example of the T- symmetric generalized eigenvalue problem.

Example 3.8.7. *Let* $\mathbf{L} \in \mathbf{L}(\mathbb{C}^{3\times 3})$ *be a* T*-symmetric matrix pencil of the form (3.1) with the following information:*

$$A_0 = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 3-\mathrm{i} & \mathrm{i} \\ -1 & \mathrm{i} & 1 \end{bmatrix}, A_1 = \begin{bmatrix} -\mathrm{i} & 0 & -1.15 \\ 0 & 3+\mathrm{i} & \mathrm{i}-0.5 \\ -1.15 & \mathrm{i}-0.5 & -1.5 \end{bmatrix},$$

let (λ_1, x_1) *and* (λ_2, x_2) *be two approximate eigenpairs of* $\mathbf{L}$*, where* $\lambda_1 = (112.001 + 3\mathrm{i}, -119.0066)$, $\lambda_2 = (13.96, -3\mathrm{i})$; $x_1 = [0.01125+0.023\mathrm{i}, 3.3, 8-\mathrm{i}]^T$, $x_2 = [1.12, 3\mathrm{i}, 2.089]^T$, *and* $(w_0, w_1)^T = (1, 1)$*. By the given information, we get that* N^ϵ *is a full row rank matrix. Then applying Theorem 3.3.6, the perturbed matrices are given by*

$$\delta A_0 = \begin{bmatrix} 0 & -1.0155 - 0.6838\mathrm{i} & 0.2128 - 0.1161\mathrm{i} \\ -1.0155 - 0.6838\mathrm{i} & -2.6100 + 1.4382\mathrm{i} & 0.0461 - 1.2377\mathrm{i} \\ 0.2128 - 0.1161\mathrm{i} & 0.0461 - 1.2377\mathrm{i} & -0.6819 + 0.0436\mathrm{i} \end{bmatrix},$$

$$\delta A_1 = \begin{bmatrix} 0.1182 + 0.0364\mathrm{i} & 0 & 0.4428 - 0.3899\mathrm{i} \\ 0 & -1.0740 - 1.2902\mathrm{i} & -0.1225 - 1.0136\mathrm{i} \\ 0.4428 - 0.3899\mathrm{i} & -0.1225 - 1.0136\mathrm{i} & 2.0809 - 0.0015\mathrm{i} \end{bmatrix},$$

and the backward error is given by $\eta^{\mathbf{S}}_{w,F}(\lambda_{1:2}, x_{1:2}, \mathbf{L}) = 5.0473$.

Remark 3.8.8. *From* [87, Theorem 3]*, we get* $\eta^{\mathbf{S}}_{w,F}(\lambda_1, x_1, \mathbf{L}) = 2.0576$*, and* $\eta^{\mathbf{S}}_{w,F}(\lambda_2, x_2, \mathbf{L}) = 4.0332$*. Results provided in* [87] *are not sufficient for obtaining the combined backward error and perturbed structured matrix pencil, which we can get by our results.*

Finally, we present an example for the T-*symmetric* case when N^1 is not a full row rank matrix.

Example 3.8.9. *Let* $\mathbf{L} \in \mathbf{L}(\mathbb{C}^{3\times 3})$ *be a* T*-symmetric matrix pencil such that* A_0 *and* A_1 *are defined in Example 3.8.7. Let* (λ_1, x_1) *and* (λ_2, x_2) *be two approximate eigenpairs of* $\mathbf{L}$*, where* $\lambda_1 = (1.23 + 2\mathrm{i}, 1.001212)$*,* $\lambda_2 = (1.23 + 2\mathrm{i}, 1.001212)$*,* $x_1 = [0.0057, 0.8899, 0.999]^T$*, and* $x_2 = [1.25, 2.121, 0.2223]^T$*. By the given information, we get that* $rank(N^1) = 5$*, which is not a full row rank. Hence by using Remark 3.3.7, we get*

$$\delta A_0 = \begin{bmatrix} 0 & -0.7315 + 0.1957\mathrm{i} & 0.8631 - 0.4995\mathrm{i} \\ -0.7315 + 0.1957\mathrm{i} & -3.5604 + 1.4716\mathrm{i} & -0.0827 - 1.0940\mathrm{i} \\ 0.8631 - 0.4995\mathrm{i} & -0.0827 - 1.0940\mathrm{i} & -0.6774 - 0.5447\mathrm{i} \end{bmatrix},$$

$$\delta A_1 = \begin{bmatrix} 0.0941 - 0.1964\mathrm{i} & 0 & 0.3742 + 0.2019\mathrm{i} \\ 0 & -1.3298 - 0.9645\mathrm{i} & 0.3789 - 0.2744\mathrm{i} \\ 0.3742 + 0.2019\mathrm{i} & 0.3789 - 0.2744\mathrm{i} & 0.0465 - 0.3677\mathrm{i} \end{bmatrix}.$$

Clearly, we get that $\delta A_0 = \delta A_0^T$*, and* $\delta A_1 = \delta A_1^T$*, which also preserve* sparsity. *In this case* $\eta^{\mathbf{S}}_{w,F}(\lambda_{1:2}, x_{1:2}, \mathbf{L}) = 4.9823$*. On the other hand,* $\eta^{\mathbf{S}}_{w,F}(\lambda_1, x_1, \mathbf{L}) = 3.5515$*, and* $\eta^{\mathbf{S}}_{w,F}(\lambda_2, x_2, \mathbf{L}) = 4.7769$. ■

CHAPTER 4

BACKWARD ERROR ANALYSIS OF SPECIFIED EIGENPAIRS FOR HANKEL AND SYMMETRIC-TOEPLITZ STRUCTURES

Abstract: In the continuation of the backward error analysis of specified eigenpairs, in this chapter, we discuss the backward error analysis for *Hankel* and *symmetric-Toeplitz* matrix pencils for one or more *specified eigenpairs* and its use in solving the *inverse eigenvalue problems.*

4.1. Introduction

Hankel and symmetric-Toeplitz matrix pencils arises in many application, see [**13**, **14**, **25**, **28**]. In particular, a *Hankel* matrix pencil arises in the shape reconstruction of the polygon from its moments [**30**] and a *symmetric-Toeplitz* matrix pencil appears in the estimation of sinusoidal signals in noise [**74**]. The coefficient matrices of *Hankel* and *symmetric-Toeplitz* pencils belong to the class of *complex-symmetric* matrices. *Hankel* and *symmetric-Toeplitz* matrices have additional properties that *complex-symmetric* matrices do not have in general. For a *Hankel* matrix, each ascending skew-diagonal from left to right is constant, while for a *symmetric-Toeplitz* matrix, each diagonal is constant. Zhang et al. [**87**] have provided the backward error formula of a single approximate eigenpair for the *complex-symmetric* matrix pencils, which also preserves sparsity. As per the knowledge of the authors, the backward error analysis of *Hankel* and *symmetric-Toeplitz* matrix pencils is not discussed in the literature. Since *Hankel* and *symmetric-Toeplitz* matrix pencils are special kinds of a *complex-symmetric* matrix pencil; hence one can apply the backward error results of *complex-symmetric* matrix pencils on *Hankel* and *symmetric-Toeplitz* matrix pencils. But this provides very unreliable backward error results, because the existing backward error results of *complex-symmetric* matrix pencils do not consider all the properties of these two structures (*Hankel* and *symmetric-Toeplitz*) during the backward error analysis. Hence to obtain the accurate backward error results, we need to take care of the structures while doing the backward error analysis because negligence in the structures of these structured matrix pencils' coefficient matrices provides false information about the computed solution, which leads to insignificant results.

Inverse eigenvalue problems deal with the construction of perturbed matrices from a given set of spectral data, which consist of one or more *eigenpairs*. Backward error analysis of *Hankel* and *symmetric-Toeplitz* matrix pencils plays an important role in providing the solution of different *inverse eigenvalue problems*. For example, consider Problem 5.2 of [**21**, Chapter 5], which requires the construction of a *symmetric-Toeplitz* matrix from given specified eigenpairs. In the same manner one can solve Problem 5.1 of [**21**, Chapter-5], which requires the construction of a *Hankel* matrix from a given set of eigenpairs (see, for example, [**19, 61, 79, 82**] for more information on *inverse eigenvalue problems*). Though in [**20**] Moody and Melissa have solved Problem 5.2 of [**21**, Chapter 5] for two specified eigenpairs in a very descriptive manner, in this chapter, we are interested in solving this problem for two or more specified eigenpairs. Moving further, we find that different authors have constructed the matrix pencil from a given set of eigenpairs, which is known as the *generalized inverse eigenvalue problem*. For example, in [**86**] the authors have solved the *generalized inverse eigenvalue problems* for Hermitian and J-Hamiltonian/skew-Hamiltonian matrices, where $J \in \mathbb{R}^{n\times n}$ such that $J^2 = -I_n$ (see, [**33, 34, 78, 84**] for more information on generalized *inverse eigenvalue problems*). In this chapter, we are also interested in solving the generalized *inverse eigenvalue problems* for *symmetric-Toeplitz* and *Hankel* matrices.

Hence for obtaining the accurate backward error results and solving the above *inverse eigenvalue problems*, we need to develop the backward error theory for one or more specified eigenpairs. In particular, for a given set of $s\,(s \leq n)$ approximate *eigenpairs* (λ_p, x_p) of an n-by-n matrix pencil, where $\lambda_p := (\lambda_{p0}, \lambda_{p1}) \in \mathbb{C}^2 \setminus \{(0,0)\}$ and $0 \neq x_p \in \mathbb{C}^n$ for $p = 1 : s$, we find the smallest structured perturbed matrix pencil with respect to the *Frobenius norm* so that given specified *eigenpairs* become exact *eigenpairs* of an appropriately perturbed problem.

4.2. Matrix pencils and preliminaries

Throughout this chapter, $\mathbf{L}$ be a matrix pencil of the form (2.1) defined in the earlier chapters. Next, we define the *Hankel* and *symmetric-Toeplitz* matrix pencils.

Definition 4.2.1. *A matrix pencil* $\mathbf{L}$ *of the form (2.1) is said to be Hankel if both the matrices associated with it are Hankel.*

Definition 4.2.2. *A matrix pencil* $\mathbf{L}$ *of the form (2.1) is said to be symmetric-Toeplitz matrix pencil if both the matrices associated with it are symmetric-Toeplitz.*

Definition 4.2.3. *A vector* $v \in \mathbb{C}^n$ *is called* symmetric *if* $J_e v = v$ *and skew-symmetric if* $J_e v = -v$*, where* J_e *is the exchange matrix, i.e., ones on the anti-diagonal and zero elsewhere.*

Throughout this chapter, $w := (w_0, w_1)^T \in \mathbb{R}^2$ be a nonnegative vector such that w_0, w_1 are nonnegative real numbers. Define $w^{-1} := (w_0^{-1}, w_1^{-1})^T$ and $w_i^{-1} = 0$ for $w_i = 0$. Next, we recall the definitions of unstructured and structured backward errors of s-approximate eigenpairs for matrix pencils.

Definition 4.2.4. *Consider* $\lambda_{1:s} := \{\lambda_1, \lambda_2, \ldots, \lambda_s\}$ *and* $x_{1:s} := \{x_1, x_2, \ldots, x_s\}$*, where* $\lambda_i \in \mathbb{C}^2 \setminus \{(0,0)\}$*, and* $0 \neq x_i \in \mathbb{C}^n$ *for* $i = 1 : s$*. Let* (λ_i, x_i) *be s-approximate eigenpairs of a matrix pencil* $\mathbf{L}$ *of the form (2.1). Then unstructured and structured backward errors of s-approximate eigenpairs* $(\lambda_i, x_i), i = 1 : s$*, are defined by*

$$\eta_{w,F}(\lambda_{1:s}, x_{1:s}, \mathbf{L}) := \inf\{\|\delta\mathbf{L}\|_{w,2}, \quad (\mathbf{L}(\lambda_i) + \delta\mathbf{L}(\lambda_i))x_i = 0; \textit{ for } 1 : s\},$$

and

$$\eta^{\mathbf{S}}_{w,F}(\lambda_{1:s}, x_{1:s}, \mathbf{L}) := \inf\{\|\delta\mathbf{L}\|_{w,2},\ \delta\mathbf{L} \in \mathbf{S},\ (\mathbf{L}(\lambda_i) + \delta\mathbf{L}(\lambda_i))x_i = 0;\ \textit{for } 1 : s\},$$

respectively, where $\delta\mathbf{L} \in \mathbf{L}(\mathbb{C}^{n\times n})$ *is of the form (2.1), and* $\|\delta\mathbf{L}\|_{w,2}$ *is given by (3.2). Here* $\mathbf{S}$ *denotes the set of structures, and we consider* $\mathbf{S} := \{$ Hankel, symmetric-Toeplitz $\}$.

Remark 4.2.5. *For* $s = 1$ *in the above definitions correspond to unstructured and structured backward errors for a single eigenpair (see, [1] for more on backward error of a single eigenpair).*

Before moving towards the main results of this chapter first, we establish some important results related to *Hankel* and *symmetric-Toeplitz* matrices.

Lemma 4.2.6. *Let $\delta A \in \mathbb{C}^{n\times n}$ be a Hankel matrix generated by $[\delta a_{11}, \dots, \delta a_{1n}, \delta a_{2n}, \dots, \delta a_{nn}]^T$. Let $x = [x^1, x^2, \dots, x^n]^T \in \mathbb{C}^n$ and $b = [b^1, b^2, \dots, b^n]^T \in \mathbb{C}^n$. Then $\delta Ax = b$ is equivalent to $X_{(x,\mathrm{Hank})}\mathrm{vec}(\delta A, \mathrm{Hank}) = b$, where $X_{(x,\mathrm{Hank})} \in \mathbb{C}^{n\times 2n-1}$ is given by*

$$X_{(x,\mathrm{Hank})} = \begin{bmatrix} x^1 & x^2 & \dots & \dots & x^n & 0 & \dots & \dots & 0 \\ 0 & x^1 & x^2 & \dots & \vdots & x^n & 0 & \dots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \dots & \ddots & \ddots & \ddots & \vdots & \dots & \ddots & 0 \\ 0 & \dots & \dots & 0 & x^1 & x^2 & \dots & \dots & x^n \end{bmatrix}.$$

Proof. Consider $\delta Ax = b$, we get

$$\begin{bmatrix} \delta a_{11} & \delta a_{12} & \dots & \delta a_{1(n-1)} & \delta a_{1n} \\ \delta a_{12} & \delta a_{13} & \dots & \delta a_{1n} & \delta a_{2n} \\ \vdots & & & & \vdots \\ \delta a_{1(n-2)} & & & & \delta a_{(n-2)n} \\ \delta a_{1(n-1)} & \dots & \dots & \delta a_{(n-2)n} & \delta a_{(n-1)n} \\ \delta a_{1n} & \dots & \delta a_{(n-2)n} & \delta a_{(n-1)n} & \delta a_{nn} \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{bmatrix} = \begin{bmatrix} b^1 \\ b^2 \\ \vdots \\ b^n \end{bmatrix}.$$

By expanding the above system, we get the following n equations:

$$\begin{aligned} \delta a_{11}x^1 + \delta a_{12}x^2 + \dots + \delta a_{1(n-1)}x^{(n-1)} + \delta a_{1n}x^n &= b^1, \\ \delta a_{12}x^1 + \delta a_{13}x^2 + \dots + \delta a_{1n}x^{(n-1)} + \delta a_{2n}x^n &= b^2, \\ \dots \qquad \dots \qquad \dots \qquad & \dots \\ \dots \qquad \dots \qquad \dots \qquad & \dots \\ \delta a_{1n}x^1 + \delta a_{2n}x^2 + \dots + \delta a_{(n-1)n}x^{(n-1)} + \delta a_{nn}x^n &= b^n. \end{aligned}$$

Further rearranging these equations by writing δA in vector form, we get $X_{(x,\mathrm{Hank})}\mathrm{vec}(\delta A, \mathrm{Hank}) = b$, which is required. ■

Lemma 4.2.7. *Let δA be a symmetric-Toeplitz matrix generated by $[\delta a_1, \delta a_2, \dots, \delta a_n]^T$. Let x and b be defined as in Lemma 4.2.6. Then $\delta Ax = b$ is equivalent to the following system*

$$X_{(x,\mathrm{symToep})}\mathrm{vec}(\delta A, \mathrm{symToep}) = b,$$

where $X_{(x,\mathrm{symToep})} \in \mathbb{C}^{n\times n}$ *is given by*

$$X_{(x,\mathrm{symToep})} = \begin{bmatrix} x^1 & 0 & 0 & \dots & \dots & 0 \\ x^2 & x^1 & 0 & \dots & \dots & 0 \\ \vdots & x^2 & x^1 & \dots & \dots & 0 \\ \vdots & 0 & x^2 & \ddots & \dots & 0 \\ \vdots & 0 & \vdots & \ddots & \ddots & 0 \\ x^n & x^{n-1} & x^{n-2} & \dots & x^2 & x^1 \end{bmatrix} + \begin{bmatrix} 0 & x^2 & x^3 & \dots & \dots & x^n \\ 0 & x^3 & x^4 & \dots & x^n & 0 \\ \vdots & \vdots & \vdots & \dots & \dots & \vdots \\ \vdots & \vdots & x^n & \dots & \dots & \vdots \\ \vdots & x^n & 0 & \dots & \dots & \vdots \\ 0 & 0 & 0 & \dots & \dots & 0 \end{bmatrix}.$$

Proof. The proof is similar to Lemma 4.2.6. ■

After establishing the preliminary results next, we present the main results of the chapter in the following sections. Next, we discuss the backward error analysis for Hankel matrix pencils.

4.3. Backward error analysis of Hankel matrix pencils

In this section, we derive the backward error formula of specified eigenpairs for a *Hankel* matrix pencil. For this derivation, we need the following matrix M whose construction is given as follows: Let $w = (w_0, w_1)^T$ be a nonnegative weight vector. Define

$$M := \begin{bmatrix} M_{10} & M_{11} \\ M_{20} & M_{21} \\ \vdots & \vdots \\ M_{s0} & M_{s1} \end{bmatrix} \in \mathbb{C}^{sn\times 4n-2},$$

where $M_{pj} = w_j^{-1}\lambda_{pj} M(j, x_p, \mathrm{Hank}) \in \mathbb{C}^{n\times 2n-1}$ for $p = 1 : s$, $j = 0, 1$. Construction of $M(j, x_p, \mathrm{Hank})$ will be done in the following manner using the approximate eigenpairs (λ_p, x_p) of *Hankel* matrix pencil $\mathbf{L} \in \mathbf{L}(\mathbb{C}^{n\times n})$ of the form (2.1), where $(\lambda_{p0}, \lambda_{p1}) = \lambda_p \in \mathbb{C}^2 \setminus \{(0, 0)\}, 0 \neq x_p \in \mathbb{C}^n$. $M(j, x_p, \mathrm{Hank}) \in \mathbb{C}^{n\times 2n-1}$ is given by

$$M(j, x_p, \mathrm{Hank}) = X_{(x_p,\mathrm{Hank})}\mathrm{diag}\left(\mathrm{vec}(\mathrm{sgn}\, A_j \circ C_H, \mathrm{Hank})\right), \tag{4.1}$$

where $X_{(x_p,\mathrm{Hank})}$ is given by Lemma 4.2.6,

$$\mathrm{vec}(\mathrm{sgn}\,A_j \circ C_H, \mathrm{Hank}) = \begin{bmatrix} \frac{1}{\sqrt{1}}\mathrm{sgn}\,a_{j,11} \\ \frac{1}{\sqrt{2}}\mathrm{sgn}\,a_{j,12} \\ \vdots \\ \frac{1}{\sqrt{n-1}}\mathrm{sgn}\,a_{j,1(n-1)} \\ \frac{1}{\sqrt{n}}\mathrm{sgn}\,a_{j,1n} \\ \frac{1}{\sqrt{n-1}}\mathrm{sgn}\,a_{j,2n} \\ \vdots \\ \frac{1}{\sqrt{2}}\mathrm{sgn}\,a_{j,(n-1)n} \\ \frac{1}{\sqrt{1}}\mathrm{sgn}\,a_{j,nn} \end{bmatrix}, \text{ and}$$

C_H, D_H are *Hankel* matrices of size n, generated by the vectors $[\frac{1}{\sqrt{1}}, \frac{1}{\sqrt{2}}, \ldots, \frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n-1}}, \ldots, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{1}}]^T$, $[\sqrt{1}, \sqrt{2}, \ldots, \sqrt{n}, \sqrt{n-1}, \ldots, \sqrt{2}, \sqrt{1}]^T$, respectively. Before moving towards the derivation of the main result of this section, we introduce the following lemma.

Lemma 4.3.1. *Let $A, \delta A \in \mathbb{C}^{n\times n}$ be Hankel matrices generated by $[a_{11}, \ldots, a_{1n}, a_{2n}, \ldots, a_{nn}]^T$ and $[\delta a_{11}, \ldots, \delta a_{1n}, \delta a_{2n}, \ldots, \delta a_{nn}]^T$, respectively. Let $x = [x^1, x^2, \ldots, x^n]^T \in \mathbb{C}^n$ and $b = [b^1, b^2, \ldots, b^n]^T \in \mathbb{C}^n$. Then we get that $(\delta A \circ \mathrm{sgn}\,A \circ C_H \circ D_H)x = b$ is equivalent to $X_{(x,\mathrm{Hank})}\mathrm{diag}(\mathrm{vec}(\mathrm{sgn}\,A \circ C_H, \mathrm{Hank}))\phi_H = b$, where $X_{(x,\mathrm{Hank})}$ is defined by Lemma 4.2.6, $\phi_H = \mathrm{vec}(\delta A \circ \mathrm{sgn}\,A \circ D_H, \mathrm{Hank})$, and C_H, D_H are defined in the beginning of this Section.*

Proof. We have $\delta A \circ \mathrm{sgn}\,A = (\delta a_{ij}\mathrm{sgn}\,a_{ij})$. On considering $(\delta A \circ \mathrm{sgn}\,A \circ C_H \circ D_H)x = b$, we get the following n equations similar to Lemma 4.2.6

$$\begin{array}{c} \delta a_{11}\mathrm{sgn}\,a_{11}x^1 + (\sqrt{2}\delta a_{12})(\frac{\mathrm{sgn}\,a_{12}}{\sqrt{2}})x^2 + \ldots + (\sqrt{n}\delta a_{1n})(\frac{\mathrm{sgn}\,a_{1n}}{\sqrt{n}})x^n = b^1, \\ (\sqrt{2}\delta a_{12})(\frac{\mathrm{sgn}\,a_{12}}{\sqrt{2}})x^1 + (\sqrt{3}\delta a_{13})(\frac{\mathrm{sgn}\,a_{13}}{\sqrt{3}})x^2 + \ldots + (\sqrt{n-1}\delta a_{2n})(\frac{\mathrm{sgn}\,a_{2n}}{\sqrt{n-1}})x^n = b^2, \\ \cdots \qquad \cdots \qquad \cdots \qquad \cdots \qquad \cdots \\ \cdots \qquad \cdots \qquad \cdots \qquad \cdots \qquad \cdots \\ (\sqrt{n}\delta a_{1n})(\frac{\mathrm{sgn}\,a_{1n}}{\sqrt{n}})x^1 + (\sqrt{n-1}\delta a_{2n})(\frac{\mathrm{sgn}\,a_{2n}}{\sqrt{n-1}})x^2 + \ldots + \delta a_{nn}\mathrm{sgn}\,a_{nn}x^n = b^n. \end{array}$$

Further rearrangement gives $X_{(x,\mathrm{Hank})}\mathrm{diag}(\mathrm{vec}(\mathrm{sgn}\,A \circ C_H, \mathrm{Hank}))\phi_H = b$, which is required. ■

Next, we derive the main result of this section.

Theorem 4.3.2. *Let $\mathbf{L} \in \mathbf{L}(\mathbb{C}^{n\times n})$ be a homogeneous Hankel matrix pencil of the form (2.1). Let (λ_p, x_p) be $s\,(s \leq n)$ approximate eigenpairs of $\mathbf{L}$, where $\lambda_p = (\lambda_{p0}, \lambda_{p1}) \in$*

$\mathbb{C}^2 \setminus \{(0,0)\}$, and $0 \neq x_p \in \mathbb{C}^n$. for $p = 1 : s$. Set $r := [r_1^T, r_2^T, \ldots, r_s^T]^T$, where $r_p = -\mathbf{L}(\lambda_p)x_p$ for $p = 1 : s$. If M (defined in the beginning of this section) is a full row rank matrix, then there exists a Hankel matrix pencil $\delta\mathbf{L} \in \mathbf{L}(\mathbb{C}^{n\times n})$ of the form $\delta\mathbf{L}(\alpha,\beta) := \alpha\delta A_0 + \beta\delta A_1$ such that $(\mathbf{L}(\lambda_p) + \delta\mathbf{L}(\lambda_p))x_p = 0$ for $p = 1 : s$, where generator vectors $[\delta a_{j,11}, \ldots, \delta a_{j,1n}, \delta a_{j,2n}, \ldots, \delta a_{j,nn}]^T$ of δA_j, for $j = 0, 1$, are given by

$$\delta a_{j,tq} = \frac{w_j^{-2}}{|t-q|+1}(\operatorname{sgn} a_{j,tq}) \sum_{p=1}^{s} \sum_{i=1}^{|t-q|+1} \overline{\lambda}_{pj} \overline{x}_p^{q-(i-1)} e_{i+t-1+(p-1)n}^T (MM^H)^{-1} r,$$

where $e_{i+t-1+(p-1)n} \in \mathbb{C}^{sn}$ and $1 \leq t, q \leq n$. Further, the backward error is given as follows:

$$\eta_{w,F}^{\mathbf{S}}(\lambda_{1:s}, x_{1:s}, \mathbf{L}) = \|M^H(MM^H)^{-1}r\|_F.$$

If M is not a full row rank matrix but $rank(M) = rank([M, r])$, then δA_j and backward error can be obtained by using singular value decomposition of M. In this case the backward error is given by $\eta_{w,F}^{\mathbf{S}}(\lambda_{1:s}, x_{1:s}, \mathbf{L}) = \|V\Sigma^{+}U^H r\|_F$, where U, V are unitary matrices of appropriate sizes and Σ is a matrix containing the singular values of M.

Proof. Corresponding to a *Hankel matrix pencil* $\mathbf{L}(\alpha,\beta) := \alpha A_0 + \beta A_1$, where A_j for $j = 0, 1$ are generated by $[a_{j,11}, a_{j,12}, \ldots, a_{j,1n}, a_{j,2n}, \ldots, a_{j,nn}]^T$, and for given approximate eigenpairs (λ_p, x_p), we need to construct a *Hankel matrix pencil* $\delta\mathbf{L}$ such that $(\mathbf{L}(\lambda_p) + \delta\mathbf{L}(\lambda_p))x_p = 0$ which preserves the sparsity. By using Proposition 1.2.23 for constructing Hankel δA_j, we consider the following generating vectors of length $(2n-1)$:

$$[\delta a_{j,11}, \delta a_{j,12}, \ldots, \delta a_{j,1n}, \delta a_{j,2n}, \ldots, \delta a_{j,nn}]^T, \ \textit{for } j = 0, 1.$$

We have $r_p = -\mathbf{L}(\lambda_p)x_p$ for $p = 1 : s$. Then $r_p = \delta\mathbf{L}(\lambda_p)x_p = \sum_{j=0}^{1} \lambda_{pj}\delta A_j x_p$. For maintaining sparsity, we replace δA_j by $(\delta A_j \circ \operatorname{sgn} A_j)$. Hence $r_p = \sum_{j=0}^{1} \lambda_{pj}(\delta A_j \circ \operatorname{sgn} A_j)x_p = \sum_{j=0}^{1} \lambda_{pj}(\delta A_j \circ \operatorname{sgn} A_j \circ D_H \circ C_H)x_p$. We get

$$r_p = \sum_{j=0}^{1} w_j^{-1} w_j \lambda_{pj}(\delta A_j \circ \operatorname{sgn} A_j \circ D_H \circ C_H)x_p. \tag{4.2}$$

On rearranging (4.2) by using Lemma 4.3.1, we get the following system

$$r_p = \sum_{j=0}^{1} w_j^{-1} \lambda_{pj} X_{(x_p,\mathrm{Hank})} \operatorname{diag}\left(\operatorname{vec}(\operatorname{sgn} A_j \circ C_H, \mathrm{Hank})\right) \Delta_j, \ \text{ where} \tag{4.3}$$

$$(4.4)\quad \Delta_j = w_j \mathrm{vec}(\delta A_j \circ \mathrm{sgn}\, A_j \circ D_H, \mathrm{Hank}) = \begin{bmatrix} \sqrt{1} w_j \delta a_{j,11} \mathrm{sgn}\, a_{j,11} \\ \sqrt{2} w_j \delta a_{j,12} \mathrm{sgn}\, a_{j,12} \\ \vdots \\ \sqrt{n-1} w_j \delta a_{j,1(n-1)} \mathrm{sgn}\, a_{j,1(n-1)} \\ \sqrt{n} w_j \delta a_{j,1n} \mathrm{sgn}\, a_{j,1n} \\ \sqrt{n-1} w_j \delta a_{j,2n} \mathrm{sgn}\, a_{j,2n} \\ \vdots \\ \sqrt{2} w_j \delta a_{j,(n-1)n} \mathrm{sgn}\, a_{j,(n-1)n} \\ \sqrt{1} w_j \delta a_{j,nn} \mathrm{sgn}\, a_{j,nn} \end{bmatrix}, j = 0, 1.$$

Using Equation 4.1 and Equation 4.3, we get

$$(4.5)\qquad r_p = \sum_{j=0}^{1} w_j^{-1} \lambda_{pj} M(j, x_p, \mathrm{Hank}) \Delta_j = \sum_{j=0}^{1} M_{pj} \Delta_j = M_p \Delta,$$

where $M_p = \begin{bmatrix} M_{p0} & M_{p1} \end{bmatrix}$, and $\Delta = \begin{bmatrix} \Delta_0^T & \Delta_1^T \end{bmatrix}^T$. Using $r_p = M_p \Delta$ for $p = 1 : s$, we get the following system of equations: $r_1 = M_1 \Delta = \begin{bmatrix} M_{10} & M_{11} \end{bmatrix} \begin{bmatrix} \Delta_0 \\ \Delta_1 \end{bmatrix}$, $r_2 = M_2 \Delta = \begin{bmatrix} M_{20} & M_{21} \end{bmatrix} \begin{bmatrix} \Delta_0 \\ \Delta_1 \end{bmatrix}, \ldots, r_s = M_s \Delta = \begin{bmatrix} M_{s0} & M_{s1} \end{bmatrix} \begin{bmatrix} \Delta_0 \\ \Delta_1 \end{bmatrix}$. Writing these s equations in the combined form, we get

$$(4.6)\qquad \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_s \end{bmatrix} = \begin{bmatrix} M_{10} & M_{11} \\ M_{20} & M_{21} \\ \vdots & \vdots \\ M_{s0} & M_{s1} \end{bmatrix} \begin{bmatrix} \Delta_0 \\ \Delta_1 \end{bmatrix}.$$

By Equation 4.6, we get $r = M\Delta$, under the assumption that M is a full row rank matrix, the minimum norm solution of $r = M\Delta$ is given by

$$(4.7)\qquad \Delta = M^H (M M^H)^{-1} r.$$

Now expanding the first M^H in (4.7) and using (4.4), we get the desired entries of perturbed matrices. The backward error is given by $\eta^{\mathbf{S}}_{w,F}(\lambda_{1:s}, x_{1:s}, \mathbf{L}) = \|\delta \mathbf{L}\|_{w,F}$, where

$$\|\delta \mathbf{L}\|_{w,F} = \sqrt{w_0^2 \|\delta A_0\|_F^2 + w_1^2 \|\delta A_1\|_F^2}.$$

Since Δ is a minimum norm solution, we get $\|\delta \mathbf{L}\|_{w,F} = \|\Delta\|_F = \|M^H (M M^H)^{-1} r\|_F$ is also minimum. Hence $\eta^{\mathbf{S}}_{w,F}(\lambda_{1:s}, x_{1:s}, \mathbf{L}) = \|\Delta\|_F = \|M^H (M M^H)^{-1} r\|_F$. Now we need to show that $(\mathbf{L}(\lambda_p) + \delta\mathbf{L}(\lambda_p))x_p = 0$ for $p = 1 : s$. For this consider

$$\begin{bmatrix} (\mathbf{L}(\lambda_1) + \delta\mathbf{L}(\lambda_1))x_1 \\ (\mathbf{L}(\lambda_2) + \delta\mathbf{L}(\lambda_2))x_2 \\ \vdots \\ (\mathbf{L}(\lambda_s) + \delta\mathbf{L}(\lambda_s))x_s \end{bmatrix} = \begin{bmatrix} -r_1 + \delta\mathbf{L}(\lambda_1)x_1 \\ -r_2 + \delta\mathbf{L}(\lambda_2)x_2 \\ \vdots \\ -r_s + \delta\mathbf{L}(\lambda_s)x_s \end{bmatrix} = \begin{bmatrix} -r_1 \\ -r_2 \\ \vdots \\ -r_s \end{bmatrix} + \begin{bmatrix} M_1 \\ M_2 \\ \vdots \\ M_s \end{bmatrix} \Delta = \begin{bmatrix} -r_1 \\ -r_2 \\ \vdots \\ -r_s \end{bmatrix} + \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

where we use $\delta\mathbf{L}(\lambda_p))x_p = M_p\Delta$ for $p = 1 : s$, $M_p = \begin{bmatrix} M_1{}^T & M_2{}^T & \dots & M_s{}^T \end{bmatrix}^T$, and $\Delta = M^H(MM^H)^{-1}r.$■

Remark 4.3.3. *When M is not a full row rank matrix but (4.6) is consistent, then using singular value decomposition, we get $M = U\Sigma V^H$. Since system (4.6) is consistent, we get at least one solution of the system (4.6), and using Theorem 1.2.6, the minimum norm solution of $r = M\Delta$ is given by $\Delta = V\Sigma^+U^Hr$, and $\eta^{\mathbf{S}}_{w,F}(\lambda_{1:s}, x_{1:s}, \mathbf{L}) = \|M^+r\|_F$, where $M^+ = V\Sigma^+U^H$. Clearly by using $\Delta = V\Sigma^+U^Hr$, we can construct the desired δA_0 and δA_1. Here U, V are unitary matrices of appropriate sizes and Σ is a matrix containing the singular values of M. We can not get the perturbed matrix entries in explicit form because singular value decomposition of M is not known explicitly.*

After obtaining the backward error result for *Hankel* matrix pencils, in the next section, we discuss the backward error analysis for *symmetric-Toeplitz* matrix pencils.

4.4. Backward error analysis of specified eigenpairs for symmetric-Toeplitz matrix pencils

This section deals with the backward error analysis of *symmetric-Toeplitz* matrix pencils. Similar to the previous section, we construct the matrix M for the *symmetric-Toeplitz* case in the following manner: Let $M := \begin{bmatrix} M_{10} & M_{11} \\ M_{20} & M_{21} \\ \vdots & \vdots \\ M_{s0} & M_{s1} \end{bmatrix} \in \mathbb{C}^{sn\times 2n}$, where $M_{pj} = w_j^{-1}\lambda_{pj}M(j, x_p, \mathrm{symToep}) \in \mathbb{C}^{n\times n}$, for $p = 1 : s$, $j = 0, 1$. Construction of $M(j, x_p, \mathrm{symToep})$ can be done in the following manner using the approximate *eigenpair* (λ_p, x_p) of a *symmetric-Toeplitz* $\mathbf{L} \in \mathbf{L}(\mathbb{C}^{n\times n})$ of the form (2.1) with nonnegative weight vector $w = (w_0, w_1)^T$, where $(\lambda_{p0},\lambda_{p1})=\lambda_p\in\mathbb{C}^2\setminus\{(0,0)\}, 0\neq x_p\in\mathbb{C}^n$. $M(j,x_p,\text{symToep})\in\mathbb{C}^{n\times n}$ is given by

$$M(j,x_p,\text{symToep})=X_{(x_p,\text{symToep})}\text{diag}(\text{vec}(\text{sgn}\,A_j\circ C_{st},\text{symToep})), \tag{4.8}$$

where $X_{(x_p,\text{symToep})}$ is given by Lemma 4.2.7 and

$$\text{diag}\,(\text{vec}(\text{sgn}\,A_j\circ C_{st},\text{symToep}))=\text{diag}\left([\frac{1}{\sqrt{n}}\text{sgn}\,a_{j,1},\frac{1}{\sqrt{2}\sqrt{n-1}}\text{sgn}a_{j,2},\ldots,\frac{1}{\sqrt{2}\sqrt{1}}\text{sgn}\,a_{j,n}]\right),$$

with C_{st},D_{st} are *symmetric-Toeplitz* matrices of size n, generated by vectors

$$[\frac{1}{\sqrt{n}},\frac{1}{\sqrt{2}\sqrt{n-1}},\ldots,\frac{1}{\sqrt{2}\sqrt{2}},\frac{1}{\sqrt{2}\sqrt{1}}]^T \text{ and } [\sqrt{n},\sqrt{2}\sqrt{n-1},\ldots,\sqrt{2}\sqrt{2},\sqrt{2}\sqrt{1}]^T,$$

respectively.

Lemma 4.4.1. *Let A and $\delta A\in\mathbb{C}^{n\times n}$ be symmetric-Toeplitz matrices generated by $[a_1,a_2,\ldots,a_n]^T$, and $[\delta a_1,\delta a_2,\ldots,\delta a_n]^T$, respectively. Let $x=[x^1,x^2,\ldots,x^n]^T\in\mathbb{C}^n$ and $b=[b^1,b^2,\ldots,b^n]^T\in\mathbb{C}^n$. Then $(\delta A\circ\text{sgn}\,A\circ C_{st}\circ D_{st})x=b$ is equivalent to $X_{(x,\text{symToep})}\text{diag}(\text{vec}(\text{sgn}\,A\circ C_{st},\text{symToep}))\phi_{st}=b$, where $X_{(x,\text{symToep})}$ is defined by Lemma 4.2.7, $\phi_{st}=\text{vec}(\delta A\circ\text{sgn}\,A\circ D_{st},\text{symToep})$ and C_{st},D_{st} are defined in the beginning of this Section.*

Proof. The proof is similar to Lemma 4.3.1. ■

Now using the above construction and Lemma 4.4.1, we derive the following main theorem of this section.

Theorem 4.4.2. *Let $\mathbf{L}\in\mathbf{L}(\mathbb{C}^{n\times n})$ be a symmetric-Toeplitz matrix pencil of the form (2.1). Suppose (λ_p,x_p) are $s\,(s\leq n)$ approximate eigenpairs of $\mathbf{L}$ for $p=1:s$, where $\lambda_p=(\lambda_{p0},\lambda_{p1})\in\mathbb{C}^2\setminus\{(0,0)\}$, and $0\neq x_p\in\mathbb{C}^n$. Set $r:=[r_1^T,r_2^T,\ldots,r_s^T]^T$, where $r_p=-\mathbf{L}(\lambda_p)x_p$ for $p=1:s$. If M (defined in the beginning of this section) is a full row rank matrix, then there exists a symmetric-Toeplitz matrix pencil $\delta\mathbf{L}$ of the form $\delta\mathbf{L}(\alpha,\beta):=\alpha\delta A_0+\beta\delta A_1$ such that $(\mathbf{L}(\lambda_p)+\delta\mathbf{L}(\lambda_p))x_p=0$ for $p=1:s$, where generator vectors $[\delta a_{j,1},\delta a_{j,2},\ldots,\delta a_{j,n}]^T$ of δA_j, for j=0,1, are given by*

$$\delta a_{j,q}=\begin{cases}\frac{w_j^{-2}}{n}(\text{sgn}\,a_{j,q})\sum_{p=1}^{s}\sum_{i=1}^{n-q+1}\overline{\lambda}_{pj}\overline{x}_p^ie_{i+q-1+(p-1)n}^T(MM^H)^{-1}r, & \text{for } q=1,\\ \frac{w_j^{-2}}{2(n-q+1)}(\text{sgn}\,a_{j,q})\sum_{p=1}^{s}\sum_{i=1}^{n-q+1}[\overline{\lambda}_{pj}\overline{x}_p^ie_{i+q-1+(p-1)n}^T+ & \\ \qquad\overline{x}_p^{q+i-1}e_{i+(p-1)n}^T](MM^H)^{-1}r, & \text{for } q=2:n,\end{cases}$$

where $e_{i+q-1+(p-1)n}\in\mathbb{C}^{sn}, e_{i+(p-1)n}\in\mathbb{C}^{sn}$. Further, the backward error is given by $\eta_{w,F}^{\mathbf{S}}(\lambda_{1:s},x_{1:s},\mathbf{L})=\|M^H(MM^H)^{-1}r\|_F$.

When M is not a full row rank matrix but rank(M) = rank$([M,r])$, then δA_j and backward error are obtained by using the singular value decomposition of M. In this case the backward error is given by $\eta^{\mathbf{S}}_{w,F}(\lambda_{1:s}, x_{1:s}, \mathbf{L}) = \|V\Sigma^+U^Hr\|_F$, where U, V are unitary matrices of appropriate sizes and Σ is a matrix containing the singular values of M.

