BLACK HOLE THERMODYNAMICS

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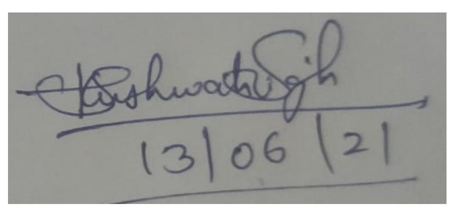
CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **"Black Hole Thermodynamics"** in the partial fulfillment of the requirements for the award of the degree of **MASTER OF SCIENCE** and submitted in the **DISCIPLINE OF PHYSICS, Indian Institute of Technology Indore,** is an authentic record of my own work carried out during the time period from july 2020 to june, 2021 under the supervision of **Dr. Manavendra N. Mahato,** Associate Professor. The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other institute.



Signature of student with date

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This is to certify that the above statement made by the candidate is correct to the best of my/our knowledge.

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Abstract

We examine here the stability of black holes, in both way first is the thermodynamic way, by using the Hessian matrix of thermodynamic variables, useful in analyzing thermodynamic instability, and second is the dynamical way by perturbing the metric of black brane/string given by Horowitz and Strominger and also perturbation of scalar fields and gauge fields in a supergravity Lagrangian using AdS/CFT arguments, if the perturbation does not vanish we found the unstable modes in the form of dynamical instability.

By comparing thermodynamical and dynamical instability it is conjectured by Gubser and Mitra in his paper that black holes with a lack of thermodynamic stability often also lack stability against small perturbation i.e, dynamical instability, which is included as the central part of the whole work.

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Chapter 1

Introduction

In recent nearly 300 years, humans realised what gravity is all about. In ancient times Greek philosophers thought that the planets and stars were part of the gods' realm and followed a "natural motion". They did not realized that gravity is involved. The Greek ideas stuck around until the $15th$ century. Beginning in the $1500s$, though, astronomers like Galileo and Tycho Brahe on their experimental basis discovered that the earth and other planets revolve around the sun. Later Kepler based on Brahe's experimental data showed that the planets move in an elliptical orbit, not a circular one. The question was why? So in seek of the answer the gravity was discovered. Sir Isaac Newton discovered gravity while thinking about the forces of nature. Newton realized that moon would fly off away from Earth in a straight line tangent to its orbit if some forces were not causing it to fall towards the earth, Newton called this force "gravity" and determined the law for the existence of gravitational force between two massive bodies called as Universal law of gravitation. Mathematically,

$$F = \frac{GM_1M_2}{r^2} \tag{1.1}$$

For nearly more than 200 years, Newtonian gravity was the basis of research and was accepted within its absolute perfection but there was some fundamental question about the definition of absoluteness of space and time and simultaneity etc, and the circular orbit of Mercury. Albert Einstein was the first to resolve such ambiguities via his special theory of relativity and the general theory of relativity where gravity was in the heart of the second one. According to Einstein, Gravity arises from the "warping" of space and time.

Einstein's new theory of gravity explains several phenomena that would violate Newton's theory, for example, light bends when passing near a massive object like the