Proof. Construction of the backward error formula and perturbed matrices for a *symmetric-Toeplitz* matrix pencil can be done similar to Theorem 4.3.2. Let sparse *symmetric-Toeplitz* A_j be generated by vectors $[a_{j,1}, a_{j,2}, \ldots, a_{j,n}]^T$ for $j = 0, 1$. Using Proposition 1.2.24, for constructing sparse *symmetric-Toeplitz* δA_j, we take the following generator vectors of length n:

$$[\delta a_{j,1}, \delta a_{j,2}, \ldots, \delta a_{j,n}]^T, \text{ for } j = 0, 1.$$

Following the steps of Theorem 4.3.2, we get

$$r_p = \sum_{j=0}^{1} w_j^{-1} w_j \lambda_{pj} (\delta A_j \circ \operatorname{sgn} A_j \circ D_{st} \circ C_{st}) x_p, \tag{4.9}$$

rearranging (4.9) by using Lemma 4.4.1, we get

$$r_p = \sum_{j=0}^{1} w_j^{-1} \lambda_{pj} X_{(x_p,\text{symToep})} \operatorname{diag}\left(\operatorname{vec}(\operatorname{sgn} A_j \circ C_{st}, \text{symToep})\right) \Delta_j, \text{ where} \tag{4.10}$$

$$\Delta_j := w_j \operatorname{vec}(\delta A_j \circ \operatorname{sgn} A_j \circ D_{st}, \text{symToep}) = \begin{bmatrix} \sqrt{n} w_j \delta a_{j,1} \operatorname{sgn} a_{j,1} \\ \sqrt{2}\sqrt{n-1} w_j \delta a_{j,2} \operatorname{sgn} a_{j,2} \\ \vdots \\ \sqrt{2}\sqrt{2} w_j \, a_{j,(n-1)} \operatorname{sgn} a_{j,(n-1)} \\ \sqrt{2}\sqrt{1} w_j \delta \, a_{j,n} \operatorname{sgn} a_{j,n} \end{bmatrix}, j = 0, 1. \tag{4.11}$$

Similar to Theorem 4.3.2, using Equation 4.8 and Equation 4.10, we get $r = M\Delta$ whose minimum norm solution, when M has *full row rank*, is given by

$$\Delta = M^H (MM^H)^{-1} r, \tag{4.12}$$

where $\Delta = \begin{bmatrix} \Delta_0^T & \Delta_1^T \end{bmatrix}^T$. By (4.11) and (4.12), we get the desired perturbation entries and backward error similar to the previous theorem. ∎

Remark 4.4.3. *Similar to Remark 4.3.3, when M is not a full row rank matrix, but $r = M\Delta$ is consistent, then using singular value decomposition, we get $M = U\Sigma V^H$. Using Theorem 1.2.6, the minimum norm solution of $r = M\Delta$ is given by $\Delta = V\Sigma^+U^Hr$,* *and* $\eta^{\mathbf{S}}_{w,F}(\lambda_{1:s}, x_{1:s}, \mathbf{L}) = \|V\Sigma^{+}U^{H}r\|_F$. *Here* U, V *are unitary matrices of appropriate sizes, and* Σ *is a matrix containing the singular values of* M.

4.5. Discussion on inverse eigenvalue problems

In this section, we establish the results for two *inverse eigenvalue problems*. First, we obtain the result for the *symmetric-Toeplitz inverse eigenvalue problem*, which comes as a corollary of Theorem 4.4.2. By this corollary, we can solve the *symmetric-Toeplitz inverse eigenvalue problem*.

Corollary 4.5.1. *Let* $A \in \mathbb{C}^{n\times n}$ *be a symmetric-Toeplitz matrix generated by* $[a_1, a_2, \ldots, a_n]^T$. *Suppose* (μ_p, v_p) *are* $s\,(s \leq n)$ *approximate eigenpairs of* A *for* $p = 1 : s$, *where* $\mu_p \in \mathbb{C}$, *and* $0 \neq v_p \in \mathbb{C}^n$. *Set* $r := [r_1^T, r_2^T, \ldots, r_s^T]^T$, *where* $r_p = -(A - \mu_p I_n)v_p$ *for* $p = 1 : s$. *If* M *(defined in the beginning of section 4.4) is a full row rank matrix, then there exists a symmetric-Toeplitz* δA *such that* $(A + \delta A - \mu_p I_n)v_p = 0$ *for* $p = 1 : s$, *where generator vector* $[\delta a_1, \delta a_2, \ldots, \delta a_n]^T$ *of* δA, *is given by*

$$\delta a_q = \begin{cases} \frac{1}{n}(\operatorname{sgn} a_q)\sum_{p=1}^{s}\sum_{i=1}^{n-q+1} \overline{v}_p^{i} e^T_{i+q-1+(p-1)n}(MM^H)^{-1}r, & \text{for } q = 1, \\ \frac{1}{2(n-q+1)}(\operatorname{sgn} a_q)\sum_{p=1}^{s}\sum_{i=1}^{n-q+1}[\overline{v}_p^{i} e^T_{i+q-1+(p-1)n} + \\ \qquad \overline{v}_p^{q+i-1} e^T_{i+(p-1)n}](MM^H)^{-1}r, & \text{for } q = 2 : n. \end{cases}$$

Further, the backward error is given by $\eta^{\mathbf{S}}_{w,F}(\mu_{1:s}, v_{1:s}, \mathbf{L}) = \|M^H(MM^H)^{-1}r\|_F$.

When M *is not a full row rank matrix but* $rank(M) = rank([M, r])$, *then* δA *and backward error are obtained by the singular value decomposition of* M. *In this case the backward error is given by* $\eta^{\mathbf{S}}_{w,F}(\mu_{1:s}, v_{1:s}, \mathbf{L}) = \|V\Sigma^{+}U^{H}r\|_F$, *where* U, V *are unitary matrices of appropriate sizes and* Σ *has the singular values of* M.

Proof. Substituting $A_0 = A, A_1 = I_n$, $w = (1, 0)^T$, and $\lambda_{p0} = 1, \lambda_{p1} = -\mu_p$, $x_p = v_p$ for $p = 1 : s$ in Theorem 4.4.2, we get the desired result. ■

Remark 4.5.2. *We observe that for establishing the above corollary, we use the fact that identity matrix* I_n, *is one type of symmetric-Toeplitz matrix. On the other hand,* I_n *is not a Hankel matrix. Hence result for the Hankel inverse eigenvalue problem is not a straight forward consequence of Theorem 4.3.2. For the Hankel case, we state the following theorem whose proof follows similar to the proof of Theorem 4.3.2.*

Before stating the theorem, we define $M := \begin{bmatrix} M_{10} \\ M_{20} \\ \vdots \\ M_{s0} \end{bmatrix} \in \mathbb{C}^{sn \times 2n-1}$, where $M_{p0} = \mu_p M(0, v_p, \text{Hank}) \in \mathbb{C}^{n \times 2n-1}$ for $p = 1 : s$. Construction of M_{p0} can be obtained by (4.1) using the approximate eigenpairs (μ_p, v_p) of a given *Hankel* matrix A.

Theorem 4.5.3. *Let $A \in \mathbb{C}^{n\times n}$ be a Hankel matrix generated by $[a_{11}, \ldots, a_{1n}, a_{2n}, \ldots, a_{nn}]^T$. Let (μ_p, v_p) be $s\,(s \leq n)$ approximate eigenpairs of A for $p = 1 : s$, where $\mu_p \in \mathbb{C}$, and $0 \neq v_p \in \mathbb{C}^n$. Set $r := [r_1^T, r_2^T, \ldots, r_s^T]^T$, where $r_p = -(A - \mu_p I_n)v_p$ for $p = 1 : s$. If M (defined as above) is a full row rank matrix, then there exists a Hankel δA such that $(A+\delta A-\mu_p I_n)v_p = 0$ for $p = 1 : s$, where generator vector $[\delta a_{11}, \ldots, \delta a_{1n}, \delta a_{2n}, \ldots, \delta a_{nn}]^T$ of δA is given by*

$$\delta a_{tq} = \frac{1}{|t-q|+1}(\operatorname{sgn} a_{tq}) \sum_{p=1}^{s} \sum_{i=1}^{|t-q|+1} \overline{x}_p^{\,q-(i-1)} e_{i+t-1+(p-1)n}^T (MM^H)^{-1} r,$$

where $e_{i+t-1+(p-1)n} \in \mathbb{C}^{sn}$, and $1 \leq t, q \leq n$. Further, the backward error is given by $\eta^{\mathbf{S}}_{w,F}(\mu_{1:s}, v_{1:s}, \mathbf{L}) = \|M^H(MM^H)^{-1}r\|_F$.

When M is not a full row rank matrix but $rank(M) = rank([M, r])$, then δA and backward error are obtained by the singular value decomposition of M. In this case backward error is given by $\eta^{\mathbf{S}}_{w,F}(\mu_{1:s}, v_{1:s}, \mathbf{L}) = \|V\Sigma^{+}U^H r\|_F$, where U, V are unitary matrices of appropriate sizes and Σ is a matrix having the singular values of M.

Proof. Following the steps of Theorem 4.3.2 along with $w = (1, 0)^T$, $\lambda_{p0} = 1, \lambda_{p1} = -\mu_p$, and $x_p = v_p$ for $p = 1 : s$, we get the desired result. ■

4.6. Numerical examples and solution of inverse eigenvalue problems

In this section, we illustrate our developed results and their necessity with numerical examples using MATLAB software. In the first three examples, we obtain the backward error of a single approximate eigenpair and corresponding perturbed matrix pencil for *Hankel* and *symmetric-Toeplitz* structures. For convenience, we take weight vector $w = (1, 1)^T$ for first three examples.

Example 4.6.1. *Let* $\mathbf{L} \in \mathbf{L}(\mathbb{C}^{3\times 3})$ *be a Hankel matrix pencil of the form (2.1), where Hankel matrices* A_0, A_1 *are given by*

$$A_0 = \begin{bmatrix} 1.02 & 0 & 5.3 \\ 0 & 5.3 & \mathrm{i} \\ 5.3 & \mathrm{i} & 1+\mathrm{i} \end{bmatrix}, \; A_1 = \begin{bmatrix} -12.78 & 6.38\mathrm{i} & 0 \\ 6.38\mathrm{i} & 0 & 59+4\mathrm{i} \\ 0 & 59+4\mathrm{i} & 79-8\mathrm{i} \end{bmatrix}.$$

Let (λ_1, x_1) *be an approximate eigenpair of* $\mathbf{L}$, *where* $\lambda_1 = (7124.001+3\mathrm{i}, -197.0066+369\mathrm{i})$ *and* $x_1 = [1+\mathrm{i}, 3+\mathrm{i}, 8-33\mathrm{i}]^T$. *Then using the results of* [87], *we get*

$$\delta A_0 = \begin{bmatrix} 0.2196+0.3727\mathrm{i} & 0 & -5.2945-0.0335\mathrm{i} \\ 0 & -0.8354-0.3170\mathrm{i} & 1.8233-4.3400\mathrm{i} \\ -5.2945-0.0335\mathrm{i} & 1.8233-4.3400\mathrm{i} & 0.7269-5.2990\mathrm{i} \end{bmatrix},$$

$$\delta A_1 = \begin{bmatrix} 0.0132-0.0217\mathrm{i} & 0.0306-0.0053\mathrm{i} & 0 \\ 0.0306-0.0053\mathrm{i} & 0 & -0.2752+0.0255\mathrm{i} \\ 0 & -0.2752+0.0255\mathrm{i} & -0.2946+0.1088\mathrm{i} \end{bmatrix}.$$

$\eta^{\mathbf{S}}_{w,F}(\lambda_1, x_1, \mathbf{L}) = 11.4120$. *On the other hand, using Theorem 4.3.2 for* $s = 1$, *we get*

$$\delta A_0 = \begin{bmatrix} 0.3277+0.5268\mathrm{i} & 0 & -5.2866-0.0341\mathrm{i} \\ 0 & -5.2866-0.0341\mathrm{i} & 1.8149-3.9261\mathrm{i} \\ -5.2866-0.0341\mathrm{i} & 1.8149-3.9261\mathrm{i} & 0.7654-5.2953\mathrm{i} \end{bmatrix},$$

$$\delta A_1 = \begin{bmatrix} 0.0182-0.0315\mathrm{i} & 0.0397-0.0132\mathrm{i} & 0 \\ 0.0397-0.0132\mathrm{i} & 0 & -0.2536+0.0145\mathrm{i} \\ 0 & -0.2536+0.0145\mathrm{i} & -0.2955+0.1067\mathrm{i} \end{bmatrix}.$$

$\eta^{\mathbf{S}}_{w,F}(\lambda_1, x_1, \mathbf{L}) = 12.2681$. *Using the results of* [87], *perturbed matrices can only preserve the complex-symmetric structure, but by our results, perturbed matrices can preserve the Hankel structure.*

Example 4.6.2. *Let* $\mathbf{L} \in \mathbf{L}(\mathbb{C}^{3\times 3})$ *be a symmetric-Toeplitz matrix pencil of the form (2.1), where symmetric-Toeplitz* A_0, A_1 *are given by*

$$A_0 = \begin{bmatrix} 0 & 1.3 & -1+\mathrm{i} \\ 1.3 & 0 & 1.3 \\ -1+\mathrm{i} & 1.3 & 0 \end{bmatrix}, \; A_1 = \begin{bmatrix} 22+3\mathrm{i} & 0 & -1-2\mathrm{i} \\ 0 & 22+3\mathrm{i} & 0 \\ -1-2\mathrm{i} & 0 & 22+3\mathrm{i} \end{bmatrix}.$$

Let (λ_1, x_1) *be an approximate eigenpair of* $\mathbf{L}$*, where* $\lambda_1 = (112.23 + 288\mathrm{i}, 1845.001212)$ *and* $x_1 = [11.25 - 0.7\mathrm{i}, 2.121 + 3\mathrm{i}, 0.2223]^T$*. Then using the results of* [87]*, we get*

$$\delta A_0 = \begin{bmatrix} 0 & -4.9231 + 0.6987\mathrm{i} & 0.3448 - 0.0096\mathrm{i} \\ -4.9231 + 0.6987\mathrm{i} & 0 & -0.0528 - 0.0975\mathrm{i} \\ 0.3448 - 0.0096\mathrm{i} & -0.0528 - 0.0975\mathrm{i} & 0 \end{bmatrix},$$

$$\delta A_1 = \begin{bmatrix} -22.0811 - 2.8157\mathrm{i} & 0 & 0.8010 + 1.8970\mathrm{i} \\ 0 & -20.1161 - 3.0188\mathrm{i} & -0 \\ 0.8010 + 1.8970\mathrm{i} & 0 & 0.0448 + 0.0727\mathrm{i} \end{bmatrix}.$$

$\eta^{\mathbf{S}}_{w,F}(\lambda_1, x_1, \mathbf{L}) = 31.1042$*. Using Theorem 4.4.2 for* $s = 1$*, we get*

$$\delta A_0 = \begin{bmatrix} 0 & -1.3 - 0\mathrm{i} & 0.3901 - 0.0608\mathrm{i} \\ -1.3 - 0\mathrm{i} & 0 & -1.3 - 0\mathrm{i} \\ 0.3901 - 0.0608\mathrm{i} & -1.3 - 0\mathrm{i} & 0 \end{bmatrix},$$

$$\delta A_1 = \begin{bmatrix} -22 - 3\mathrm{i} & 0 & 1.1837 + 2.0381\mathrm{i} \\ 0 & -22 - 3\mathrm{i} & -0 \\ 1.1837 + 2.0381\mathrm{i} & 0 & -22 - 3\mathrm{i} \end{bmatrix}.$$

$\eta^{\mathbf{S}}_{w,F}(\lambda_1, x_1, \mathbf{L}) = 38.6934$*. We can also see that* $(\mathbf{L}(\lambda_1) + \delta\mathbf{L}(\lambda_1))x_1 = 0$.

Remark 4.6.3. *Though the backward error obtained by our method is higher than the backward error of* [87] *but it is the actual backward error when we consider additional properties during backward error analysis of Hankel and symmetric-Toeplitz matrix pencils. Matrices* δA_0 *and* δA_1 *obtained by our method respect the required structures which are not possible by* [87]*. Hence the development of our results is very much essential to understand the real structured backward error analysis.*

Finally, for a *Hankel* matrix pencil, when M is not a full row rank matrix, but system (4.6) is consistent, we illustrate this situation by an example. In this case, using Remark 4.3.3, we get the required backward error and perturbed matrix pencil which preserve the sparsity.

Example 4.6.4. *Let* $\mathbf{L} \in \mathbf{L}(\mathbb{C}^{3\times 3})$ *be a Hankel matrix pencil of the form (2.1), where Hankel matrices* A_0 *and* A_1 *are given by*

$$A_0 = \begin{bmatrix} 0 & 0 & 11.0000 \\ 0 & 11.0000 & 0 + 1.0000\mathrm{i} \\ 11.0000 & 0 + 1.0000\mathrm{i} & 0 \end{bmatrix}, \; A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 59.0000 + 4.0000\mathrm{i} \\ 0 & 59.0000 + 4.0000\mathrm{i} & 79.0000 - 8.0000\mathrm{i} \end{bmatrix}.$$

Let (λ_1, x_1) *be an approximate eigenpair of* $\mathbf{L}$, *where* $\lambda_1 = (7124.001+3\mathrm{i}, -197.0066+369\mathrm{i})$, $x_1 = [1, 1.2 + \mathrm{i}, 0]^T$. *Clearly,* M *is not a full row rank matrix as rank*$(M) = 2 < 3$. *Then using the results of* [**87**], *we get*

$$\delta A_0 = \begin{bmatrix} 0 & 0 & -1.4061 - 0.8403\mathrm{i} \\ 0 & -11.0000 - 0.0000\mathrm{i} & -2.5277 + 0.3977\mathrm{i} \\ -1.4061 - 0.8403\mathrm{i} & -2.5277 + 0.3977\mathrm{i} & 0 \end{bmatrix},$$

$$\delta A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0.0904 + 0.1200\mathrm{i} \\ 0 & 0.0904 + 0.1200\mathrm{i} & 0 \end{bmatrix}.$$

$\eta^{\mathbf{S}}_{w,F}(\lambda_1, x_1, \mathbf{L}) = 11.8113$. *On the other hand, our results provide the following perturbed matrices*

$$\delta A_0 = \begin{bmatrix} 0 & 0 & -11.0000 - 0.0000\mathrm{i} \\ 0 & -11.0000 - 0.0000\mathrm{i} & 1.8312 - 3.9326\mathrm{i} \\ -11.0000 - 0.0000\mathrm{i} & 1.8312 - 3.9326\mathrm{i} & 0 \end{bmatrix},$$

$$\delta A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -0.2543 + 0.0138\mathrm{i} \\ 0 & -0.2543 + 0.0138\mathrm{i} & 0 \end{bmatrix}.$$

The backward error is given by $\eta^{\mathbf{S}}_{w,F}(\lambda_1, x_1, \mathbf{L}) = 20.0192$. *Clearly, our method preserves the Hankel structure.*

Our target is to solve [**21**, Problem 5.2], which asks to construct a *symmetric-Toeplitz* matrix $T \in \mathbb{C}^{n\times n}$ from a given set of real orthonormal eigenvectors $\{v_1, v_2, \ldots, v_s\}$, where each v_i is *symmetric* or *skew-symmetric*, and a set of real numbers $\{\mu_1, \mu_2, \ldots, \mu_s\}$. Before discussing that how we apply Corollary 4.5.1 to get the desired *symmetric-Toeplitz* matrix T, we define the matrix of ones,

$$H_n = \begin{bmatrix} 1 & 1 & \ldots & \ldots & 1 \\ 1 & 1 & \ldots & \ldots & 1 \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ 1 & 1 & \ldots & \ldots & 1 \end{bmatrix}. \tag{4.13}$$

To apply the Corollary 4.5.1, we need matrix A. Since H_n is a *symmetric-Toeplitz* matrix, we set $A = H_n$ be an arbitrary *symmetric-Toeplitz* matrix in Corollary 4.5.1. Then

$T = A + \delta A$ is the desired *symmetric-Toeplitz* matrix. Now, we illustrate this problem for $n = 3$ and two specified *eigenpairs* ($s = 2$).

Example 4.6.5. *Let* (μ_1, v_1) *and* (μ_2, v_2) *be two specified eigenpairs, where* $\mu_1 = -20, \mu_2 = 89.23$ *and* $v_1 = [\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}]^T, v_2 = [-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}]^T$. *Construct symmetric-Toeplitz* $T \in \mathbb{C}^{3\times 3}$ *such that* $Tv_i = \mu_i v_i$ *for* $i = 1, 2$. *By setting* $A = H_3$, *and substituting the values of* μ_1, μ_2, v_1, v_2 *in Corollary 4.5.1, we get* δA *as follows:*

$$\delta A = \begin{bmatrix} 51.8200 & -37.4100 & -37.4100 \\ -37.4100 & 51.8200 & -37.4100 \\ -37.4100 & -37.4100 & 51.8200 \end{bmatrix}. \textit{ Then}$$

$$T = A + \delta A = \begin{bmatrix} 52.8200 & -36.4100 & -36.4100 \\ -36.4100 & 52.8200 & -36.4100 \\ -36.4100 & -36.4100 & 52.8200 \end{bmatrix},$$ *is the desired symmetric-Toeplitz matrix.*

$$\textit{Also } Tv_1 = \mu_1 v_1 = \begin{bmatrix} -11.5470 \\ -11.5470 \\ -11.5470 \end{bmatrix}, \textit{ and } Tv_2 = \mu_2 v_2 = \begin{bmatrix} -63.0951 \\ 0 \\ 63.0951 \end{bmatrix}.$$

Remark 4.6.6. *One can set arbitrary* symmetric-Toeplitz A *instead of* H_n *to get the required symmetric-Toeplitz matrix* T.

Next target is to solve [**21**, Problem 5.1], which asks to construct a *Hankel* matrix $G \in \mathbb{C}^{n\times n}$ from a set of real orthonormal eigenvectors $\{v_1, v_2, \ldots, v_s\}$, and a set of real numbers $\{\mu_1, \mu_2, \ldots, \mu_s\}$. Since H_n is also a *Hankel* matrix, hence similar to *symmetric-Toeplitz* case, we set $A = H_n$ in Theorem 4.5.3 to obtain the desired $G = A + \delta A$. Now, we illustrate this problem for $n = 3$ and two specified eigenpairs ($s = 2$).

Example 4.6.7. *Let* (μ_1, v_1) *and* (μ_2, v_2) *be two specified eigenpairs, where* $\mu_1 = 10.3, \mu_2 = -53.27$ *and* $v_1 = [\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}]^T, v_2 = [-\frac{2}{3}, \frac{1}{3}, \frac{2}{3},]^T$. *Construct Hankel* $G \in \mathbb{C}^{3\times 3}$ *such that* $Gv_i = \mu_i v_i$ *for* $i = 1, 2$. *By setting* $A = H_3$, *and substituting the values of* μ_1, μ_2, v_1, v_2 *in Theorem 4.5.3, we get* δA *as follows:*

$$\delta A = \begin{bmatrix} -17.4818 & 19.0129 & 25.7818 \\ 19.0129 & 25.7818 & -21.0129 \\ 25.7818 & -21.0129 & -17.4818 \end{bmatrix}.$$

Then

$$G = A + \delta A = \begin{bmatrix} -16.4818 & 20.0129 & 26.7818 \\ 20.0129 & 26.7818 & -20.0129 \\ 26.7818 & -20.0129 & -16.4818 \end{bmatrix}, \textit{ is the desired Hankel matrix.}$$

Also $Gv_1 = \mu_1 v_1 = \begin{bmatrix} 7.2832 \\ 0 \\ 7.2832 \end{bmatrix}$, *and* $Gv_2 = \mu_2 v_2 = \begin{bmatrix} 35.5133 \\ -17.7567 \\ -35.5133 \end{bmatrix}$. *Similar to the matrix inverse eigenvalue problem, one can solve the generalized inverse eigenvalue problem. We illustrate the generalized inverse eigenvalue problem for symmetric-Toeplitz structure by the following example.*

Example 4.6.8. *Let* (λ_1, x_1) *and* (λ_2, x_2) *be two specified eigenpairs, where* $\lambda_1 = (1, -25 + \mathrm{i})$, $\lambda_2 = (1 + \mathrm{i}, 9.89)$ *and* $x_1 = [\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}]^T, x_2 = [0, 1, 0]^T$. *Construct symmetric-Toeplitz matrices* $T_0, T_1 \in \mathbb{C}^{3\times 3}$ *such that* $\lambda_{i0} T_0 x_i + \lambda_{i1} T_1 x_i = 0$ *for* $i = 1, 2$. *For constructing* T_0 *and* T_1, *set* $A_0 = A_1 = H_3$, *in Theorem 4.4.2. Then using the given eigenpairs,* δA_0 *and* δA_1 *are given by*

$$\delta A_0 = \begin{bmatrix} -0.0498 + 0.0339\mathrm{i} & -1.0000 - 0.0000\mathrm{i} & 0.0329 - 0.0055\mathrm{i} \\ -1.0000 - 0.0000\mathrm{i} & -0.0498 + 0.0339\mathrm{i} & -1.0000 - 0.0000\mathrm{i} \\ 0.0329 - 0.0055\mathrm{i} & -1.0000 - 0.0000\mathrm{i} & -0.0498 + 0.0339\mathrm{i} \end{bmatrix},$$

$$\delta A_1 = \begin{bmatrix} -1.0926 - 0.0995\mathrm{i} & -1.0000 - 0.0000\mathrm{i} & -0.8282 + 0.1038\mathrm{i} \\ -1.0000 - 0.0000\mathrm{i} & -1.0926 - 0.0995\mathrm{i} & -1.0000 - 0.0000\mathrm{i} \\ -0.8282 + 0.1038\mathrm{i} & -1.0000 - 0.0000\mathrm{i} & -1.0926 - 0.0995\mathrm{i} \end{bmatrix}.$$

Finally,

$$T_0 = \begin{bmatrix} 0.9502 + 0.0339\mathrm{i} & 0.0000 - 0.0000\mathrm{i} & 1.0329 - 0.0055\mathrm{i} \\ 0.0000 - 0.0000\mathrm{i} & 0.9502 + 0.0339\mathrm{i} & 0.0000 - 0.0000\mathrm{i} \\ 1.0329 - 0.0055\mathrm{i} & 0.0000 - 0.0000\mathrm{i} & 0.9502 + 0.0339\mathrm{i} \end{bmatrix},$$

$$T_1 = \begin{bmatrix} -0.0926 - 0.0995\mathrm{i} & -0.0000 - 0.0000\mathrm{i} & 0.1718 + 0.1038\mathrm{i} \\ -0.0000 - 0.0000\mathrm{i} & -0.0926 - 0.0995\mathrm{i} & -0.0000 - 0.0000\mathrm{i} \\ 0.1718 + 0.1038\mathrm{i} & -0.0000 - 0.0000\mathrm{i} & -0.0926 - 0.0995\mathrm{i} \end{bmatrix}.$$

Clearly, $\lambda_{i0} T_0 x_i + \lambda_{i1} T_1 x_i = 0$ *for* $i = 1, 2$.

Remark 4.6.9. *Since* A_0 *and* A_1 *are arbitrarily chosen symmetric-Toeplitz matrices, hence obtained symmetric-Toeplitz matrix pencil is not unique.*

Remark 4.6.10. *Similar to the generalized inverse eigenvalue problem of symmetric-Toeplitz matrices, we can solve the generalized inverse eigenvalue problem for Hankel matrices by using Theorem 4.3.2.* ■

CHAPTER 5

PERTURBATION ANALYSIS OF SPECIFIED EIGENPAIRS FOR STRUCTURED MATRIX POLYNOMIALS

Abstract: This chapter discusses the backward error analysis of specified eigenpairs for structured and unstructured matrix polynomials. We generalize the methodology of Chapter 3 to obtain the backward error results for matrix polynomials. In particular, for palindromic matrix polynomials, our results generalized the results of [**47**]. Further, the backward error results developed in this chapter allow us to solve the different kinds of quadratic and polynomial inverse eigenvalue problems without linearization.

5.1. Introduction

The term *matrix polynomial* is a well-known in numerical linear algebra which refers to the expression $\mathbf{P}(\lambda) = \sum_{j=0}^{l} \lambda^j A_j$, where $A_j, j = 0 : l$, are $n \times n$ matrices, λ is a complex scalar, and non-negative integer l is known as the degree of the matrix polynomial (see, [**23, 51, 72**] for more on matrix polynomials). Matrix polynomials with special structures occur in numerous applications in mechanics, control theory, linear systems theory and computer-aided graphic design, see [**8, 9**]. In particular, palindromic matrix polynomials arise in the mathematical modelling and numerical simulation of surface acoustic wave filters and vibration analysis of railway tracks excited by high-speed trains, see [**17, 51**]. For obtaining the eigenvalues and eigenvectors of matrix polynomials, the most widely used approach is to linearize the given matrix polynomial into a bigger size matrix pencil (see, [**37**] for more on linearization). In practice, the eigenpairs of the linearized matrix pencil are approximate due to the rounding errors, and truncation errors of the iterative methods. Hence the obtained eigenvalues and eigenvectors may contain a huge amount of error which can leads to insignificant results. Since the backward error analysis tells us that how much accurate these obtained eigenpairs for a matrix polynomial, the role of backward error analysis of these obtained eigenpairs with respect to matrix polynomials become very much crucial to understand the reliability of these obtained eigenpairs.

Though in Chapter 3, we have discussed the detailed backward error analysis of one or more eigenpairs for matrix pencils but results for matrix pencils are not enough to cover the *backward error* analysis of one or more specified eigenpairs for structured and unstructured matrix polynomials. In [**87**] the authors have obtained the structured backward error formulas of one eigenpair for *T-symmetric, T-skew-symmetric, Hermitian, and skew-Hermitian* matrix polynomials which also preserve sparsity. In [**2, 8, 9**] the authors have obtained the backward error of a single eigenpair for different structured matrix polynomials but the literature of backward error analysis of more than one eigenpairs is still open for development. Hence in this chapter, we are concerned to obtain the *backward error* formulas of the given specified eigenpairs and corresponding perturbed matrix polynomials for different structured as well as unstructured matrix polynomials which also preserve sparsity. These results will give a more realistic picture of the backward error analysis of eigenpairs.

Next, a given $n \times n$ matrix polynomial $\mathbf{P}(\lambda)$ of degree l can have up to ln eigenpairs. Hence during the backward error analysis of each structured matrix polynomial, we need to find the cap on the maximum number of approximate eigenpairs. This challenge has not arisen during the backward error analysis of a single eigenpair. In particular, if $\mathbf{P}(\lambda)$ is a matrix polynomial of degree l, and $((c_i, d_i), x_i), i = 1 : s \, (s \leq nl)$, are given s approximate *eigenpairs*, where $(c_i, d_i) \in \mathbb{C}^2 \setminus \{(0,0)\}$ and $0 \neq x_i \in \mathbb{C}^n$, we provide the upper bound on s for each structure. We have adopted and extended the vectorization methodology of Chapter 3 to obtain the desired backward error and corresponding perturbed matrix polynomial. We develop a general framework such that our perturbed matrices preserved sparsity in addition to the structure. The structures we consider include T-symmetric, T-skew-symmetric, Hermitian, skew-Hermitian, H-even, H-odd, T-even, T-odd, T-palindromic, T-anti-palindromic, H-palindromic, and H-anti-palindromic matrix polynomials. In particular, if we consider the palindromic structure, the authors in [**47**] have obtained the backward error of one specified eigenpair of palindromic matrix polynomials provided the corresponding minimization problem is solvable. We generalize the work of [**47**] from backward error of one specified eigenpair to backward error of one or more eigenpairs and discuss the comparisons in detail.

Next, we discuss in detail that how one can apply the backward error results of one or more eigenpairs to solve the quadratic inverse eigenvalue problems. In particular, we have discussed and solved the symmetric quadratic inverse eigenvalue problem [21, Problem 5.4]. We will also discuss and solve the T-palindromic quadratic inverse eigenvalue problem [88]. We illustrate both the quadratic inverse eigenvalue problems with suitable examples. One can also solve the other palindromic quadratic inverse eigenvalue problems of [88]. In the similar manner, one can also solve the other structured inverse eigenvalue problems for the above mentioned structures.

5.2. Matrix polynomials and definitions

Let us recall the definition of a matrix polynomial. Let $\mathbf{P}_l(\mathbb{C}^{n\times n})$ be the space of matrix polynomials up to degree l and a matrix polynomial $\mathbf{P} \in \mathbf{P}_l(\mathbb{C}^{n\times n})$ be of the following form:

$$\mathbf{P}(\alpha,\beta) := \alpha^l A_0 + \alpha^{l-1}\beta A_1 + \cdots + \beta^l A_l,\ A_i \in \mathbb{C}^{n\times n} \quad \text{for } i = 0,\ldots,l. \tag{5.1}$$

$\mathbf{P}(\alpha,\beta)$ defined in (5.1) is called the matrix polynomial in $(\alpha,\beta) \in \mathbb{C}^2$. We denote (5.1) by $\mathbf{P}$. Finding $(c,d) \in \mathbb{C}^2 \setminus \{(0,0)\}$, $0 \neq x \in \mathbb{C}^n$ such that $\mathbf{P}(c,d)x = 0$ is called the polynomial eigenvalue problem. $((c,d),x)$ is called the eigenpair of matrix polynomial $\mathbf{P}$.

Definition 5.2.1. *A matrix polynomial* $\mathbf{P}(\alpha,\beta) = \sum_{i=0}^{l} \alpha^{l-i}\beta^i A_i$ *is said to be regular if and only if* $\det(\mathbf{P}(c,d)) \neq 0$ *for some* $(c,d) \in \mathbb{C}^2 \setminus \{(0,0)\}$, *otherwise it is called the singular matrix polynomial.*

Spectrum of *matrix polynomial* $\mathbf{P} \in \mathbf{P}_l(\mathbb{C}^{n\times n})$ is defined as follows:

$$\Lambda(\mathbf{P}) := \{(\lambda,\mu) \in \mathbb{C}^2 \setminus \{(0,0)\} : \det(\mathbf{P}(\lambda,\mu)) = 0\}.$$

Throughout this chapter, $w := (w_0,\ldots,w_l)^T \in \mathbb{R}^{l+1}$ be a nonnegative vector such that each w_i is a nonnegative real number. Define $w^{-1} := (w_0^{-1},\ldots,w_l^{-1})^T$ and $w_i^{-1} = 0$ for $w_i = 0$. For a given nonnegative weight vector $w := (w_0,\ldots,w_l)^T \in \mathbb{R}^{l+1}$, define the matrix polynomial norm as follows:

$$\|\mathbf{P}\|_{w,2} := \|(w_0\|A_0\|,\ldots,w_l\|A_l\|)\|_2 = (\sum_{i=0}^{l} w_i^2\|A_i\|^2)^{1/2}, \tag{5.2}$$

where $\|.\|$ is the *Frobenius norm.*

Next, based on the different properties of the coefficient matrices $A_j, j = 0 : l$, of matrix polynomial $\mathbf{P}$ of the form (5.1), we define different kind of structured matrix polynomials by Table 5.1.

Now, we generalize the definition of *backward error* from one eigenpair to s eigenpairs for unstructured and structured matrix polynomials.

Definition 5.2.2. *Consider* $\lambda_{1:s} = \{\lambda_1, \lambda_2, \dots, \lambda_s\}$ *and* $x_{1:s} = \{x_1, x_2, \dots, x_s\}$*, where* $(c_i, d_i) = \lambda_i \in \mathbb{C}^2 \setminus \{(0,0)\}$ *and* $0 \neq x_i \in \mathbb{C}^n$*. Let* $(\lambda_i, x_i), i = 1 : s$*, are approximate eigenpairs of matrix polynomial* $\mathbf{P}$ *of the form (5.1). Then we define unstructured and structured backward errors for* s *approximate eigenpairs* (λ_i, x_i) *by*

$$\eta_{w,F}(\lambda_{1:s}, x_{1:s}, \mathbf{P}) := \inf\{\|\delta\mathbf{P}\|_{w,2}, \quad (\mathbf{P}(\lambda_i) + \delta\mathbf{P}(\lambda_i))x_i = 0; \ \textit{for } 1 \leq i \leq s\}, \ \textit{and}$$

$$\eta^{\mathbf{S}}_{w,F}(\lambda_{1:s}, x_{1:s}, \mathbf{P}) := \inf\{\|\delta\mathbf{P}\|_{w,2}, \ \delta\mathbf{P} \in \mathbf{S}, \ (\mathbf{P}(\lambda_i) + \delta\mathbf{P}(\lambda_i))x_i = 0; \ \textit{for } 1 \leq i \leq s\},$$

respectively, where $\delta\mathbf{P}$ is of the form (5.1), $\|\delta\mathbf{P}\|_{w,2}$ is given by (5.2), and

$\mathbf{S}$=\{*T-symmetric, T-skew- symmetric, Hermitian, skew Hermitian, T-even, T-odd, H-even, H-odd, T-palindromic, T-anti-palindromic, H-palindromic H-anti-palindromic*\}.

S	**Matrix structure**
T-symmetric	$A_j = A_j^T$
T-skew-symmetric	$A_j = -A_j^T$
Hermitian	$A_j = A_j^H$
skew-Hermitian	$A_j = -A_j^H$
T-even	$A_j = A_j^T$ for j even, $A_j = -A_j^T$ for j odd
T-odd	$A_j = -A_j^T$ for j even, $A_j = A_j^T$ for j odd
H-even	$A_j = A_j^H$ for j even, $A_j = -A_j^H$ for j odd
H-odd	$A_j = -A_j^H$ for j even, $A_j = A_j^H$ for j odd

S	**Matrix structure**
T-palindromic	$A_j = A_{l-j}^T$
T-anti-palindromic	$A_j = -A_{l-j}^T$
H-palindromic	$A_j = A_{l-j}^H$
H-anti-palindromic	$A_j = -A_{l-j}^H$

TABLE 5.1. An overview for structured matrix polynomials

Remark 5.2.3. *For $s = 1$ in the above definitions correspond to unstructured and structured backward error for a single eigenpair (see,* [8]*).*

Remark 5.2.4. *Let $\mathbf{P} \in \mathbf{P}_l(\mathbb{C}^{n\times n})$ be a matrix polynomial of the form (5.1). Then similar to a matrix pencil, the following relations are also hold for a matrix polynomial and are immediate consequences of the definitions of backward error:*

$$\eta_{w,F}(\lambda_{1:s}, x_{1:s}, \mathbf{P}) \leq \eta^{\mathbf{S}}_{w,F}(\lambda_{1:s}, x_{1:s}, \mathbf{P}), \ \max_{p=1:s} \eta_{w,F}(\lambda_p, x_p, \mathbf{P}) \leq \eta_{w,F}(\lambda_{1:s}, x_{1:s}, \mathbf{P}), \ \ \textit{and}$$

$$\max_{p=1:s} \eta^{\mathbf{S}}_{w,F}(\lambda_p, x_p, \mathbf{P}) \leq \eta^{\mathbf{S}}_{w,F}(\lambda_{1:s}, x_{1:s}, \mathbf{P}).$$

Now, we discuss the backward error analysis for T-symmetric and T-skew-symmetric matrix polynomials.

5.3. Perturbation of T-symmetric/T-skew-symmetric matrix polynomials

This section deals with the backward error analysis of T-symmetric and T-skew-symmetric matrix polynomials. Before stating the theorem, let $0 \neq x_p \in \mathbb{C}^n$ and $0 \neq \lambda_p = (c_p, d_p) \in \mathbb{C}^2 \setminus \{(0,0)\}$ for $p = 1 : s$, and $w = (w_0, \ldots, w_l)^T$ be a non-negative vector. We define

$$N^\epsilon = \begin{bmatrix} N^\epsilon_{10} & \ldots & N^\epsilon_{1l} \\ N^\epsilon_{20} & \ldots & N^\epsilon_{2l} \\ \vdots & \ldots & \vdots \\ N^\epsilon_{s0} & \ldots & N^\epsilon_{sl} \end{bmatrix} \in \mathbb{C}^{sn\,\times\,(l+1)(n^2+\epsilon n)/2},$$

where $N^\epsilon_{pj} = w_j^{-1} c_p^{l-j} d_p^j N^\epsilon(x_p) \mathrm{diag}(\mathrm{vec}(\mathrm{sgn}\, A_j \circ C, \epsilon)), j = 0 : l, p = 1 : s$, are constructed by (3.3). Now, we state and prove the main result of this section. Throughout this section, $\epsilon = 1$ represents the T-symmetric case and $\epsilon = -1$ represents the T-skew-symmetric case. The upper bound on the number of approximate eigenpairs "s" for T-symmetric and T-skew-symmetric matrix polynomials is capped by Table 5.2. Now, we state and prove the main result of this section.

Theorem 5.3.1. *Let $\mathbf{P} \in \mathbf{P}_l(\mathbb{C}^{n\times n})$ be a T-symmetric/T-skew-symmetric matrix polynomial of the form (5.1). Let $((c_p, d_p), x_p)$ be s approximate eigenpairs of $\mathbf{P}$, where $0 \neq x_p \in \mathbb{C}^n$ and $0 \neq \lambda_p = (c_p, d_p)$ for $p = 1 : s$. Set $r := \begin{bmatrix} r_1^T & r_2^T & \ldots & r_s^T \end{bmatrix}^T$ where*

$r_p := -\mathbf{P}(c_p, d_p)x_p$ for $p = 1 : s$. If N^ϵ (defined as above) is a full row rank matrix, then there exists a minimizing T-symmetric/T-skew symmetric $\delta\mathbf{P} \in \mathbf{P}_l(\mathbb{C}^{n\times n})$ of the form $\delta\mathbf{P}(\alpha,\beta) := \sum_{i=0}^{l} \alpha^{l-i}\beta^i \delta A_i$, where $\delta A_j = (\delta a_{j,tk})$ for $j = 0, 1, \ldots, l$ are given by

$$\delta a_{j,tk} = \begin{cases} \sum_{p=1}^{s} \frac{1}{2} w_j^{-2} \overline{c_p^{l-j} d_p^j} (\operatorname{sgn} a_{j,tk}) (\overline{x}_p^k e_{t+(p-1)n}^T + \epsilon \overline{x}_p^t e_{k+(p-1)n}^T)(N^\epsilon N^{\epsilon H})^{-1} r & \text{for } \quad t \neq k \\ \sum_{p=1}^{s} \frac{1+\epsilon}{2} w_j^{-2} \overline{c_p^{l-j} d_p^j} (\operatorname{sgn} a_{j,tk}) \overline{x}_p^k e_{t+(p-1)n}^T (N^\epsilon N^{\epsilon H})^{-1} r & \text{for } \quad t = k. \end{cases}$$

Then $(\mathbf{P}(c_p, d_p) + \delta\mathbf{P}(c_p, d_p))x_p = 0$ and the backward error is given by

$$\eta_{w,F}^{\mathbf{S}}(\lambda_{1:s}, x_{1:s}, \mathbf{P}) = \|N^{\epsilon H}(N^\epsilon N^{\epsilon H})^{-1} r\|_F.$$

If N^ϵ is not full a rank matrix but $rank(N^\epsilon) = rank([N^\epsilon, r])$, then the perturbed matrices are obtained by singular value decomposition of N^ϵ and the backward error is given by

$$\eta_{w,F}^{\mathbf{S}}(\lambda_{1:s}, x_{1:s}, \mathbf{P}) = \|V^\epsilon D^{\epsilon +} U^{\epsilon H} r\|_F,$$

where U^ϵ, V^ϵ are unitary matrices and D^ϵ is a diagonal matrix with singular values of N^ϵ.

Proof. The proof for T-symmetric/T-skew-symmetric matrix polynomials follows similar to the proof of T-symmetric/T-skew-symmetric matrix pencils. But for the sake of completeness, we recall the proof so one can easily understand the changes for the polynomial version. Corresponding to a given T-symmetric/T-skew symmetric $\mathbf{P} \in \mathbf{P}_l(\mathbb{C}^{n\times n})$, the given approximate eigenvalues are (c_p, d_p) and corresponding eigenvectors are x_p for $p = 1 : s$. We need to construct structured $\delta\mathbf{P} \in \mathbf{P}_l(\mathbb{C}^{n\times n})$ which has sparsity such that $(\mathbf{P}(\lambda_p) + \delta\mathbf{P}(\lambda_p))x_p = 0$. By assumption $\mathbf{P}(\lambda_p)x_p + r_p = 0$ for $p = 1 : s$. Then $r_p = \delta\mathbf{P}(\lambda_p)x_p = \sum_{j=0}^{l} c_p^{l-j} d_p^j \delta A_j x_p$. For maintaining sparsity replace δA_j by $(\delta A_j \circ \operatorname{sgn} A_j)$ which gives $r_p = \sum_{j=0}^{l} c_p^{l-j} d_p^j (\delta A_j \circ \operatorname{sgn} A_j)x_p$. Finally, we get

$$r_p = \sum_{j=0}^{l} c_p^{l-j} d_p^j (\delta A_j \circ \operatorname{sgn} A_j \circ D \circ C)x_p,$$

where C, D are defined by (3.4).

Structure	**upper bound on number of eigenpairs** "s"
T-symmetric	$s \leq (l+1)(\frac{n+1}{2})$
T-skew-symmetric	$s \leq (l+1)(\frac{n-1}{2})$

TABLE 5.2. Upper bound on eigenpairs for T-symmetric and T-skew-symmetric matrix polynomials

Let $\Delta_j^\epsilon = w_j \mathrm{vec}(\delta A_j \circ \mathrm{sgn} A_j \circ D)$. Then using Lemma 3.3.3, we get

$$r_p = \sum_{j=0}^{l} w_j^{-1} c_p^{l-j} d_p^j N^\epsilon(x_p)\mathrm{diag}([\mathrm{vec}(\mathrm{sgn}(A_j)\circ C, \epsilon)])\Delta_j^\epsilon = \sum_{j=0}^{l} N_{pj}^\epsilon \Delta_j^\epsilon = N_p^\epsilon \Delta^\epsilon, \tag{5.3}$$

where $N_p^\epsilon = \begin{bmatrix} N_{p0}^\epsilon & \dots & N_{pl}^\epsilon \end{bmatrix}$, $\Delta^\epsilon = \begin{bmatrix} \Delta_0^{\epsilon T} & \dots & \Delta_l^{\epsilon T} \end{bmatrix}^T$ for $p = 1 : s$. From (5.3) we have $r_p = N_p^\epsilon \Delta^\epsilon$ for $p = 1 : s$. Further, writing these s equations in the combined form we get the following system

$$r = N^\epsilon \Delta^\epsilon, \tag{5.4}$$

$$\begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_s \end{bmatrix} = \begin{bmatrix} N_{10}^\epsilon & \dots & N_{1l}^\epsilon \\ N_{20}^\epsilon & \dots & N_{2l}^\epsilon \\ \vdots & \dots & \vdots \\ N_{s0}^\epsilon & \dots & N_{sl}^\epsilon \end{bmatrix} \begin{bmatrix} \Delta_0^\epsilon \\ \Delta_1^\epsilon \\ \vdots \\ \Delta_l^\epsilon \end{bmatrix}. \tag{5.5}$$

If in (5.4) N^ϵ is a full row rank matrix, then the minimum norm solution of the system (5.5) is given by

$$\Delta^\epsilon = N^{\epsilon H}(N^\epsilon N^{\epsilon H})^{-1} r, \tag{5.6}$$

and expanding the first $N^{\epsilon H}$ in (5.6), we get the desired entries of perturbed matrices δA_j for $j = 0 : l$. In this case the backward error in *Frobenius* norm is given by $\eta_{w,F}^{\mathbf{S}}(\lambda_{1:s}, x_{1:s}, \mathbf{P}) = \|\delta \mathbf{P}\|_{w,F}$, where

$$\|\delta \mathbf{P}\|_{w,F}^2 = \sum_{i=0}^{l} w_i^2 \|\delta A_i\|_F^2 = \|\Delta^\epsilon\|_{w,F}^2 = \|N^{\epsilon H}(N^\epsilon N^{\epsilon H})^{-1} r\|_F^2.$$

We get the *backward error* when N^ϵ is a *full rank* matrix as follows:

$$\eta_{w,F}^{\mathbf{S}}(\lambda_{1:s}, x_{1:s}, \mathbf{P}) = \|\Delta^\epsilon\|_F = \|N^{\epsilon H}(N^\epsilon N^{\epsilon H})^{-1} r\|_F.$$

Now, we need to show that $(\mathbf{P}(c_p, d_p) + \delta\mathbf{P}(c_p, d_p))x_p = 0$ for $p = 1 : s$, for this consider

$$\begin{bmatrix} (\mathbf{P}(c_1,d_1)+\delta\mathbf{P}(c_1,d_1))x_1 \\ (\mathbf{P}(c_2,d_2)+\delta\mathbf{P}(c_2,d_2))x_2 \\ \vdots \\ (\mathbf{P}(c_s,d_s)+\delta\mathbf{P}(c_s,d_s))x_s \end{bmatrix} = \begin{bmatrix} -r_1 + \delta\mathbf{P}(c_1,d_1)x_1 \\ -r_2 + \delta\mathbf{P}(c_2,d_2)x_2 \\ \vdots \\ -r_s + \delta\mathbf{P}(c_s,d_s)x_s \end{bmatrix} = \begin{bmatrix} -r_1 \\ -r_2 \\ \vdots \\ -r_s \end{bmatrix} + \begin{bmatrix} N_1^\epsilon \\ N_2^\epsilon \\ \vdots \\ N_s^\epsilon \end{bmatrix} \Delta^\epsilon = \begin{bmatrix} -r_1 \\ -r_2 \\ \vdots \\ -r_s \end{bmatrix} + \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_s \end{bmatrix} = 0,$$

where we use $\delta\mathbf{P}(\lambda_p))x_p = N_p^\epsilon \Delta^\epsilon$, for $p = 1 : s, N^\epsilon = \begin{bmatrix} N_1^{\epsilon T} & N_2^{\epsilon T} & \dots & N_s^{\epsilon T} \end{bmatrix}^T$, and $\Delta^\epsilon = N^{\epsilon H}(N^\epsilon N^{\epsilon H})^{-1} r$. ■

If N^ϵ is not a *full row rank* matrix but system (5.5) is consistent, then using Theorem 1.1, we get $N^\epsilon = U^\epsilon \Sigma^\epsilon V^{\epsilon H}$, and using Theorem 1.2.6, we get $\Delta^\epsilon = V^\epsilon \Sigma^{\epsilon +} U^{\epsilon H} r$, and

$$\eta^{\mathbf{S}}_{w,F}(\lambda_{1:s}, x_{1:s}, \mathbf{P}) = \|V^\epsilon \Sigma^{\epsilon +} U^{\epsilon H} r\|_F.$$

Here U^ϵ, V^ϵ are unitary matrices and Σ^ϵ has the singular values of N^ϵ. In this case, we can not get the perturbed matrix entries in explicit form because the singular value decomposition of N^ϵ is not known explicitly in terms of the given information.

Remark 5.3.2. *When $rank(N^\epsilon) \neq rank([N^\epsilon, r])$, then $\eta^{\mathbf{S}}_{w,F}(\lambda_{1:s}, x_{1:s}, \mathbf{P}) = \infty$.*

Remark 5.3.3. *If one is interested in obtaining the backward error formula without sparsity, then by following the above procedure with $\operatorname{sgn} A_j = \operatorname{sgn} H_n$ for $j = 0 : l$, where H_n is the matrix of all ones, we get the backward error result without sparsity. Hence this method is valuable for obtaining the backward error with sparsity as well as for backward error without sparsity.*

Remark 5.3.4. *By substituting $s = 1$ in Theorem 5.3.1, we get the result of backward error of a single approximate eigenpair for T-symmetric/T-skew-symmetric matrix polynomials* [87, Theorem 2].■

5.4. Backward error of *Hermitian/skew-Hermitian* matrix polynomials

In this section, we discuss the backward error analysis for Hermitian and skew-Hermitian matrix polynomials. Throughout this section, $\epsilon = 1$ represents the Hermitian case and $\epsilon = -1$ represents the skew-Hermitian case. The upper bound on the number of approximate eigenpairs s for Hermitian and skew-Hermitian matrix polynomials is capped by Table 5.3. Next, we discuss the main theorem of this section as follows.

Structure	**upper bound on number of eigenpairs** "s"
Hermitian	$s \leq (\frac{l+1}{2})n$
skew-Hermitian	$s \leq (\frac{l+1}{2})n$

TABLE 5.3. Upper bound on eigenpairs for Hermitian and skew-Hermitian matrix polynomials

Theorem 5.4.1. *Let $\mathbf{P} \in \mathbf{P}_l(\mathbb{C}^{n\times n})$ be a Hermitian/skew-Hermitian homogeneous matrix polynomial of the form (5.1). Let $((c_p, d_p), x_p)$ be s approximate eigenpairs of $\mathbf{P}$, where $0 \neq x_p \in \mathbb{C}^n$ and $0 \neq \lambda_p = (c_p, d_p)$. Set $r := \begin{bmatrix} \Re(r_1)^T & \Im(r_1)^T & \dots & \Re(r_s)^T & \Im(r_s)^T \end{bmatrix}^T$, where $r_p := -\mathbf{P}(c_p, d_p)x_p$ for $p = 1 : s$. If N^ϵ (defined as below) is a full row rank matrix, then there exists a minimizing Hermitian/skew-Hermitian $\delta\mathbf{P} \in \mathbf{P}_l(\mathbb{C}^{n\times n})$ of the form $\delta\mathbf{P}(\alpha, \beta) := \sum_{j=0}^{l} \alpha^{l-j}\beta^j \delta A_j$, where δA_j for $j = 0 : l$ are given by*

$$\delta a_{j,tk} = \begin{cases} (\operatorname{sgn} a_{j,tt}) \displaystyle\sum_{p=1}^{s} \sqrt{\epsilon} w_j^{-2} \big(g_{pj}^t e^T_{t+\frac{(1-\epsilon)}{2}n(2p-1)+\frac{(1+\epsilon)}{2}2n(p-1)} + \epsilon h_{pj}^t \\ e^T_{t+\frac{(1+\epsilon)}{2}n(2p-1)+\frac{(1-\epsilon)}{2}2n(p-1)} \big) (N^\epsilon N^{\epsilon T})^{-1} r, & \text{for } t = k, \\ (\operatorname{sgn} a_{j,tk}) \displaystyle\sum_{p=1}^{s} \tfrac{1}{2} w_j^{-2} f_{pj,tk} (N^\epsilon N^{\epsilon T})^{-1} r, & \text{for } \ t \neq k, \end{cases}$$

with $f_{pj,tk} = (e_{t+2n(p-1)} + \mathrm{i}e_{t+n+2n(p-1)})^T (g_{pj}^k - \mathrm{i}h_{pj}^k) + (e_{k+2n(p-1)} - \mathrm{i}e_{k+n+2n(p-1)})^T (\epsilon g_{pj}^t + \mathrm{i}\epsilon h_{pj}^t)$, $g_{pj} = \Re(c_p^{l-j} d_p^j x_p)$, $h_{pj} = \Im(c_p^{l-j} d_p^j x_p)$, $g_{pj}^t = \Re(c_p^{l-j} d_p^j x_p^t)$, $h_{pj}^t = \Im(c_p^{l-j} d_p^j x_p^t)$ for $p = 1 : s$, $j = 0 : l$, and $t, k = 1 : n$. Then $(\mathbf{P}(c_p, d_p) + \delta\mathbf{P}(c_p, d_p))x_p = 0$ and the backward error is given by

$$\eta^{\mathbf{S}}_{w,F}(\lambda_{1:s}, x_{1:s}, \mathbf{P}) = \|N^{\epsilon T}(N^\epsilon N^{\epsilon T})^{-1} r\|_F, \quad \textit{where}$$

$$N^\epsilon = \begin{bmatrix} N^\epsilon_{10} & \dots & N^\epsilon_{1l} \\ N^\epsilon_{20} & \dots & N^\epsilon_{2l} \\ \vdots & \dots & \vdots \\ N^\epsilon_{s0} & \dots & N^\epsilon_{sl} \end{bmatrix} \in \mathbb{C}^{2sn\times(l+1)n^2} \quad \textit{and}$$

$$N^\epsilon_{pj} = w_j^{-1} \begin{bmatrix} N^\epsilon(g_{pj}) & -N^{-\epsilon}(h_{pj}) \\ N^\epsilon(h_{pj}) & N^{-\epsilon}(g_{pj}) \end{bmatrix} \operatorname{diag}\left(\begin{bmatrix} \operatorname{vec}(\operatorname{sgn} A_j \circ C, \epsilon) \\ \operatorname{vec}(\operatorname{sgn} A_j \circ C, -\epsilon) \end{bmatrix} \right)$$

for $j = 0 : l$ are constructed by Equation 3.3. If N^ϵ is not full rank matrix but $\operatorname{rank}(N^\epsilon) =$ $\operatorname{rank}([N^\epsilon, r])$, *then the backward error is given by*

$$\eta^{\mathbf{S}}_{w,F}(\lambda_{1:s}, x_{1:s}, \mathbf{P}) = \|V^\epsilon D^{\epsilon +} U^{\epsilon H} r\|_F,$$

where U^ϵ, V^ϵ are unitary matrices and D^ϵ is a diagonal matrix with singular values of N^ϵ. Here $\epsilon = 1$ stands for Hermitian case and $\epsilon = -1$ stands for skew-Hermitian case.

Proof. Proof can be obtained by taking the summation from 0 to l instead of 0 to 1, and to replace λ_{pj} with $c_p^{l-j} d_p^j$ in Theorem 3.4.1. Hence, we are omitting the proof.

Remark 5.4.2. *By substituting $s = 1$ in Theorem 5.4.1, we get the backward error result of a single approximate eigenpair for homogeneous matrix polynomial case of* [87, Theorem 3].■

Structure	upper bound on number of eigenpairs "s"
T-even	$s \leq (\frac{l}{2})n + (\frac{n+1}{2})$, when l is even
T-odd	$s \leq (\frac{l}{2})n + (\frac{n-1}{2})$, when l is even
T-even	$s \leq (\frac{l+1}{2})n$, when l is odd
T-odd	$s \leq (\frac{l+1}{2})n$, when l is odd

TABLE 5.4. Upper bound on eigenpairs for T-even and T-odd matrix polynomials

5.5. Backward error analysis for T-even/T-odd matrix polynomials

In this section, we discuss the backward error analysis of matrix polynomials whose coefficient matrices are alternatively symmetric and skew-symmetric. Throughout this section, $\epsilon = 1$ represents the T-even case and $\epsilon = -1$ represents the T-odd case. The upper bound on the number of eigenpairs s for T-even and T-odd matrix polynomials is capped by Table 5.4. Now, we state the main theorem of this section.

Theorem 5.5.1. *Let $\mathbf{P} \in \mathbf{P}_l(\mathbb{C}^{n\times n})$ be a T-even/T-odd homogeneous matrix polynomial of the form (5.1). Set $((c_p, d_p), x_p)$ be s approximate eigenpairs of $\mathbf{P}$ with $0 \neq x_p \in \mathbb{C}^n$ and let $0 \neq \lambda_p = (c_p, d_p)$. Set $r := \begin{bmatrix} r_1^T & r_2^T & \dots & r_s^T \end{bmatrix}^T$ where $r_p := -\mathbf{P}(\lambda_p)x_p$ for $p = 1 : s$. If N^ϵ (defined as below) is has full row rank, then there exists a minimizing T-even/T-odd $\delta\mathbf{P} \in \mathbf{P}_l(\mathbb{C}^{n\times n})$ of the form $\delta\mathbf{P}(c, s) := \sum_{i=0}^{l} c^{l-i}s^i\delta A_i$, where $\delta A_j = (\delta a_{j,tk}), j = 0 : l$, for even j are given by*

$$\delta a_{j,tk} = \begin{cases} \sum_{p=1}^{s} \frac{1}{2} w_j^{-2} \overline{c_p^{l-j} d_p^j} (\operatorname{sgn} a_{j,tk}) (\overline{x}_p^k e_{t+(p-1)n}^T + \epsilon \overline{x}_p^t e_{k+(p-1)n}^T)(N^\epsilon N^{\epsilon H})^{-1} r & \text{for} \quad t \neq k \\ \sum_{p=1}^{s} \frac{1+\epsilon}{2} w_j^{-2} \overline{c_p^{l-j} d_p^j} (\operatorname{sgn} a_{j,tk}) \overline{x}_p^k e_{t+(p-1)n}^T (N^\epsilon N^{\epsilon H})^{-1} r & \text{for} \quad t = k, \end{cases}$$

and $\delta A_j = (\delta a_{j,tk})$ for odd j are given as follows

$$\delta a_{j,tk} = \begin{cases} \sum_{p=1}^{s} \frac{1}{2} w_j^{-2} \overline{c_p^{l-j} d_p^j} (\operatorname{sgn} a_{j,tk}) (\overline{x}_p^k e_{t+(p-1)n}^T - \epsilon \overline{x}_p^t e_{k+(p-1)n}^T)(N^\epsilon N^{\epsilon H})^{-1} r & \text{for} \quad t \neq k \\ \sum_{p=1}^{s} \frac{1-\epsilon}{2} w_j^{-2} \overline{c_p^{l-j} d_p^j} (\operatorname{sgn} a_{j,tk}) \overline{x}_p^k e_{t+(p-1)n}^T (N^\epsilon N^{\epsilon H})^{-1} r & \text{for} \quad t = k. \end{cases}$$

Then $(\mathbf{P}(c_p, d_p) + \delta\mathbf{P}(c_p, d_p))x_p = 0$ and the backward error is given by

$$\eta_{w,F}^{\mathbf{S}}(\lambda_{1:s}, x_{1:s}, \mathbf{P}) = \|N^{\epsilon H}(N^\epsilon N^{\epsilon H})^{-1} r\|_F, \quad \textit{where}$$

$$N^{\epsilon} = \begin{bmatrix} N_{10}^{\epsilon} & N_{11}^{-\epsilon} & \dots & N_{1l}^{(-1)^l\epsilon} \\ N_{20}^{\epsilon} & N_{21}^{-\epsilon} & \dots & N_{2l}^{(-1)^l\epsilon} \\ \vdots & \vdots & \dots & \vdots \\ N_{s0}^{\epsilon} & N_{s1}^{-\epsilon} & \dots & N_{sl}^{(-1)^l\epsilon} \end{bmatrix} \textit{ such that } N_{pj}^{\epsilon} = w_j^{-1}c_p^{l-j}d_p^jN^{\epsilon}(x_p)\mathrm{diag}(\mathrm{vec}(\mathrm{sgn}\, A_j \circ C, \epsilon)),$$

and $N_{pj}^{-\epsilon} = w_j^{-1}c_p^{l-j}d_p^jN^{\epsilon}(x_p)\mathrm{diag}(\mathrm{vec}(\mathrm{sgn}\, A_j \circ C, -\epsilon))$ *for* $j = 0 : l$ *are defined in Equation 3.3. Here size of* N^{ϵ} *is* $sn \times (\frac{l}{2}n^2 + \frac{n^2+\epsilon n}{2})$ *when* l *is an even integer, and* $sn \times (\frac{l+1}{2})n^2$ *for odd* l.

Proof. Corresponding to T-even/T-odd matrix polynomial $\mathbf{P}$, given approximate eigenpairs are (λ_p, x_p) for $p = 1 : s$. We need to construct minimal norm $\delta\mathbf{P}$ such that $(\mathbf{P}(\lambda_p)+\delta\mathbf{P}(\lambda_p))x_p = 0$. Since $\mathbf{P}(\lambda_p)x_p + r_p = 0$. Then $r_p = \delta\mathbf{P}(\lambda_p)x_p = \sum_{j=0}^{l} c_p^{l-j}d_p^j\delta A_jx_p = \sum_{j=0}^{l} c_p^{l-j}d_p^j(\delta A_j \circ \mathrm{sgn}A_j)x_p = \sum_{j=0}^{l} c_p^{l-j}d_p^j(\delta A_j \circ \mathrm{sgn}A_j \circ D \circ C)x_p$. Let $\Delta_j^{\epsilon} = w_j\mathrm{vec}(\delta A_j \circ \mathrm{sgn}A_j \circ D, \epsilon)$ and $\Delta_j^{-\epsilon} = w_j\mathrm{vec}(\delta A_j \circ \mathrm{sgn}A_j \circ D, -\epsilon)$. Then $r_p = \sum_{j=0, j=even}^{l} w_j^{-1}c_p^{l-j}d_p^jN^{\epsilon}(x_p)$ $\mathrm{diag}([\mathrm{vec}(\mathrm{sgn}A_j \circ C, \epsilon)])\Delta_j^{\epsilon} + \sum_{j=0,j=odd}^{l} w_j^{-1}c_p^{l-j}d_p^jN^{-\epsilon}(x_p)\mathrm{diag}([\mathrm{vec}(\mathrm{sgn}A_j \circ C, \epsilon)])\Delta_j^{-\epsilon}$. Further, we get

$$r_p = \sum_{j=0,j=even}^{l} N_{pj}^{\epsilon}\Delta_j^{\epsilon} + \sum_{j=0,j=odd}^{l} N_{pj}^{-\epsilon}\Delta_j^{-\epsilon} = N_p^{\epsilon}\Delta^{\epsilon} \ for \ p = 1 : s,$$

where $N_p^{\epsilon} = \begin{bmatrix} N_{p0}^{\epsilon} & N_{p1}^{-\epsilon} & \dots & N_{pl}^{\epsilon(-1)^l} \end{bmatrix}$, $\Delta^{\epsilon} = \begin{bmatrix} \Delta_0^{\epsilon T} & \Delta_1^{-\epsilon T} & \dots & \Delta_l^{\epsilon(-1)^l T} \end{bmatrix}^T$. Combining the s equations, we get

$$r = N^{\epsilon}\Delta^{\epsilon}, \ \ where$$

$$r = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_s \end{bmatrix}, N^{\epsilon} = \begin{bmatrix} N_{10}^{\epsilon} & N_{11}^{-\epsilon} & \dots & N_{1l}^{(-1)^l\epsilon} \\ N_{20}^{\epsilon} & N_{21}^{-\epsilon} & \dots & N_{2l}^{(-1)^l\epsilon} \\ \vdots & \vdots & \dots & \vdots \\ N_{s0}^{\epsilon} & N_{s1}^{-\epsilon} & \dots & N_{sl}^{(-1)^l\epsilon} \end{bmatrix}.$$

Perturbation matrices and backward error formula can be obtained in the same manner as we get in Theorem 5.3.1. ■

5.6. Backward error analysis for H-even/H-odd matrix polynomials

In this section, we discuss the backward error for H-even/H-odd matrix polynomials. For this first we define the basic terminology. Let $x_p \in \mathbb{C}^n, (c_p, d_p) \in \mathbb{C}^2 \setminus \{(0,0)\}$ for

$p = 1 : s$, and $e_{t+2n(p-1)}, e_{k+2n(p-1)}, e_{k+n+2n(p-1)}, e_{t+n+2n(p-1)} \in \mathbb{C}^{2sn}$ for $t, k = 1 : n$. Deine $f_{pj,tk} := (e_{t+2n(p-1)} + \mathrm{i}e_{t+n+2n(p-1)})^T(g^k_{pj} - \mathrm{i}h^k_{pj}) + (e_{k+2n(p-1)} - \mathrm{i}e_{k+n+2n(p-1)})^T(\epsilon g^t_{pj} + \mathrm{i}\epsilon h^t_{pj})$ when j is even and $z_{pj,tk} := (e_{t+2n(p-1)} + \mathrm{i}e_{t+n+2n(p-1)})^T(g^k_{pj} - \mathrm{i}h^k_{pj}) + (e_{k+2n(p-1)} - \mathrm{i}e_{k+n+2n(p-1)})^T(-\epsilon g^t_{pj} - \mathrm{i}\epsilon h^t_{pj})$ when j is odd, where $g_{pj} := \Re(c_p^{l-j}d_p^j x_p)$, $h_{pj} = \Im(c_p^{l-j}d_p^j x_p)$, $g^t_{pj} = \Re(c_p^{l-j}d_p^j x^t_p)$, $h^t_{pj} = \Im(c_p^{l-j}d_p^j x^t_p)$ for $p = 1 : s$, $j = 0 : l$, and $t, k = 1 : n$. Next, define

$$N^\epsilon = \begin{bmatrix} N^\epsilon_{10} & N^{-\epsilon}_{11} & \dots & N^{(-1)^l\epsilon}_{1l} \\ N^\epsilon_{20} & N^{-\epsilon}_{21} & \dots & N^{(-1)^l\epsilon}_{2l} \\ \vdots & \vdots & \dots & \vdots \\ N^\epsilon_{s0} & N^{-\epsilon}_{s1} & \dots & N^{(-1)^l\epsilon}_{sl} \end{bmatrix} \in \mathbb{C}^{2sn\times(l+1)n^2},$$

where

$$N^\epsilon_{pj} = w_j^{-1}\begin{bmatrix} N^\epsilon(g_{pj}) & -N^{-\epsilon}(h_{pj}) \\ N^\epsilon(h_{pj}) & N^{-\epsilon}(g_{pj}) \end{bmatrix} \mathrm{diag}\left(\begin{bmatrix} \mathrm{vec}(\mathrm{sgn}A_j \circ C, \epsilon) \\ \mathrm{vec}(\mathrm{sgn}A_j \circ C, -\epsilon) \end{bmatrix}\right), \text{ for } j = 0 : l$$

are defined by Equation 3.3.

Throughout this section, $\epsilon = 1$ represents the H-even case and $\epsilon = -1$ represents the H-odd case. The upper bound on the number of approximate eigenpairs s for H-even and H-odd matrix polynomials are capped by Table 5.5.

Structure	**upper bound on number of eigenpairs** "s"
H-even	$s \leq (\frac{l+1}{2})n$ for all l
H-odd	$s \leq (\frac{l+1}{2})n$ for all l

TABLE 5.5. Upper bound on eigenpairs for H-even and H-odd matrix polynomials

Now, we state the main theorem of this section.

Theorem 5.6.1. *Let $\mathbf{P} \in \mathbf{P}_l(\mathbb{C}^{n\times n})$ be H-even/H-odd homogeneous matrix polynomial of the form (5.1). Let $((c_p, d_p), x_p)$ be s approximate eigenpairs of $\mathbf{P}$ with $0 \neq x_p \in \mathbb{C}^n$ and $0 \neq \lambda_p = (c_p, s_p)$ for $p = 1 : s$. Set $r := \begin{bmatrix} \Re(r_1)^T & \Im(r_1)^T & \dots & \Re(r_s)^T & \Im(r_s)^T \end{bmatrix}^T$, where $r_p := -\mathbf{P}(\lambda_p)x_p$ for $p = 1 : s$. If N^ϵ (defined as above) is a full row rank matrix, then there exists a minimizing H-even/H-odd matrix polynomial $\delta\mathbf{P} \in \mathbf{P}_l(\mathbb{C}^{n\times n})$ of the* *form* $\delta\mathbf{P}(\alpha,\beta) := \sum_{j=0}^{l} \alpha^{l-j}\beta^{j}\delta A_j$, *where* $\delta A_j = (\delta a_{j,tk}), j = 0:l$, *are given by*

$$\delta a_{j,tk} = \begin{cases} (\operatorname{sgn} a_{j,tt}) \displaystyle\sum_{p=1}^{s} \sqrt{\epsilon} w_j^{-2} \big(g_{pj}^{t} e^{T}_{t+\frac{1-\epsilon}{2}n(2p-1)+\frac{1+\epsilon}{2}2n(p-1)} + \epsilon h_{pj}^{t} \\ e^{T}_{t+\frac{1+\epsilon}{2}n(2p-1)+\frac{1-\epsilon}{2}2n(p-1)} \big) (N^{\epsilon}N^{\epsilon T})^{-1} r & \text{for } t = k \\ (\operatorname{sgn} a_{j,tk}) \displaystyle\sum_{p=1}^{s} \frac{1}{2} w_j^{-2} f_{pj,tk} (N^{\epsilon}N^{\epsilon T})^{-1} r & \text{for } t \neq k, \end{cases}$$

for even j, *and* $\delta A_j = (\delta a_j, tk), j = 0:l$, *are given by*

$$\delta a_{j,tk} = \begin{cases} (\operatorname{sgn} a_{j,tt}) \displaystyle\sum_{p=1}^{s} \sqrt{-\epsilon} w_j^{-2} \big(g_{pj}^{t} e^{T}_{t+\frac{1-\epsilon}{2}n(2p-1)+\frac{1+\epsilon}{2}2n(p-1)} - \epsilon h_{pj}^{t} \\ e^{T}_{t+\frac{1+\epsilon}{2}n(2p-1)+\frac{1-\epsilon}{2}2n(p-1)} \big) (N^{\epsilon}N^{\epsilon T})^{-1} r & \text{for } t = k \\ (\operatorname{sgn} a_{j,tk}) \displaystyle\sum_{p=1}^{s} \frac{1}{2} w_j^{-2} z_{pj,tk} (N^{\epsilon}N^{\epsilon T})^{-1} r & \text{for } t \neq k, \end{cases}$$

for odd j. *Here* $e_i \in \mathbb{C}^{2sn}$ *for every* $i \in \mathbb{N}$. *Then* $(\mathbf{P}(c_p, d_p) + \delta\mathbf{P}(c_p, d_p))x_p = 0$ *and the backward error is given by*

$$\eta^{\mathbf{S}}_{w,F}(\lambda_{1:s}, x_{1:s}, \mathbf{P}) = \|N^{\epsilon T}(N^{\epsilon}N^{\epsilon T})^{-1} r\|_F.$$

Proof. Proof is computational and follows similar to Theorem 5.4.1 and Theorem 5.5.1.

5.7. Backward error for T-palindromic/ T-anti-palindromic matrix polynomials

To understand the backward error analysis and perturbation theory of *palindromic* matrix polynomials, we need to understand the construction of matrix $M^{\epsilon}((c_p, d_p), j, y_p)$, which is obtained by $((c_p, d_p), y_p) \in \mathbb{C}^2 \setminus \{(0,0)\} \times \mathbb{C}^n$. For construction of $M^{\epsilon}((c_p, d_p), j, y_p)$, we need to understand the construction of matrices $M^{\epsilon}(y_p)$ for $\epsilon = 1, -1$, where $M^{1}(y_p) \in \mathbb{C}^{n \times n^2}$ and $M^{-1}(y_p) \in \mathbb{C}^{n \times n^2}$.

Remark 5.7.1. *Superscript "−1" in* $M^{-1}(y_p)$ *is only for notational point of view. It should not be mismatched with the inverse of* $M(y_p)$.

Throughout this section $w := (w_0, w_1, \ldots, w_l)^T \in \mathbb{R}^{l+1}$ be a nonzero and nonnegative vector such that $w_j = w_{l-j}$ for $j = 0 : l$. For deriving the *backward error* formulas of specified *eigenpairs*, we need the construction of the matrices $M^1(y_p), M^{-1}(y_p)$ and $M^\epsilon((c_p, d_p), j, y_p)$. We define

$$M^1(y_p) = \begin{bmatrix} M_1^1(y_p) & \ldots & M_n^1(y_p) \end{bmatrix}, \; M^{-1}(y_p) = \begin{bmatrix} M_1^{-1}(y_p) & \ldots & M_n^{-1}(y_p) \end{bmatrix}, \text{ and}$$

$$M^\epsilon((c_p, d_p), j, y_p) = \begin{bmatrix} c_p^{l-j} d_p^j M_1^1(y_p) + \epsilon c_p^j d_p^{l-j} M_1^{-1}(y_p) & \ldots & \ldots & c_p^{l-j} d_p^j M_n^1(y_p) + \epsilon c_p^j d_p^{l-j} M_n^{-1}(y_p) \end{bmatrix},$$

for $p = 1 : s$, where

$$M_1^1(y_p) = \begin{bmatrix} y_p^1 & y_p^2 & y_p^3 & \ldots & y_p^n \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}, M_2^1(y_p) = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \\ y_p^1 & y_p^2 & y_p^3 & \ldots & y_p^n \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}, \text{ and}$$

$$M_n^1(y_p) = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \\ y_p^1 & y_p^2 & y_p^3 & \ldots & y_p^n \end{bmatrix}, \text{ and}$$

$$M_z^{-1}(y_p) = \mathrm{diag}([y_p^z, \ldots, y_p^z]^T) \in \mathbb{C}^{n \times n}, z = 1, \ldots, n.$$

Suppose $A_j = (a_{j,tk}), \delta A_j = (\delta a_{j,tk}) \in \mathbb{C}^{n \times n}$. Define $\Delta_j := \begin{bmatrix} \Delta_{j1} \\ \Delta_{j2} \\ \vdots \\ \Delta_{jn} \end{bmatrix}$, where $\Delta_{ji} = \begin{bmatrix} w_j \delta a_{j,i1} \mathrm{sgn}\, a_{j,i1} \\ \vdots \\ w_j \delta a_{j,in} \mathrm{sgn}\, a_{j,1n} \end{bmatrix}$

and w_j is a nonnegative real number for $j = 0 : l$. Before stating the theorem define

$$M^\epsilon = \begin{bmatrix} M_{10}^\epsilon & M_{11}^\epsilon & \ldots & M_{1\tilde{l}}^\epsilon \\ M_{20}^\epsilon & M_{21}^\epsilon & \ldots & M_{2\tilde{l}}^\epsilon \\ \vdots & & & \vdots \\ M_{s0}^\epsilon & M_{s1}^\epsilon & \ldots & M_{s\tilde{l}}^\epsilon \end{bmatrix}, G^\epsilon = \begin{bmatrix} M_{10}^\epsilon & M_{11}^\epsilon & \ldots & M_{1\tilde{l}}^\epsilon & N_{1\frac{l}{2}}^\epsilon \\ M_{20}^\epsilon & M_{21}^\epsilon & \ldots & M_{2\tilde{l}}^\epsilon & N_{2\frac{l}{2}}^\epsilon \\ \vdots & & & \vdots & \vdots \\ M_{s0}^\epsilon & M_{s1}^\epsilon & \ldots & M_{s\tilde{l}}^\epsilon & N_{s\frac{l}{2}}^\epsilon \end{bmatrix}, \text{ where}$$

$$\tilde{l} = \begin{cases} \frac{l-1}{2}, & \text{when } l \text{ is odd} \\ \frac{l}{2} - 1, & \text{when } l \text{ is even} \end{cases}$$

$M^{\epsilon}_{pj} = (w_j^{-1} c_p^{l-j} d_p^j M^1(y_p) + \epsilon w_j^{-1} c_p^j d_p^{l-j} M^{-1}(y_p))\text{diag}(\text{vec}(\text{sgn}\, A_j))$ is defined as above and

$$N^{\epsilon}_{p\frac{l}{2}} = w_{\frac{l}{2}}^{-1} c_p^{\frac{l}{2}} d_p^{\frac{l}{2}} N^{\epsilon}(y_p)\, \text{diag}(\text{vec}(\text{sgn} A_{\frac{l}{2}} \circ C, \epsilon))$$

is defined in Section 3.3. M^{ϵ} is of size "$sn \times (\tilde{l}+1)n^2$" and G^{ϵ} is of size "$sn \times ((\tilde{l}+1)n^2 + (n^2+\epsilon n)/2)$".

Throughout this section, $\epsilon = 1$ represents the T-palindromic case and $\epsilon = -1$ represents the T-anti-palindromic case.

Remark 5.7.2. *For a palindromic matrix polynomial, backward error of an approximate eigenpair (μ, x), where $\mu \in \mathbb{C}, x \in \mathbb{C}^n$, is defined by Li et al. [47] in the following manner:*

$$\min\{\sqrt{\sum_{i=0}^{\lfloor l/2 \rfloor} w_i^2 \|\delta A_i\|_F^2} : (\mathbf{P}(\mu) + \delta\mathbf{P}(\mu))x = 0\}.$$

From now onwards, for palindromic structures we calculate the backward error of one or more specified eigenpairs with respect to the above definition, i.e., if $((c_i, d_i), x_i)$ for $i = 1 : s$ are s approximate eigenpairs of a palindromic matrix polynomial $\mathbf{P}$, then

$$\eta^{\mathbf{S}}_{w,F}(\lambda_{1:s}, x_{1:s}, \mathbf{P}) = \min\{\sqrt{\sum_{i=0}^{\lfloor l/2 \rfloor} w_i^2 \|\delta A_i\|_F^2} : (\mathbf{P}(c_i, d_i) + \delta\mathbf{P}(c_i, d_i))x_i = 0 \text{ for } i = 1 : s\}.$$

By using the above definition, we can easily compare ours and backward errors of Li et al. [47] . Since we are also providing the perturbed matrices together with the backward error formula, so one can also calculate the backward error according to definition 5.2.2.

The upper bound on the number of eigenpairs s for T-palindromic and T-anti-palindromic matrix polynomials are capped by Table 5.6

Structure	**upper bound on number of eigenpairs** "s"
T-palindromic	$s \leq (\frac{l}{2})n + \frac{n+1}{2}$, when l is even
T-palindromic	$s < (\frac{l+1}{2})n$, when l is odd
T-anti-palindromic	$s \leq (\frac{l}{2})n + \frac{n-1}{2}$, when l is even
T-anti-palindromic	$s < (\frac{l+1}{2})n$, when l is odd

TABLE 5.6. Upper bound on eigenpairs for T-palindromic and T-anti-palindromic polynomials

Now, we state and prove the theorem for T-*palindromic* and T-*anti-palindromic* matrix polynomials.

Theorem 5.7.3. *Let $\mathbf{P} \in \mathbf{P}_l(\mathbb{C}^{n\times n})$ be a T-palindromic/T-anti-palindromic homogeneous matrix polynomial of the form (5.1). Let $((c_p, d_p), x_p), p = 1 : s$ be s approximate eigenpairs of $\mathbf{P}$, where $0 \neq x_p \in \mathbb{C}^n$ and $0 \neq \lambda_p = (c_p, d_p)$. Set $r := \begin{bmatrix} r_1^T & r_2^T & \dots & r_s^T \end{bmatrix}^T$, where $r_p := -\mathbf{P}(\lambda_p)x_p$ for $p = 1 : s$. Then*

Case-1: *If l is odd, and M^ϵ (defined as above) is a full row rank matrix, then there exists a minimizing T-palindromic/T-anti-palindromic $\delta\mathbf{P} \in \mathbf{P}_l(\mathbb{C}^{n\times n})$ of the form $\delta\mathbf{P}(\alpha,\beta) := \sum_{j=0}^{l} \alpha^{l-j}\beta^i \delta A_j$, where $\delta A_j = (\delta a_{j,tk})$ for $j = 0 : 1$ are given by*

$$\delta a_{j,tk} = \begin{cases} \sum_{p=1}^{s} \mathrm{sgn}(a_{j,tk})(w_j^{-2}\overline{c_p^{l-j} d_p^j} \overline{x}_p^k e_{t+(p-1)n}^T + \\ \epsilon w_{l-j}^{-2}\overline{c_p^j d_p^{l-j}}\overline{x}_p^t e_{k+(p-1)n}^T)(M^\epsilon M^{\epsilon H})^{-1} r, & \text{for } t \neq k, \\ \sum_{p=1}^{s} \mathrm{sgn}(a_{j,tk})(w_j^{-2}\overline{c_p^{l-j} d_p^j}\overline{x}_p^t + \epsilon w_{l-j}^{-2}\overline{c_p^j d_p^{l-j}}\overline{x}_p^t) e_{t+(p-1)n}^T (M^\epsilon M^{\epsilon H})^{-1} r, & \text{for } t = k. \end{cases}$$

Here $e_{t+(p-1)n} \in \mathbb{C}^{sn}$, $e_{k+(p-1)n} \in \mathbb{C}^{sn}$. Then $(\mathbf{P}(c_p, d_p) + \delta\mathbf{P}(c_p, d_p))x_p = 0$ for $p = 1 : s$, and the backward error is given by

$$\eta_{w,F}^{\mathbf{S}}(\lambda_{1:s}, x_{1:s}, \mathbf{P}) = \|M^{\epsilon H}(M^\epsilon M^{\epsilon H})^{-1} r\|_F.$$

If M^ϵ is not a full rank matrix but $rank(M^\epsilon) = rank([M^\epsilon, r])$, then the backward error is given by

$$\eta_{w,F}^{\mathbf{S}}(\lambda_{1:s}, x_{1:s}, \mathbf{P}) = \|V^\epsilon D^{\epsilon +} U^{\epsilon H} r\|_F,$$

where U^ϵ, V^ϵ are unitary matrices and D^ϵ is a diagonal matrix with singular values of M^ϵ.