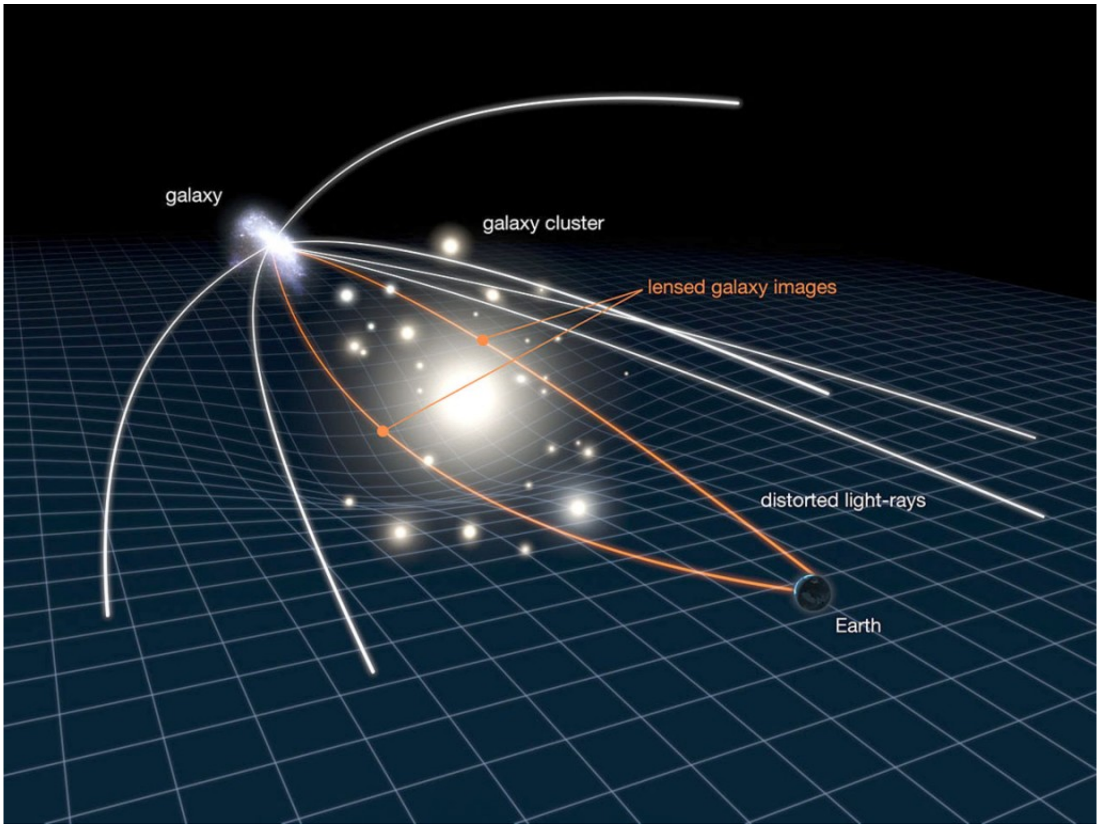

Figure 1.1: Gravitational lensing

sun. The phenomenon called Gravitation lensing.

1.1 Thermodynamic evolution of black holes:

From Einstein's general theory of relativity, we get the black hole as an ordinary body with certain metric but with no thermodynamic behavior, but in later years many physicists like Hawking, Bekenstein, etc tried a different aspect called the quantum aspect of black holes and showed the thermal nature of black holes by analysing the entropy and temperature of black holes and also derived four thermodynamics laws of black holes given in chapter 2, which are closely similar to the thermodynamical laws of the ordinary body. The relevance in thermodynamical laws of the black hole and that of the ordinary body can be seen in the table given above:

Laws of Thermodynamics	Laws of Black Hole Mechanics
Temperature is constant throughout a body at equilibrium. T= constant.	Surface gravity is constant on the event horizon. κ =constant.
Energy is conserved. $dE = TdS + \mu dQ + \Omega dJ.$	Energy is conserved. $dM = \frac{\kappa}{8\pi} dA + \mu dQ + \Omega dJ.$
Entropy never decrease. $\Delta S \geq 0.$	Area never decreases. $\Delta A \geq 0.$

Figure 1.2: Relevance between thermodynamics of the black hole and the ordinary body

1.2 Classical stabiltiy of Black holes

The first attempt to evaluate whether the black holes are stable or not was done by Regge and Wheeler but they were unable to solve a differential equation found as a result of metric perturbation called as Regge-Wheeler equation. Later it was solved by C.V. Vishweshwara and proven that Schwarzschild black holes are stable against small perturbations.

The basic question is that what is meant by instability in small perturbation? To answer that question consider a system like for our cases we have black holes, perturb their metric up to linear order and then put them in the parent equation from where we got the parent metric. Then we will get a differential equation for the perturbation. Then the solution of this equation will give us whether the system is stable or not. If the solution is oscillating about the initial state then we say that system is stable otherwise the system is said to be unstable. This procedure can be seen in ref[16] to check the dynamical stability of Schwarzschild black holes.

1.3 Classical stability vs thermodynamics stability

There is a question that has never been answered more satisfactory, What is more, fundamental thermodynamics of the body or dynamics of the body? If we can answer this we can give solidarity that which way for instability/stability is more fundamental. Thermodynamic instability is very easy to evaluate using the basic postulates and laws of thermodynamics. In opposite to that dynamic stability, calculations are very cumbersome via linear perturbation.

Greggory and Laflamme conjectured about the existence of linear stability in black strings/black branes, and this concept is adopted by Gubser and Mitra to evaluate that black holes which lack local thermodynamic stability often also lack stability against small perturbations[7].

Chapter 2

Black Holes and their Thermodynamics

The story of black holes is far more latest to start with, so let's start with discussing more basic connections which gave rise to such bodies like black holes, warm hole, white holes, etc.