Case-2: *If l is even, and G^ϵ (defined as above) is a full row rank matrices, then there exists a minimizing T-palindromic/T-anti-palindromic $\delta\mathbf{P} \in \mathbf{P}_l(\mathbb{C}^{n\times n})$ of the form $\delta\mathbf{P}(\alpha,\beta) := \sum_{j=0}^{l} \alpha^{l-j}\beta^i \delta A_j$, where $\delta A_j = (\delta a_{j,tk})$ for $j = 0 : l$, $j \neq \frac{l}{2}$ are given by*

$$\delta a_{j,tk} = \begin{cases} \sum_{p=1}^{s} \mathrm{sgn}(a_{j,tk})(w_j^{-2}\overline{c_p^{l-j} d_p^j} \overline{x}_p^k e_{t+(p-1)n}^T + \\ \epsilon w_{l-j}^{-2}\overline{c_p^j d_p^{l-j}}\overline{x}_p^t e_{k+(p-1)n}^T)(G^\epsilon G^{\epsilon H})^{-1} r, & \text{for } t \neq k, \\ \sum_{p=1}^{s} \mathrm{sgn}(a_{j,tk})(w_j^{-2}\overline{c_p^{l-j} d_p^j}\overline{x}_p^t + \epsilon w_{l-j}^{-2}\overline{c_p^j d_p^{l-j}}\overline{x}_p^t) e_{t+(p-1)n}^T (G^\epsilon G^{\epsilon H})^{-1} r, & \text{for } t = k, \end{cases}$$

and $\delta A_{\frac{l}{2}} = (\delta a_{\frac{l}{2},tk})$ is given by

$$\delta a_{\frac{l}{2},tk} = \begin{cases} \sum_{p=1}^{s} \frac{1}{2} w_{\frac{l}{2}}^{-2}\overline{c_p^{\frac{l}{2}} d_p^{\frac{l}{2}}}(\mathrm{sgn}\, a_{\frac{l}{2},tk})(\overline{x}_p^k e_{t+(p-1)n}^T + \epsilon \overline{x}_p^t e_{k+(p-1)n}^T)(G^\epsilon G^{\epsilon H})^{-1} r & \text{for} \quad t \neq k \\ \sum_{p=1}^{s} \frac{1+\epsilon}{2} w_{\frac{l}{2}}^{-2}\overline{c_p^{\frac{l}{2}} d_p^j}(\mathrm{sgn}\, a_{\frac{l}{2},tk})\overline{x}_p^k e_{t+(p-1)n}^T (G^\epsilon G^{\epsilon H})^{-1} r & \text{for} \quad t = k. \end{cases}$$

Here $e_{t+(p-1)n} \in \mathbb{C}^{sn}$, $e_{k+(p-1)n} \in \mathbb{C}^{sn}$. *Then* $(\mathbf{P}(c_p, d_p) + \delta\mathbf{P}(c_p, d_p))x_p = 0$ *for* $p = 1 : s$, *and the backward error is given by*

$$\eta^{\mathbf{S}}_{w,F}(\lambda_{1:s}, x_{1:s}, \mathbf{P}) = \|G^{\epsilon H}(G^{\epsilon}G^{\epsilon H})^{-1}r\|_F.$$

Proof. **Case-1:** When $\mathbf{P} \in \mathbf{P}_l(\mathbb{C}^{n\times n})$ is a *T-palindromic/T-anti-palindromic* matrix polynomial of the form (5.1) and l is an odd natural number.

Corresponding to a given *T-palindromic/anti-palindromic* $\mathbf{P}$, its given approximate *eigenpairs* are $((c_p, d_p), x_p)$ for $p = 1 : s$. We need to construct structured $\delta\mathbf{P}$ such that $(\mathbf{P}(c_p, d_p) + \delta\mathbf{P}(c_p, d_p))x_p = 0$, which also preserve sparsity. By assumption $\mathbf{P}(c_p, d_p)x_p + r_p = 0$ for $p = 1 : s$. Then $r_p = \delta\mathbf{P}(c_p, d_p)x_p = \sum_{j=0}^{l} c_p^{l-j} d_p^j \delta A_j x_p = \sum_{j=0}^{\tilde{l}} (w_j w_j^{-1} c_p^{l-j} d_p^j \delta A_j + w_{l-j} w_{l-j}^{-1} c_p^j d_p^{l-j} \delta A_{l-j}) x_p$. Since $\delta A_{l-j} = \epsilon \delta A_j^T$, and $w_j = w_{l-j}$, we get $r_p = \sum_{j=0}^{\tilde{l}} (w_j w_j^{-1} c_p^{l-j} d_p^j \delta A_j + \epsilon w_j w_j^{-1} c_p^j d_p^{l-j} \delta A_j^T) x_p$. Further, for maintaining the sparsity, we get

$$r_p = \sum_{j=0}^{\tilde{l}} (w_j w_j^{-1} c_p^{l-j} d_p^j \delta A_j \circ \operatorname{sgn} A_j + \epsilon w_j w_j^{-1} c_p^j d_p^{l-j} \delta A_j^T \circ \operatorname{sgn} A_j^T) x_p.$$

Let $\Delta_j = w_j \mathrm{vec}(\delta A_j \circ \operatorname{sgn} A_j)$. Then $r_p = \sum_{j=0}^{\tilde{l}} (w_j^{-1} c_p^{l-j} d_p^j M^1(x_p) + \epsilon w_j^{-1} c_p^j d_p^{l-j} M^{-1}(x_p))$ $\mathrm{diag}(\mathrm{vec}(\operatorname{sgn} A_j))\Delta_j = \sum_{j=0}^{\tilde{l}} w_j^{-1} M^{\epsilon}((c_p, d_p), j, x_p)\mathrm{diag}(\mathrm{vec}(\operatorname{sgn} A_j))\Delta_j = \sum_{j=0}^{\tilde{l}} M^{\epsilon}_{pj}\Delta_j = M^{\epsilon}_p \Delta$, where

$$M^{\epsilon}_p = \begin{bmatrix} M^{\epsilon}_{p0} & M^{\epsilon}_{p1} & \dots & M^{\epsilon}_{p\frac{l-1}{2}} \end{bmatrix}, \Delta = \begin{bmatrix} \Delta_0^T & \Delta_1^T & \dots & \Delta_{\tilde{l}}^T \end{bmatrix}^T, \text{ and}$$
$$M^{\epsilon}_{pj} = (w_j^{-1} c_p^{l-j} d_p^j M^1(x_p) + \epsilon w_j^{-1} c_p^j d_p^{l-j} M^{-1}(x_p))\mathrm{diag}(\mathrm{vec}(\operatorname{sgn} A_j)).$$

Finally, we get $r_p = M^{\epsilon}_p \Delta$ for $p = 1 : s$. Writing s equations in combined form we get

$$r = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_s \end{bmatrix} = \begin{bmatrix} M^{\epsilon}_{10} & M^{\epsilon}_{11} & \dots & M^{\epsilon}_{1\tilde{l}} \\ M^{\epsilon}_{20} & M^{\epsilon}_{21} & \dots & M^{\epsilon}_{2\tilde{l}} \\ \vdots & & & \vdots \\ M^{\epsilon}_{s0} & M^{\epsilon}_{s1} & \dots & M^{\epsilon}_{s\tilde{l}} \end{bmatrix} \begin{bmatrix} \Delta_0 \\ \Delta_1 \\ \vdots \\ \Delta_{\frac{l-1}{2}} \end{bmatrix} = \begin{bmatrix} M^{\epsilon}_1 \\ M^{\epsilon}_2 \\ \vdots \\ M^{\epsilon}_s \end{bmatrix} \Delta = M^{\epsilon}\Delta. \tag{5.7}$$

If M^{ϵ} is a full row rank matrix, then from (5.7), minimal norm solution of $r = M^{\epsilon}\Delta$ is given by $\Delta = M^{\epsilon H}(M^{\epsilon}M^{\epsilon H})^{-1}r$. Now using equation $\Delta = M^{\epsilon H}(M^{\epsilon}M^{\epsilon H})^{-1}r$ and expanding the first $M^{\epsilon H}$, we get the desired entry-wise perturbations. If M^{ϵ} has not full rank but system $M^{\epsilon}\Delta = r$ is consistent, then $\Delta = V^{\epsilon}D^{\epsilon +}U^{\epsilon H}r$. Here $U^{\epsilon}, V^{\epsilon}$ are unitary matrices of appropriate sizes and $D^{\epsilon +}$ is pseudoinverse of D^{ϵ}. Backward error in *Frobenius* norm case is given by $\eta^{\mathbf{S}}_{w,F}(\lambda_{1:s}, x_{1:s}, \mathbf{P}) = \|\delta\mathbf{P}\|_{w,F}$, where $\|\delta\mathbf{P}\|^2_{w,F} = \sum_{i=0}^{\lfloor l/2 \rfloor} w_i^2 \|\delta A_i\|_F^2$.

But $\sum_{i=0}^{\lfloor l/2 \rfloor} w_i^2 \|\delta A_i\|_F^2 = \|\Delta\|_F^2 = \|M^{\epsilon H}(M^\epsilon M^{\epsilon H})^{-1} r\|_F^2$. Hence backward error when M^ϵ is full rank is given by

$$\eta_{w,F}^{\mathbf{S}}(\lambda_{1:s}, x_{1:s}, \mathbf{P}) = \|M^{\epsilon H}(M^\epsilon M^{\epsilon H})^{-1} r\|_F.$$

When M^ϵ is not a full rank matrix but system is consistent, then the backward error is $\eta_{w,F}^{\mathbf{S}}(\lambda_{1:s}, x_{1:s}, \mathbf{P}) = \|V^\epsilon D^{\epsilon+} U^{\epsilon H} r\|$.

Case-2: When $\mathbf{P} \in \mathbf{P}_l(\mathbb{C}^{n\times n})$ is a *T-palindromic/T-anti-palindromic* matrix polynomial of the form (5.1) such that l is even natural number. To construct $\delta\mathbf{P}$ such that $(\mathbf{P}(c_p,d_p) + \delta\mathbf{P}(c_p,d_p))x_p = 0$ for $p = 1:s$, by following the process of case-1, we get $r_p = \delta\mathbf{P}(c_p,d_p)x_p = \sum_{j=0}^{l} c_p^{l-j} d_p^j \delta A_j x_p = \sum_{j=0}^{\tilde{l}} (w_j w_j^{-1} c_p^{l-j} d_p^j \delta A_j + w_{l-j} w_{l-j}^{-1} c_p^j d_p^{l-j} \delta A_{l-j}) x_p + w_{\frac{l}{2}} w_{\frac{l}{2}}^{-1} c_p^{\frac{l}{2}} d_p^{\frac{l}{2}} \delta A_{\frac{l}{2}} x_p$. Since $\delta A_{l-j} = \epsilon \delta A_j^T$, and $w_j = w_{l-j}$, we get $r_p = \sum_{j=0}^{\tilde{l}} (w_j w_j^{-1} c_p^{l-j} d_p^j \delta A_j \circ \operatorname{sgn} A_j + \epsilon w_j w_j^{-1} c_p^j d_p^{l-j} \delta A_j^T \circ \operatorname{sgn} A_j^T) x_p + w_{\frac{l}{2}} w_{\frac{l}{2}}^{-1} c_p^{\frac{l}{2}} d_p^{\frac{l}{2}} (\delta A_{\frac{l}{2}} \circ \operatorname{sgn} A_{\frac{l}{2}} \circ C \circ D) x_p$ where C, D are defined by 3.4.

Let $\Delta_j = w_j \operatorname{vec}(\delta A_j \circ \operatorname{sgn} A_j)$, and let $\Delta_{\frac{l}{2}}^\epsilon = w_{\frac{l}{2}} \operatorname{vec}(\delta A_{\frac{l}{2}} \circ \operatorname{sgn} A_{\frac{l}{2}} \circ D, \epsilon)$ is defined at Section 3.2.1. Then $r_p = \sum_{j=0}^{\tilde{l}} (w_j^{-1} c_p^{l-j} d_p^j M^1(x_p) + \epsilon w_j^{-1} c_p^j d_p^{l-j} M^{-1}(x_p)) \operatorname{diag}(\operatorname{vec}(\operatorname{sgn} A_j)) \Delta_j + (w_{\frac{l}{2}}^{-1} c_p^{\frac{l}{2}} d_p^{\frac{l}{2}} N^\epsilon(x_p) \operatorname{diag}(\operatorname{vec}(\operatorname{sgn} A_{\frac{l}{2}} \circ C, \epsilon)) \Delta_{\frac{l}{2}}^\epsilon$, where $N^\epsilon(x_p)$ is defined at Subsection 3.2.1. Further simplification gives

$r_p = \sum_{j=0}^{\tilde{l}} w_j^{-1} M^\epsilon((c_p,d_p), j, x_p) \operatorname{diag}(\operatorname{vec}(\operatorname{sgn} A_j)) \Delta_j + N_{p\frac{l}{2}}^\epsilon \Delta_{\frac{l}{2}}^\epsilon$, where $M^\epsilon((c_p,d_p), j, x_p)$ is defined at the beginning of this section and $N_{p\frac{l}{2}}^\epsilon = w_{\frac{l}{2}}^{-1} c_p^{\frac{l}{2}} d_p^{\frac{l}{2}} N^\epsilon(x_p) \operatorname{diag}(\operatorname{vec}(\operatorname{sgn} A_{\frac{l}{2}} \circ C, \epsilon))$. Similar to Case-1, we get $r_p = \sum_{j=0}^{\tilde{l}} M_{pj}^\epsilon \Delta_j + N_{p\frac{l}{2}}^\epsilon \Delta_{\frac{l}{2}}^\epsilon = G_p^\epsilon \Delta^\epsilon$, where $G_p^\epsilon = \begin{bmatrix} M_{p0}^\epsilon & M_{p1}^\epsilon & \dots & M_{p\tilde{l}}^\epsilon & N_{p\frac{l}{2}}^\epsilon \end{bmatrix}, \Delta = \begin{bmatrix} \Delta_0^T & \Delta_1^T & \dots & \Delta_{\tilde{l}}^T & \Delta_{\frac{l}{2}}^T \end{bmatrix}^T$. Finally, we get $r_p = G_p^\epsilon \Delta$ for $p = 1:s$. Writing s equations in the combined form, we get

$$r = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_s \end{bmatrix} = \begin{bmatrix} M_{10}^\epsilon & M_{11}^\epsilon & \dots & M_{1\tilde{l}}^\epsilon & N_{1\frac{l}{2}}^\epsilon \\ M_{20}^\epsilon & M_{21}^\epsilon & \dots & M_{2\tilde{l}}^\epsilon & N_{2\frac{l}{2}}^\epsilon \\ \vdots & \vdots & & \vdots & \vdots \\ M_{s0}^\epsilon & M_{s1}^\epsilon & \dots & M_{s\tilde{l}}^\epsilon & N_{s\frac{l}{2}}^\epsilon \end{bmatrix} \begin{bmatrix} \Delta_0 \\ \Delta_1 \\ \vdots \\ \Delta_{\tilde{l}} \\ \Delta_{\frac{l}{2}}^\epsilon \end{bmatrix} = \begin{bmatrix} G_1^\epsilon \\ G_2^\epsilon \\ \vdots \\ G_s^\epsilon \end{bmatrix} \Delta^\epsilon = G^\epsilon \Delta^\epsilon. \tag{5.8}$$

Similar to the previous case, using system $r = G^\epsilon \Delta^\epsilon$, we can obtain the desired entry wise perturbation and backward error formula.

Remark 5.7.4. *For Case-1, when l is an odd natural number, we get the even number of coefficient matrices δA_j in the perturbed matrix polynomial $\delta\mathbf{P}$, which can be paired with* *the property $\delta A_j = \epsilon \delta A_{l-j}$ for $j = 0 : l$, and we get the backward error results in this case. But when l is an even natural number, then all the coefficient matrices can be paired similar to Case-1 except $\delta A_{\frac{l}{2}}$ which satisfied $\delta A_{\frac{l}{2}} = \epsilon \delta A^T_{\frac{l}{2}}$. Now to tackle this coefficient matrix $\delta A_{\frac{l}{2}}$, and to obtain the backward error formula, we use the construction of Section 3.2.1 as described in Case-2.*

Remark 5.7.5. *If* $\operatorname{rank}(M^\epsilon) \neq \operatorname{rank}([M^\epsilon, r])$, *then the backward error* $\eta^{\mathbf{S}}_{w,F}(\lambda_{1:s}, x_{1:s}, \mathbf{P}) = \infty$.

Remark 5.7.6. *If* $\operatorname{rank}(G^\epsilon) \neq \operatorname{rank}([G^\epsilon, r])$, *then the backward error* $\eta^{\mathbf{S}}_{w,F}(\lambda_{1:s}, x_{1:s}, \mathbf{P}) = \infty$.

5.8. Backward error analysis for H-palindromic/H-anti-palindromic matrix polynomials

In this section, we state the theorem and related important terminologies for constructing the backward error formulas for H-palindromic/H-anti-palindromic matrix polynomials. Before stating the theorem, let $x_p \in \mathbb{C}^n$, $\lambda_p = (c_p, d_p) \in \mathbb{C}^2 \setminus \{(0,0)\}$ and $e_i \in \mathbb{C}^{2sn}$ for any $i \in \mathbb{N}$ and $s \leq n$. Define $g_{pj} := \Re(c_p^{l-j} d_p^j x_p)$, $h_{pj} := \Im(c_p^{l-j} d_p^j x_p)$, $g^t_{pj} = \Re(c_p^{l-j} d_p^j x^t_p)$, $h^t_{pj} = \Im(c_p^{l-j} d_p^j x^t_p)$ for $p = 1 : s$, $j = 0 : l$, and $t = 1 : n$. Define

$$M^\epsilon(g_{pj}) = \begin{bmatrix} M^1_1(g_{pj}) + \epsilon M^{-1}_1(g_{p(l-j)}) & \dots & M^1_n(g_{pj}) + \epsilon M^{-1}_n(g_{p(l-j)}) \end{bmatrix},$$

$$M^\epsilon(h_{pj}) = \begin{bmatrix} M^1_1(h_{pj}) + \epsilon M^{-1}_1(h_{p(l-j)}) & \dots & M^1_n(h_{pj}) + \epsilon M^{-1}_n(h_{p(l-j)}) \end{bmatrix},$$

$$M^\epsilon = \begin{bmatrix} M^\epsilon_{10} & M^\epsilon_{11} & \dots & M^\epsilon_{1\tilde{l}} \\ M^\epsilon_{20} & M^\epsilon_{21} & \dots & M^\epsilon_{2\tilde{l}} \\ \vdots & & & \vdots \\ M^\epsilon_{s0} & M^\epsilon_{s1} & \dots & M^\epsilon_{s\tilde{l}} \end{bmatrix}, \; G^\epsilon = \begin{bmatrix} M^\epsilon_{10} & M^\epsilon_{11} & \dots & M^\epsilon_{1\tilde{l}} & N^\epsilon_{1\frac{l}{2}} \\ M^\epsilon_{20} & M^\epsilon_{21} & \dots & M^\epsilon_{2\tilde{l}} & N^\epsilon_{2\frac{l}{2}} \\ \vdots & & & \vdots & \vdots \\ M^\epsilon_{s0} & M^\epsilon_{s1} & \dots & M^\epsilon_{s\tilde{l}} & N^\epsilon_{s\frac{l}{2}} \end{bmatrix}, \text{ where}$$

$$\tilde{l} = \left\{ \begin{array}{ll} \frac{l-1}{2}, & \text{when } l \text{ is odd} \\ \frac{l}{2} - 1, & \text{when } l \text{ is even} \end{array} \right\}, M^\epsilon_{pj} = w_j^{-1} \begin{bmatrix} M^\epsilon(g_{pj}) & -M^{-\epsilon}(h_{pj}) \\ M^\epsilon(h_{pj}) & M^{-\epsilon}(g_{pj}) \end{bmatrix} \operatorname{diag}\left(\begin{bmatrix} \operatorname{vec}(\operatorname{sgn} A_j) \\ \operatorname{vec}(\operatorname{sgn} A_j) \end{bmatrix} \right),$$

and $N^\epsilon_{p\frac{l}{2}} = w^{-1}_{\frac{l}{2}} \begin{bmatrix} N^\epsilon(g_{p\frac{l}{2}}) & -N^{-\epsilon}(h_{p\frac{l}{2}}) \\ N^\epsilon(h_{p\frac{l}{2}}) & N^{-\epsilon}(g_{p\frac{l}{2}}) \end{bmatrix} \operatorname{diag}\left(\begin{bmatrix} \operatorname{vec}(\operatorname{sgn}(A_{\frac{l}{2}}) \circ C, \epsilon) \\ \operatorname{vec}(\operatorname{sgn}(A_{\frac{l}{2}}) \circ C, -\epsilon) \end{bmatrix} \right)$ is defined in Section 3.4. Further M^ϵ is of size "$2sn \times 2(\tilde{l}+1)n^2$" and G^ϵ is of size "$2sn \times (2(\tilde{l}+1)n^2 + n^2)$"

for *H-palindromic/anti-palindromic* cases. Throughout this section, $\epsilon = 1$ represents the H-even case and $\epsilon = -1$ represents the H-palindromic case. The upper bound on the number of eigenpairs s for H-anti-palindromic and H-anti-palindromic matrix polynomials are capped by Table 5.7.

Structure	**upper bound on number of eigenpairs** "s"
H-palindromic	$s \leq (\frac{l}{2})n + \frac{n}{2}$, when l is even
H-palindromic	$s \leq (\frac{l+1}{2})n$, when l is odd
H-anti-palindromic	$s \leq (\frac{l}{2})n + \frac{n}{2}$, when l is even
H-anti-palindromic	$s \leq (\frac{l+1}{2})n$, when l is odd

TABLE 5.7. Upper bound on eigenpairs for H-palindromic and T-anti-palindromic matrix polynomials

Next, we prove the main result of this section by the following theorem.

Theorem 5.8.1. *Let $\mathbf{P} \in \mathbf{P}_l(\mathbb{C}^{n\times n})$ be a H-palindromic/H-anti-palindromic matrix polynomial of the form (5.1). Let $((c_p, d_p), x_p)$ be s-approximate eigenpairs of $\mathbf{P}$, where $0 \neq x_p \in \mathbb{C}^n$ and $0 \neq \lambda_p = (c_p, d_p)$ for $p = 1 : s$. Set $r := \begin{bmatrix} \Re(r_1)^T & \Im(r_1)^T & \dots & \Re(r_s)^T & \Im(r_s)^T \end{bmatrix}^T$, where $r_p := -\mathbf{P}(\lambda_p)x_p$ for $p = 1 : s$. Then*

Case-1 : *If l is odd, and M^ϵ (defined as above) is a full row rank matrices, then there exists a minimizing H-palindromic/H-anti-palindromic $\delta\mathbf{P} \in \mathbf{P}_l(\mathbb{C}^{n\times n})$ of the form $\delta\mathbf{P}(\alpha, \beta) := \sum_{j=0}^{l} \alpha^{l-j}\beta^i \delta A_j$, where $\delta A_j = (\delta a_{j,tk})$ for $j = 0 : l$, $j \neq \frac{l}{2}$ are given by*

$$\delta a_{j,tk} = \begin{cases} \sum_{p=1}^{s}(\operatorname{sgn} a_{j,tk})[w_j^{-2} g_{pj}^k e_{t+2(p-1)n}^T + \epsilon w_{l-j}^{-2} g_{p(l-j)}^t e_{k+2(p-1)n}^T + \\ w_j^{-2} h_{pj}^k e_{t+(2p-1)n}^T + \epsilon w_{l-j}^{-2} h_{p(l-j)}^t e_{k+(2p-1)n}^T + \mathrm{i}(-w_j^{-2} h_{pj}^k e_{t+2(p-1)n}^T + \\ \epsilon w_{l-j}^{-2} h_{p(l-j)}^t e_{k+2(p-1)n}^T) + \\ \mathrm{i}(w_j^{-2} g_{pj}^k e_{t+(2p-1)n}^T - \epsilon w_{l-j}^{-2} g_{p(l-j)}^t e_{k+(2p-1)n}^T)](M^\epsilon M^{\epsilon T})^{-1} r, & \text{for } t \neq k, \\ \sum_{p=1}^{s}(\operatorname{sgn} a_{j,tk})[(w_j^{-2} g_{pj}^t + \epsilon w_{l-j}^{-2} g_{p(l-j)}^t) e_{t+2(p-1)n}^T + (w_j^{-2} h_{pj}^t + \\ \epsilon w_{l-j}^{-2} h_{p(l-j)}^t) e_{t+(2p-1)n}^T + \mathrm{i}(-w_j^{-2} h_{pj}^t + \epsilon w_{l-j}^{-2} h_{p(l-j)}^t) e_{t+2(p-1)n}^T + \\ \mathrm{i}(w_j^{-2} g_{pj}^t - \epsilon w_{l-j}^{-2} g_{p(l-j)}^t) e_{t+(2p-1)n}^T](M^\epsilon M^{\epsilon T})^{-1} r, & \text{for } t = k. \end{cases}$$

Here $e_{t+(p-1)n} \in \mathbb{C}^{2sn}$, $e_{k+(p-1)n} \in \mathbb{C}^{sn}$. Then $(\mathbf{P}(c_p, d_p) + \delta\mathbf{P}(c_p, d_p))x_p = 0$ for $p = 1 : s$, and the backward error is given by

$$\eta_{w,F}^{\mathbf{S}}(\lambda_{1:s}, x_{1:s}, \mathbf{P}) = \|M^{\epsilon T}(M^\epsilon M^{\epsilon T})^{-1} r\|_F.$$

If M^ϵ is not full rank matrix but $rank(M^\epsilon) = rank([M^\epsilon, r])$, then the backward error is given by

$$\eta^{\mathbf{S}}_{w,F}(\lambda_{1:s}, x_{1:s}, \mathbf{P}) = \|V^\epsilon D^{\epsilon+} U^{\epsilon H} r\|_F,$$

where U^ϵ, V^ϵ are unitary matrices and D^ϵ is a diagonal matrix with singular values of M^ϵ.

Case-2 : *If l is even, and G^ϵ (defined as above) is a full row rank matrices, then there exists a minimizing H-palindromic/H-anti-palindromic $\delta\mathbf{P} \in \mathbf{P}_l(\mathbb{C}^{n\times n})$ of the form $\delta\mathbf{P}(\alpha,\beta) := \sum_{j=0}^{l} \alpha^{l-j}\beta^i \delta A_j$, where $\delta A_j = (\delta a_{j,tk})$ for $j = 0 : l$, $j \neq \frac{l}{2}$ are given by*

$$\delta a_{j,tk} = \begin{cases} \sum_{p=1}^{s} (\operatorname{sgn} a_{j,tk})[w_j^{-2} g_{pj}^k e^T_{t+2(p-1)n} + \epsilon w_{l-j}^{-2} g^t_{p(l-j)} e^T_{k+2(p-1)n} + \\ w_j^{-2} h^k_{pj} e^T_{t+(2p-1)n} + \epsilon w_{l-j}^{-2} h^t_{p(l-j)} e^T_{k+(2p-1)n} + \\ \mathrm{i}(-w_j^{-2} h^k_{pj} e^T_{t+2(p-1)n} + \epsilon w_{l-j}^{-2} h^t_{p(l-j)} e^T_{k+2(p-1)n}) + \\ \mathrm{i}(w_j^{-2} g^k_{pj} e^T_{t+(2p-1)n} - \epsilon w_{l-j}^{-2} g^t_{p(l-j)} e^T_{k+(2p-1)n})](G^\epsilon G^{\epsilon T})^{-1} r, & \text{for } t \neq k, \\ \sum_{p=1}^{s} (\operatorname{sgn} a_{j,tk})[(w_j^{-2} g^t_{pj} + \epsilon w_{l-j}^{-2} g^t_{p(l-j)}) e^T_{t+2(p-1)n} + \\ (w_j^{-2} h^t_{pj} + \epsilon w_{l-j}^{-2} h^t_{p(l-j)}) e^T_{t+(2p-1)n} + \\ \mathrm{i}(-w_j^{-2} h^t_{pj} + \epsilon w_{l-j}^{-2} h^t_{p(l-j)}) e^T_{t+2(p-1)n} + \\ \mathrm{i}(w_j^{-2} g^t_{pj} - \epsilon w_{l-j}^{-2} g^t_{p(l-j)}) e^T_{t+(2p-1)n}](G^\epsilon G^{\epsilon T})^{-1} r, & \text{for } t = k, \end{cases}$$

and $\delta A_{\frac{l}{2}} = (\delta a_{\frac{l}{2},tk})$ is given by

$$\delta a_{\frac{l}{2},tk} = \begin{cases} (\operatorname{sgn} a_{\frac{l}{2},tt}) \sum_{p=1}^{s} \sqrt{\epsilon} w_{\frac{l}{2}}^{-2} \big(g^t_{p\frac{l}{2}} e^T_{t+\frac{1-\epsilon}{2}n(2p-1)+\frac{1+\epsilon}{2}2n(p-1)} + \epsilon h^t_{p\frac{l}{2}} \\ e^T_{t+\frac{1+\epsilon}{2}n(2p-1)+\frac{1-\epsilon}{2}2n(p-1)}\big)(G^\epsilon G^{\epsilon T})^{-1} r, & \text{for } t = k, \\ (\operatorname{sgn} a_{\frac{l}{2},tk}) \sum_{p=1}^{s} \frac{1}{2} w_j^{-2} c_{pj,tk} (G^\epsilon G^{\epsilon T})^{-1} r, & \text{for } t \neq k. \end{cases}$$

Here

$$c_{p\frac{l}{2},tk} = (e_{t+2n(p-1)} + \mathrm{i}e_{t+n+2n(p-1)})^T (g^k_{p\frac{l}{2}} - ih^k_{p\frac{l}{2}}) + (e_{k+2n(p-1)} - \mathrm{i}e_{k+n+2n(p-1)})^T (\epsilon g^t_{p\frac{l}{2}} + \mathrm{i}\epsilon h^t_{p\frac{l}{2}}),$$

for $p = 1 : s$, and $t, k = 1 : n$. Then $(\mathbf{P}(c_p, d_p) + \delta\mathbf{P}(c_p, d_p))x_p = 0$ for $p = 1 : s$, and the backward error is given by

$$\eta^{\mathbf{S}}_{w,F}(\lambda_{1:s}, x_{1:s}, \mathbf{P}) = \|G^{\epsilon T}(G^\epsilon G^{\epsilon T})^{-1} r\|_F.$$

If G^ϵ is a not full rank matrix but $\mathrm{rank}(G^\epsilon) = rank([G^\epsilon, r])$, then the backward error is given by

$$\eta^{\mathbf{S}}_{w,F}(\lambda_{1:s}, x_{1:s}, \mathbf{P}) = \|V^\epsilon D^{\epsilon+} U^{\epsilon H} r\|_F,$$

where U^ϵ, V^ϵ are unitary matrices and D^ϵ is a diagonal matrix with singular values of G^ϵ. For H-palindromic case we take $\epsilon = 1$ and for H-anti palindromic case $\epsilon = -1$.

Proof. Proof is computational and follows from Theorem 5.7.3, and Theorem 3.7.5.

Next, we obtain the backward error formula of one or more approximate eigenpairs for unstructured matrix polynomials.

5.9. Backward error analysis for unstructured matrix polynomials

In this section, we are interested in finding the backward error of approximate *eigenpairs* for unstructured matrix polynomials. To obtain the unstructured backward error, we ignore any kind of structure while doing the analysis. Before going to the main result of this section, we construct matrices K by using the given approximate *eigenpairs* $((c_p, d_p), x_p) \in \mathbb{C}^2 \setminus \{(0,0)\} \times \mathbb{C}^n, p = 1 : s$, of a matrix polynomial $\mathbf{P}$ of the form (5.1). Let

$$K = \begin{bmatrix} K_{10} & \dots & K_{1l} \\ K_{20} & \dots & K_{2l} \\ \vdots & \dots & \vdots \\ K_{s0} & \dots & K_{sl} \end{bmatrix} \in \mathbb{C}^{sn \times (l+1)n^2}, \text{ where } K_{pj} = w_j^{-1} c_p^{l-j} d_p^j K(x_p) \mathrm{diag}(\mathrm{vec}(\mathrm{sgn}\, A_j)),$$

with

$$K(x_p) = \begin{bmatrix} x_p^1 & \dots & x_p^n & \dots & \dots & \dots & \dots & 0 & \dots & 0 \\ 0 & \dots & 0 & x_p^1 & \dots & x_p^n & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & 0 & \dots & 0 & \dots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 & \dots & x_p^1 & \dots & x_p^n \end{bmatrix}$$

for $p = 1 : s$ and $j = 0 : l$.

Now, we state the main theorem of this section as follows:

Theorem 5.9.1. *Let $\mathbf{P} \in \mathbf{P}_l(\mathbb{C}^{n\times n})$ be a homogeneous matrix polynomial of the form (5.1). Let $((c_p, d_p), x_p)$ be s $(s \leq nl)$ approximate eigenpairs of $\mathbf{P}$ with $0 \neq x_p \in \mathbb{C}^n$ and $0 \neq \lambda_p = (c_p, d_p)$. Set $r := \begin{bmatrix} r_1^T & r_2^T & \dots & r_s^T \end{bmatrix}^T$ where $r_p := -\mathbf{P}(\lambda_p)x_p$ for $p = 1 : s$. If K (defined as above) is a full row rank matrix, then there exists a minimizing $\delta\mathbf{P} \in \mathbf{P}_l(\mathbb{C}^{n\times n})$*

of the form $\delta\mathbf{P}(\alpha,\beta) := \sum_{j=0}^{l} \alpha^{l-j}\beta^{j}\delta A_j$, *where* $\delta A_j = (\delta a_{j,tk})$ *for* $j = 0 : l$ *are given by*

$$\delta a_{j,tk} = \sum_{p=1}^{s} w_j^{-2}\overline{c_p^{l-j}d_p^{j}}(\operatorname{sgn} a_{j,tk})(\overline{x}_p^{k} e_{t+(p-1)n}^{T})(KK^{H})^{-1}r,$$

and $e_{t+(p-1)n} \in \mathbb{R}^{sn}$. *The backward error is given by*

$$(\eta_{w,F}^{\mathbf{S}}(\lambda_{1:s}, x_{1:s}, \mathbf{P}))^2 = \|K^T(KK^T)^{-1}r\|_F.$$

If K *is not a full rank matrix but* $rank(K) = rank([K, r])$, *then the backward error is given by*

$$(\eta_{w,F}^{\mathbf{S}}(\lambda_{1:s}, x_{1:s}, \mathbf{P}))^2 = \|V^{\epsilon}D^{\epsilon+}U^{\epsilon H}r\|_F,$$

where $U^{\epsilon}, V^{\epsilon}$ *are unitary matrices and* D^{ϵ} *is a diagonal matrix with singular values of* K.

Proof. Corresponding to a given matrix polynomial $\mathbf{P}$, its given approximate eigenpairs are (λ_p, x_p), $p = 1 : s$. We need to construct unstructured $\delta\mathbf{P}$ which has sparsity such that $(\mathbf{P}(\lambda_p) + \delta\mathbf{P}(\lambda_p))x_p = 0$ for $p = 1 : s$. By assumption $\mathbf{P}(\lambda_p)x_p + r_p = 0$ for $p = 1 : s$. Then $r_p = \delta\mathbf{P}(\lambda_p)x_p = \sum_{j=0}^{l} c_p^{l-j}d_p^{j}\delta A_i x_p$, for maintaining sparsity replace δA_j by $(\delta A_j \circ \operatorname{sgn} A_j)$, hence, we get

$$r_p = \sum_{j=0}^{l} c^{l-j}d^{j}(\delta A_j \circ \operatorname{sgn} A_j)x_p.$$

Let $\Delta_j = w_j \operatorname{vec}(\delta V_j \circ \operatorname{sgn} A_j)$ for $j = 0 : l$. Then similar to Theorem 5.3.1, we get

$$r_p = \sum_{j=0}^{l} w_j^{-1}c^{l-j}d_p^{j}K(x_p)\operatorname{diag}([\operatorname{vec}(\operatorname{sgn}(A_j))])\Delta_j^{\epsilon} = \sum_{j=0}^{l} K_{pj}\Delta_j = K_p\Delta, \tag{5.9}$$

where $K_p = \begin{bmatrix} K_{p0} & \dots & K_{pl} \end{bmatrix}$, $\Delta = \begin{bmatrix} \Delta_0{}^T & \dots & \Delta_l{}^T \end{bmatrix}^T$ for $p = 1 : s$. Rest of the proof follows similar to Theorem 5.3.1.

Further, in this chapter, we are also interested in solving the real symmetric quadratic inverse eigenvalue problem. This problem asks to construct a matrix polynomial with real symmetric coefficient matrices from a given set of approximate eigenpairs. These eigenpairs can be real as well as complex. These matrix polynomials are known as real symmetric matrix polynomials. Hence for solving this problem next, we discuss the backward error analysis for real symmetric/skew-symmetric matrix polynomial.

5.10. Backward error of real symmetric/skew-symmetric matrix polynomials

In this section, we are interested in the backward error analysis of approximate eigenpairs for real *symmetric/skew-symmetric* matrix polynomials. Let $\mathbf{P} \in \mathbf{P}_l(\mathbb{C}^{n\times n})$ be a real symmetric/skew-symmetric matrix polynomial, i.e., $A_j \in \mathbb{R}^{n\times n}$ for $j = 0 : l$. Let $((c_p, d_p), x_p), p = 1 : s$ be s approximate eigenpairs of $\mathbf{P}$, where $(c_p, d_p) \in \mathbb{C}^2 \setminus \{(0,0)\}$, and $x_p \in \mathbb{C}^n$. Using approximate eigenpairs, we define the matrix N^ϵ as follows:

$$N^\epsilon = \begin{bmatrix} N^\epsilon_{10} & \dots & N^\epsilon_{1l} \\ N^\epsilon_{20} & \dots & N^\epsilon_{2l} \\ \vdots & \dots & \vdots \\ N^\epsilon_{s0} & \dots & N^\epsilon_{sl} \end{bmatrix} \in \mathbb{C}^{2sn\times(l+1)(n^2+\epsilon n)/2},$$

where

$$N^\epsilon_{pj} = w_j^{-1} \begin{bmatrix} N^\epsilon(g_{pj}) \\ N^\epsilon(h_{pj}) \end{bmatrix} \operatorname{diag}\left(\left[\operatorname{vec}(\operatorname{sgn} A_j \circ C, \epsilon)\right]\right), \ for\ j = 0 : l.$$

Throughout this section, $\epsilon = 1$ represents a real symmetric matrix polynomial and $\epsilon = -1$ represents a real skew-symmetric matrix polynomial. The upper bound on the number of eigenpairs s is $s \leq \frac{(l+1)}{2}(n+\epsilon)$.

Now, we state the main theorem of this section. Since the proof of the theorem is similar to Theorem 5.4.1, we recall only the main steps of the proof.

Theorem 5.10.1. *Let $\mathbf{P} \in \mathbf{P}_l(\mathbb{C}^{n\times n})$ be a real symmetric/skew-symmetric matrix polynomial of the form (5.1). Let $((c_p, d_p), x_p)$ be s approximate eigenpairs of $\mathbf{P}$, where $0 \neq x_p \in \mathbb{C}^n$ and $0 \neq \lambda_p = (c_p, d_p)$ for $p = 1 : s$. Set $r := \begin{bmatrix} \Re(r_1)^T & \Im(r_1)^T & \dots & \Re(r_s)^T & \Im(r_s)^T \end{bmatrix}^T$, where $r_p = -\mathbf{P}(c_p, d_p)x_p$ for $p = 1 : s$. If N^ϵ (defined as above) is a full row rank matrix, then there exists a minimizing real symmetric/skew-symmetric $\delta\mathbf{P} \in \mathbf{P}_l(\mathbb{C}^{n\times n})$ of the form $\delta\mathbf{P}(\alpha, \beta) = \sum_{j=0}^{l} \alpha^{l-j}\beta^j \delta A_j$, where $\delta A_j = (\delta a_{j,tk})$ for $j = 0 : l$ are given by*

$$\delta a_{j,tk} = \begin{cases} (\operatorname{sgn} a_{j,tt}) \displaystyle\sum_{p=1}^{s} \tfrac{(1+\epsilon)}{2} w_j^{-2} \big(g^t_{pj} e^T_{t+\frac{(1-\epsilon)}{2}n(2p-1)+\frac{(1+\epsilon)}{2}2n(p-1)} + \epsilon h^t_{pj} \\ e^T_{t+\frac{(1+\epsilon)}{2}n(2p-1)+\frac{(1-\epsilon)}{2}2n(p-1)}\big)(N^\epsilon N^{\epsilon T})^{-1} r, & for\ t = k, \\ (\operatorname{sgn} a_{j,tk}) \displaystyle\sum_{p=1}^{s} \tfrac{1}{2} w_j^{-2} f_{pj,tk} (N^\epsilon N^{\epsilon T})^{-1} r, & for\ t \neq k, \end{cases}$$

and

$f_{pj,tk} = g_{pj}^k e_{t+2n(p-1)}^T + h_{pj}^k e_{t+n+2n(p-1)}^T + \epsilon g_{pj}^t e_{k+2n(p-1)}^T + \epsilon h_{pj}^t e_{k+n+2n(p-1)}^T$, $g_{pj} = \Re(c_p^{l-j} d_p^j x_p)$, $h_{pj} = \Im(c_p^{l-j} d_p^j x_p)$, $g_{pj}^t = \Re(c_p^{l-j} d_p^j x_p^t)$, $h_{pj}^t = \Im(c_p^{l-j} d_p^j x_p^t)$ *for* $p = 1 : s$, $j = 0 : l$, *and* $t, k = 1 : n$. *Then* $(\mathbf{P}(c_p, d_p) + \delta\mathbf{P}(c_p, d_p))x_p = 0$, *and the backward error is given by*

$$\eta_{w,F}^{\mathbf{S}}(\lambda_{1:s}, x_{1:s}, \mathbf{P}) = \|N^{\epsilon T}(N^{\epsilon} N^{\epsilon T})^{-1} r\|_F.$$

If N^ϵ *is not a full rank matrix but* $rank(N^\epsilon) = rank([N^\epsilon, r])$, *then the backward error is given by*

$$\eta_{w,F}^{\mathbf{S}}(\lambda_{1:s}, x_{1:s}, \mathbf{P}) = \|V^\epsilon D^{\epsilon+} U^{\epsilon H} r\|_F,$$

where U^ϵ, V^ϵ *are unitary matrices and* D^ϵ *is a diagonal matrix with singular values of* N^ϵ.

Proof. Corresponding to a real symmetric/skew-symmetric $\mathbf{P} \in \mathbf{P}_l(\mathbb{C}^{n\times n})$, its given approximate eigenvalues are λ_p and corresponding eigenvector are x_p for $p = 1 : s$. We have to construct structured minimal norm sparse $\delta\mathbf{P} \in \mathbf{P}_l(\mathbb{C}^{n\times n})$ such that $(\mathbf{P}(\lambda_p) + \delta\mathbf{P}(\lambda_p))x_p = 0$. By assumption $\mathbf{P}(\lambda_p)x_p + r_p = 0$, for $p = 1 : s$. Then $r_p = \delta\mathbf{P}(\lambda_p)x_p = \sum_{j=0}^{l} c_p^{l-j} d_p^j \delta A_j x_p = \sum_{j=0}^{l} \delta A_j(\Re(c_p^{l-j} d_p^j x_p) + \mathrm{i}\Im(c_p^{l-j} d_p^j x_p))$. For maintaining sparsity, we replace δA_j by $(\delta A_j \circ \operatorname{sgn} A_j)$, we get $r_p = \sum_{j=0}^{l} \delta A_j \circ \operatorname{sgn} A_j(\Re(c_p^{l-j} d_p^j x_p)) + \mathrm{i}\delta A_j \circ \operatorname{sgn} A_j(\Im(c_p^{l-j} d_p^j x_p))$. Finally we get

$r_p = \left[\sum_{j=0}^{l} \delta A_j \circ \operatorname{sgn} A_j(\Re(c_p^{l-j} d_p^j x_p)) + \mathrm{i}\delta A_j \circ \operatorname{sgn} A_j(\Im(c_p^{l-j} d_p^j x_p))\right] \circ D \circ C = \Re(r_p) + \mathrm{i}\Im(r_p)$, where

$$\Re(r_p) = \sum_{j=0}^{l} [\delta A_j \circ \operatorname{sgn} A_j g_{pj}] \circ D \circ C$$

$$\Im(r_p) = \sum_{j=0}^{l} [\delta A_j \circ \operatorname{sgn} A_j h_{pj}] \circ D \circ C$$

for $p = 1 : s$. Now separating the unknown and known variables, we get the following system for $p = 1 : s$

$$(5.10)\begin{bmatrix} \Re(r_p) \\ \Im(r_p) \end{bmatrix} = \sum_{j=0}^{l} w_j^{-1} \begin{bmatrix} N^\epsilon(g_{pj}) \\ N^\epsilon(h_{pj}) \end{bmatrix} \operatorname{diag}\left(\left[\operatorname{vec}(\operatorname{sgn} A_j \circ C, \epsilon)\right]\right) \Delta_j^\epsilon = \sum_{j=0}^{l} N_{pj}^\epsilon \Delta_j^\epsilon,$$

where

$$\Delta_j^\epsilon = w_j \operatorname{vec}(\delta A_j \circ \operatorname{sgn} A_j \circ D, \epsilon), \ \ for\ j = 0, 1, \ldots, l.$$

Rest of the proof will follow similar to the proof of Theorem 5.4.1.

After obtaining the result for the general case, we discuss the backward error result for quadratic matrix polynomial by the following corollary.

Corollary 5.10.2. *Let $Q(\beta) = A_0 + \beta A_1 + \beta^2 A_2$, be a non-homogeneous real Hermitian/skew-Hermitian quadratic matrix polynomial such that $A_j \in \mathbb{R}^{n\times n}$ for $j = 0 : 2$. Let (μ_p, x_p) be s approximate eigenpairs of $Q(\beta)$, where $\mu_p \in \mathbb{C}, x_p \in \mathbb{C}^n \setminus \{0\}$ for $p = 1 : s$. Set $r := \begin{bmatrix} \Re(r_1)^T & \Im(r_1)^T & \dots & \Re(r_s)^T & \Im(r_s)^T \end{bmatrix}^T$, where $r_p := -Q(\mu_p)x_p$ for $p = 1 : s$. If N^ϵ is a full row rank matrix, then there exists $\delta Q(\beta) = \delta A_0 + \beta \delta A_1 + \beta^2 \delta A_2$ such that $(Q(\mu_p) + \delta Q(\mu_p))x_p = 0$, where $\delta A_j = (\delta a_{j,tk}), j = 0 : 2$ are given by*

$$\delta a_{j,tk} = \begin{cases} (\operatorname{sgn} a_{j,tt}) \displaystyle\sum_{p=1}^{s} \frac{(1+\epsilon)}{2} w_j^{-2} \big(g_{pj}^t e^T_{t+\frac{(1-\epsilon)}{2}n(2p-1)+\frac{(1+\epsilon)}{2}2n(p-1)} + \epsilon h_{pj}^t \\ e^T_{t+\frac{(1+\epsilon)}{2}n(2p-1)+\frac{(1-\epsilon)}{2}2n(p-1)} \big) (N^\epsilon N^{\epsilon T})^{-1} r, & \text{for } t = k, \\ (\operatorname{sgn} a_{j,tk}) \displaystyle\sum_{p=1}^{s} \frac{1}{2} w_j^{-2} d_{pj,tk} (N^\epsilon N^{\epsilon T})^{-1} r, & \text{for } t \neq k, \end{cases}$$

where $d_{pj,tk} = g_{pj}^k e^T_{t+2n(p-1)} + h_{pj}^k e^T_{t+n+2n(p-1)} + \epsilon g_{pj}^t e^T_{k+2n(p-1)} + \epsilon h_{pj}^t e^T_{k+n+2n(p-1)}$, $g_{pj} = \Re(\mu_p^j x_p)$, $h_{pj} = \Im(\mu_p^j x_p)$, $g_{pj}^t = \Re(\mu_p^j x_p^t)$, $h_{pj}^t = \Im(\mu_p^j x_p^t)$ for $p = 1 : s$, $j = 0 : 2$, and $t, k = 1 : n$. The backward error is given by

$$\eta^{\mathbf{S}}_{w,F}(\lambda_{1:s}, x_{1:s}, \mathbf{P}) = \|N^{\epsilon T}(N^\epsilon N^{\epsilon T})^{-1} r\|_F.$$

If N^ϵ is not full rank matrix but $rank(N^\epsilon) = rank([N^\epsilon, r])$, then δA_j are constructed using singular values decomposition of N^ϵ.

Proof. Substituting $l = 2, c_p = 1, d_p = \mu_p$ in Theorem 5.10.1, we get the desired result.

Remark 5.10.3. *We know that for a real symmetric matrix polynomial if (λ, x) is an eigenpair, where $\lambda \in \mathbb{C}^2 \setminus \{(0,0)\}, x \in \mathbb{C}^n$, then $(\overline{\lambda}, \overline{x})$ is also an eigenpair. Rest of the eigenpairs are real. Using this information, and size of the matrix N^ϵ, we get that $s \leq \frac{(l+1)}{2}(n+\epsilon)$.*

5.11. Numerical examples and discussion of quadratic inverse eigenvalue problems

In this section, we illustrate our theory with suitable examples and graphs. We start our discussion with the solution of quadratic inverse eigenvalue problems. In particular, [**21**, Problem 5.4] ask to construct $C, K \in \mathbb{R}^{n\times n}$ from the given specified eigenpairs (Λ, X) such that

$$X\Lambda^2 + CX\Lambda + KX = 0, \tag{5.11}$$

where $\Lambda \in \mathbb{C}^{s\times s}$ has specified eigenvalues $\mu_i \in \mathbb{C}$ on its diagonal and $X \in \mathbb{C}^{n\times s}$ has corresponding eigenvector $x_i \in \mathbb{C}^n$ as its column. We need to construct $C, K \in \mathbb{R}^{n\times n}$ with $C = C^T$ and $K = K^T$, so that Equation 5.11 is satisfied.

Quadratic inverse eigenvalue Problem 5.11 is equivalent to solving $\mu_i^2 I x_i + \mu_i C x_i + K x_i = 0$ for $i = 1 : s$. Since we need to construct $C, K \in \mathbb{R}^{n\times n}$, we set $C = A_1 + \delta A_1, K = A_0 + \delta A_0$, where A_0, A_1 are fixed but arbitrarily chosen real symmetric matrices. Then applying Corollary 5.10.2 with weight vector $w = (1, 1, 0)$, we get $\delta A_0, \delta A_1$ and hence desired C, K which satisfied $\mu_i^2 I x_i + \mu_i C x_i + K x_i = 0$ for $i = 1 : s$. We will illustrate it by an example for $s = 3$.

Example 5.11.1. *Let* (μ_i, x_i) *for* $i = 1 : 3$ *be specified* eigenpairs *where* $\mu_1 = \overline{\mu}_2 = -0.2168-4.3159\mathrm{i}, \mu_3 = -0.3064$, $x_1 = \overline{x}_2 = [-0.4132+5.2801\mathrm{i}, -4.3518+3.2758\mathrm{i}, -0.1336-4.0588i, -5.1414+4.4003\mathrm{i}, 8.6146-4.0112\mathrm{i}]^T$, *and* $x_3 = [-9.6715, -9.1357, -4.4715, -6.9659, -4.4708]^T$. *Choose*

$$A_0 = \begin{bmatrix} 0.2028 & 0.107 & 0.5112 & 0.55515 & 0.3508 \\ 0.107 & 0.7468 & 0.64565 & 0.4757 & 0.58775 \\ 0.5112 & 0.64565 & 0.5252 & 0.44195 & 0.5505 \\ 0.55515 & 0.4757 & 0.44195 & 0.3795 & 0.5682 \\ 0.3508 & 0.58775 & 0.5505 & 0.5682 & 0.1897 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 0.9501 & 0.4966 & 0.6111 & 0.44585 & 0.4746 \\ 0.4966 & 0.4565 & 0.4052 & 0.87845 & 0.3988 \\ 0.6111 & 0.4052 & 0.9218 & 0.82755 & 0.49475 \\ 0.44585 & 0.87845 & 0.82755 & 0.4103 & 0.45175 \\ 0.4746 & 0.3988 & 0.49475 & 0.45175 & 0.1389 \end{bmatrix}.$$

Then applying Corollary 5.10.2, we get

$$C = \begin{bmatrix} -1.6514 & -0.7099 & -0.1769 & 0.0671 & -2.6998 \\ -0.7099 & 0.3340 & 1.2028 & 0.7282 & -0.3518 \\ -0.1769 & 1.2028 & 3.6546 & 0.7699 & 1.5992 \\ 0.0671 & 0.7282 & 0.7699 & 0.3928 & -0.1274 \\ -2.6998 & -0.3518 & 1.5992 & -0.1274 & -1.8210 \end{bmatrix},$$

$$K = \begin{bmatrix} 0.7075 & -0.8079 & -1.9246 & 0.2321 & -0.9079 \\ -0.8079 & 1.8371 & -0.2280 & 2.2767 & -5.1703 \\ -1.9246 & -0.2280 & 5.0627 & -1.9317 & 5.0950 \\ 0.2321 & 2.2767 & -1.9317 & 3.5307 & -7.9849 \\ -0.9079 & -5.1703 & 5.0950 & -7.9849 & 17.6420 \end{bmatrix},$$

which satisfy $\mu_i^2 I x_i + \mu_i C x_i + K x_i = 0$ *for* $i = 1 : 3$. *Clearly* $C = C^T, K = K^T$.

Remark 5.11.2. *Since we can choose different* A_0, A_1 *so* C, K *are not unique.*

Next, we discuss the quadratic *inverse eigenvalue problem* of the following form:

$$MX\Lambda^2 + CX\Lambda + KX = 0, \tag{5.12}$$

where we need to construct matrices $M, C, K \in \mathbb{C}^{n\times n}$ such that $M = M^H, C = C^H$ and $K = K^H$. Quadratic inverse eigenvalue problem 5.12 is equivalent to solving $\mu_i^2 M x_i + \mu_i C x_i + K x_i = 0$ for $i = 1 : s$.

In this quadratic inverse eigenvalue problem matrices are not restricted to real entries, it means for the given approximate eigenpairs we can construct the matrices M, C and K from the complex field. We set $K = A_0 + \delta A_0, C = A_1 + \delta A_1$, and $M = A_2 + \delta A_2$ where A_0, A_1 and A_2 are fixed but arbitrarily chosen Hermitian matrices. Similar to previous inverse eigenvalue problem, here for finding $\delta A_0, \delta A_1$ and δA_2, we use Theorem 5.4.1 with $l = 2$. We will illustrate this inverse eigenvalue problem by the following example for $s = 3$.

Example 5.11.3. *Let* (μ_i, x_i) *for* $i = 1 : 3$ *are specified eigenpairs defined in the previous example. Choose*

$$A_0 = \begin{bmatrix} 0.3244 + 0.0000\mathrm{i} & 1.3963 + 0.3470\mathrm{i} & 0.7618 - 0.2637\mathrm{i} & 1.3544 + 0.5932\mathrm{i} & 0.2723 - 0.7241\mathrm{i} \\ 1.3963 - 0.3470\mathrm{i} & 0.5259 + 0.0000\mathrm{i} & 0.7379 + 0.6874\mathrm{i} & 1.2276 + 0.3583\mathrm{i} & 1.7100 + 0.0944\mathrm{i} \\ 0.7618 + 0.2637\mathrm{i} & 0.7379 - 0.6874\mathrm{i} & 0.4580 + 0.0000\mathrm{i} & 1.9095 - 0.1989\mathrm{i} & 0.1570 - 0.0216\mathrm{i} \\ 1.3544 - 0.5932\mathrm{i} & 1.2276 - 0.3583\mathrm{i} & 1.9095 + 0.1989\mathrm{i} & 0.1564 + 0.0000\mathrm{i} & 1.2176 - 0.2179\mathrm{i} \\ 0.2723 + 0.7241\mathrm{i} & 1.7100 - 0.0944\mathrm{i} & 0.1570 + 0.0216\mathrm{i} & 1.2176 + 0.2179\mathrm{i} & 1.6346 + 0.0000\mathrm{i} \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 1.6294 + 0.0000\mathrm{i} & 1.0033 - 0.0371\mathrm{i} & 0.2846 + 0.4312\mathrm{i} & 1.0553 - 0.2167\mathrm{i} & 1.2881 + 0.3186\mathrm{i} \\ 1.0033 + 0.0371\mathrm{i} & 0.5570 + 0.0000\mathrm{i} & 1.5175 + 0.4179\mathrm{i} & 1.3793 + 0.3354\mathrm{i} & 1.0006 + 0.3485\mathrm{i} \\ 0.2846 - 0.4312\mathrm{i} & 1.5175 - 0.4179\mathrm{i} & 1.9143 + 0.0000\mathrm{i} & 1.4011 - 0.1847\mathrm{i} & 1.6494 + 0.6119\mathrm{i} \\ 1.0553 + 0.2167\mathrm{i} & 1.3793 - 0.3354\mathrm{i} & 1.4011 + 0.1847\mathrm{i} & 1.5844 + 0.0000\mathrm{i} & 1.8935 + 0.5225\mathrm{i} \\ 1.2881 - 0.3186\mathrm{i} & 1.0006 - 0.3485\mathrm{i} & 1.6494 - 0.6119\mathrm{i} & 1.8935 - 0.5225\mathrm{i} & 1.3575 + 0.0000\mathrm{i} \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.5521+0.0000\mathrm{i} & 1.1781+0.1551\mathrm{i} & 1.4064+0.0348\mathrm{i} & 1.1219-0.5402\mathrm{i} & 0.9597-0.3434\mathrm{i} \\ 1.1781-0.1551\mathrm{i} & 1.9195+0.0000\mathrm{i} & 0.5955+0.1719\mathrm{i} & 1.1325-0.4958\mathrm{i} & 0.4781-0.3484\mathrm{i} \\ 1.4064-0.0348\mathrm{i} & 0.5955-0.1719\mathrm{i} & 1.0119+0.0000\mathrm{i} & 0.8377+0.1504\mathrm{i} & 1.7052-0.0984\mathrm{i} \\ 1.1219+0.5402\mathrm{i} & 1.1325+0.4958\mathrm{i} & 0.8377-0.1504\mathrm{i} & 0.2986+0.0000\mathrm{i} & 0.5010-0.9221\mathrm{i} \\ 0.9597+0.3434\mathrm{i} & 0.4781+0.3484\mathrm{i} & 1.7052+0.0984\mathrm{i} & 0.5010+0.9221\mathrm{i} & 1.8585+0.0000\mathrm{i} \end{bmatrix}.$$

Then by above discussion, we get

$$M = \begin{bmatrix} 0.0178+0.0000\mathrm{i} & 0.3304-0.1560\mathrm{i} & 0.3074+0.1910\mathrm{i} & 0.6939+0.0991\mathrm{i} & 0.8037-0.0687\mathrm{i} \\ 0.3304+0.1560\mathrm{i} & 0.5566+0.0000\mathrm{i} & 0.5918+0.4660\mathrm{i} & 1.2508+0.3542\mathrm{i} & 1.1863+0.2165\mathrm{i} \\ 0.3074-0.1910\mathrm{i} & 0.5918-0.4660\mathrm{i} & 1.0174+0.0000\mathrm{i} & 1.2786-0.0331\mathrm{i} & 1.1548-0.3680\mathrm{i} \\ 0.6939-0.0991\mathrm{i} & 1.2508-0.3542\mathrm{i} & 1.2786+0.0331\mathrm{i} & 0.2492+0.0000\mathrm{i} & 0.9270-0.2494\mathrm{i} \\ 0.8037+0.0687\mathrm{i} & 1.1863-0.2165\mathrm{i} & 1.1548+0.3680\mathrm{i} & 0.9270+0.2494\mathrm{i} & 1.5546+0.0000\mathrm{i} \end{bmatrix},$$

$$C = \begin{bmatrix} 1.5608+0.0000\mathrm{i} & 1.3287+0.0333\mathrm{i} & 0.5771+0.3665\mathrm{i} & 1.1114-0.2769\mathrm{i} & 1.5979+0.0747\mathrm{i} \\ 1.3287-0.0333\mathrm{i} & 1.2126+0.0000\mathrm{i} & 1.7469+0.2964\mathrm{i} & 1.7729+0.2093\mathrm{i} & 1.0639+0.2205\mathrm{i} \\ 0.5771-0.3665\mathrm{i} & 1.7469-0.2964\mathrm{i} & 2.1851+0.0000\mathrm{i} & 1.6049-0.0627\mathrm{i} & 1.6561+0.3479\mathrm{i} \\ 1.1114+0.2769\mathrm{i} & 1.7729-0.2093\mathrm{i} & 1.6049+0.0627\mathrm{i} & 1.6961+0.0000\mathrm{i} & 2.1438+0.4383\mathrm{i} \\ 1.5979-0.0747\mathrm{i} & 1.0639-0.2205\mathrm{i} & 1.6561-0.3479\mathrm{i} & 2.1438-0.4383\mathrm{i} & 1.1219+0.0000\mathrm{i} \end{bmatrix},$$

$$K = \begin{bmatrix} -0.0361+0.000\mathrm{i} & 0.3696+0.0763\mathrm{i} & 0.6438+0.0605\mathrm{i} & 0.7567-0.2953\mathrm{i} & 0.2906+0.2714\mathrm{i} \\ 0.3696-0.0763\mathrm{i} & 0.8224+0.0000\mathrm{i} & -0.2709+0.2805\mathrm{i} & 0.5394-0.2127\mathrm{i} & -0.2257+0.3047\mathrm{i} \\ 0.6438-0.0605\mathrm{i} & -0.2709-0.2805\mathrm{i} & 0.3838+0.0000\mathrm{i} & 0.3221+0.2041\mathrm{i} & 1.1276+0.2017\mathrm{i} \\ 0.7567+0.2953\mathrm{i} & 0.5394+0.2127\mathrm{i} & 0.3221-0.2041\mathrm{i} & 0.0304+0.0000\mathrm{i} & 0.0718-0.5548\mathrm{i} \\ 0.2906-0.2714\mathrm{i} & -0.2257-0.3047\mathrm{i} & 1.1276-0.2017\mathrm{i} & 0.0718+0.5548\mathrm{i} & 1.4121+0.0000\mathrm{i} \end{bmatrix},$$

which satisfy $\mu_i^2 M x_i + \mu_i C x_i + K x_i = 0$ *for* $i = 1:3$. *Clearly* $M = M^H, C = C^H, K = K^H$.

Next, we discuss the quadratic T-palindromic inverse eigenvalue problem. For the given specified eigenpairs $(\mu_i, x_i), i = 1:s$, the quadratic T-palindromic inverse eigenvalue problem is to construct the matrices D_0, D_1, and D_2 such that $D_0 = D_2^T$, $D_1 = D_1^T$ and $(D_0 + \mu_i D_1 + \mu_i^2 D_2)x_i = 0$, where $\mu_i \in \mathbb{C}, x_i \in \mathbb{C}^n$ and $s \leq 2n$.

We use Theorem 5.7.3 for solving the quadratic T-Palindromic inverse eigenvalue problem. For getting the solution let $D_i = A_i + \delta A_i$ for $i = 0:2$, where $A_i, i = 0:2$ are the known matrices, and δA_i are the unknown matrices to be obtained using Theorem 5.7.3. Set $A_i = H_3$, where H_n is defined by (4.13).

To illustrate the problem let us consider the eigenpairs information from [**88**, Example 4.3] which asks to construct a T-palindromic quadratic matrix polynomials of size 3×3

from the given set of eigenpairs $(\mu_i, x_i), i = 1:5$. By Table 5.6, we know that the maximum value of s for quadratic T-palindromic matrix polynomial $(l = 2)$ can be $(3n+1)/2$, which can be maximum 5 for $n = 3$. Let the eigenpairs are given as follows: $\mu_1 = \mathrm{i}, \mu_2 = 1/\mathrm{i}, \mu_3 = 1+\mathrm{i}, \mu_4 = 1/(1+\mathrm{i}), \mu_5 = 2\mathrm{i}$, and $x_1 = [1,0,0]^T, x_2 = [0,1,0]^T, x_3 = [0,0,1]^T, x_4 = [1,1,1]^T, x_5 = [-1,0,1]^T$.

By applying Theorem 5.7.3 for $l = 2$ on the given eigenpairs $(\mu_i, x_i), i = 1:5$, we get

$$\delta A_0 = \begin{bmatrix} -1.0046 - 0.0180\mathrm{i} & -1.0061 + 0.1054\mathrm{i} & -1.0180 + 0.0046\mathrm{i} \\ -1.0475 - 0.0960\mathrm{i} & -1.1861 + 0.3393\mathrm{i} & -1.1147 - 0.0597\mathrm{i} \\ -1.0113 - 0.0067\mathrm{i} & -0.9725 + 0.0872\mathrm{i} & -1.0067 + 0.0113\mathrm{i} \end{bmatrix},$$

$$\delta A_1 = \begin{bmatrix} -1.0000 + 0.0000\mathrm{i} & -0.7986 - 0.0414\mathrm{i} & -0.9887 + 0.0067\mathrm{i} \\ -0.7986 - 0.0414\mathrm{i} & -1.0000 + 0.0000\mathrm{i} & -0.8531 - 0.1422\mathrm{i} \\ -0.9887 + 0.0067\mathrm{i} & -0.8531 - 0.1422\mathrm{i} & -0.9843 - 0.0136\mathrm{i} \end{bmatrix},$$

$$\delta A_2 = \begin{bmatrix} -1.0046 - 0.0180\mathrm{i} & -1.0475 - 0.0960\mathrm{i} & -1.0113 - 0.0067\mathrm{i} \\ -1.0061 + 0.1054\mathrm{i} & -1.1861 + 0.3393\mathrm{i} & -0.9725 + 0.0872\mathrm{i} \\ -1.0180 + 0.0046\mathrm{i} & -1.1147 - 0.0597\mathrm{i} & -1.0067 + 0.0113\mathrm{i} \end{bmatrix}.$$

Finally, we get the required D_0, D_1, and D_2 as follows:

$$D_0 = A_0 + \delta A_0 = \begin{bmatrix} -0.0046 - 0.0180\mathrm{i} & -0.0061 + 0.1054\mathrm{i} & -0.0180 + 0.0046\mathrm{i} \\ -0.0475 - 0.0960\mathrm{i} & -0.1861 + 0.3393\mathrm{i} & -0.1147 - 0.0597\mathrm{i} \\ -0.0113 - 0.0067\mathrm{i} & 0.0275 + 0.0872\mathrm{i} & -0.0067 + 0.0113\mathrm{i} \end{bmatrix},$$

$$D_1 = A_1 + \delta A_1 = \begin{bmatrix} -0.0000 + 0.0000\mathrm{i} & 0.2014 - 0.0414\mathrm{i} & 0.0113 + 0.0067\mathrm{i} \\ 0.2014 - 0.0414\mathrm{i} & 0.0000 + 0.0000\mathrm{i} & 0.1469 - 0.1422\mathrm{i} \\ 0.0113 + 0.0067\mathrm{i} & 0.1469 - 0.1422\mathrm{i} & 0.0157 - 0.0136\mathrm{i} \end{bmatrix},$$

$$D_2 = A_2 + \delta A_2 = \begin{bmatrix} -0.0046 - 0.0180\mathrm{i} & -0.0475 - 0.0960\mathrm{i} & -0.0113 - 0.0067\mathrm{i} \\ -0.0061 + 0.1054\mathrm{i} & -0.1861 + 0.3393\mathrm{i} & 0.0275 + 0.0872\mathrm{i} \\ -0.0180 + 0.0046\mathrm{i} & -0.1147 - 0.0597\mathrm{i} & -0.0067 + 0.0113\mathrm{i} \end{bmatrix}.$$

Clearly, one can see that $(D_0 + \mu_i D_1 + \mu_i^2 D_2)x_i = 0$ for $i = 1:5$. Also, $D_0 = D_2^T$, and $D_1 = D_1^T$.

Remark 5.11.4. *Similar to the above quadratic inverse eigenvalue problems, one can also solve the different kind of palindromic inverse eigenvalue problems of* [88] *by using our developed backward error theory.*

Remark 5.11.5. *Theory of [88] works under the assumption that the approximate eigenvalues should be nonzero and distinct (Condition A2 of [88]). On the other hand by using our theory one can also solve the palindromic quadratic inverse eigenvalue problems for repeated as well as for zero eigenvalues.*

Moving further, we find that Li et al. [47] have developed the backward error formulas of a single approximate eigenpair for different kind of palindromic matrix polynomials. On the other hand, we have developed backward error of one or more approximate eigenpairs for the palindromic matrix polynomials. Hence at this point, we want to numerically compare the backward error results of a single approximate eignepair of [47] with our results. For this comparison, we have performed several numerical runs for arbitrary specified eigenpair (λ, x) for H-palindromic quadratic matrix polynomial, where $\lambda \in \mathbb{C}$, and $x \in \mathbb{C}^n$. We found that for a single specified eigenpair, backward error obtained by our method (without sparsity) is equal to the backward error obtained by Li et al. when $|\lambda| = 1$. For $|\lambda| \neq 1$, we have obtained the Figure 5.1 which shows the comparison between structured backward error obtained by our method (with and without sparsity) and backward error bounds obtained by Li et al. [47].

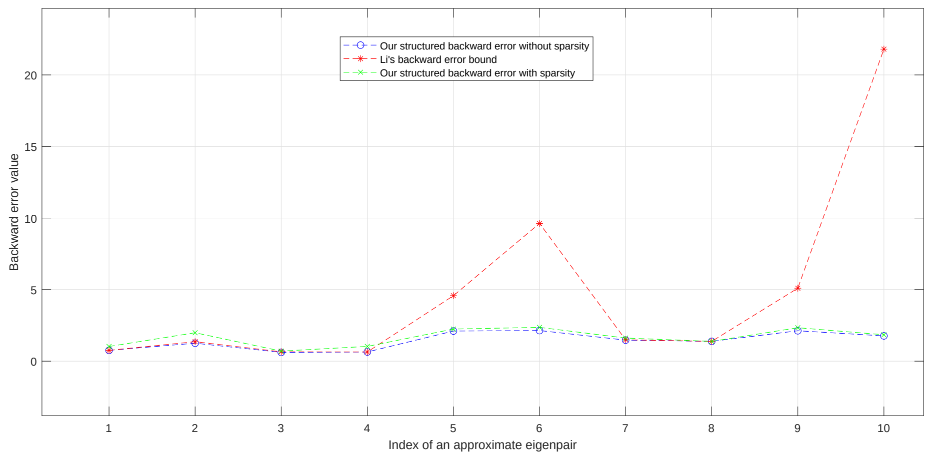

FIGURE 5.1. Backward errors comparison of a single eigenpair for H-palindromic matrix polynomial

From Figure 5.1, one can easily see that whenever we consider both sparsity and H-palindromic structure for obtaining the backward error of a single approximate eigenpair, the backward error bounds obtained by Li et al. [47] give quite large values, which seem unreasonable. For example, suppose we consider indexes 6, 9 and 10 of Figure 5.1 and compare the different backward error values. We can see that the backward error bound of Li et al. gives a higher value even when we consider both sparsity and H-palindromic structure together for calculating the backward error. This comparison shows that though the authors of [**47**] provide the upper bound for the backward error of a single eigenpair but this upper bound is quite far from the exact backward error value.

Next, Figure 5.2 provides the comparison between the structured and unstructured sparse as well as non-sparse backward errors for two specified eigenpairs with respect to the definition 5.2.2 (See Remark 5.3.3 for obtaining the backward error without sparsity). By Figure 5.2, we can easily understand that unstructured backward error is always the lower bound for all the backward errors and structured backward error with sparsity is always an upper bound when we consider the backward error for more than one eigenpairs. Theoretically it is easy to verify. Interestingly by the figure, we observe that the graph of structured backward error (H*-palindromic*) and the graph of unstructured backward error with sparsity cut each other. This shows that the "sparsity structure" and "H*-palindromic* structure" are theoretically incomparable. To obtain this graph, we perform several numerical experiments with H*-palindromic* matrix polynomial and run the numerical experiment with arbitrary set of two specified *eigenpairs*.

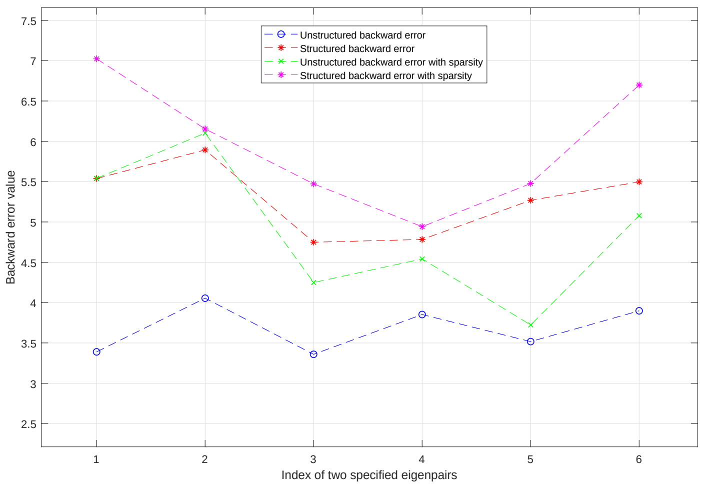

FIGURE 5.2. Backward error comparison of two eigenpairs for H-palindromic quadratic polynomial

Remark 5.11.6. *Similar to Figure 5.2 for* H*-palindromic structure, one can also obtain the similar figures for other structures.* ■

CHAPTER 6

BACKWARD ERROR ANALYSIS OF SPECIFIED EIGENPAIRS FOR TWO-PARAMETER EIGENVALUE PROBLEMS

Abstract: In the continuation of the detailed study of backward error analysis of specified eigenpairs, this chapter is dedicated for the structured and unstructured backward error analysis of two approximate eigenpairs of a double semisimple eigenvalue for two-parameter eigenvalues problems. We work with different structures such as *complex symmetric, complex skew-symmetric*, *Hermitian*, *skew-Hermitian*, T*-even alternating*, T-odd alternating, H*-even alternating*, and H-odd alternating two-parameter matrix systems with respect to *Frobenius norm*. Further, we illustrate the developed theory with the help of numerical experiments.

6.1. Introduction

Backward error analysis is one of the most important topics in numerical linear algebra. The term backward error analysis is discussed by different authors in different aspects and is continuously developing. If we briefly recall the development, we can find that the backward error analysis of approximate solutions for linear systems is already discussed by different authors and is well developed (see, for example, [**35**, **36**, **63**] and the references therein). For the matrix case, Dief [**24**] has discussed the backward error analysis for a single approximate eigenpair, and this work is further extended for structured matrices by Tisseur [**71**] for one or more eigenpairs. Backward error analysis of a single and more approximate eigenpairs for matrix pencils is well developed but it is limited to a single approximate eigenpair for matrix polynomials (see, for example, [**1**, **7**, **8**, **9**, **10**]). More specifically in [**6**] the authors have discussed the backward error analysis of two approximate eigenpairs of a double-semisimple eigenvalue for structured and unstructured matrix pencils. Next, in [42] Hochstenbach and Plestenjak have found the backward error of an approximate eigenpair for unstructured multiparameter eigenvalue problems. In the same paper, they have also obtained the backward error of an approximate eigenpair for Hermite multiparameter eigenvalue problems provided the given approximate eigenvalue is real. In [50], the author has extended the work of [42] and obtained the backward error of an approximate eigenpair for a Hermite multiparameter eigenvalue problem provided the approximate eigenvalue is complex.

A given multiparameter eigenvalue problem (MEP) can have more than one eigenpairs in general. Hence the backward error analysis of approximate eigenpairs can not be limited to a single eigenpair. Next step in the backward error analysis for MPE is to investigate the backward error formula of two approximate eigenpairs. Situations for two eigenpairs are not as similar as for the case of a single eigenpair. For the given two approximate eigenpairs, one can face different situations. For example, the given approximate eigenvalue can be semisimple. The given approximate eigenvalue can be defective, i.e., both the eigenvalues are same and eigenvectors are linearly dependent. It may also possible that both the eigenvalues are distinct, but eigenvectors are linearly independent. Further, these obtained eigenpairs are approximate, not exact. This happens due to roundoff errors. Backward error is an essential tool to understand the quality of computed approximate solutions. A two-parameter eigenvalue problem is the most widely discussed form of the MEP (see, for example, [11, 12, 15, 22, 29, 55] for more on two-parameter eigenvalue problems). Two-parameter eigenvalue problems arise in many applications, particularly in mathematical physics when the method of separation of variables is used to solve boundary value problems (see, [39, 73] and the references therein). In this chapter, we are interested in the backward error analysis for a two-parameter eigenvalue problem.

From the above discussion, we know that the backward error analysis of a single approximate eigenpair is well discussed for two-parameter eigenvalue problems, but the backward error analysis of two approximate eigenpairs is unanswered even for a semisimple eigenvalue (see, [27, 42] for more on backward error analysis of a single eigenpair). Hence a natural question arises that what will be the backward error of two approximate eigenpairs of a given two-parameter eigenvalue problem when the given approximate eigenvalue is semisimple with multiplicity two? We answer the above question with respect to the

Frobenius norm for unstructured and structured two-parameter eigenvalue problems. We work with complex symmetric, complex skew-symmetric, Hermitian, and skew-Hermitian two-parameter eigenvalue problems (see, [**27, 42**] for more details on structured two-parameter eigenvalue problems).

6.2. Two-parameter matrix system and its classification

Let us start this section by recalling the definition of a two-parameter matrix system. Let $\mathbb{K}$ be the space of two-parameter matrix systems. A two parameter matrix system is defined in the following manner:

$$W(\alpha) := (W_1(\alpha), W_2(\alpha)), \text{ where } W_i(\alpha) := \alpha_0 V_{i0} + \alpha_1 V_{i1} + \alpha_2 V_{i2}, \ i = 1:2, \tag{6.1}$$

where $V_{ij} \in \mathbb{C}^{n_i \times n_i}$ for $i = 1:2, j = 0:2$, and $\alpha = (\alpha_0, \alpha_1, \alpha_2) \in \mathbb{C}^3$. We denote the system (6.1) by $W := (W_1, W_2) \in \mathbb{C}^{n_1 \times n_1} \times \mathbb{C}^{n_2 \times n_2}$. Finding $(\lambda_0, \lambda_1, \lambda_2) \in \mathbb{C}^3 \setminus \{(0,0,0)\}$, and non zero vectors $x_i \in \mathbb{C}^{n_i}$ such that $W_i(\lambda)x_i = 0$ for $i = 1:2$ is called a two parameter eigenvalue problem (TEP). Further, $(\lambda_0, \lambda_1, \lambda_2) = \lambda \in \mathbb{C}^3 \setminus \{(0,0,0)\}$ is called an eigenvalue of (6.1), and the pair (x_1, x_2) is called an eigenvector of W corresponding to λ.

Remark 6.2.1. *If λ is an eigenvalue then $a\lambda$ is also an eigenvalue of for each nonzero $a \in \mathbb{C}$. Hence, we consider the normalized eigenvalue $\lambda \in \mathbb{C}^3 \setminus \{(0,0,0)\}$ for our analysis, i.e., $|\lambda_0|^2 + |\lambda_1|^2 + |\lambda_2|^2 = 1$.*

Remark 6.2.2. *Let $(\lambda_0, \lambda_1, \lambda_2) \in \mathbb{C}^3 \setminus \{(0,0,0)\}$ be an eigenvalue of W. Then $\lambda_0 = 0$ corresponds to an infinite eigenvalue and $\lambda_0 \neq 0$ corresponds to a finite eigenvalue.*

Remark 6.2.3. *By fixing $\alpha_0 = 1$ in (6.1), we can get the non-homogeneous form of a two-parameter matrix system. In that case by fixing $\lambda_0 = 1$ in a homogeneous eigenvalue $(\lambda_0, \lambda_1, \lambda_2)$, we can easily get the corresponding non-homogeneous eigenvalue $(1, \lambda_1, \lambda_2)$. In the non-homogeneous case for simplicity we denote an eigenvalue by (λ_1, λ_2) instead of $(1, \lambda_1, \lambda_2)$.*

Definition 6.2.4. *Let W be a two-parameter matrix system of the form (6.1). Then the set of eigenvalues of W is defined as*

$$\Lambda(W) = \{\lambda \in \mathbb{C}^3 \setminus \{(0,0,0)\} : \det(W_i(\lambda)) = 0 \ \textit{for}\ i = 1, 2\}.$$

At this point, we present an example.

Example 6.2.5. *Let W be a two-parameter matrix system of the form (6.1), where $W_i(\alpha) := \alpha_0 V_{i0} + \alpha_1 V_{i1} + \alpha_2 V_{i2},\ i = 1:2,$ such that*

$V_{10} = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}, V_{11} = \begin{bmatrix} 7 & 0 \\ 0 & 9 \end{bmatrix}, V_{12} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}; V_{20} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, V_{21} = \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix}, V_{22} = \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix}$. *Then*

$$\Lambda(W) = \{(\frac{6}{\sqrt{53}}, -\frac{4}{\sqrt{53}}, -\frac{1}{\sqrt{53}}), (\frac{69}{\sqrt{5117}}, -\frac{10}{\sqrt{5117}}, -\frac{16}{\sqrt{5117}}), (\frac{3}{\sqrt{35}}, \frac{1}{\sqrt{35}}, -\frac{5}{\sqrt{35}}), (0, \frac{2}{\sqrt{53}}, -\frac{7}{\sqrt{53}})\}.$$

In this chapter, we are interested in the backward error analysis of two approximate eigenpairs, especially when the approximate eigenvalue is semisimple. At this point we need to understand the definitions of geometric and algebraic multiplicities of an eigenvalue $\lambda = (\lambda_0, \lambda_1, \lambda_2)$ of a two-parameter matrix system W.

Definition 6.2.6. [42] *The geometric multiplicity (G.M.) of an eigenvalue $\lambda = (\lambda_0, \lambda_1, \lambda_2)$ of a two-parameter W is defined in the following manner:*

$$G.M. = \dim(\ker(W_1(\lambda))) \times \dim(\ker(W_2(\lambda))).$$

Definition 6.2.7. [60] *The algebraic multiplicity (A.M.) of $\lambda = (\lambda_0, \lambda_1, \lambda_2)$ is equal to the intersection multiplicity of two curves $w_1 = 0$ and $w_2 = 0$ at λ. Here $w_i = \det(W_i(\alpha))$ for $i = 1, 2$.*

Definition 6.2.8. *An eigenvalue $\lambda = (\lambda_0, \lambda_1, \lambda_2)$ of W is semisimple if its algebraic and geometric multiplicity coincide.*

Definition 6.2.9. *An eigenvalue $\lambda = (\lambda_0, \lambda_1, \lambda_2)$ is said to be a double-semisimple eigenvalue if it is semisimple and its geometric multiplicity is two.*

Next, we present an example from [73] to understand the above definitions.

Example 6.2.10. *Let W be a two parameter matrix system of the form (6.1) such that*

$$V_{10} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, V_{11} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix}, V_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix};$$

$$V_{20} = \begin{bmatrix} 20 & 0 \\ 0 & 0 \end{bmatrix}, V_{21} = \begin{bmatrix} 0 & \sqrt{3} \\ \sqrt{3} & 0 \end{bmatrix}, V_{22} = \begin{bmatrix} 7 & 0 \\ 0 & 1 \end{bmatrix}.$$

Here $w_1 = (4\alpha_0 + 2\alpha_1)\alpha_2^2 - 6(4\alpha_0 + \alpha_1)\alpha_2^2$ *and* $w_2 = 3\alpha_1^2 - 20\alpha_0\alpha_2 - 7\alpha_1^2$*. We have* $(1,0,0)$ *as one of the eigenvalues of* W*. Eigenvalue* $(1,0,0)$ *has multiplicity two. We get* $\dim(\ker(W_1(1,0,0))) = 2$ *and* $\dim(\ker(W_2(1,0,0))) = 1$*. Hence G.M. and A.M. of* $(1,0,0)$ *is equal to two and* $(1,0,0)$ *is a double-semisimple eigenvalue of* W*.*

Remark 6.2.11. *If* $(\lambda_0, \lambda_1, \lambda_2)$ *is a double-semisimple eigenvalue of a two-parameter matrix system* W*, then either* $\dim(\ker(W_1(\lambda_0, \lambda_1, \lambda_2)) = 2$ *and* $\dim(\ker(W_2(\lambda_0, \lambda_1, \lambda_2)) = 1$ *or* $\dim(\ker(W_2(\lambda_0, \lambda_1, \lambda_2)) = 2$ *and* $\dim(\ker(W_1(\lambda_0, \lambda_1, \lambda_2)) = 1$*. Without loss of generality we assume that* $\dim(\ker(W_1(\lambda_0, \lambda_1, \lambda_2)) = 2$ *and* $\dim(\ker(W_2(\lambda_0, \lambda_1, \lambda_2)) = 1$ *for the backward error analysis.*

Next, we classify the two-parameter matrix systems based on the normal rank which is defined as follows: Let W be a two-parameter matrix system of the form (6.1). Then we define the normal rank of W_i for $i = 1:2$ by

$$\operatorname{Nrank}(W_i) = \max_{\lambda \in \mathbb{C}^3 \setminus \{(0,0,0)\}} \operatorname{rank}(W_i(\lambda)).$$

Based on the normal rank, let us consider the following examples:

Example 6.2.12. *Let* W *be a two-parameter matrix system of the form (6.1), where*

$$W_1(\alpha) = \begin{bmatrix} 1 + \alpha_1 + \alpha_2 & 0 \\ 0 & 7 + \alpha_1 + \alpha_2 \end{bmatrix}, W_2(\alpha) = \begin{bmatrix} 3 + \alpha_1 + \alpha_2 & 0 \\ 0 & 4 + \alpha_1 + \alpha_2 \end{bmatrix}.$$

Then $\operatorname{Nrank}(W_1) = 2$ *and* $\operatorname{Nrank}(W_2) = 2$*. One can see easily that the above two-parameter matrix system has no eigenvalue.*

Example 6.2.13. *Let* W *be a two-parameter matrix system of the form (6.1), where*

$$W_1(\alpha) = \begin{bmatrix} \alpha_0 + \alpha_1 + \alpha_2 & 0 \\ 0 & 0 \end{bmatrix}, W_2(\alpha) = \begin{bmatrix} \alpha_1 - \alpha_2 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then Nrank $(W_1) = 1$ *and* Nrank $(W_1) = 1$. *On the other hand, one can check easily that the spectrum of* W *is* $C^3 \setminus \{(0,0,0)\}$.