Black holes are one of the most important topics of research in the field area of general relativity. The story of black holes centrally concerns gravity. As an ordinary definition of black holes, the black holes are the objects whose gravity is so large that even light can not escape through them. So the central question will be, What is gravity? There are two different answers regarding this based on two different approaches towards gravity, one is merely an approximation of the other which are as follows:

a)Newtonian theory of gravity

b)Einstein's theory of gravity.

Newton's theory of gravity was based on assumption that space and time are absolute and independent of each other. According to Newton's Principia

For time:

"Absolute, true, and mathematical time, from its own nature, passes equably without relation to anything external, and thus without reference to any change or way of measuring of time (e.g., the hour, day, month, or year)".

For Space:

> "Absolute, true, and mathematical space remains similar and immovable without relation to anything external. (The specific meaning of this will become clearer below from the way it contrasts with Descartes' concept of space.) Relative spaces are measures of absolute space defined with reference to some system of bodies or another, and thus a relative space may, and likely will, be in motion"[14].

Let's ask a question to Newton why space and time are absolute. Newton has no answer about it, but these absoluteness concepts were fundamentals for more than two centuries. Later Albert Einstein questioned these fundamental concepts, Why space and time are absolute? It is more perfectly answered after his special theory of relativity in 1905. But special relativity was the theory only for inertial frames. What if the frames are non-inertial? One best example of it was the body falling under gravity. This was answered after his general theory of relativity in 1915, where he explained that the fundamental of gravity is not related to force but some peculiar curvature attained by the spacetime due to heavy massive body explained by metric $g_{\mu\nu}$, which is derived from Einstein's field equation

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{2.1}$$

Here left-hand side represents second-order non-linear partial differentials of metric $g_{\mu\nu}$, and the right-hand side $T_{\mu\nu}$ represents energy-momentum tensor which gives us the matter content present.

2.1 The Schwarzschild Black Hole

The most obvious application of the Einstein theory of gravity is to find the solution for a spherically symmetric gravitational field. This would be relevant to describe, for example, the field created by the Earth or sun (to a good approximation). Our concern is with the exterior solutions i.e, the empty space surrounding a gravitating body, these solutions are also known as vacuum solution of Einstein Field equation which is just vanishing of Ricci tensor.

In GR, the unique spherically symmetric vacuum solution is Schwarzschild metric. It is in spherical coordinates also called Schwarzschild coordinates. The metric is given by [1],

$$ds^2 = -(1 - \frac{2GM}{r})dt^2 + (1 - \frac{2GM}{r})^{-1}dr^2 + r^2 d\Omega^2 \tag{2.2}$$

where

$$d\Omega^2 = d\theta^2 + sin^2\theta d\phi^2 \tag{2.3}$$

and M is the mass of the gravitating object. Its whole derivation can be found in [1].

The uniqueness of the equation(2.2) is given by Birkhoff's theorem [1], which can be stated as, " The Schwarzschild metric is the unique vacuum solution with spherical symmetry (and in particular, that there are no independent solutions of this form)".

The specialty of the Schwarzschild metric is that it contains singularity at $r = 2GM$ and $r = 0$ which can be easily seen from the metric, but the question is that, whether are they actual singularity? It means if somehow we can remove the singularity by any coordinate transformation then it is just merely a coordinate singularity but not the feature of spacetime otherwise there is another alternative way to find that whether singularity exists at any particular point or not, is by constructing a scalar at that point. If it exists that means there is no absolute singularity there i.e, we have chosen a bad coordinate system. But if the scalar blows up at that point it means that the point has an absolute singularity at that point.

Now that we will see some conformal diagram of Schwarzschild black holes;

The description of above figure (2.1) as: $r = 0$, represents the absolute singularity of black hole,

$r = 2GM$, represents the event horizon of black hole

i^0, resents spacelike infinity.

$i^{\pm}$ represents the future and past like infinity respectively, and

$I^{\pm}$ represents future and past null infinity respectively.