Example 6.2.14. *Let* W *be a two-parameter matrix system of the form (6.1), where*

$$W_1(\alpha) = \begin{bmatrix} \alpha_0 + \alpha_1 + \alpha_2 & 0 \\ 0 & \alpha_0 + \alpha_1 + \alpha_2 \end{bmatrix}, W_2(\alpha) = \begin{bmatrix} \alpha_1 + \alpha_2 & 0 \\ 0 & \alpha_0 + \alpha_2 \end{bmatrix}.$$

Then Nrank $(W_1) = 2$ *and* Nrank $(W_2) = 2$. *Also we have* $\Lambda(W) = \{(0, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}), (\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}})\}$.

Example 6.2.15. *Let* W *be a two-parameter matrix system of the form (6.1), where*

$$W_1(\alpha) = \begin{bmatrix} \alpha_0 + \alpha_1 + \alpha_2 & 0 \\ 0 & \alpha_1 - \alpha_2 \end{bmatrix}, W_2(\alpha) = \begin{bmatrix} \alpha_1 - 2\alpha_2 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then Nrank $(W_1) = 2$ *and* Nrank $(W_2) = 1$. *But the spectrum of* W *is nonempty.*

As we can see from the above examples that not every two-parameter matrix system needs to have the eigenvalues. In general, a given two-parameter matrix system need not have a common root, which leads to an interesting observation that unlike to a matrix, matrix pencil, and matrix polynomial where we always get a solution (solution means an eigenvalue), a two-parameter matrix system may not have an eigenvalue at all. Based on this observation, we can categorize two-parameter matrix systems in the following two categories: regular and irregular. Further, each class can be divided into two categories, namely weakly and strongly.

A two-parameter matrix system W of the form (6.1) is said to be regular if Nrank $(W_1) = n_1$ and Nrank $(W_2) = n_2$ (see, [55]). Otherwise, we called a two-parameter is irregular. We further classify the two-parameter matrix systems in the following categories:

(1) A two-parameter matrix system W is said to be **weakly regular** if Nrank $(W_1) = n_1$, Nrank $(W_2) = n_2$, and the spectrum of W is empty.

(2) A two-parameter matrix system W is said to be **strongly regular** if Nrank $(W_1) = n_1$, Nrank $(W_2) = n_2$, and has a nonempty spectrum.

(3) A two-parameter matrix system W is said to be **weakly irregular** if either Nrank $(W_1) < n_1$ or Nrank $(W_2) < n_2$ and its spectrum $\Lambda(W) = \mathbb{C}^3 \setminus \{(0,0,0)\}$.

(4) A two-parameter matrix system W is said to be **strongly irregular** if either $\text{Nrank}\,(W_1) < n_1$ or $\text{Nrank}\,(W_2) < n_2$ and its nonempty spectrum is a proper subset of $\mathbb{C}^3 \setminus \{(0,0,0)\}$.

Next, we discuss an important lemma as follows.

Lemma 6.2.16. *Suppose $\lambda = (\lambda_0, \lambda_1, \lambda_2) \in \mathbb{C}^3 \setminus \{(0,0,0)\}$ is a double-semisimple eigenvalue of a two-parameter matrix system W. Then there exists orthonormal vectors $x_1, x_2 \in \mathbb{C}^{n_1}$ such that $(W_1(\lambda))x_i = 0$ for $i = 1:2$ and $y_1 \in \mathbb{C}^{n_2}$ such that $(W_2(\lambda))y_1 = 0$.*

Proof. Let $\lambda = (\lambda_0, \lambda_1, \lambda_2) \in \mathbb{C}^3 \setminus \{(0,0,0)\}$ be a *double-semisimple eigenvalue* of W. It implies that its algebraic multiplicity and geometric multiplicity are equal to two. Then Without loss of generality we say that $\dim(ker(W_1(\lambda))) = 2$ and $\dim(ker(W_2(\lambda))) = 1$. Now $\dim(ker(W_1(\lambda))) = 2$ implies that there exists two linearly independent eigenvectors $z_1, z_2 \in \mathbb{C}^{n_1}$ such that $(W_1(\lambda))z_i = 0$ for $i = 1:2$, and there exists $y_1 \in \mathbb{C}^{n_2}$ such that $(W_2(\lambda))y_1 = 0$.

By *Gram-Schmidt* process, we can set $x_1 = z_1$ and $x_2 = z_2 - \gamma z_1$, where $\gamma = \frac{z_1^H z_2}{z_1^H z_1} \in \mathbb{C}$. We can easily see that $(W_1(\lambda))x_i = 0$ for $i = 1:2$, and x_1, x_2 are orthogonal, in particular orthonormal. ■

Next, based on the properties of matrices $V_{ij}, i = 1:2, j = 0:2$ of a two-parameter matrix system W of the form (6.1), we present Table 6.1 to classify the two-parameter matrix systems.

S	**Matrix structure**
Complex *symmetric*	$V_{ij} = V_{ij}^T$ for $i = 1:2, j = 0:2$
Complex *skew-symmetric*	$V_{ij} = -V_{ij}^T$ for $i = 1:2, j = 0:2$
Hermitian	$V_{ij} = V_{ij}^H$ for $i = 1:2, j = 0:2$
Skew-Hermitian	$V_{ij} = -V_{ij}^H$ for $i = 1:2, j = 0:2$
T-even alternating	$V_{ij} = V_{ij}^T$ for $i = 1:2, j = 0,2$ and $V_{i1} = -V_{i1}^T$ for $i = 1:2$.
T-odd alternating	$V_{ij} = -V_{ij}^T$ for $i = 1:2, j = 0,2$ and $V_{i1} = V_{i1}^T$ for $i = 1:2$.
H-even alternating	$V_{ij} = V_{ij}^H$ for $i = 1:2, j = 0,2$ and $V_{i1} = -V_{i1}^H$ for $i = 1:2$.
H-odd alternating	$V_{ij} = -V_{ij}^H$ for $i = 1:2, j = 0,2$ and $V_{i1} = V_{i1}^H$ for $i = 1:2$.

TABLE 6.1. An overview for structured two parameter matrix systems

Throughout this chapter $w = \begin{bmatrix} w_1^T \\ w_2^T \end{bmatrix} \in \mathbb{R}^{2\times 3}$ be a nonnegative matrix, where $w_i = [w_{i0}, w_{i1}, w_{i2}]^T \in \mathbb{R}^3 \setminus \{(0,0,0)\}$ and $w_{ij}, i = 1 : l, j = 0 : l$ are nonnegative real numbers. For a nonnegative matrix w we define the component-wise inverse via $w^{-1} = \begin{bmatrix} w_{10}^{-1} & w_{11}^{-1} & w_{12}^{-1} \\ w_{20}^{-1} & w_{21}^{-1} & w_{22}^{-1} \end{bmatrix}$, where we use the convention that $w_{ij}^{-1} = 0$ if $w_{ij} = 0$. For a nonnegative vector $v = \begin{bmatrix} v_i \end{bmatrix} \in \mathbb{R}^n$ and $x = \begin{bmatrix} x_j \end{bmatrix} \in \mathbb{C}^n$ we define the weighted 2-norm/seminorm by $\|x\|_{v,2} = (\sum_{i=0}^{n} v_i^2 |x_i|^2)^{1/2}$. If v is strictly positive, i.e., each component of v is positive, then this is a norm, and if it has at least one zero component then it is a seminorm.

Next, we define the unstructured and structured backward errors for two approximate eigenpairs. Let $(\lambda, x_1 \otimes y_1)$ and $(\lambda, x_2 \otimes y_1)$ be two approximate eigenpairs of W, where $\lambda \in \mathbb{C}^3 \setminus \{(0,0,0)\}$ is a semisimple eigenvalue, $x_1, x_2 \in \mathbb{C}^{n_1}$ and $y_1 \in \mathbb{C}^{n_2}$. Then unstructured and structured backward errors are given by

$$\begin{aligned}
\eta_{w,F}(\lambda, x_{1:2}, y_1, W) = \inf\{|\!|\!|(\delta W_1, \delta W_2)|\!|\!|_{w,2} : (W_1(\lambda) + \delta W_1(\lambda))x_i = 0, i = 1:2; \\
(W_2(\lambda) + \delta W_2(\lambda))y_1 = 0\}, \\
\eta^{\mathbf{S}}_{w,F}(\lambda, x_{1:2}, y_1, W) = \inf\{|\!|\!|(\delta W_1, \delta W_2)|\!|\!|_{w,2} : \delta W \in \mathbf{S}, (W_1(\lambda) + \delta W_1(\lambda))x_i = 0, i = 1:2; \\
(W_2(\lambda) + \delta W_2(\lambda))y_1 = 0\},
\end{aligned}$$

respectively, where $\delta W_i, i = 1 : 2$, are of the form (6.1) such that $\delta W_i(\alpha) := \sum_{j=0}^{2} \alpha_j \delta V_{ij}$, $w \in \mathbb{R}^{2\times 3}$ is a nonnegative matrix, $|\!|\!|(\delta W_1, \delta W_2)|\!|\!|^2_{w,2} = \|\delta W_1\|^2_{w_1,2} + \|\delta W_2\|^2_{w_2,2}$ with $\|\delta W_i\|^2_{w_i,2} = \sum_{j=0}^{2} w_{ij}^2 \|\delta V_{ij}\|^2$ for $i = 1 : 2$, and

$$\begin{aligned}
\mathbf{S} := \{&\text{complex symmetric, complex skew-symmetric, Hermitian, skew-Hermitian,} \\
&T\text{-even alternating, T-odd alternating, } H\text{-even alternating, } H\text{-odd alternating}\}.
\end{aligned}$$

Remark 6.2.17. *One can see Chapter-2 and Chapter-6 of* [27] *to obtain structured and unstructured backward error formulas of a single approximate eigenpair.*

To derive the backward error formulas, we will recall the concept of derivative of the map from [9]. Let $z, v \in \mathbb{C}^3$. The partial gardient $\nabla_i \|z\|_{v,2}$ of the map $\mathbb{C}^3 \to \mathbb{R}$, $z \to \|z\|_{v,2}$ which is just the derivative of the map $\mathbb{C} \to \mathbb{R}$, $z_i \to \|[z_0, z_1, z_2]^T\|_{v,2}$ with the variable $z_0, z_{i-1}, z_{i+1}, \ldots, z_2$ are fixed as constants.

Let $\lambda = [\lambda_0, \lambda_1, \lambda_2]^T \in \mathbb{C}^3 \setminus \{(0,0,0)\}$ and $w_p = [w_{p0}, w_{p1}, w_{p2}]^T \in \mathbb{R}^3 \setminus \{(0,0,0)\}$ for $p = 1 : 2$ are nonnegative vectors. Define $H^2_{w_p^{-1},2}(\lambda) = w_{p0}^{-2}|\lambda_0|^2 + w_{p1}^{-2}|\lambda_1|^2 + w_{p2}^{-2}|\lambda_2|^2$.

Now introducing

$$z_{A_{pi}} = \frac{\nabla_{pi} H_{w_p^{-1},2}}{H_{w_p^{-1},2}(\lambda)}, \quad i = 0:2,$$

where the partial gradient is evaluated at $[\lambda_0, \lambda_1, \lambda_2]^T$ and is given by $\nabla_{pi} H_{w_p^{-1},2} = \frac{w_{pi}^{-2}\lambda_i}{H_{w_p^{-1},2}(\lambda)}$.

Lemma 6.2.18. [9] *Let* $\lambda = [\lambda_0, \lambda_1, \lambda_2]^T \in \mathbb{C}^3 \setminus \{(0,0,0)\}$, $w_p = [w_{p0}, w_{p1}, w_{p2}]^T \in \mathbb{R}^3 \setminus \{(0,0,0)\}$, $p = 1:2$, *be nonnegative vectors and* $\nabla_{pi} H_{w_p^{-1},2} = \frac{w_{pi}^{-2}\lambda_i}{H_{w_p^{-1},2}(\lambda)}, p = 1:2, i = 0:2.$ *Then*

$$\sum_{i=0}^{2} w_{pi}^2 |\nabla_{pi} H_{w_p^{-1},2}|^2 = 1.$$

Proof. Proof of the above equality is given in the following manner:

$$\sum_{i=0}^{2} w_{pi}^2 |\nabla_{pi} H_{w_p^{-1},2}|^2 = \sum_{i=0}^{2} w_{pi}^2 \frac{w_{pi}^{-4}|\lambda_i|^2}{H^2_{w_p^{-1},2}(\lambda)} = \sum_{i=0}^{2} \frac{w_{pi}^{-2}|\lambda_i|^2}{H^2_{w_p^{-1},2}(\lambda)} = 1.$$

Next, for $\epsilon = \pm 1$, we define $G^2_{\epsilon\, w_p^{-1},2}(\lambda) = \frac{(1+\epsilon)}{2} w_{p0}^{-2}|\lambda_0|^2 + \frac{(1-\epsilon)}{2} w_{p1}^{-2}|\lambda_1|^2 + \frac{(1+\epsilon)}{2} w_{p2}^{-2}|\lambda_2|^2.$

Lemma 6.2.19. [9] *Let* $\lambda = [\lambda_0, \lambda_1, \lambda_2]^T \in \mathbb{C}^3 \setminus \{(0,0,0)\}$, $w_p = [w_{p0}, w_{p1}, w_{p2}]^T \in \mathbb{R}^3 \setminus \{(0,0,0)\}$, $p = 1:2$, *be nonnegative vectors and* $\nabla_{pi} G_{\epsilon\, w_p^{-1},2} = \frac{(1+\epsilon)}{2} \frac{w_{pi}^{-2}\lambda_i}{G_{\epsilon\, w_p^{-1},2}(\lambda)}, p = 1:2, i = 0, 2,$ *and* $\nabla_{pi} G_{\epsilon, w_p^{-1},2} = \frac{(1-\epsilon)}{2} \frac{w_{pi}^{-2}\lambda_i}{G_{\epsilon\, w_p^{-1},2}(\lambda)}, p = 1:2.$ *Then*

$$\sum_{i=0}^{2} w_{pi}^2 |\nabla_{pi} G_{\epsilon\, w_p^{-1},2}|^2 = 1.$$

Proof. For $\epsilon = \pm 1$, we have $\frac{(1+\epsilon)^2}{4} = \frac{(1+\epsilon)}{2}$, and $\frac{(1-\epsilon)^2}{4} = \frac{(1-\epsilon)}{2}$. Proof of the above equality is given in the following manner:

$$\sum_{i=0}^{2} w_{pi}^2 |\nabla_{pi} G_{\epsilon\, w_p^{-1},2}|^2 = \frac{(1+\epsilon)}{2} w_{p0}^2 \frac{w_{p0}^{-4}|\lambda_0|^2}{G^2_{\epsilon\, w_p^{-1},2}(\lambda)} + \frac{(1-\epsilon)}{2} w_{p1}^2 \frac{w_{p1}^{-4}|\lambda_1|^2}{G^2_{\epsilon\, w_p^{-1},2}(\lambda)} + \frac{(1+\epsilon)}{2} w_{p2}^2 \frac{w_{p2}^{-4}|\lambda_2|^2}{G^2_{\epsilon\, w_p^{-1},2}(\lambda)}.$$

Then

$$\sum_{i=0}^{2} w_{pi}^2 |\nabla_{pi} G_{\epsilon\, w_p^{-1},2}|^2 = \frac{(1+\epsilon)}{2} \frac{w_{p0}^{-2}|\lambda_0|^2}{G^2_{\epsilon\, w_p^{-1},2}(\lambda)} + \frac{(1-\epsilon)}{2} \frac{w_{p1}^{-2}|\lambda_1|^2}{G^2_{\epsilon\, w_p^{-1},2}(\lambda)} + \frac{(1+\epsilon)}{2} \frac{w_{p2}^{-2}|\lambda_2|^2}{G^2_{\epsilon\, w_p^{-1},2}(\lambda)} = 1.$$

Now, in the light of Lemma 6.2.16, we derive the backward error formulas for unstructured and structured two-parameter matrix systems. Note that throughout this chapter, a two-parameter matrix system can be either regular or irregular.

6.3. Backward error analysis for unstructured two-parameter eigenvalue problems

In this section, we derive the backward error formula for two approximate eigenpairs of a double-semisimple eigenvalue for an unstructured two-parameter matrix system.

Theorem 6.3.1. *Let W be a two-parameter matrix system of the form (6.1). Let $(\lambda, x_1 \otimes y_1)$ and $(\lambda, x_2 \otimes y_1)$ be two approximate eigenpairs of W, where $\lambda = (\lambda_0, \lambda_1, \lambda_2) \in \mathbb{C}^3 \setminus \{(0,0,0)\}$ is a double-semisimple eigenvalue, $x_1, x_2 \in \mathbb{C}^{n_1}$ are orthonormal vectors, and $y_1 \in \mathbb{C}^{n_2}$ such that $y_1^H y_1 = 1$. Set $r_{t1} := -W_1(\lambda)x_t$ for $t = 1:2$ and $r_{12} := -W_2(\lambda)y_1$. Then the unstructured backward error of approximate eigenpairs is given by*

$$(\eta_{w,F}(\lambda, x_{1:2}, y_1, W))^2 = \sum_{t=1}^{2} \frac{\|r_{t1}\|_2^2}{H^2_{w_1^{-1},2}(\lambda)} + \frac{\|r_{12}\|_2^2}{H^2_{w_2^{-1},2}(\lambda)},$$

where $H^2_{w_i^{-1},2}(\lambda) = w_{i0}^{-2}|\lambda_0|^2 + w_{i1}^{-2}|\lambda_1|^2 + w_{i2}^{-2}|\lambda_2|^2$, $i = 1:2$, and $w = \begin{bmatrix} w_1^T \\ w_2^T \end{bmatrix} \in \mathbb{R}^{2\times 3}$ be a nonnegative matrix with $w_i = [w_{i0}, w_{i1}, w_{i2}]^T \in \mathbb{R}^3 \setminus \{(0,0,0)\}$.

Proof. For constructing the unstructured backward error formula, we need the minimum *Frobenius* norm values of $\delta V_{ij}, i = 1:2, j = 0:2$. For this purpose, we consider

$$\widetilde{\delta V_{1j}} = U_1^T \delta V_{1j} U_1 = \begin{array}{c} \\ 2 \\ n-2 \end{array} \begin{array}{c} \begin{array}{cc} 2 & n-2 \end{array} \\ \left[\begin{array}{c|c} \widehat{\delta V_{1j}} & \delta {C_{1j}}^T \\ \hline \delta B_{1j} & \delta D_{1j} \end{array}\right] \end{array}, \tag{6.2}$$

where $\widehat{\delta V_{1j}} = \begin{bmatrix} \delta v_{1j,11} & \delta v_{1j,12} \\ \delta v_{1j,21} & \delta v_{1j,22} \end{bmatrix}$, $\delta B_{1j} = \begin{bmatrix} b_{1j,1} & b_{1j,2} \end{bmatrix}$, $\delta C_{1j} = \begin{bmatrix} c_{1j,1} & c_{1j,2} \end{bmatrix}$, for $j = 0:2$, and $U_1 \in \mathbb{C}^{n_1 \times n_1}$ is a unitary matrix such that $U_1 = \begin{bmatrix} U_{11} & U_{21} \end{bmatrix}$ with $U_{11} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \in \mathbb{C}^{n_1 \times 2}$, and

$$\widetilde{\delta V_{2j}} = U_2^T \delta V_{2j} U_2 = \begin{array}{c} \\ 2 \\ n-2 \end{array} \begin{array}{c} \begin{array}{cc} 2 & n-2 \end{array} \\ \left[\begin{array}{c|c} \delta v_{2j,11} & {c_{2j,1}}^T \\ \hline b_{2j,1} & \delta D_{2j} \end{array}\right] \end{array}, \tag{6.3}$$

where $U_2 = \begin{bmatrix} U_{12} & U_{22} \end{bmatrix}$ with $U_{12} = \begin{bmatrix} y_1 \end{bmatrix} \in \mathbb{C}^{n_2 \times 1}$. It is given that $r_{t1} := -W_1(\lambda)x_t$ for $t = 1:2$ and $r_{12} := -W_2(\lambda)y_1$. Then using the unstructured backward error definition, we get $r_{t1} := \delta W_1(\lambda)x_t$ for $t = 1:2$ and $r_{12} := \delta W_2(\lambda)y_1$. From (6.2) we have $\widetilde{\delta W_i}(\lambda) = {U_i}^T \delta W_i(\lambda) U_i$. Further, we get $\widetilde{\delta W_1}(\lambda){U_1}^H x_t = {U_1}^T \delta W_1(\lambda)x_t = {U_1}^T r_{t1}$ for $t = 1:2$. This implies

$$\sum_{j=0}^{2} w_{1j}w_{1j}^{-1}\lambda_j \begin{bmatrix} \widehat{\delta V_{1j}} & \delta C_{1j}^T \\ \delta B_{1j} & \delta D_{1j} \end{bmatrix} \begin{bmatrix} e_t \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} U_{11}^T r_{t1} \\ U_{21}^T r_{t1} \end{bmatrix}, \text{ further simplification gives}$$

$$\begin{bmatrix} (w_{10}w_{10}^{-1}\lambda_0\widehat{\delta V_{10}} + w_{11}w_{1j}^{-1}\lambda_1\widehat{\delta V_{11}} + w_{12}w_{12}^{-1}\lambda_2\widehat{\delta V_{12}})e_t \\ (w_{10}w_{10}^{-1}\lambda_0\delta B_{10} + w_{11}w_{11}^{-1}\lambda_1\delta B_{11} + w_{12}w_{12}^{-1}\lambda_2\delta B_{12})e_t \end{bmatrix} = \begin{bmatrix} U_{11}^T r_{t1} \\ U_{21}^T r_{t1} \end{bmatrix}. \tag{6.4}$$

Also, from (6.3) we have $\widetilde{\delta W_2}(\lambda){U_2}^H y_1 = {U_2}^T \delta W_2(\lambda)y_1 = {U_2}^T r_{12}$. This implies

$$\sum_{j=0}^{2} w_{2j}w_{2j}^{-1}\lambda_j \begin{bmatrix} \delta v_{2j,11} & c_{2j,1}^T \\ b_{2j,1} & \delta D_{2j} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} U_{12}^T r_{12} \\ U_{22}^T r_{12} \end{bmatrix}, \text{ further simplification gives}$$

$$\begin{bmatrix} (w_{20}w_{20}^{-1}\lambda_0\delta v_{20,11} + w_{21}w_{21}^{-1}\lambda_1\delta v_{21,11} + w_{22}w_{12}^{-1}\lambda_2\delta v_{22,11}) \\ (w_{20}w_{20}^{-1}\lambda_0 b_{20,1} + w_{21}w_{21}^{-1}\lambda_1 b_{21,1} + w_{22}w_{22}^{-1}\lambda_2 b_{22,1}) \end{bmatrix} = \begin{bmatrix} U_{12}^T r_{12} \\ U_{22}^T r_{12} \end{bmatrix}, \tag{6.5}$$

where $e_t \in \mathbb{C}^2$ is a vector having 1 at t^{th} position and 0 elsewhere. From (6.4), we get the following equations

$$w_{10}w_{10}^{-1}\lambda_0\delta v_{10,tt} + w_{11}w_{11}^{-1}\lambda_1\delta v_{11,tt} + w_{12}w_{12}^{-1}\lambda_2\delta v_{12,tt} = x_t^T r_{t1}, \;\; t = 1,2, \tag{6.6}$$

$$w_{10}w_{10}^{-1}\lambda_0 b_{10,t} + w_{11}w_{11}^{-1}\lambda_1 b_{11,t} + w_{12}w_{12}^{-1}\lambda_2 b_{12,t} = U_{21}^T r_{t1}, \;\; t = 1,2. \tag{6.7}$$

From (6.5), we get the following equations

$$w_{20}w_{20}^{-1}\lambda_0\delta v_{20,11} + w_{21}w_{21}^{-1}\lambda_1\delta v_{21,11} + w_{22}w_{22}^{-1}\lambda_2\delta v_{22,11} = y_1^T r_{12}, \tag{6.8}$$

$$w_{20}w_{20}^{-1}\lambda_0 b_{20,1} + w_{21}w_{21}^{-1}\lambda_1 b_{21,1} + w_{22}w_{22}^{-1}\lambda_2 b_{22,1} = U_{22}^T r_{12}. \tag{6.9}$$

The minimum norm solutions of (6.6) and (6.7) are given by

$$\delta v_{10,tt} = \overline{z}_{A_{10}} x_t^T r_{t1}, \delta v_{11,tt} = \overline{z}_{A_{11}} x_t^T r_{t1}, \delta v_{12,tt} = \overline{z}_{A_{12}} x_t^T r_{t1};$$

$$b_{10,t} = \overline{z}_{A_{10}} U_{21}^T r_{t1}, b_{11,t} = \overline{z}_{A_{11}} U_{21}^T r_{t1}, b_{12,t} = \overline{z}_{A_{12}} U_{21}^T r_{t1}.$$

The minimum norm solutions of (6.8) and (6.9) are given by

$$\delta v_{20,11} = \overline{z}_{A_{20}} y_1^T r_{12},\; \delta v_{21,11} = \overline{z}_{A_{21}} y_1^T r_{12},\; \delta v_{22,11} = \overline{z}_{A_{22}} y_1^T r_{12};$$

$$b_{20,1} = \overline{z}_{A_{20}} U_{12}^T r_{12},\; b_{21,1} = \overline{z}_{A_{21}} U_{12}^T r_{12},\; b_{22,1} = \overline{z}_{A_{22}} U_{12}^T r_{12}.$$

Further from (6.4), we get the following two equations

$$w_{10}w_{10}^{-1}\delta v_{10,21} + w_{11}w_{11}^{-1}\lambda_1\delta v_{11,21} + w_{12}w_{12}^{-1}\lambda_2\delta v_{12,21} = x_2^T r_{11}, \tag{6.10}$$

$$w_{10}w_{10}^{-1}\delta v_{10,12} + w_{11}w_{11}^{-1}\lambda_1\delta v_{11,12} + w_{12}w_{12}^{-1}\lambda_2\delta v_{12,12} = x_1^T r_{21}. \tag{6.11}$$

The minimum norm solutions of (6.10) and (6.11) are given by

$$\delta v_{10,21} = \overline{z}_{A_{10}}x_2^T r_{11},\ \delta v_{11,21} = \overline{z}_{A_{11}}x_2^T r_{11},\ \delta v_{12,21} = \overline{z}_{A_{12}}x_2^T r_{11};$$

$$\delta v_{10,12} = \overline{z}_{A_{10}}x_1^T r_{21},\ \delta v_{11,12} = \overline{z}_{A_{11}}x_1^T r_{21}, \delta v_{12,12} = \overline{z}_{A_{12}}x_1^T r_{21}.$$

The backward error is given by $(\eta_F(\lambda, x_{1:2}, y_1, W))^2 = \sum_{i=1}^2\sum_{j=0}^2 w_{ij}^2\|\delta V_{ij}\|^2$, where $w_{1j}^2\|\delta V_{1j}\|^2 = w_{1j}^2\|\widehat{\delta V_{1j}}\|^2 + w_{1j}^2\|\delta B_{1j}\|^2 + w_{1j}^2\|\delta C_{1j}\|^2 + w_{1j}^2\|\delta D_{1j}\|^2$, and $w_{2j}^2\|\delta V_{2j}\|^2 = w_{2j}^2|\delta v_{2j,11}|^2 + w_{2j}^2\|c_{2j,1}\|^2 + w_{2j}^2\|b_{2j,1}\|^2 + w_{2j}^2\|\delta D_{2j}\|^2$.

We have $\sum_{j=0}^2 w_{1j}^2\|\delta V_{1j}\|^2 = \sum_{j=0}^2\sum_{t=1}^2 w_{1j}^2|\delta v_{1j,tt}|^2 + w_{1j}^2|\delta v_{1j,12}|^2 + w_{1j}^2|\delta v_{1j,21}|^2 + w_{1j}^2\|b_{1j,t}\|^2 + w_{1j}^2\|c_{1j,t}\|^2 + \|\delta D_{1j}\|^2 = \sum_{j=0}^2\sum_{t=1}^2 w_{1j}^2|z_{A_{1j}}|^2|x_t^T r_{t1}|^2 + |z_{A_{1j}}|^2||x_1^T r_{21}|^2 + |z_{A_{1j}}|^2||x_2^T r_{11}|^2 + |z_{A_{1j}}|^2|\|U_{21}^T r_{t1}\|^2 + w_{1j}^2\|c_{1j,t}\|^2 + \|\delta D_{1j}\|^2$. Since using Lemma 6.2.18, we get

$$\sum_{j=0}^2\sum_{t=1}^2 w_{1j}^2|z_{A_{1j}}|^2|x_t^T r_{t1}|^2 = \sum_{t=1}^2\sum_{j=0}^2 w_{1j}^2\frac{|\nabla_{1i}H_{w_p^{-1},2}|^2|x_t^T r_{t1}|^2}{H^2_{w_1^{-1},2}(\lambda)} = \sum_{t=1}^2\frac{|x_t^T r_{t1}|^2}{H^2_{w_1^{-1},2}(\lambda)},$$

$$\sum_{j=0}^2\sum_{t=1}^2 w_{1j}^2|z_{A_{1j}}|^2|x_1^T r_{21}|^2 = \sum_{t=1}^2\sum_{j=0}^2 w_{1j}^2\frac{|\nabla_{1i}H_{w_1^{-1},2}|^2|x_1^T r_{21}|^2}{H^2_{w_1^{-1},2}(\lambda)} = \sum_{t=1}^2\frac{|x_t^T r_{t1}|^2}{H^2_{w_1^{-1},2}(\lambda)},$$

$$\sum_{j=0}^2\sum_{t=1}^2 w_{1j}^2|z_{A_{1j}}|^2|x_2^T r_{11}|^2 = \sum_{t=1}^2\sum_{j=0}^2 w_{1j}^2\frac{|\nabla_{1i}H_{w_1^{-1},2}|^2|x_2^T r_{11}|^2}{H^2_{w_1^{-2},2}(\lambda)} = \sum_{t=1}^2\frac{|x_t^T r_{t1}|^2}{H^2_{w_1^{-2},2}(\lambda)},$$

$$\sum_{j=0}^2\sum_{t=1}^2 w_{1j}^2|z_{A_{1j}}|^2\|U_{21}^T r_{t1}\|^2 = \sum_{t=1}^2\sum_{j=0}^2 w_{1j}^2\frac{|\nabla_{1i}H_{w_1^{-1},2}|^2\|U_{21}^T r_{t1}\|^2}{H^2_{w_1^{-1},2}(\lambda)} = \sum_{t=1}^2\frac{|x_t^T r_{t1}|^2}{H^2_{w_1^{-1},2}(\lambda)}.$$

Finally, we get $\sum_{j=0}^2 w_{1j}^2\|\delta V_{1j}\|^2 = \sum_{t=1}^2\frac{|x_t^T r_{t1}|^2}{H^2_{w_1^{-1},2}(\lambda)} + \frac{|x_1^T r_{21}|^2}{H^2_{w_1^{-1},2}(\lambda)} + \frac{|x_2^T r_{11}|^2}{H^2_{w_1^{-1},2}(\lambda)} + \frac{\|U_{21}^T r_{t1}\|^2}{H^2_{w_1^{-1},2}(\lambda)} + \sum_{j=0}^2 w_{1j}^2\|c_{1j,t}\|^2 + \sum_{j=0}^2 w_{1j}^2\|\delta D_{1j}\|^2$, where $\|U_{21}^T r_{t1}\|^2 = \|r_{t1}\|^2 - |x_1^T r_{t1}|^2 - |x_2^T r_{t1}|^2$. Since we need minimum norm solution hence setting $c_{1j,t} = 0$ and $\delta D_{1j} = 0$, we get

$$\sum_{j=0}^2 w_{1j}^2\|\delta V_{1j}\|^2 = \frac{\|r_{11}\|^2 + \|r_{21}\|^2}{H^2_{w_1^{-1},2}(\lambda)}. \tag{6.12}$$

In the same manner by using Lemma 6.2.18, we have $\sum_{j=0}^{2} w_{2j}^2\|\delta V_{2j}\|^2 = \sum_{j=0}^{2} w_{2j}^2|\delta v_{2j,11}|^2 + w_{2j}^2\|b_{2j,1}\|^2 + w_{2j}^2\|c_{2j,1}\|^2 + w_{2j}^2\|\delta D_{2j}\|^2 = \frac{|y_1^T r_{12}|^2}{H^2_{w_2^{-1},2}(\lambda)} + \frac{\|U_{12}r_{12}\|^2}{H^2_{w_2^{-1},2}(\lambda)}$, where $\|U_{12}r_{12}\|^2 = \|r_{12}\|^2 - |y_1^T r_{12}|^2$. Similar to (6.12), we get

$$\sum_{j=0}^{2} w_{2j}^2\|\delta V_{2j}\|^2 = \frac{\|r_{12}\|^2}{H^2_{w_2^{-1},2}(\lambda)}. \tag{6.13}$$

Using (6.12) and (6.13), we get

$$(\eta_{w,F}(\lambda, x_{1:2}, y_1, W))^2 = \frac{\|r_{11}\|^2 + \|r_{21}\|^2}{H^2_{w_1^{-1},2}(\lambda)} + \frac{\|r_{12}\|^2}{H^2_{w_2^{-1},2}(\lambda)}. \blacksquare \tag{6.14}$$

Remark 6.3.2. *Substituting back all the obtained entries in (6.2) and (6.3), we get the desired perturbed matrices.*

Next, we state the backward error result for complex symmetric and complex skew-symmetric two-parameter matrix systems.

6.4. Backward error analysis for complex symmetric/complex skew-symmetric two-parameter eigenvalue problems

In this section first we discuss the backward error analysis for complex symmetric and complex skew-symmetric two-parameter matrix systems. Next, we establish a relationship between unstructured and structured backward errors. Throughout this section, $\epsilon = 1$ represents a complex symmetric two-parameter matrix system and $\epsilon = -1$ represents a complex skew-symmetric two-parameter matrix system.

Theorem 6.4.1. *Let W be a complex symmetric/ complex skew-symmetric two-parameter matrix system of the form (6.1). Let $(\lambda, x_1 \otimes y_1)$ and $(\lambda, x_2 \otimes y_1)$ be two approximate eigenpairs of W, where $\lambda = (\lambda_0, \lambda_1, \lambda_2) \in \mathbb{C}^3 \setminus \{(0,0,0)\}$ is a double-semisimple eigenvalue, $x_1, x_2 \in \mathbb{C}^{n_1}$ are orthonormal vectors, and $y_1 \in \mathbb{C}^{n_2}$ such that $y_1^H y_1 = 1$. Set $r_{t1} := -W_1(\lambda)x_t$ for $t = 1:2$ and $r_{12} := -W_2(\lambda)y_1$. Then the backward error of approximate eigenpairs is given by*

$$(\eta^{\mathbf{S}}_{w,F}(\lambda, x_{1:2}, y_1, W))^2 = \sum_{t=1}^{2}\left(\frac{2\|r_{t1}\|_2^2 - (1+\epsilon)/2|x_t^T r_{t1}|^2}{H^2_{w_1^{-1},2}(\lambda)}\right) + \left(\frac{2\|r_{12}\|_2^2 - (1+\epsilon)/2|y_1^T r_{12}|^2}{H^2_{w_2^{-1},2}(\lambda)}\right) - 2\frac{|x_2^T r_{11}|^2}{H^2_{w_1^{-1},2}(\lambda)},$$

where $H^2_{w_i^{-1},2}(\lambda) = w_{i0}^{-2}|\lambda_0|^2 + w_{i1}^{-2}|\lambda_1|^2 + w_{i2}^{-2}|\lambda_2|^2$, $i = 1:2$, *and* $w = \begin{bmatrix} w_1^T \\ w_2^T \end{bmatrix} \in \mathbb{R}^{2\times 3}$ *be a nonnegative matrix with* $w_i = [w_{i0}, w_{i1}, w_{i2}]^T \in \mathbb{R}^3 \setminus \{(0,0,0)\}$.

Proof. For constructing the structured backward error formula, we need the minimum *Frobenius* norm values of δV_{ij} such that $\delta V_{ij} = \epsilon \delta V_{ij}^T, i = 1:2, j = 0:2$. For this purpose, we consider

$$\widetilde{\delta V_{1j}} = U_1^T \delta V_{1j} U_1 = \begin{array}{c c} & \begin{array}{cc} 2 & n-2 \end{array} \\ \begin{array}{c} 2 \\ n-2 \end{array} & \left[\begin{array}{c|c} \widehat{\delta V_{1j}} & \epsilon \delta {B_{1j}}^T \\ \hline \delta B_{1j} & \delta D_{1j} \end{array}\right] \end{array}, \tag{6.15}$$

where $\delta V_{1j} = \epsilon \delta V_{1j}^T, j = 0:2$, $\widehat{\delta V_{1j}} = \begin{bmatrix} \frac{(1+\epsilon)}{2}\delta v_{1j,11} & \epsilon \delta v_{1j,12} \\ \delta v_{1j,12} & \frac{(1+\epsilon)}{2}\delta v_{1j,22} \end{bmatrix}$, $\delta B_{1j} = \begin{bmatrix} b_{1j,1} & b_{1j,2} \end{bmatrix}$, $\delta C_{1j} = \begin{bmatrix} c_{1j,1} & c_{1j,2} \end{bmatrix}$, $\delta D_{1j} = \epsilon \delta {D_{1j}}^T$ for $j = 0:2$, and $U_1 \in \mathbb{C}^{n_1 \times n_1}$ is a unitary matrix such that $U_1 = \begin{bmatrix} U_{11} & U_{21} \end{bmatrix}$ with $U_{11} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \in \mathbb{C}^{n_1 \times 2}$. Further

$$\widetilde{\delta V_{2j}} = U_2^T \delta V_{2j} U_2 = \begin{array}{c c} & \begin{array}{cc} 2 & n-2 \end{array} \\ \begin{array}{c} 2 \\ n-2 \end{array} & \left[\begin{array}{c|c} \frac{(1+\epsilon)}{2}\delta v_{2j,11} & \epsilon {b_{2j,1}}^T \\ \hline b_{2j,1} & \delta D_{2j} \end{array}\right] \end{array}, \tag{6.16}$$

where $\delta D_{2j} = \epsilon \delta {D_{2j}}^T$ for $j = 0:2$, and $U_2 = \begin{bmatrix} U_{12} & U_{22} \end{bmatrix}$ with $U_{12} = \begin{bmatrix} y_1 \end{bmatrix} \in \mathbb{C}^{n_2 \times 1}$.

It is given that $r_{t1} := -W_1(\lambda)x_t$ for $t = 1:2$ and $r_{12} := -W_2(\lambda)y_1$. Then using the structured backward error definition, we get $r_{t1} := \delta W_1(\lambda)x_t$ for $t = 1:2$ and $r_{12} := \delta W_2(\lambda)y_1$. From (6.15) we have $\widetilde{\delta W_i}(\lambda) = {U_i}^T \delta W_i(\lambda) U_i$. Further, we get $\widetilde{\delta W_1}(\lambda){U_1}^H x_t = {U_1}^T \delta W_1(\lambda) x_t = {U_1}^T r_{t1}$ for $t = 1:2$. This implies

$$\sum_{j=0}^{2} w_{1j} w_{1j}^{-1} \lambda_j \begin{bmatrix} \widehat{\delta V_{1j}} & \epsilon \delta B_{1j}^T \\ \delta B_{1j} & \delta D_{1j} \end{bmatrix} \begin{bmatrix} e_t \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} U_{11}^T r_{t1} \\ U_{21}^T r_{t1} \end{bmatrix}, \text{ further simplification gives}$$

$$\begin{bmatrix} (w_{10}w_{10}^{-1}\lambda_0 \widehat{\delta V_{10}} + w_{11}w_{11}^{-1}\lambda_1 \widehat{\delta V_{11}} + w_{12}w_{12}^{-1}\lambda_2 \widehat{\delta V_{12}})e_t \\ (w_{10}w_{10}^{-1}\lambda_0 \delta B_{10} + w_{11}w_{11}^{-1}\lambda_1 \delta B_{11} + w_{12}w_{12}^{-1}\lambda_2 \delta B_{12})e_t \end{bmatrix} = \begin{bmatrix} U_{11}^T r_{t1} \\ U_{21}^T r_{t1} \end{bmatrix}. \tag{6.17}$$

Also, from (6.16) we have $\widetilde{\delta W_2}(\lambda){U_2}^H y_1 = {U_2}^T \delta W_2(\lambda) y_1 = {U_2}^T r_{12}$. This implies

$$\sum_{j=1}^{2} w_{2j} w_{2j}^{-1} \lambda_j \begin{bmatrix} \frac{(1+\epsilon)}{2}\delta v_{2j,11} & \epsilon b_{2j,1}^T \\ b_{2j,1} & \delta D_{2j} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} U_{12}^T r_{12} \\ U_{22}^T r_{12} \end{bmatrix}, \text{ further simplification gives}$$

$$\begin{bmatrix} \frac{(1+\epsilon)}{2}(w_{20}w_{20}^{-1}\lambda_0\delta v_{20,11} + w_{21}w_{21}^{-1}\lambda_1\delta v_{21,11} + w_{22}w_{22}^{-1}\lambda_2\delta v_{22,11}) \\ (w_{20}w_{20}^{-1}\lambda_0\delta b_{20,1} + w_{21}w_{21}^{-1}\lambda_1\delta b_{21,1} + w_{22}w_{22}^{-1}\lambda_2\delta b_{22,1}) \end{bmatrix} = \begin{bmatrix} U_{12}^T r_{12} \\ U_{22}^T r_{12} \end{bmatrix}, \tag{6.18}$$

where $e_t \in \mathbb{C}^2$ is a vector having 1 at t^{th} position and 0 elsewhere. From (6.17), we get the following equations for $t = 1, 2$

$$\frac{(1+\epsilon)}{2} w_{10}w_{10}^{-1}\lambda_0\delta v_{10,tt} + \frac{(1+\epsilon)}{2} w_{11}w_{11}^{-1}\lambda_1\delta v_{11,tt} + \frac{(1+\epsilon)}{2} w_{12}w_{12}^{-1}\lambda_2\delta v_{12,tt} = x_t^T r_{t1}, \tag{6.19}$$

$$w_{10}w_{10}^{-1}\lambda_0 b_{10,t} + w_{11}w_{11}^{-1}\lambda_1 b_{11,t} + w_{12}w_{12}^{-1}\lambda_2 b_{12,t} = U_{21}^T r_{t1}. \tag{6.20}$$

From (6.18), we get the following equations

$$\frac{(1+\epsilon)}{2} w_{20}w_{20}^{-1}\lambda_0\delta v_{20,11} + \frac{(1+\epsilon)}{2} w_{21}w_{21}^{-1}\lambda_1\delta v_{21,11} + \frac{(1+\epsilon)}{2} w_{22}w_{22}^{-1}\lambda_2\delta v_{22,11} = y_1^T r_{12}, \tag{6.21}$$

$$w_{20}w_{20}^{-1}\lambda_0 b_{20,1} + w_{21}w_{21}^{-1}\lambda_1 b_{21,1} + w_{22}w_{22}^{-1}\lambda_2 b_{22,1} = U_{22}^T r_{12}. \tag{6.22}$$

The minimum norm solutions of (6.19) and (6.20) are given by

$$\begin{aligned} \delta v_{10,tt} &= \frac{(1+\epsilon)}{2}\overline{z}_{A_{10}} x_t^T r_{t1}, \delta v_{11,tt} = \frac{(1+\epsilon)}{2}\overline{z}_{A_{11}} x_t^T r_{t1},\ \delta v_{12,tt} = \frac{(1+\epsilon)}{2}\overline{z}_{A_{12}} x_t^T r_{t1} \\ b_{10,t} &= \overline{z}_{A_{10}} U_{21}^T r_{t1}, b_{11,t} = \overline{z}_{A_{11}} U_{21}^T r_{t1}, b_{12,t} = \overline{z}_{A_{12}} U_{21}^T r_{t1}. \end{aligned}$$

The minimum norm solutions of (6.21) and (6.22) are given by

$$\delta v_{20,11} = \frac{(1+\epsilon)}{2}\overline{z}_{A_{10}} y_1^T r_{12}, \delta v_{21,11} = \frac{(1+\epsilon)}{2}\overline{z}_{A_{11}} y_1^T r_{12},\ \delta v_{22,11} = \frac{(1+\epsilon)}{2}\overline{z}_{A_{12}} y_1^T r_{12};$$

$$b_{20,1} = w_{20}^{-2}\overline{z}_{A_{20}} U_{12}^T r_{12}, b_{12,1} = w_{21}^{-2}\overline{z}_{A_{21}} U_{12}^T r_{12}, b_{22,1} = w_{22}^{-2}\overline{z}_{A_{22}} U_{12}^T r_{12}.$$

Further from (6.17), we get the following two equations

$$w_{10}w_{10}^{-1}\lambda_0\delta v_{10,12} + w_{11}w_{11}^{-1}\lambda_1\delta v_{11,12} + w_{12}w_{12}^{-1}\lambda_2\delta v_{12,12} = x_2^T r_{11}, \tag{6.23}$$

$$w_{10}w_{10}^{-1}\lambda_0\delta v_{10,12} + w_{11}w_{11}^{-1}\lambda_1\delta v_{11,12} + w_{12}w_{12}^{-1}\lambda_2\delta v_{12,12} = \epsilon x_1^T r_{21}. \tag{6.24}$$

Equation 6.23 and Equation 6.24 are equal. The minimum norm solutions of (6.23) is given by

$$\delta v_{10,21} = \epsilon\overline{z}_{A_{10}} x_1^T r_{21},\ \delta v_{11,21} = \epsilon\overline{z}_{A_{11}} x_1^T r_{21},\ \delta v_{12,21} = \epsilon\overline{z}_{A_{12}} x_1^T r_{21}.$$

The backward error is given by $(\eta_F(\lambda, x_{1:2}, y_1, W))^2 = \sum_{i=1}^2 \sum_{j=0}^2 w_{ij}^2 \|\delta V_{ij}\|^2$, where $w_{1j}^2\|\delta V_{1j}\|^2$ $= w_{1j}^2\|\widehat{\delta V_{1j}}\|^2 + 2w_{1j}^2\|\delta B_{1j}\|^2 + w_{1j}^2\|\delta D_{1j}\|^2$, and $w_{2j}^2\|\delta V_{2j}\|^2 = w_{2j}^2|\delta v_{2j,11}|^2 + w_{2j}^2\|c_{2j,1}\|^2 + w_{2j}^2\|b_{2j,1}\|^2 + w_{2j}^2\|\delta D_{2j}\|^2$.

Similar to Theorem 6.3.1 by using Lemma 6.2.18, we have $\sum_{j=0}^2 w_{1j}^2\|\delta V_{1j}\|^2 = \sum_{j=0}^2 \sum_{t=1}^2 ((1+\epsilon)/2) w_{1j}^2 |\delta v_{1j,tt}|^2 + 2w_{1j}^2|\delta v_{1j,12}|^2 + 2w_{1j}^2\|b_{1j,t}\|^2 + w_{1j}^2\|\delta D_{1j}\|^2 = \sum_{t=1}^2((1+\epsilon)/2)\frac{|x_t^T r_{t1}|^2}{H^2_{w_1^{-1},2}(\lambda)} + 2\frac{|x_1^T r_{21}|^2}{H^2_{w_1^{-1},2}(\lambda)} + 2\frac{\|U_{21}^T r_{t1}\|^2}{H^2_{w_1^{-1},2}(\lambda)} + \sum_{j=0}^2 w_{1j}^2\|\delta D_{1j}\|^2$, where $\|U_{21}^T r_{t1}\|^2 = \|r_{t1}\|^2 - |x_1^T r_{t1}|^2 - |x_2^T r_{t1}|^2$. Since we need the minimum norm solution hence setting $\delta D_{1j} = 0$, we get

$$\sum_{j=0}^{2} w_{1j}^2\|\delta V_{1j}\|^2 = \sum_{t=1}^{2}\left(\frac{2\|r_{t1}\|_2^2 - ((1+\epsilon)/2)|x_t^T r_{t1}|^2}{H^2_{w_1^{-1},2}(\lambda)}\right) - 2\frac{|x_2^T r_{11}|^2}{H^2_{w_1^{-1},2}(\lambda)}. \tag{6.25}$$

In the same manner, we have $\sum_{j=0}^2 w_{2j}^2\|\delta V_{2j}\|^2 = \sum_{j=0}^2 w_{2j}^2|\delta v_{2j,11}|^2 + 2w_{2j}^2\|b_{2j,1}\|^2 + w_{2j}^2\|\delta D_{2j}\|^2 = \frac{|y_1^T r_{12}|^2}{H^2_{w_2^{-1},2}(\lambda)} + 2\frac{\|U_{12} r_{12}\|^2}{H^2_{w_2^{-1},2}(\lambda)}$, where $\|U_{12} r_{12}\|^2 = \|r_{12}\|^2 - |y_1^T r_{12}|^2$. Similar to (6.25), we get

$$\sum_{j=0}^{2} w_{2j}^2\|\delta V_{2j}\|^2 = \frac{2\|r_{12}\|_2^2 - ((1+\epsilon)/2)|y_1^T r_{12}|^2}{H^2_{w_2^{-1},2}(\lambda)}. \tag{6.26}$$

Using (6.25) and (6.26), we get

$$(\eta^{\mathbf{S}}_{w,F}(\lambda, x_{1:2}, y_1, W))^2 = \sum_{t=1}^2\left(\frac{2\|r_{t1}\|_2^2 - ((1+\epsilon)/2)|x_t^T r_{t1}|^2}{H^2_{w_1^{-1},2}(\lambda)}\right) + \left(\frac{2\|r_{12}\|_2^2 - ((1+\epsilon)/2)|y_1^T r_{12}|^2}{H^2_{w_2^{-1},2}(\lambda)}\right) - 2\frac{|x_2^T r_{11}|^2}{H^2_{w_1^{-1},2}(\lambda)}.$$

Now we present the relation between complex symmetric/complex skew-symmetric and unstructured backward errors.

Lemma 6.4.2. *Let W be a complex symmetric/ complex skew-symmetric two-parameter matrix system of the form (6.1). Let $(\lambda, x_1 \otimes y_1)$ and $(\lambda, x_2 \otimes y_1)$ be two approximate eigenpairs of W, where $\lambda = (\lambda_0, \lambda_1, \lambda_2) \in \mathbb{C}^3 \setminus \{(0,0,0)\}$ is a double-semisimple eigenvalue, $x_1, x_2 \in \mathbb{C}^{n_1}$ are orthonormal vectors, and $y_1 \in \mathbb{C}^{n_2}$ such that $y_1^H y_1 = 1$. Set $r_{t1} := -W_1(\lambda)x_t$ for $t = 1:2$ and $r_{12} := -W_2(\lambda)y_1$. Then*

$$(\eta^{\mathbf{S}}_F(\lambda, x_{1:2}, y_1, W)) \leq \sqrt{2}(\eta_F(\lambda, x_{1:2}, y_1, W)).$$

Proof. From Theorem 6.4.1, we have

$$(\eta_F^{\mathbf{S}}(\lambda, x_{1:2}, y_1, W))^2 = \sum_{t=1}^2 (\tfrac{2\|r_{t1}\|_2^2 - ((1+\epsilon)/2)|x_t^T r_{t1}|^2}{H^2_{w_1^{-1},2}(\lambda)}) + \tfrac{2\|r_{12}\|_2^2 - ((1+\epsilon)/2)|y_1^T r_{12}|^2}{H^2_{w_2^{-1},2}(\lambda)} - 2\tfrac{|x_2^T r_{11}|^2}{H^2_{w_1^{-1},2}(\lambda)}.$$

Since $\dfrac{|x_t^T r_{t1}|^2}{H^2_{w_1^{-1},2}(\lambda)} \geq 0$, $\dfrac{|y_1^T r_{12}|^2}{H^2_{w_2^{-1},2}(\lambda)} \geq 0$, and $\dfrac{|x_2^T r_{11}|^2}{H^2_{w_1^{-1},2}(\lambda)} \geq 0$, we get

$$(\eta_F^{\mathbf{S}}(\lambda, x_{1:2}, y_1, W))^2 \leq \sum_{t=1}^2 \frac{2\|r_{t1}\|_2^2}{H^2_{w_1^{-1},2}(\lambda)} + \frac{2\|r_{12}\|_2^2}{H^2_{w_2^{-1},2}(\lambda)}.$$

Now, using the backward error expression of Theorem 6.3.1 and above inequality, we get the desired result.

Next, we present the backward error analysis for Hermitian/skew-Hermitian two-parameter matrix systems.

6.5. Backward error for Hermitian/skew-Hermitian two-parameter eigenvalue problems

This section deals with the backward error analysis of Hermitian and skew-Hermitian two-parameter matrix systems. For this backward error analysis first we define the following terminologies.

Let W be a Hermitian/skew-Hermitian two-parameter matrix system of the form (6.1). Let $\lambda = (\lambda_0, \lambda_1, \lambda_2) \in \mathbb{C}^3 \setminus \{(0,0,0)\}$, and $x_1, x_2 \in \mathbb{C}^{n_1}$ be orthonormal vectors, and $y_1 \in \mathbb{C}^{n_2}$ such that $y_1^H y_1 = 1$. Introduce $r_{t1} := -W_1(\lambda)x_t$ for $t = 1:2$ and $r_{12} := -W_2(\lambda)y_1$, and and $w = \begin{bmatrix} w_1^T \\ w_2^T \end{bmatrix} \in \mathbb{R}^{2\times 3}$ be a nonnegative matrix with $w_i = [w_{i0}, w_{i1}, w_{i2}]^T \in \mathbb{R}^3 \setminus \{(0,0,0)\}$. Furthermore, define

$$G := \begin{bmatrix} \frac{\Re(\sqrt{\epsilon}\lambda_0)}{w_{10}} & \frac{\Re(\sqrt{\epsilon}\lambda_1)}{w_{11}} & \frac{\Re(\sqrt{\epsilon}\lambda_2)}{w_{12}} \\ \frac{\Im(\sqrt{\epsilon}\lambda_0)}{w_{10}} & \frac{\Im(\sqrt{\epsilon}\lambda_1)}{w_{11}} & \frac{\Im(\sqrt{\epsilon}\lambda_2)}{w_{12}} \end{bmatrix}, H := \begin{bmatrix} \frac{\Re(\sqrt{\epsilon}\lambda_0)}{w_{20}} & \frac{\Re(\sqrt{\epsilon}\lambda_1)}{w_{21}} & \frac{\Re(\sqrt{\epsilon}\lambda_2)}{w_{22}} \\ \frac{\Im(\sqrt{\epsilon}\lambda_0)}{w_{20}} & \frac{\Im(\sqrt{\epsilon}\lambda_1)}{w_{21}} & \frac{\Im(\sqrt{\epsilon}\lambda_2)}{w_{22}} \end{bmatrix}, K := \begin{bmatrix} \frac{\overline{\lambda_0}}{w_{10}} & \frac{\overline{\lambda_1}}{w_{11}} & \frac{\overline{\lambda_2}}{w_{12}} \\ \frac{\lambda_0}{w_{10}} & \frac{\lambda_1}{w_{11}} & \frac{\lambda_2}{w_{12}} \end{bmatrix}.$$

For $t = 1:2$ set

$$g_t := G^+ \begin{bmatrix} \Re(x_t^H r_{t1}) \\ \Im(x_t^H r_{t1}) \end{bmatrix}; h = H^+ \begin{bmatrix} \Re(y_1^H r_{12}) \\ \Im(y_1^H r_{12}) \end{bmatrix}; k := K^+ \begin{bmatrix} \epsilon\overline{x_2^H r_{11}} \\ x_1^H r_{21} \end{bmatrix},$$

where $g_t := [g_{t0}, g_{t1}, g_{t2}]^T; h = [h_0, h_1, h_2]^T; k := [k_0, k_1, k_2]^T$. Now, we derive the main result of this section. Throughout this section, $\epsilon = 1$ represents a Hermitian two-parameter matrix system and $\epsilon = -1$ represents a skew-Hermitian two-parameter matrix system.

Theorem 6.5.1. *Let W be a Hermitian/skew-Hermitian matrix two-parameter matrix system of the form (6.1). Let $(\lambda, x_1 \otimes y_1)$ and $(\lambda, x_2 \otimes y_1)$ be two approximate eigenpairs of W, where $\lambda = (\lambda_0, \lambda_1, \lambda_2) \in \mathbb{C}^3 \setminus \{(0,0,0)\}$ is a double-semisimple eigenvalue, $x_1, x_2 \in \mathbb{C}^{n_1}$ are orthonormal vectors, and $y_1 \in \mathbb{C}^{n_2}$ such that $y_1^H y_1 = 1$. Set $r_{t1} := -W_1(\lambda)x_t$ for $t = 1:2$ and $r_{12} := -W_2(\lambda)y_1$. Then we have*

Case-1: *When $(\lambda_0, \lambda_1, \lambda_2) \in \mathbb{R}^3$, then the backward error of approximate eigenpairs is given by*

$$(\eta^{\mathbf{S}}_{w,F}(\lambda, x_{1:2}, y_1, W))^2 = \frac{\sum_{t=1}^2 2\|r_{t1}\|^2 - \sum_{t=1}^2 |x_t^H r_{t1}|^2}{H^2_{w_1^{-1},2}(\lambda)} - 2\frac{|x_1^H r_{21}|^2}{H^2_{w_1^{-1},2}(\lambda)} + \frac{2\|r_{12}\|^2 - |y_1^H r_{12}|^2}{H^2_{w_2^{-1},2}(\lambda)}.$$

Case-2: *When $(\lambda_0, \lambda_1, \lambda_2) \in \mathbb{C}^3 \setminus \mathbb{R}^3$, then the backward error of approximate eigenpairs is given by*

$$(\eta^{\mathbf{S}}_{w,F}(\lambda, x_{1:2}, y_1, W))^2 = \sum_{j=0}^2 \sum_{t=1}^2 |\frac{g_{tj}}{w_{1j}}|^2 + \sum_{j=0}^2 |\frac{h_j}{w_{2j}}|^2 + 2\sum_{j=0}^2 |\frac{k_j}{w_{1j}}|^2 +$$
$$2(\frac{\sum_{t=1}^2 \|r_{t1}\|^2 - \sum_{t=1}^2 \sum_{i=1}^2 |x_i^H r_{t1}|^2}{H^2_{w_1^{-1},2}(\lambda)}) + 2(\frac{\|r_{12}\|^2 - |y_1^H r_{12}|^2}{H^2_{w_2^{-1},2}(\lambda)}),$$

where $H^2_{w_i^{-1},2}(\lambda) = w_{i0}^{-2}|\lambda_0|^2 + w_{i1}^{-2}|\lambda_1|^2 + w_{i2}^{-2}|\lambda_2|^2$, $i = 1:2$, and $w = \begin{bmatrix} w_1^T \\ w_2^T \end{bmatrix} \in \mathbb{R}^{2\times 3}$ be a nonnegative matrix with $w_i = [w_{i0}, w_{i1}, w_{i2}]^T \in \mathbb{R}^3 \setminus \{(0,0,0)\}$.

Proof. For constructing the structured backward error formula, we need the minimum *Frobenius* norm values of δV_{ij} such that $\delta V_{ij} = \epsilon \delta V_{ij}^H, i = 1:2, j = 0:2$. For this purpose, we consider

$$\widetilde{\delta V_{1j}} = U_1^H \delta V_{1j} U_1 = \begin{array}{c c} & \begin{array}{cc} 2 & n-2 \end{array} \\ \begin{array}{c} 2 \\ n-2 \end{array} & \left[\begin{array}{c|c} \widehat{\delta V_{1j}} & \epsilon \delta B_{1j}{}^H \\ \hline \delta B_{1j} & \delta D_{1j} \end{array}\right] \end{array}, \tag{6.27}$$

where $\widehat{\delta V_{1j}} = \begin{bmatrix} \sqrt{\epsilon}\delta v_{1j,11} & \delta v_{1j,12} \\ \epsilon\overline{\delta v_{1j,12}} & \sqrt{\epsilon}\delta v_{1j,22} \end{bmatrix}$ with $\delta v_{1j,tt} \in \mathbb{R}$, $\delta B_{1j} = [b_{1j,1}\ b_{1j,2}]$, $\delta C_{1j} = [c_{1j,1}\ c_{1j,2}]$, $\delta D_{1j} = \epsilon \delta D_{1j}{}^H$ for $j = 0:2$, and $U_1 \in \mathbb{C}^{n_1 \times n_1}$ is a unitary matrix such that $U_1 = [U_{11}\ U_{21}]$

with $U_{11} = [x_1,\ x_2] \in \mathbb{C}^{n_1\times 2}$.

$$\widetilde{\delta V_{2j}} = U_2^H \delta V_{2j} U_2 = \begin{array}{c} \\ 2 \\ n-2 \end{array}\begin{array}{c} \begin{array}{cc} 2 & n-2 \end{array} \\ \left[\begin{array}{c|c} \sqrt{\epsilon}\delta v_{2j,11} & \epsilon {b_{2j,1}}^H \\ \hline b_{2j,1} & \delta D_{2j} \end{array}\right] \end{array}, \tag{6.28}$$

where $\delta v_{2j,11} \in \mathbb{R}$, $\delta D_{2j} = \epsilon {\delta D_{2j}}^H$ for $j = 0:2$, and $U_2 = [U_{12}\ U_{22}]$ with $U_{12} = y_1 \in \mathbb{C}^{n_2\times 1}$. It is given that $r_{t1} := -W_1(\lambda)x_t$ for $t = 1:2$ and $r_{12} := -W_2(\lambda)y_1$. Then using the structured backward error definition, we get $r_{t1} := \delta W_1(\lambda)x_t$ for $t = 1:2$ and $r_{12} := \delta W_2(\lambda)y_1$. From (6.27), we have $\widetilde{\delta W_i}(\lambda) = {U_i}^H \delta W_i(\lambda) U_i$. Further, we get $\widetilde{\delta W_1}(\lambda){U_1}^H x_t = {U_1}^H \delta W_1(\lambda)x_t = {U_1}^H r_{t1}$ for $t = 1:2$. This implies

$$\sum_{j=0}^{2} w_{1j} w_{1j}^{-1} \lambda_j \begin{bmatrix} \widehat{\delta V_{1j}} & \epsilon \delta B_{1j}^H \\ \delta B_{1j} & \delta D_{1j} \end{bmatrix} \begin{bmatrix} e_t \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} U_{11}^H r_{t1} \\ U_{21}^T r_{t1} \end{bmatrix},$$ further simplification gives

$$\begin{bmatrix} (w_{10}w_{10}^{-1}\lambda_0\widehat{\delta V_{10}} + w_{11}w_{11}^{-1}\lambda_1\widehat{\delta V_{11}} + w_{12}w_{12}^{-1}\lambda_2\widehat{\delta V_{12}})e_t \\ (w_{10}w_{10}^{-1}\lambda_0\delta B_{10} + w_{11}w_{11}^{-1}\lambda_1\delta B_{11} + w_{12}w_{12}^{-1}\lambda_2\delta B_{12})e_t \end{bmatrix} = \begin{bmatrix} U_{11}^H r_{t1} \\ U_{21}^H r_{t1} \end{bmatrix}, \tag{6.29}$$

Also, from (6.28) we have $\widetilde{\delta W_2}(\lambda)U_2^H y_1 = {U_2}^H \delta W_2(\lambda)y_1 = {U_2}^H r_{12}$. This implies

$$\sum_{j=0}^{2} w_{2j} w_{2j}^{-1} \lambda_j \begin{bmatrix} \delta v_{2j,11} & b_{2j,1}^H \\ b_{2j,1} & \delta D_{2j} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} U_{12}^H r_{12} \\ U_{22}^H r_{12} \end{bmatrix},$$ further simplification gives

$$\begin{bmatrix} \sqrt{\epsilon}(w_{20}w_{20}^{-1}\lambda_0\delta v_{20,11} + w_{21}w_{21}^{-1}\lambda_1\delta v_{21,11} + w_{21}w_{21}^{-1}\lambda_2\delta v_{22,11}) \\ (w_{20}w_{20}^{-1}\lambda_0\delta b_{20,1} + w_{21}w_{21}^{-1}\lambda_1\delta b_{21,1} + w_{22}w_{22}^{-1}\lambda_2\delta b_{22,1}) \end{bmatrix} = \begin{bmatrix} U_{12}^H r_{12} \\ U_{22}^H r_{12} \end{bmatrix}, \tag{6.30}$$

where $e_t \in \mathbb{C}^2$ is a vector having 1 at t^{th} position and 0 elsewhere. From (6.29), we get the following equations

$$\sqrt{\epsilon}w_{10}w_{10}^{-1}\lambda_0\delta v_{10,tt} + \sqrt{\epsilon}w_{11}w_{11}^{-1}\lambda_1\delta v_{11,tt} + \sqrt{\epsilon}w_{12}w_{12}^{-1}\lambda_2\delta v_{12,tt} = x_t^H r_{t1},\ \ t = 1, 2, \tag{6.31}$$

$$w_{10}w_{10}^{-1}\lambda_0 b_{10,t} + w_{11}w_{11}^{-1}\lambda_1 b_{11,t} + w_{12}w_{12}^{-1}\lambda_2 b_{12,t} = U_{21}^H r_{t1},\ \ t = 1, 2. \tag{6.32}$$

From (6.30), we get the following equations

$$\sqrt{\epsilon}w_{20}w_{20}^{-1}\lambda_0\delta v_{20,11} + \sqrt{\epsilon}w_{21}w_{21}^{-1}\lambda_1\delta v_{21,11} + \sqrt{\epsilon}w_{22}w_{22}^{-1}\lambda_2\delta v_{22,11} = y_1^H r_{12}, \tag{6.33}$$

$$w_{20}w_{20}^{-1}\lambda_0 b_{20,1} + w_{21}w_{21}^{-1}\lambda_1 b_{21,1} + w_{22}w_{22}^{-1}\lambda_2 b_{22,1} = U_{22}^H r_{12}. \tag{6.34}$$

When $(\lambda_0, \lambda_1, \lambda_2) \in \mathbb{R}^3$, then the minimum norm solution of (6.31) is given by

$$\delta v_{10,tt} = \overline{z}_{A_{10}} \overline{\sqrt{\epsilon}} x_t^H r_{t1}, \delta v_{11,tt} = \overline{z}_{A_{11}} \overline{\sqrt{\epsilon}} x_t^H r_{t1}, \delta v_{12,tt} = \overline{z}_{A_{12}} \overline{\sqrt{\epsilon}} x_t^H r_{t1}.$$

On the other hand when $(\lambda_0, \lambda_1, \lambda_2) \in \mathbb{C}^3 \setminus \mathbb{R}^3$, we can rewrite (6.31) in the following form

$$\sum_{j=0}^{2} w_{1j} w_{1j}^{-1} \Re(\sqrt{\epsilon}\lambda_j)\, \delta v_{1j,tt} = \Re(x_t^H r_{t1})$$

$$\sum_{j=0}^{2} w_{1j} w_{1j}^{-1} \Im(\sqrt{\epsilon}\lambda_j)\, \delta v_{1j,tt} = \Im(x_t^H r_{t1})$$

The minimum norm solution of the above two equations in the combined form is given by

$$\begin{bmatrix} w_{10}\delta v_{10,tt} \\ w_{11}\delta v_{11,tt} \\ w_{12}\delta v_{12,tt} \end{bmatrix} = \begin{bmatrix} \frac{\Re(\sqrt{\epsilon}\lambda_0)}{w_{10}} & \frac{\Re(\sqrt{\epsilon}\lambda_1)}{w_{11}} & \frac{\Re(\sqrt{\epsilon}\lambda_2)}{w_{12}} \\ \frac{\Im(\sqrt{\epsilon}\lambda_0)}{w_{10}} & \frac{\Im(\sqrt{\epsilon}\lambda_1)}{w_{11}} & \frac{\Im(\sqrt{\epsilon}\lambda_2)}{w_{12}} \end{bmatrix}^+ \begin{bmatrix} \Re(x_t^H r_{t1}) \\ \Im(x_t^H r_{t1}) \end{bmatrix} = [g_{t0}, g_{t1}, g_{t2}]^T = g. \tag{6.35}$$

The minimum norm solution of (6.32) is given by

$$b_{10,t} = \overline{z}_{A_{10}} U_{21}^H r_{t1}, b_{11,t} = \overline{z}_{A_{11}} U_{21}^H r_{t1}, b_{12,t} = \overline{z}_{A_{12}} U_{21}^H r_{t1}.$$

When $(\lambda_0, \lambda_1, \lambda_2) \in \mathbb{R}^3$, then the minimum norm solution of (6.33) is given by

$$\delta v_{20,11} = \overline{z}_{A_{20}} \overline{\sqrt{\epsilon}} y_1^H r_{12}, \delta v_{21,11} = \overline{z}_{A_{21}} \overline{\sqrt{\epsilon}} y_1^H r_{12}, \delta v_{22,11} = \overline{z}_{A_{22}} \overline{\sqrt{\epsilon}} y_1^H r_{12}.$$

When $(\lambda_1, \lambda_1, \lambda_2) \in \mathbb{C}^3 \setminus \mathbb{R}^3$, separating (6.33) in real and imaginary parts, we get the following two equation

$$\sum_{j=0}^{2} \sqrt{\epsilon}\, w_{2j} w_{2j}^{-1} \Re(\lambda_j)\, \delta v_{2j,11} = \Re(y_1^H r_{12}) \tag{6.36}$$

$$\sum_{j=0}^{2} w_{2j} w_{2j}^{-1} \sqrt{\epsilon}\, \Im(\lambda_j)\, \delta v_{2j,tt} = \Im(y_1^H r_{12}) \tag{6.37}$$

The minimum norm solutions of (6.36) and (6.37) in the combined form is given by

$$\begin{bmatrix} w_{20}\delta v_{20,11} \\ w_{21}\delta v_{21,11} \\ w_{22}\delta v_{22,11} \end{bmatrix} = \begin{bmatrix} \frac{\Re(\sqrt{\epsilon}\lambda_0)}{w_{20}} & \frac{\Re(\sqrt{\epsilon}\lambda_1)}{w_{21}} & \frac{\Re(\sqrt{\epsilon}\lambda_2)}{w_{22}} \\ \frac{\Im(\sqrt{\epsilon}\lambda_0)}{w_{20}} & \frac{\Im(\sqrt{\epsilon}\lambda_1)}{w_{21}} & \frac{\Im(\sqrt{\epsilon}\lambda_2)}{w_{22}} \end{bmatrix}^+ \begin{bmatrix} \Re(y_1^H r_{12}) \\ \Im(y_1^H r_{12}) \end{bmatrix} = [h_0, h_1, h_2]^T = h. \tag{6.38}$$

The minimum norm solution of (6.34) is given by

$$b_{20,1} = \overline{z}_{A_{20}} U_{12}^H r_{12},\ b_{21,1} = \overline{z}_{A_{21}} U_{12}^H r_{12},\ b_{22,1} = \overline{z}_{A_{22}} U_{12}^H r_{12}.$$

Further from (6.29), we get the following two equations

$$w_{10}w_{10}^{-1}\overline{\lambda}_0\delta v_{10,12}+w_{11}w_{11}^{-1}\overline{\lambda}_1\delta v_{11,12}+w_{12}w_{12}^{-1}\overline{\lambda}_2\delta v_{12,12}=\overline{\epsilon x_2^H r_{11}}, \tag{6.39}$$

$$w_{10}w_{10}^{-1}\lambda_0\delta v_{10,12}+w_{11}w_{11}^{-1}\lambda_1\delta v_{11,12}+w_{12}w_{12}^{-1}\lambda_2\delta v_{12,12}=x_1^H r_{21}. \tag{6.40}$$

When $(\lambda_0,\lambda_1,\lambda_2)\in\mathbb{R}^3$, we know that $\overline{\epsilon x_2^H r_{11}}=x_1^H r_{21}$. Hence (6.39) and (6.40) are the same and the minimum norm solution (6.40) is given by

$$\delta v_{10,12}=\overline{z}_{A_{10}}\overline{\sqrt{\epsilon}}x_1^H r_{21},\delta v_{11,12}=\overline{z}_{A_{11}}\overline{\sqrt{\epsilon}}x_1^H r_{21},\delta v_{12,12}=\overline{z}_{A_{12}}\overline{\sqrt{\epsilon}}x_1^H r_{21}.$$

On the other hand if $(\lambda_0,\lambda_1,\lambda_2)\in\mathbb{C}^3\setminus\mathbb{R}^3$, from (6.39) and (6.40), we get the minimum norm solution as follows:

$$\begin{bmatrix} w_{10}\delta v_{10,12}\\ w_{11}\delta v_{11,12}\\ w_{12}\delta v_{12,12}\end{bmatrix}=\begin{bmatrix}\frac{\overline{\lambda_0}}{w_{10}} & \frac{\overline{\lambda_1}}{w_{11}} & \frac{\overline{\lambda_2}}{w_{12}}\\ \frac{\lambda_0}{w_{10}} & \frac{\lambda_1}{w_{11}} & \frac{\lambda_2}{w_{12}}\end{bmatrix}^{+}\begin{bmatrix}\overline{\epsilon x_2^H r_{11}}\\ x_1^H r_{21}\end{bmatrix}=[k_0,k_1,k_2]^T=:k. \tag{6.41}$$

The backward error is given by $(\eta_F(\lambda,x_{1:2},y_1,W))^2=\sum_{i=1}^2\sum_{j=0}^2 w_{ij}^2\|\delta V_{ij}\|^2$, where $w_{1j}^2\|\delta V_{1j}\|^2=w_{1j}^2\|\widehat{\delta V_{1j}}\|^2+2w_{1j}^2\|\delta B_{1j}\|^2+w_{1j}^2\|\delta D_{1j}\|^2$, and $w_{2j}^2\|\delta V_{2j}\|^2=w_{2j}^2|\delta v_{2j,11}|^2+w_{2j}^2\|c_{2j,1}\|^2+w_{2j}^2\|b_{2j,1}\|^2+w_{2j}^2\|\delta D_{2j}\|^2$. When $(\lambda_0,\lambda_1,\lambda_2)\in\mathbb{R}^3$, similar to Theorem 6.3.1 by using Lemma 6.2.18, we get $\sum_{j=0}^2 w_{1j}^2\|\delta V_{1j}\|^2=\sum_{j=0}^2\sum_{t=1}^2 w_{1j}^2|\delta v_{1j,tt}|^2+2w_{1j}^2|\delta v_{1j,12}|^2+2w_{1j}^2\|b_{1j,t}\|^2+w_{1j}^2\|\delta D_{1j}\|^2=\sum_{t=1}^2\frac{|x_t^H r_{t1}|^2}{H^2_{w_1^{-1},2}(\lambda)}+2\sum_{j=0}^2\frac{|x_1^H r_{21}|^2}{H^2_{w_1^{-1},2}(\lambda)}+2\frac{\|U_{21}^H r_{t1}\|^2}{H^2_{w_1^{-1},2}(\lambda)}+\sum_{j=0}^2 w_{1j}^2\|\delta D_{1j}\|^2$, where $\|U_{21}^H r_{t1}\|^2=\|r_{t1}\|^2-|x_1^H r_{t1}|^2-|x_2^H r_{t1}|^2$. Since we need the minimum norm solution hence setting $\delta D_{1j}=0$, we get

$$\sum_{j=0}^2 w_{1j}^2\|\delta V_{1j}\|^2=\frac{\sum_{t=1}^2 2\|r_{t1}\|^2-\sum_{t=1}^2|x_t^H r_{t1}|^2}{H^2_{w_1^{-1},2}(\lambda)}-2\frac{|x_1^H r_{21}|^2}{H^2_{w_1^{-1},2}(\lambda)}. \tag{6.42}$$

When $(\lambda_0,\lambda_1,\lambda_2)\in\mathbb{C}^3\backslash\mathbb{R}^3$, we get $\sum_{j=0}^2 w_{1j}^2\|\delta V_{1j}\|^2=\sum_{j=0}^2\sum_{t=1}^2 w_{1j}^2|\delta v_{1j,tt}|^2+2w_{1j}^2|\delta v_{1j,12}|^2+2w_{1j}^2\|b_{1j,t}\|^2+w_{1j}^2\|\delta D_{1j}\|^2=\sum_{j=0}^2\sum_{t=1}^2|\frac{g_{tj}}{w_{1j}}|^2+2|\frac{k_j}{w_{1j}}|^2+2\frac{\|U_{21}^H r_{t1}\|^2}{H^2_{w_1^{-1},2}(\lambda)}+w_{1j}^2\|\delta D_{1j}\|^2$, where $\|U_{21}^H r_{t1}\|^2=\|r_{t1}\|^2-|x_1^H r_{t1}|^2-|x_2^H r_{t1}|^2$. Since we need the minimum norm solution hence setting $\delta D_{1j}=0$, we get

$$\sum_{j=0}^2 w_{1j}^2\|\delta V_{1j}\|^2=\sum_{j=0}^2\sum_{t=1}^2|\frac{g_{tj}}{w_{1j}}|^2+2\sum_{j=0}^2|\frac{k_j}{w_{1j}}|^2+2(\frac{\sum_{t=1}^2\|r_{t1}\|^2-\sum_{t=1}^2\sum_{i=1}^2|x_i^H r_{t1}|^2}{H^2_{w_1^{-1},2}(\lambda)}). \tag{6.43}$$

In the same manner, when $(\lambda_0,\lambda_1,\lambda_2)\in\mathbb{R}^3$, we have $\sum_{j=0}^2 w_{2j}^2\|\delta V_{2j}\|^2=\sum_{j=0}^2 w_{2j}^2|\delta v_{2j,11}|^2+2w_{2j}^2\|b_{2j,1}\|^2+w_{2j}^2\|\delta D_{2j}\|^2=\frac{|y_1^T r_{12}|^2}{H^2_{w_2^{-1},2}(\lambda)}+2\frac{\|U_{12}r_{12}\|^2}{H^2_{w_2^{-1},2}(\lambda)}+w_{2j}^2\|\delta D_{2j}\|^2$, where $\|U_{12}r_{12}\|^2=$

$\|r_{12}\|^2 - |y_1^H r_{12}|^2$. Similar to (6.42), we get

$$\sum_{j=0}^{2} w_{2j}^2 \|\delta V_{2j}\|^2 = \left(\frac{2\|r_{12}\|^2 - |y_1^H r_{12}|^2}{H^2_{w_2^{-1},2}(\lambda)}\right). \tag{6.44}$$

On the other hand, when $(\lambda_0, \lambda_1, \lambda_2) \in \mathbb{C}^3 \backslash \mathbb{R}^3$, we have $\sum_{j=0}^{2} w_{2j}^2 \|\delta V_{2j}\|^2 = \sum_{j=0}^{2} w_{2j}^2 |\delta v_{2j,11}|^2$ $+ 2w_{2j}^2 \|b_{2j,1}\|^2 + w_{2j}^2 \|\delta D_{2j}\|^2 = \sum_{j=0}^{2} |\frac{h_j}{w_{2j}}|^2 + 2\frac{\|U_{12} r_{12}\|^2}{H^2_{w_2^{-1},2}(\lambda)} + w_{2j}^2 \|\delta D_{2j}\|^2$, where $\|U_{12} r_{12}\|^2 =$ $\|r_{12}\|^2 - |y_1^H r_{12}|^2$. Similar to (6.43), we get

$$\sum_{j=0}^{2} w_{2j}^2 \|\delta V_{2j}\|^2 = \sum_{j=0}^{2} |\frac{h_j}{w_{2j}}|^2 + 2\left(\frac{\|r_{12}\|^2 - |y_1^H r_{12}|^2}{H^2_{w_2^{-1},2}(\lambda)}\right). \tag{6.45}$$

For $(\lambda_0, \lambda_1, \lambda_2) \in \mathbb{R}^3$ by using (6.42) and (6.44), we get

(6.46)

$$(\eta_F^{\mathbf{S}}(\lambda, x_{1:2}, y_1, W))^2 = \frac{\sum_{t=1}^{2} 2\|r_{t1}\|^2 - \sum_{t=1}^{2} |x_t^H r_{t1}|^2}{H^2_{w_1^{-1},2}(\lambda)} - 2\frac{|x_1^H r_{21}|^2}{H^2_{w_1^{-1},2}(\lambda)} + \frac{2\|r_{12}\|^2 - |y_1^H r_{12}|^2}{H^2_{w_2^{-1},2}(\lambda)}.$$

Similarly for $(\lambda_0, \lambda_1, \lambda_2) \in \mathbb{C}^3 \setminus \mathbb{R}^3$ by using (6.43) and (6.45), we get the desired backward error. ■

6.6. Backward error for T-even/T-odd alternating two-parameter eigenvalue problem

In this section we discuss the backward error analysis for T-even alternating and T-odd alternating two-parameter matrix systems. For this backward error analysis first we define the following terminologies.

Let W be a T-even/T-odd alternating two-parameter matrix system of the form (6.1). Let $\lambda = (\lambda_0, \lambda_1, \lambda_2) \in \mathbb{C}^3 \setminus \{(0, 0, 0)\}$, $x_1, x_2 \in \mathbb{C}^{n_1}$ are orthonormal vectors, and $y_1 \in \mathbb{C}^{n_2}$ such that $y_1^H y_1 = 1$. Introduce $r_{t1} := -W_1(\lambda) x_t$ for $t = 1 : 2$ and $r_{12} := -W_2(\lambda) y_1$, and $w = \begin{bmatrix} w_1^T \\ w_2^T \end{bmatrix} \in \mathbb{R}^{2\times 3}$ be a nonnegative matrix with $w_i = [w_{i0}, w_{i1}, w_{i2}]^T \in \mathbb{R}^3 \setminus \{(0, 0, 0)\}$. Furthermore, define

$$A := \begin{bmatrix} \frac{\lambda_0}{w_{10}} & \frac{\lambda_1}{w_{11}} & \frac{\lambda_2}{w_{12}} \\ \frac{\lambda_0}{w_{10}} & -\frac{\lambda_1}{w_{11}} & \frac{\lambda_2}{w_{12}} \end{bmatrix},$$

and set

$$a := \begin{bmatrix} \frac{\lambda_0}{w_{10}} & \frac{\lambda_1}{w_{11}} & \frac{\lambda_2}{w_{12}} \\ \frac{\lambda_0}{w_{10}} & -\frac{\lambda_1}{w_{11}} & \frac{\lambda_2}{w_{12}} \end{bmatrix}^{+} \begin{bmatrix} x_2^T r_{11} \\ \epsilon x_1^T r_{21} \end{bmatrix},$$

where $a := [a_0, a_1, a_2]^T$.

Throughout this section, $\epsilon = 1$ represents a T-even alternating two-parameter matrix system and $\epsilon = -1$ represents a T-odd alternating two-parameter matrix system.