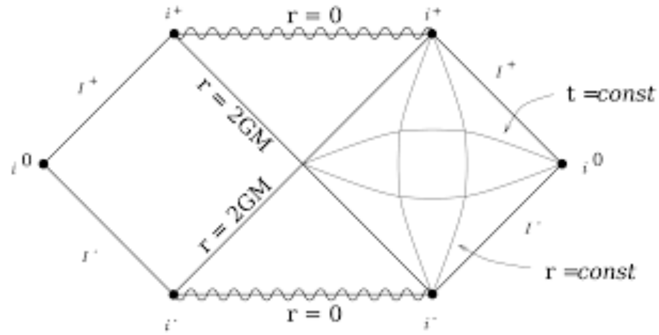

Figure 2.1: Confomal diagram of Scharzschild black hole

2.2 The Reissner-Nordstrom Black Hole

This black hole has one more feature that it's not a neutral one but it does contain charge which can be considered as either electric or magnetic (via considering the existence of magnetic monopoles). This solution of Einstein field equation is known as the charged spherically symmetric vacuum solution. The metric of Reissner-Nordstrom black hole can be given as,

$$ds^2 = -f dt^2 + \frac{1}{f} dr^2 + r^2 d\Omega^2 \tag{2.4}$$

where,

$$f = 1 - \frac{2GM}{r} + \frac{G(Q^2 + P^2)}{r^2} \tag{2.5}$$

where, M is the mass of the gravitating body.

Q is the electric charge and

P is the charge of magnetic monopoles, as magnetic monopoles have never been observed so here for our convenience we can also take $P = 0$, by denying, for now, the existence of magnetic monopoles. But if they exist then $P \neq 0$.

The electromagnetic field associated with this solution is given by,

$$E_r = F_{rt} = \frac{Q}{r^2}$$
$$B_r = \frac{F_{\theta\phi}}{r^2 sin\theta} = \frac{P}{r^2} \tag{2.6}$$

The Reissner-Nordstrom metric has absolute curvature singularity at $r = 0$, as can be checked by computing the invariant scalar curvature $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$. The horizon structure here is a little complicated than in Schwarzschild one.

The event horizon of the metric can be found as,

$$g^{rr} = 1 - \frac{2GM}{r} + \frac{G(Q^2 + P^2)}{r^2} = 0 \tag{2.7}$$

This will occur at

$$r_{\pm} = GM \pm \sqrt{G^2M^2 - G(Q^2 + P^2)} \tag{2.8}$$

From the above equation, we can classify three cases which are as follows:

1. $GM^2 < Q^2 + P^2$
2. $GM^2 > Q^2 + P^2$
3. $GM^2 = Q^2 + P^2$

The three different cases as above stated stand as the very important cases as classification,

Case one:$GM^2 < Q^2 + P^2$

In this case from equation(2.7) we can see that Δ will always be positive. Which means $g^{rr} \neq 0$, i.e, there is no event horizon for this case. But by looking at the metric equation in the context of this case we find the metric is not regular only at $r = 0$. It is the only singularity present.

In this case as Δ is positive so coefficient of dt^2 and the coefficient of dr^2 is in usual signature $-, +, +, +$. So the t is timelike coordinate and r is space like coordinate. Its conformal diagram will be very similar to Minkowskian at infinities telling its asymptotic flat behavior.

In this case, the nakedness of the singularity at $r = 0$ violates the Penrose's cosmic censorship conjecture. We should never find a black hole with $GM^2 < Q^2 + P^2$ as the result of gravitational collapse. Roughly manner this condition states that the total

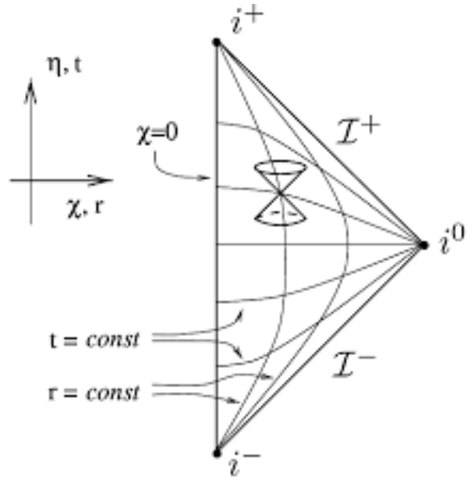

Figure 2.2: Confomal diagram of case 1

energy(i.e, GM^2) of the hole is less than the contribution to the energy from the electromagnetic fields alone. This means the mass of the matter that carried charge would have to be negative, which is absurd, So this solution is generally considered unphysical.
Note: There are no Cauchy surfaces in this spacetime since time-like lines can begin and end at the singularity, see figure2.2.

Case2 : $GM^2 > Q^2 + P^2$:
In contrast to the above case, this case can be applied in realistic gravitational collapse because the gravitational energy or matter content (i,e. GM^2) is greater than the electromagnetic field energy (i.e, $Q^2 + P^2$).