Theorem 6.6.1. *Let W be a T-even/T-odd alternating two-parameter matrix system of the form (6.1). Let $(\lambda, x_1 \otimes y_1)$ and $(\lambda, x_2 \otimes y_1)$ be two approximate eigenpairs of W, where $\lambda = (\lambda_0, \lambda_1, \lambda_2) \in \mathbb{C}^3 \setminus \{(0,0,0)\}$ is a double-semisimple eigenvalue, $x_1, x_2 \in \mathbb{C}^{n_1}$ are orthonormal vectors, and $y_1 \in \mathbb{C}^{n_2}$ such that $y_1^H y_1 = 1$. Set $r_{t1} := -W_1(\lambda)x_t$ for $t = 1:2$ and $r_{12} := -W_2(\lambda)y_1$. Then the backward error of approximate eigenpairs is given by*

$$(\eta^{\mathbf{S}}_{w,F}(\lambda, x_{1:2}, y_1, W))^2 = \sum_{t=1}^{2}\left(\frac{1}{G^2_{\epsilon\, w_1^{-1},2}(\lambda)} - \frac{2}{H^2_{w_1^{-1},2}(\lambda)})|x_t^T r_{t1}|^2 - 2\frac{|x_2^T r_{11}|^2 + |x_1^T r_{21}|^2}{H^2_{w_1^{-1},2}(\lambda)}\right) + \sum_{t=1}^{2}(\frac{2\|r_{t1}\|_2^2}{H^2_{w_1^{-1},2}(\lambda)}) + 2\sum_{j=0}^{2}|\frac{a_j}{w_{1j}}|^2 + \frac{2\|r_{12}\|_2^2}{H^2_{w_2^{-1},2}(\lambda)} + (\frac{1}{G^2_{\epsilon\, w_2^{-1},2}(\lambda)} - \frac{2}{H^2_{w_2^{-1},2}(\lambda)})|y_1^T r_{12}|^2,$$

where $H^2_{w_i^{-1},2}(\lambda) = w_{i0}^{-2}|\lambda_0|^2 + w_{i1}^{-2}|\lambda_1|^2 + w_{i2}^{-2}|\lambda_2|^2$, and $G^2_{\epsilon\, w_i^{-1},2}(\lambda) = \frac{(1+\epsilon)}{2}w_{i0}^{-2}|\lambda_0|^2 + \frac{(1-\epsilon)}{2}w_{i1}^{-2}|\lambda_1|^2 + \frac{(1+\epsilon)}{2}w_{i2}^{-2}|\lambda_2|^2$ for $i = 1:2$, and and $w = \begin{bmatrix} w_1^T \\ w_2^T \end{bmatrix} \in \mathbb{R}^{2\times 3}$ be a nonnegative matrix with $w_i = [w_{i0}, w_{i1}, w_{i2}]^T \in \mathbb{R}^3 \setminus \{(0,0,0)\}$.

Proof. For constructing the structured backward error formula, we need the minimum *Frobenius* norm values of δV_{ij} such that $\delta V_{ij} = \epsilon \delta V_{ij}^T, i = 1:2, j = 0,2$. For this purpose, we consider

$$\widetilde{\delta V_{1j}} = U_1^T \delta V_{1j} U_1 = \begin{array}{c} \\ 2 \\ n-2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} \quad 2 \quad & \quad n-2 \end{array} \\ \left[\begin{array}{c|c} \widehat{\delta V_{1j}} & \epsilon \delta B_{1j}{}^T \\ \hline \delta B_{1j} & \delta D_{1j} \end{array}\right] \end{array}, \tag{6.47}$$

where $\widehat{\delta V_{1j}} = \begin{bmatrix} \frac{(1+\epsilon)}{2}\delta v_{1j,11} & \epsilon \delta v_{1j,12} \\ \delta v_{1j,12} & \frac{(1+\epsilon)}{2}\delta v_{1j,22} \end{bmatrix}$, $\delta B_{1j} = \begin{bmatrix} b_{1j,1} & b_{1j,2} \end{bmatrix}$, $\delta C_{1j} = \begin{bmatrix} c_{1j,1} & c_{1j,2} \end{bmatrix}$, $\delta D_{1j} = \epsilon \delta D_{1j}{}^T$ for $j = 0,2$, and $U_1 \in \mathbb{C}^{n_1 \times n_1}$ is a unitary matrix such that $U_1 = \begin{bmatrix} U_{11} & U_{21} \end{bmatrix}$ with $U_{11} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \in \mathbb{C}^{n_1 \times 2}$. Next, we consider

$$\widetilde{\delta V_{2j}} = U_2^T \delta V_{2j} U_2 = \begin{array}{c} \\ 2 \\ n-2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} \quad 2 \quad & \quad n-2 \end{array} \\ \left[\begin{array}{c|c} \frac{(1+\epsilon)}{2}\delta v_{2j,11} & \epsilon b_{2j,1}{}^T \\ \hline b_{2j,1} & \delta D_{2j} \end{array}\right] \end{array}, \tag{6.48}$$

where $\delta D_{2j} = \epsilon \delta {D_{2j}}^T$ for $j = 0, 2$, and $U_2 = \begin{bmatrix} U_{12} & U_{22} \end{bmatrix}$ with $U_{12} = \begin{bmatrix} y_1 \end{bmatrix} \in \mathbb{C}^{n_2 \times 1}$.

Further to get $\delta V_{11} = -\epsilon \delta V_{11}^T$, we consider

$$\widetilde{\delta V_{11}} = U_1^T \delta V_{11} U_1 = \begin{array}{c c} & \begin{array}{cc} 2 & n-2 \end{array} \\ \begin{array}{c} 2 \\ n-2 \end{array} & \left[\begin{array}{c|c} \widehat{\delta V_{11}} & \epsilon \delta {B_{11}}^T \\ \hline \delta B_{1j} & \delta D_{11} \end{array}\right] \end{array}, \tag{6.49}$$

where $\delta V_{11} = -\epsilon \delta V_{11}^T, i = 1:2$, $\widehat{\delta V_{11}} = \begin{bmatrix} \frac{(1-\epsilon)}{2} \delta v_{11,11} & -\epsilon \delta v_{11,12} \\ \delta v_{11,12} & \frac{(1+\epsilon)}{2} \delta v_{11,22} \end{bmatrix}$, $\delta B_{11} = \begin{bmatrix} b_{11,1} & b_{11,2} \end{bmatrix}$, $\delta C_{11} = \begin{bmatrix} c_{11,1} & c_{11,2} \end{bmatrix}$, $\delta D_{11} = -\epsilon \delta {D_{11}}^T$. Next, we consider

$$\widetilde{\delta V_{21}} = U_2^T \delta V_{21} U_2 = \begin{array}{c c} & \begin{array}{cc} 2 & n-2 \end{array} \\ \begin{array}{c} 2 \\ n-2 \end{array} & \left[\begin{array}{c|c} \frac{(1-\epsilon)}{2} \delta v_{21,11} & -\epsilon {b_{21,1}}^T \\ \hline b_{21,1} & \delta D_{21} \end{array}\right] \end{array}, \tag{6.50}$$

where $\delta D_{21} = -\epsilon \delta {D_{21}}^T$, and $U_2 = \begin{bmatrix} U_{12} & U_{22} \end{bmatrix}$ with $U_{12} = \begin{bmatrix} y_1 \end{bmatrix} \in \mathbb{C}^{n_2 \times 1}$.

It is given that $r_{t1} := -W_1(\lambda) x_t$ for $t = 1:2$ and $r_{12} := -W_2(\lambda) y_1$. Then using the structured backward error definition, we get $r_{t1} := \delta W_1(\lambda) x_t$ for $t = 1:2$ and $r_{12} := \delta W_2(\lambda) y_1$. From (6.49) we have $\widetilde{\delta W_i}(\lambda) = {U_i}^T \delta W_i(\lambda) U_i$. Further, we get $\widetilde{\delta W_1}(\lambda) {U_1}^H x_t = {U_1}^T \delta W_1(\lambda) x_t = {U_1}^T r_{t1}$ for $t = 1:2$. This implies

$$\left(w_{10} w_{10}^{-1} \lambda_0 \begin{bmatrix} \widehat{\delta V_{10}} & \epsilon \delta B_{10}^T \\ \delta B_{10} & \delta D_{10} \end{bmatrix} + w_{11} w_{11}^{-1} \lambda_1 \begin{bmatrix} \widehat{\delta V_{11}} & -\epsilon \delta B_{11}^T \\ \delta B_{11} & \delta D_{1j} \end{bmatrix} \right) \begin{bmatrix} e_t \\ \mathbf{0} \end{bmatrix} + \left(w_{12} w_{12}^{-1} \lambda_2 \begin{bmatrix} \widehat{\delta V_{12}} & \epsilon \delta B_{12}^T \\ \delta B_{12} & \delta D_{12} \end{bmatrix} \right) \begin{bmatrix} e_t \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} U_{11}^T r_{t1} \\ U_{21}^T r_{t1} \end{bmatrix},$$

further simplification gives

$$\begin{bmatrix} (w_{10} w_{10}^{-1} \lambda_0 \widehat{\delta V_{10}} + w_{11} w_{11}^{-1} \lambda_1 \widehat{\delta V_{11}} + w_{12} w_{12}^{-1} \lambda_2 \widehat{\delta V_{12}}) e_t \\ (w_{10} w_{10}^{-1} \lambda_0 \delta B_{10} + w_{11} w_{11}^{-1} \lambda_1 \delta B_{11} + w_{12} w_{12}^{-1} \lambda_2 \delta B_{12}) e_t \end{bmatrix} = \begin{bmatrix} U_{11}^T r_{t1} \\ U_{21}^T r_{t1} \end{bmatrix}. \tag{6.51}$$

Also, from (6.48) and (6.50) we have $\widetilde{\delta W_2}(\lambda) {U_2}^H y_1 = {U_2}^T \delta W_2(\lambda) y_1 = {U_2}^T r_{12}$. This implies

$$\left(w_{20} w_{20}^{-1} \lambda_0 \begin{bmatrix} \frac{(1+\epsilon)}{2} \delta v_{20,11} & \epsilon b_{20,1}^T \\ b_{20,1} & \delta D_{20} \end{bmatrix} + w_{21} w_{21}^{-1} \lambda_1 \begin{bmatrix} \frac{(1-\epsilon)}{2} \delta v_{21,11} & -\epsilon b_{21,1}^T \\ b_{21,1} & \delta D_{21} \end{bmatrix} \right) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \left(w_{22} w_{22}^{-1} \lambda_2 \begin{bmatrix} \frac{(1+\epsilon)}{2} \delta v_{22,11} & \epsilon b_{22,1}^T \\ b_{22,1} & \delta D_{22} \end{bmatrix} \right) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} U_{12}^T r_{12} \\ U_{22}^T r_{12} \end{bmatrix}.$$

Further simplification gives

(6.52)
$$\begin{bmatrix} \frac{(1+\epsilon)}{2} w_{20}w_{20}^{-1}\lambda_0\delta v_{20,11} + \frac{(1-\epsilon)}{2} w_{21}w_{21}^{-1}\lambda_1\delta v_{21,11} + \frac{(1+\epsilon)}{2} w_{22}w_{22}^{-1}\lambda_2\delta v_{22,11} \\ (w_{20}w_{20}^{-1}\lambda_0\delta b_{20,1} + w_{21}w_{21}^{-1}\lambda_1\delta b_{21,1} + w_{22}w_{22}^{-1}\lambda_2\delta b_{22,1}) \end{bmatrix} = \begin{bmatrix} U_{12}^T r_{12} \\ U_{22}^T r_{12} \end{bmatrix},$$

where $e_t \in \mathbb{C}^2$ is a vector having 1 at t^{th} position and 0 elsewhere. From (6.51), we get the following equations for $t = 1:2$

$$\frac{(1+\epsilon)}{2} w_{10}w_{10}^{-1}\lambda_0\delta v_{10,tt} + \frac{(1-\epsilon)}{2} w_{11}w_{11}^{-1}\lambda_1\delta v_{11,tt} + \frac{(1+\epsilon)}{2} w_{12}w_{12}^{-1}\lambda_2\delta v_{12,tt} = x_t^T r_{t1}, \tag{6.53}$$

$$w_{10}w_{10}^{-1}\lambda_0 b_{10,t} + w_{11}w_{11}^{-1}\lambda_1 b_{11,t} + w_{12}w_{12}^{-1}\lambda_2 b_{12,t} = U_{21}^T r_{t1}. \tag{6.54}$$

From (6.52), we get the following equations

(6.55)
$$\frac{(1+\epsilon)}{2} w_{20}w_{20}^{-1}\lambda_0\delta v_{20,11} + \frac{(1-\epsilon)}{2} w_{21}w_{21}^{-1}\lambda_1\delta v_{21,11} + \frac{(1+\epsilon)}{2} w_{22}w_{22}^{-1}\lambda_2\delta v_{22,11} = y_1^T r_{12},$$

$$w_{20}w_{20}^{-1}\lambda_0 b_{20,1} + w_{21}w_{21}^{-1}\lambda_1 b_{21,1} + w_{22}w_{22}^{-1}\lambda_2 b_{22,1} = U_{22}^T r_{12}. \tag{6.56}$$

The minimum norm solutions of (6.53) and (6.54) are given by

$$\begin{aligned} \delta v_{10,tt} &= \frac{(1+\epsilon)}{2}\overline{z}_{A_{10}} x_t^T r_{t1}, \delta v_{11,tt} = \frac{(1-\epsilon)}{2}\overline{z}_{A_{11}} x_t^T r_{t1},\ \delta v_{12,tt} = \frac{(1+\epsilon)}{2}\overline{z}_{A_{12}} x_t^T r_{t1} \\ b_{10,t} &= \overline{z}_{A_{10}} U_{21}^T r_{t1}, b_{11,t} = \overline{z}_{A_{11}} U_{21}^T r_{t1}, b_{12,t} = \overline{z}_{A_{12}} U_{21}^T r_{t1}. \end{aligned}$$

The minimum norm solutions of (6.55) and (6.56) are given by

$$\delta v_{20,11} = ((1+\epsilon)/2)\overline{z}_{A_{10}} y_1^T r_{12}, \delta v_{21,11} = ((1-\epsilon)/2)\overline{z}_{A_{11}} y_1^T r_{12},\ \delta v_{22,11} = ((1+\epsilon)/2)\overline{z}_{A_{12}} y_1^T r_{12};$$

$$b_{20,1} = w_{20}^{-2}\overline{z}_{A_{20}} U_{12}^T r_{12}, b_{12,1} = w_{21}^{-2}\overline{z}_{A_{21}} U_{12}^T r_{12}, b_{22,1} = w_{22}^{-2}\overline{z}_{A_{22}} U_{12}^T r_{12}.$$

Further from (6.51), we get the following two equations

$$w_{10}w_{10}^{-1}\lambda_0\delta v_{10,12} + w_{11}w_{11}^{-1}\lambda_1\delta v_{11,12} + w_{12}w_{12}^{-1}\lambda_2\delta v_{12,12} = x_2^T r_{11}, \tag{6.57}$$

$$w_{10}w_{10}^{-1}\lambda_0\delta v_{10,12} - w_{11}w_{11}^{-1}\lambda_1\delta v_{11,12} + w_{12}w_{12}^{-1}\lambda_2\delta v_{12,12} = \epsilon x_1^T r_{21}. \tag{6.58}$$

From (6.57) and (6.58) the minimum norm solution is given by

$$\begin{bmatrix} w_{10}\delta v_{10,12} \\ w_{11}\delta v_{11,12} \\ w_{12}\delta v_{12,12} \end{bmatrix} = \begin{bmatrix} \frac{\lambda_0}{w_{10}} & \frac{\lambda_1}{w_{11}} & \frac{\lambda_2}{w_{12}} \\ \frac{\lambda_0}{w_{10}} & -\frac{\lambda_1}{w_{11}} & \frac{\lambda_2}{w_{12}} \end{bmatrix}^+ \begin{bmatrix} x_2^T r_{11} \\ \epsilon x_1^T r_{21} \end{bmatrix} = a. \tag{6.59}$$

Backward error is given by $(\eta_F(\lambda, x_{1:2}, y_1, W))^2 = \sum_{i=1}^2 \sum_{j=0}^2 w_{ij}^2 \|\delta V_{ij}\|^2$, where $w_{1j}^2\|\delta V_{1j}\|^2$ $= w_{1j}^2\|\widehat{\delta V_{1j}}\|^2 + 2w_{1j}^2\|\delta B_{1j}\|^2 + w_{1j}^2\|\delta D_{1j}\|^2$, and $w_{2j}^2\|\delta V_{2j}\|^2 = w_{2j}^2|\delta v_{2j,11}|^2 + w_{2j}^2\|c_{2j,1}\|^2 +$ $w_{2j}^2\|b_{2j,1}\|^2 + w_{2j}^2\|\delta D_{2j}\|^2$.

Similar to Theorem 6.3.1 by using Lemma 6.2.19, we have $\sum_{j=0}^{2} w_{1j}^2\|\delta V_{1j}\|^2 = \sum_{j=0}^{2}\sum_{t=1}^{2}\frac{(1+\epsilon)}{2}w_{1j}^2|\delta v_{1j,tt}|^2 + 2w_{1j}^2|\delta v_{1j,12}|^2 + 2w_{1j}^2\|b_{1j,t}\|^2 + w_{1j}^2\|\delta D_{1j}\|^2 = \sum_{t=1}^{2}\frac{|x_t^T r_{t1}|^2}{G^2_{\epsilon\, w_1^{-1},2}(\lambda)} + 2\frac{\|U_{21}^T r_{t1}\|^2}{H^2_{w_1^{-1},2}(\lambda)} + 2\sum_{j=0}^{2}|\frac{a_j}{w_{1j}}|^2 + \sum_{j=0}^{2} w_{1j}^2\|\delta D_{1j}\|^2$, where $\|U_{21}^T r_{t1}\|^2 = \|r_{t1}\|^2 - |x_1^T r_{t1}|^2 - |x_2^T r_{t1}|^2$. Since we need the minimum norm solution hence setting $\delta D_{1j} = 0$, we get

$$\sum_{j=0}^{2} w_{1j}^2\|\delta V_{1j}\|^2 = \sum_{t=1}^{2}\Big(\frac{2\|r_{t1}\|_2^2}{H^2_{w_1^{-1},2}(\lambda)} + \Big(\frac{1}{G^2_{\epsilon\, w_1^{-1},2}(\lambda)} - \frac{2}{H^2_{w_1^{-1},2}(\lambda)}\Big)|x_t^T r_{t1}|^2 - 2\frac{|x_2^T r_{11}|^2 + |x_1^T r_{21}|^2}{H^2_{w_1^{-1},2}(\lambda)}\Big) + 2\sum_{j=0}^{2}|\frac{a_j}{w_{1j}}|^2. \tag{6.60}$$

In the same manner, we have $\sum_{j=0}^{2} w_{2j}^2\|\delta V_{2j}\|^2 = \sum_{j=0}^{2} w_{2j}^2|\delta v_{2j,11}|^2 + 2w_{2j}^2\|b_{2j,1}\|^2 + w_{2j}^2\|\delta D_{2j}\|^2 = \frac{|y_1^T r_{12}|^2}{G^2_{\epsilon\, w_2^{-1},2}(\lambda)} + 2\frac{\|U_{12} r_{12}\|^2}{H^2_{w_2^{-1},2}(\lambda)}$, where $\|U_{12}r_{12}\|^2 = \|r_{12}\|^2 - |y_1^T r_{12}|^2$. Similar to (6.60), we get

$$\sum_{j=0}^{2} w_{2j}^2\|\delta V_{2j}\|^2 = \frac{2\|r_{12}\|_2^2}{H^2_{w_2^{-1},2}(\lambda)} + \Big(\frac{1}{G^2_{\epsilon\, w_2^{-1},2}(\lambda)} - \frac{2}{H^2_{w_2^{-1},2}(\lambda)}\Big)|y_1^T r_{12}|^2. \tag{6.61}$$

Using (6.60) and (6.61), we get

$$(\eta^{\mathbf{S}}_{w,F}(\lambda, x_{1:2}, y_1, W))^2 = \sum_{t=1}^{2}\left(\frac{1}{G^2_{\epsilon\, w_1^{-1},2}(\lambda)} - \frac{2}{H^2_{w_1^{-1},2}(\lambda)})|x_t^T r_{t1}|^2 - 2\frac{|x_2^T r_{11}|^2 + |x_1^T r_{21}|^2}{H^2_{w_1^{-1},2}(\lambda)}\right) + \sum_{t=1}^{2}\Big(\frac{2\|r_{t1}\|_2^2}{H^2_{w_1^{-1},2}(\lambda)}\Big) + 2\sum_{j=0}^{2}|\frac{a_j}{w_{1j}}|^2 + \frac{2\|r_{12}\|_2^2}{H^2_{w_2^{-1},2}(\lambda)} + \Big(\frac{1}{G^2_{\epsilon\, w_2^{-1},2}(\lambda)} - \frac{2}{H^2_{w_2^{-1},2}(\lambda)}\Big)|y_1^T r_{12}|^2. \blacksquare$$

6.7. Backward error for H-even/H-odd alternating two-parameter eigenvalue problems

This section deals with the backward error analysis of H-even alternating and H-odd alternating two-parameter matrix systems. For this backward error analysis first we define the following terminologies.

Let W be a H-even/H-odd alternating two-parameter matrix system of the form (6.1). Let $\lambda = (\lambda_0, \lambda_1, \lambda_2) \in \mathbb{C}^3 \setminus \{(0,0,0)\}$, and $x_1, x_2 \in \mathbb{C}^{n_1}$ be orthonormal vectors, and $y_1 \in \mathbb{C}^{n_2}$ such that $y_1^H y_1 = 1$. Introduce $r_{t1} := -W_1(\lambda)x_t$ for $t = 1:2$ and $r_{12} := -W_2(\lambda)y_1$, and

and $w = \begin{bmatrix} w_1^T \\ w_2^T \end{bmatrix} \in \mathbb{R}^{2\times 3}$ be a nonnegative matrix with $w_i = [w_{i0}, w_{i1}, w_{i2}]^T \in \mathbb{R}^3 \backslash \{(0,0,0)\}$. Furthermore, define

$$L := \begin{bmatrix} \frac{\Re(\sqrt{\epsilon}\lambda_0)}{w_{10}} & \frac{\Re(\sqrt{-\epsilon}\lambda_1)}{w_{11}} & \frac{\Re(\sqrt{\epsilon}\lambda_2)}{w_{12}} \\ \frac{\Im(\sqrt{\epsilon}\lambda_0)}{w_{10}} & \frac{\Im(\sqrt{-\epsilon}\lambda_1)}{w_{11}} & \frac{\Im(\sqrt{\epsilon}\lambda_2)}{w_{12}} \end{bmatrix}, Q := \begin{bmatrix} \frac{\Re(\sqrt{\epsilon}\lambda_0)}{w_{20}} & \frac{\Re(\sqrt{-\epsilon}\lambda_1)}{w_{21}} & \frac{\Re(\sqrt{\epsilon}\lambda_2)}{w_{22}} \\ \frac{\Im(\sqrt{\epsilon}\lambda_0)}{w_{20}} & \frac{\Im(\sqrt{-\epsilon}\lambda_1)}{w_{21}} & \frac{\Im(\sqrt{\epsilon}\lambda_2)}{w_{22}} \end{bmatrix},$$

$$Z := \begin{bmatrix} \frac{\overline{\lambda_0}}{w_{10}} & -\frac{\overline{\lambda_1}}{w_{11}} & \frac{\overline{\lambda_2}}{w_{12}} \\ \frac{\lambda_0}{w_{10}} & \frac{\lambda_1}{w_{11}} & \frac{\lambda_2}{w_{12}} \end{bmatrix}.$$

For $t = 1:2$ set

$$l_t := L^+ \begin{bmatrix} \Re(x_t^H r_{t1}) \\ \Im(x_t^H r_{t1}) \end{bmatrix}; q = Q^+ \begin{bmatrix} \Re(y_1^H r_{12}) \\ \Im(y_1^H r_{12}) \end{bmatrix}; z := Z^+ \begin{bmatrix} \overline{\epsilon x_2^H r_{11}} \\ x_1^H r_{21} \end{bmatrix},$$

where $l_t := [l_{t0}, l_{t1}, l_{t2}]^T; q := [q_0, q_1, q_2]^T; z := [z_0, z_1, z_2]^T$. Now, we derive the main result of this section. Throughout this section, $\epsilon = 1$ represents a H-even alternating two-parameter matrix system and $\epsilon = -1$ represents a H-odd alternating two-parameter matrix system.

Theorem 6.7.1. *Let W be a H-even/H-odd alternating matrix two-parameter matrix system of the form (6.1). Let $(\lambda, x_1 \otimes y_1)$ and $(\lambda, x_2 \otimes y_1)$ be two approximate eigenpairs of W, where $\lambda = (\lambda_0, \lambda_1, \lambda_2) \in \mathbb{C}^3 \setminus \{(0,0,0)\}$ is a double-semisimple eigenvalue, $x_1, x_2 \in \mathbb{C}^{n_1}$ are orthonormal vectors, and $y_1 \in \mathbb{C}^{n_2}$ such that $y_1^H y_1 = 1$. Set $r_{t1} := -W_1(\lambda)x_t$ for $t = 1:2$ and $r_{12} := -W_2(\lambda)y_1$. Then we have*

Case-1: *When $\lambda_0 \in \sqrt{\epsilon}\mathbb{R}, \lambda_1 \in \sqrt{-\epsilon}\mathbb{R}, \lambda_2 \in \sqrt{\epsilon}\mathbb{R}$, then the backward error of approximate eigenpairs is given by*

$$(\eta^{\mathbf{S}}_{w,F}(\lambda, x_{1:2}, y_1, W))^2 = \frac{\sum_{t=1}^2 [2\|r_{t1}\|^2 - |x_t^H r_{t1}|^2]}{H^2_{w_1^{-1},2}(\lambda)} - 2\frac{|x_1^H r_{21}|^2]}{H^2_{w_1^{-1},2}(\lambda)} + \frac{2\|r_{12}\|^2 - |y_1^H r_{12}|^2}{H^2_{w_2^{-1},2}(\lambda)}.$$

Case-2: *Otherwise, the backward error of approximate eigenpairs is given by*

$$\begin{aligned}(\eta^{\mathbf{S}}_{w,F}(\lambda, x_{1:2}, y_1, W))^2 = & \sum_{j=0}^{2}\sum_{t=1}^{2} |\frac{l_{tj}}{w_{1j}}|^2 + \sum_{j=0}^{2} |\frac{q_j}{w_{2j}}|^2 + 2\sum_{j=0}^{2} |\frac{z_j}{w_{1j}}|^2 + \\ & 2(\frac{\sum_{t=1}^2 \|r_{t1}\|^2 - \sum_{t=1}^2\sum_{i=1}^2 |x_i^H r_{t1}|^2}{H^2_{w_1^{-1},2}(\lambda)}) + 2(\frac{\|r_{12}\|^2 - |y_1^H r_{12}|^2}{H^2_{w_2^{-1},2}(\lambda)}),\end{aligned}$$

where $H^2_{w_i^{-1},2}(\lambda) = w_{i0}^{-2}|\lambda_0|^2 + w_{i1}^{-2}|\lambda_1|^2 + w_{i2}^{-2}|\lambda_2|^2$, $i = 1:2$, and $w = \begin{bmatrix} w_1^T \\ w_2^T \end{bmatrix} \in \mathbb{R}^{2\times 3}$ be a nonnegative matrix with $w_i = [w_{i0}, w_{i1}, w_{i2}]^T \in \mathbb{R}^3 \setminus \{(0,0,0)\}$.

Proof. Proof is similar to Theorem 6.5.1. We present only the main steps of the proof. For constructing the structured backward error formula, we need the minimum *Frobenius* norm values of δV_{ij} such that $\delta V_{ij} = \epsilon \delta V_{ij}^H, i = 1:2, j = 0:2$. For this purpose, we consider

$$\widetilde{\delta V_{1j}} = U_1^H \delta V_{1j} U_1 = \begin{array}{c} \\ 2 \\ n-2 \end{array} \!\!\begin{array}{c} \begin{array}{cc} 2 & n-2 \end{array} \\ \left[\begin{array}{c|c} \widehat{\delta V_{1j}} & \epsilon \delta B_{1j}{}^H \\ \hline \delta B_{1j} & \delta D_{1j} \end{array}\right] \end{array}, \tag{6.62}$$

where $\widehat{\delta V_{1j}} = \begin{bmatrix} \sqrt{\epsilon}\delta v_{1j,11} & \delta v_{1j,12} \\ \epsilon \overline{\delta v_{1j,12}} & \sqrt{\epsilon}\delta v_{1j,22} \end{bmatrix}$ with $\delta v_{1j,tt} \in \mathbb{R}$, $\delta B_{1j} = [b_{1j,1}\ b_{1j,2}]$, $\delta C_{1j} = [c_{1j,1}\ c_{1j,2}]$, $\delta D_{1j} = \epsilon \delta D_{1j}{}^H$ for $j = 0, 2$, and $U_1 \in \mathbb{C}^{n_1 \times n_1}$ is a unitary matrix such that $U_1 = \begin{bmatrix} U_{11} & U_{21} \end{bmatrix}$ with $U_{11} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \in \mathbb{C}^{n_1 \times 2}$. Next, we consider

$$\widetilde{\delta V_{2j}} = U_2^H \delta V_{2j} U_2 = \begin{array}{c} \\ 2 \\ n-2 \end{array} \!\!\begin{array}{c} \begin{array}{cc} 2 & n-2 \end{array} \\ \left[\begin{array}{c|c} \sqrt{\epsilon}\delta v_{2j,11} & \epsilon b_{2j,1}{}^H \\ \hline b_{2j,1} & \delta D_{2j} \end{array}\right] \end{array}, \tag{6.63}$$

where $\delta D_{2j} = \epsilon \delta D_{2j}{}^H$ for $j = 0, 2$, and $U_2 = \begin{bmatrix} U_{12} & U_{22} \end{bmatrix}$ with $U_{12} = \begin{bmatrix} y_1 \end{bmatrix} \in \mathbb{C}^{n_2 \times 1}$. Further to get $\delta V_{11} = -\epsilon \delta V_{11}^H$, we consider

$$\widetilde{\delta V_{11}} = U_1^H \delta V_{11} U_1 = \begin{array}{c} \\ 2 \\ n-2 \end{array} \!\!\begin{array}{c} \begin{array}{cc} 2 & n-2 \end{array} \\ \left[\begin{array}{c|c} \widehat{\delta V_{11}} & -\epsilon \delta B_{11}{}^H \\ \hline \delta B_{1j} & \delta D_{11} \end{array}\right] \end{array}, \tag{6.64}$$

where $\widehat{\delta V_{11}} = \begin{bmatrix} \sqrt{-\epsilon}\delta v_{11,11} & \delta v_{11,12} \\ -\epsilon \overline{\delta v_{11,12}} & \sqrt{\epsilon}\delta v_{11,22} \end{bmatrix}$, $\delta B_{11} = \begin{bmatrix} b_{11,1} & b_{11,2} \end{bmatrix}$, $\delta C_{11} = \begin{bmatrix} c_{11,1} & c_{11,2} \end{bmatrix}$, $\delta D_{11} = -\epsilon \delta D_{11}{}^H$. for $j = 0:2$, and $U_1 \in \mathbb{C}^{n_1 \times n_1}$ is a unitary matrix such that $U_1 = [U_{11}\ U_{21}]$ with $U_{11} = [x_1,\ x_2] \in \mathbb{C}^{n_1 \times 2}$. Also, to get $\delta V_{21} = -\epsilon \delta V_{21}^H$, we consider

$$\widetilde{\delta V_{21}} = U_2^H \delta V_{21} U_2 = \begin{array}{c} \\ 2 \\ n-2 \end{array} \!\!\begin{array}{c} \begin{array}{cc} 2 & n-2 \end{array} \\ \left[\begin{array}{c|c} \sqrt{-\epsilon}\delta v_{21,11} & -\epsilon b_{21,1}{}^H \\ \hline b_{21,1} & \delta D_{21} \end{array}\right] \end{array}, \tag{6.65}$$

where $\delta v_{2j,11} \in \mathbb{R}$, $\delta D_{21} = \epsilon \delta D_{21}{}^H$, and $U_2 = [U_{12}\ U_{22}]$ with $U_{12} = y_1 \in \mathbb{C}^{n_2 \times 1}$. From now onwards, rest of the proof is similar to Theorem 6.5.1.

6.8. Numerical experiments

In this section, we discuss the behaviour of structured and unstructured backward errors of a single and two approximate eigenpairs of a double semisimple eigenvalue through numerical experiments. For example, by using Matlab software, we have generated several random Hermitian and complex symmetric two-parameter matrix systems of the form (6.1). For these structured two-parameter matrix systems, we present two tables. In these tables, we have compared the structured (complex symmetric or Hermitian) and unstructured backward errors of a single approximate eigenpair, and structured and unstructured backward errors of two approximate eigenpairs of a double-semisimple eigenvalue. From Table 6.2 and Table 6.3, we have found that there is a large difference between the backward error of a single eigenpair and the backward error of two approximate eigenpairs of a double-semisimple eigenvalue. These tables show that the existing study of the backward error analysis of a single eigenpair [**27**] is not sufficient for getting the true backward error results. The development of our results is quite important for getting a real picture of the backward error analysis.

$\eta_F(\lambda, x_{1:2}, y_1, W)$	$\eta_F(\lambda, x_1, y_1, W)$	$\eta_F(\lambda, x_2, y_1, W)$	$\eta_F^{\mathbf{S}}(\lambda, x_{1:2}, y_1, W)$	$\eta_F^{\mathbf{S}}(\lambda, x_1, y_1, W)$	$\eta_F^{\mathbf{S}}(\lambda, x_2, y_1, W)$
12.3023	11.6804	10.3821	12.9837	12.7720	11.5145
23.7252	10.4937	22.7006	27.8680	12.8338	27.4516
26.8988	9.8124	26.1355	31.3653	11.2592	30.7277
10.8864	10.8408	7.1434	12.2760	12.2509	8.8901
31.6428	11.0866	30.9074	36.3550	12.9398	36.1764

TABLE 6.2. Difference between the backward error of a single eigenpair and the backward error of two approximate eigenpairs of a double-semisimple eigenvalue of complex symmetric case

$\eta_F(\lambda, x_{1:2}, y_1, W)$	$\eta_F(\lambda, x_1, y_1, W)$	$\eta_F(\lambda, x_2, y_1, W)$	$\eta_F^{\mathbf{S}}(\lambda, x_{1:2}, y_1, W)$	$\eta_F^{\mathbf{S}}(\lambda, x_1, y_1, W)$	$\eta_F^{\mathbf{S}}(\lambda, x_2, y_1, W)$
46.0795	37.4226	27.5172	62.9120	51.1588	38.3118
62.4263	22.6460	58.3689	96.0504	32.1382	89.0737
90.4605	48.6684	76.5280	108.7541	64.7518	97.0628
77.5618	64.4508	43.6031	103.7190	87.0330	60.2110
64.3337	37.0634	52.9468	82.9928	49.6117	70.7678

TABLE 6.3. Difference between the backward error of a single eigenpair and the backward error of two approximate eigenpairs of a double-semisimple eigenvalue for Hermitian case

Remark 6.8.1. *For rest of the structures discussed in this chapter, one can easily obtain the similar tables to show the importance of backward error analysis.*

Remark 6.8.2. *We have borrowed the backward error formulas for a single approximate eigenpair from* [**27**].

In [**50**] authors have plotted the graph which represents the ratio between structured (Hermitian) and unstructured backward error of a single eigenpair for two-parameter eigenvalue problem. They found that the majority of ratio lies in interval [1, 4]. In the similar manner, next we present two graphs which represents the ratio between structured (Hermitian or complex symmetric) and unstructured backward errors of two specified eigenpairs of a double semisimple eigenvalue. To obtain these graphs, we have generated several random Hermitian and complex symmetric two-parameter matrix systems of the form (6.1) by using Matlab software. From the several numerical experiments, we have found that the majority of the ratios usually distributed in $[1, 2]$. From the several numerical experiments, we have taken 100 random numerical values to plot the graphs: Figure 6.1 and Figure 6.2.

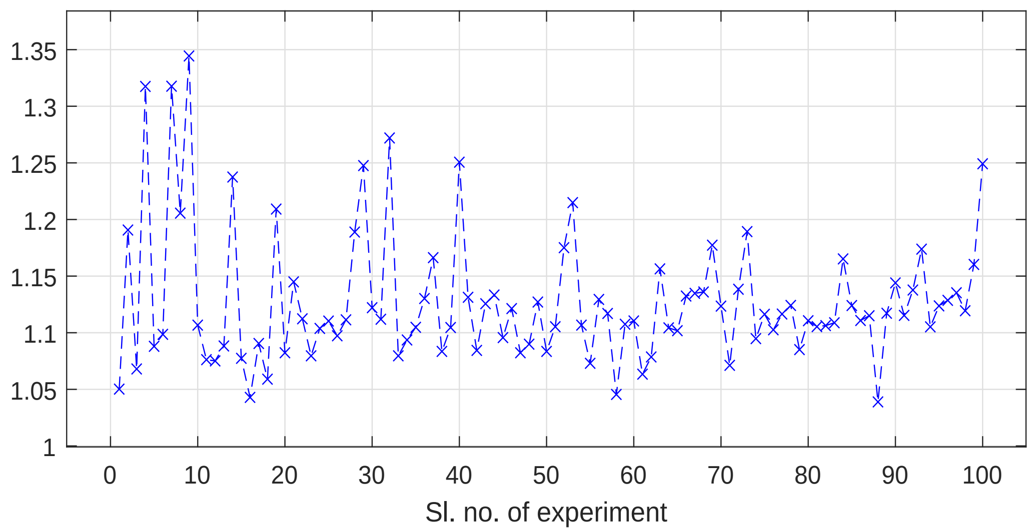

FIGURE 6.1. Ratio of complex symmetric backward error and unstructured backward error.

Remark 6.8.3. *Similar to Figure 6.1 and Figure 6.2, one can also plot the graphs for the rest of the structures to obtain the respective intervals.*

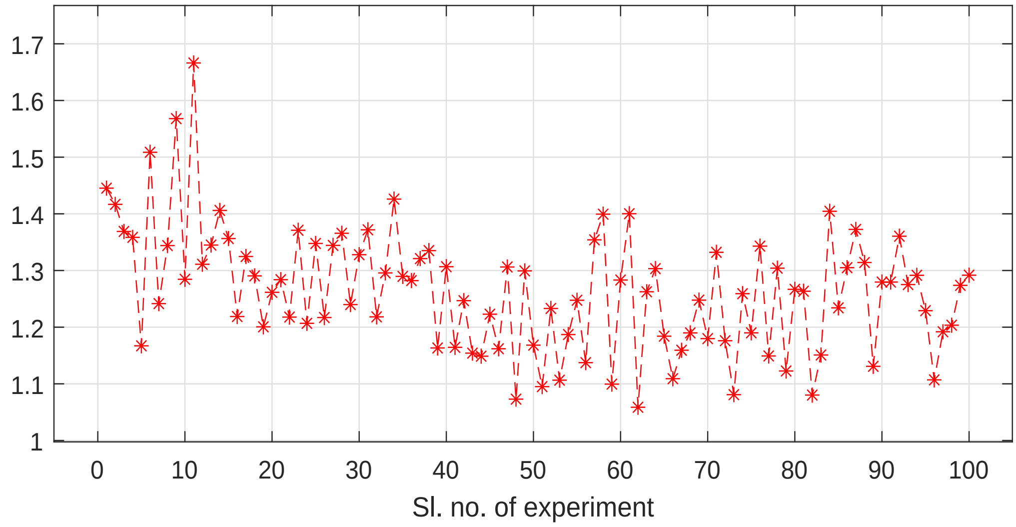

FIGURE 6.2. Ratio of Hermitian backward error and unstructured backward error.

Remark 6.8.4. *From Table 6.2 and Table 6.3 one can get the following relations between the structured and unstructured backward errors of a single approximate eigenpair and two approximate eigenpairs of a double-semisimple eigenvalue whose proofs are immediate from the respective definitions:*

$$\eta_F(\lambda, x_1, y_1, W), \eta_F(\lambda, x_2, y_1, W) \leq \eta_F(\lambda, x_{1:2}, y_1, W),$$
$$\eta_F^{\mathbf{S}}(\lambda, x_1, y_1, W), \eta_F^{\mathbf{S}}(\lambda, x_2, y_1, W) \leq \eta_F^{\mathbf{S}}(\lambda, x_{1:2}, y_1, W),$$
$$\eta_F(\lambda, x_{1:2}, y_1, W) \leq \eta_F^{\mathbf{S}}(\lambda, x_{1:2}, y_1, W). \blacksquare$$

CHAPTER 7

CONCLUSION AND SCOPE FOR FUTURE WORK

This thesis is revolving around the backward error analysis of one or more specified eigenpairs of structured and unstructured matrix pencils, matrix polynomials, and two-parameter matrix systems. A general framework on backward error analysis is established for specified eigenpairs in such a way that the different kinds of inverse eigenvalue problems can be solved using our developed results. In particular, in Chapter 2, we have studied the structured and unstructured backward error analysis of two specified eigenpairs of a double-semisimple eigenvalue for matrix pencils. We have also obtained the relationships between the unstructured backward error of a single eigenpair, structured backward error of two eigenpairs of a double-semisimple eigenvalue, and structured backward error of a single approximate eigenpair.

In Chapter 3, we have obtained the backward error formulas of one or more specified eigenpairs for structured matrix pencils. We have also obtained the minimal *Frobenius* norm perturbed matrix pencils, which also preserve the sparsity. Further, we have used our backward error results in such a way that the different kinds of inverse eigenvalue problems are also solvable. In Chapter 4, we have established the backward error results for Hankel and symmetric-Toeplitz matrix pencils. We have further used these backward error results to solve the matrix inverse eigenvalue problems and generalized inverse eigenvalue problems of both the structures.

Next, in Chapter 5, we have extended the backward error results from matrix pencils to matrix polynomials. For each structured matrix polynomial, we have provided the upper bound on the maximum number of approximate eigenpairs whose backward error analysis can be done simultaneously. We have also obtained the unstructured backward error of one or more specified eigenpairs. Further, we have used the developed backward error results in solving the different kinds of *quadratic inverse eigenvalue problems*. Finally, in chapter 6, we have classified the two-parameter matrix systems on the basis of normal rank definition. We have further found the backward error formulas of two approximate eigenpairs for structured and unstructured two-parameter matrix systems.

Though we have discussed a detailed backward error analysis for one or more specified eigenpairs, there are many questions that are still open and need to be answered to further develop the literature of backward error analysis of more than one approximate eigenpairs. Some of the unanswered questions are summarized by the following points:

- What is the the backward error of two approximate eigenpairs of a double-semisimple eigenvalue for $*$-palindromic and $*$-anti-palindromic matrix pencils ? Here $* \in \{T, H\}$.
- Can we develop the backward error analysis for two approximate eigenpairs of a double-semisimple eigenvalue for matrix polynomials ?
- What is the backward error for three or more eigenpairs for two -parameter matrix systems and for multi-parameter matrix systems ?
- Can we develop the backward error analysis for multi-parameter matrix systems in such a way that inverse eigenvalue problems for multi-parameter matrix systems are also solvable from those results ?

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