In this case, the metric coefficient vanishes at two points r_+ and r_- which are

$$r_\pm = GM \pm \sqrt{G^2M^2 - G(Q^2 + P^2)} \tag{2.9}$$

The metric has coordinate singularity both at r_+ and r_-. It is same as we have seen in the Schwarzschild case and they both are just coordinate singularity. So they are removable by some suitable coordinate transformation. So r_+ and r_- are two

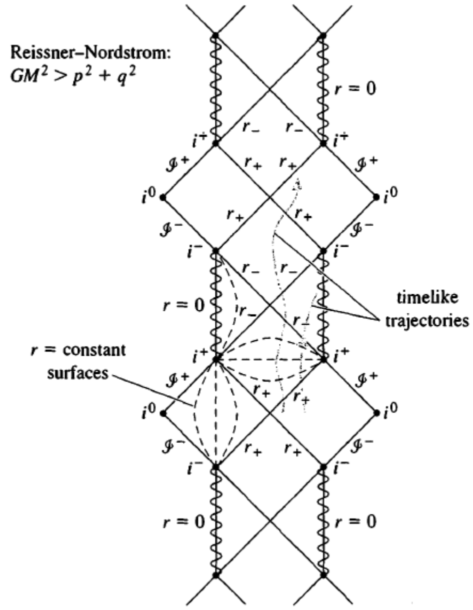

Figure 2.3: Confomal diagram of case 2

event horizons one is the outer event horizon and the other is the inner event horizon respectively. The conformal diagram of this case is given in figure(2.3).

Case 3: $GM^2 = Q^2 + P^2$

This case is known as the extremal Reissner-Nordstrom solution. This particular case has importance in the studies of the black hole in quantum gravity. In supersymmetric theories, extremal black holes can leave certain symmetries unbroken, which is of considerable aid in calculations.

The extremal black holes have $\Delta(r) = 0$, only at $r = GM$. This represents an event horizon, but the coordinate r is never timelike. It becomes null at r=GM and spacelike on either side. The singularity at $r = 0$ is the timelike line, as in the above all cases.

One of the most interesting features of this case is that the mass is in some sense balanced by the charge. More specifically, two extremal black holes with the same sign charges will repel each other electromagnetically and attract each other gravitationally. It turns out that these effects precisely cancel. The calculation regarding exact solution to the coupled Einstein-Maxwell equations representing any number of such black holes in a stationary configuration is given in reference[1].

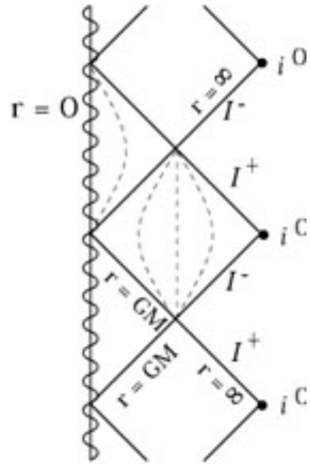

Figure 2.4: Confomal diagram of case 3

2.3 Thermodynamics of Black Holes

2.3.1 Thermodynamical laws of Black holes

The black hole thermodynamical laws are mainly the work of Bekenstein, Carter, and Bardeen. The four basic laws of black hole thermodynamics are as follows;

Zeroth law: The horizon has constant surface gravity for a stationary black hole.

First law: For perturbations of stationary black holes, the change of energy is related to change of area, angular momentum, and electric charge by

$$dE = \frac{\kappa}{8\pi} dA + \Omega dJ + \phi dQ \tag{2.10}$$

where, E is the energy, κ is the surface gravity, A is the area of the event horizon, is the angular velocity J is the angular momentum, ϕ is the electric potential and Q is the electric charge.

Second law: The area of the event horizon, assuming the weak energy condition, is a non-decreasing function of time, i.e,

$$\frac{dA}{dt} \geq 0 \tag{2.11}$$

This "law" was superseded by Hawking's discovery that black holes radiate, which causes both the black hole's mass and the area of its horizon to decrease over time.

Third law: It is not possible to form a black hole with vanishing surface gravity. That is, $\kappa = 0$ cannot be achieved.

Chapter 3

Instability of Black Strings and p-Branes

In general relativity, black holes are among the most perplexing things. They conceal a singularity below their horizon: a point that denotes the theory's doom. The region around this singularity has extraordinarily intense gravity and is most likely explained by quantum gravity. Black holes are stable in four spacetime dimensions, for Schwarzschild black hole see reference C.V. Vishweshara[16]. Once the black hole is formed, they settle down to a state solely described by their charge, mass, and angular momentum which is from the no-hairs theorem, therefore the singularities remain hidden from distant observers. This classical stability of black holes led to Penrose's cosmic censorship Hypothesis according to which[11]:

"In an asymptotically flat spacetime obeying the dominating energy condition, naked singularities cannot develop in gravitational collapse from generic, initially non-singular states.".

Black holes are equivalent to a thermal system in terms of quantum mechanics. The surface of black holes behaves like entropy, and they can even be linked to temperature because Hawking demonstrated that they radiate thermally. Hawking, on the other hand, hypothesized that a black hole generated from a pure quantum state would radiate out, leaving a mixed state of radiation. In fundamentals of quantum mechanics, determinism and reversibility mean that for a wave function, its past, present, and future can be uniquely determined by an evolution operator i.e, the information must be preserved. But according to Hawking's calculations black hole evaporation via Hawking radiation does not preserve information or leaves a mixed state, conclusively the information is lost. This is known as Hawking's black hole information loss paradox.

From above, the final stage of black hole evaporation is difficult to comprehend because general relativity is expected to break down at Planckian curvatures. But if quantum gravity is to preserve unitarity and information, it must do so well before the black hole reaches Planckian curvature, otherwise, there is simply not enough energy left in a Planck mass black hole to emit all the information stored in a macroscopic black hole.

Recently, there has been a resurgence of interest in this subject, owing to the development of string theory as a candidate in the weak gravity domain, with researchers looking into the implications of low energy string theory on black hole structure. Some of these findings have already piqued our interest. In Einstein's gravity, charged black holes (Reissner-Nordstrom black holes) have an unfortunate weakness. As well as an outer event horizon they contain an inner Cauchy horizon which is unstable to matter perturbation in the exterior spacetime. However, there is no static charge black hole solution in Einstein gravity with only one horizon and a spacelike singularity.

On the other hand, in low energy string theory gravity acquires a dilaton which greatly changes the causal structure of charged black holes making them like Schwarzschild with one event horizon and a spacelike singularity [1]. This structure is generic even if the dilaton has a mass[9], as it must do to keep in line with the principle of equivalence.

A particularly amusing aspect of these black holes is that in the extreme limit of a magnetically charged black hole, the spacetime acquires an internal "scri" at $r = 2M$ which is an infinite volume "throat" in which much information can be stored. In four dimensions an event horizon must be topologically spherical, but in higher dimensions, this is not necessarily the case, where we can $S^2 \times R^6$.[6]

The goal here is to emphasize that a huge subset of these black holes is unstable when subjected to minor disturbances. This is a trait that differs significantly from its four-dimensional counterpart.
However, there is a heuristic argument to show this is reasonable. Consider a five-dimensional black string, $Sch \times R$, a portion of length L has mass $\mathcal{M} = ML$,

$$entropy \propto (\frac{\mathcal{M}}{L}) \tag{3.1}$$

But For, five-dimensional black hole , $entropy \propto \mathcal{M}^{3/2}$. Thus for a large length of the horizon, the mass contained within the horizon contributes a much lower entropy than if it were in a hyperspherical black hole. This indicates that for large wavelength perturbations in the fifth dimension, we might expect instability.

3.1 Proving the Gregory Laflamme Conjecture

An investigation of the perturbation equations, with appropriate references to gauge and boundary conditions, is necessary to demonstrate the linear instability. Despite the fact that this is a lengthy and complicated procedure.

We're particularly interested in the black branes described by Horowitz and Strominger in ten-dimensional low-energy string theory with the form metric.[6]

$$ds^2 = -Vdt^2 + \frac{1}{V}dr^2 + r^2 d\omega^2_{D-2} + dx^i dx_i \tag{3.2}$$

where,
$V = 1 - (\frac{r_+}{r})^{D-3}$, $D = 4, ..., 10$
The index i runes from 1 to $10 - D$

Because we're only dealing with uncharged black holes here, perturbations to the Einstein equations will suffice, as the dilaton and gauge perturbations are decoupled and may be set to zero [6]. We write a metric perturbation in the usual way:

$$g_{ab} \rightarrow g_{ab} + h_{ab} \tag{3.3}$$

whereas we use the transverse trace free (de Donder) gauge for h_{ab}:

$$h^a_a = 0 = h^a_{b;a} \tag{3.4}$$

From Einstein vacuum field equation we have

$$R_{\mu\nu}(g) = 0 \tag{3.5}$$

and,

$$R_{ab}(g+h) = o \tag{3.6}$$

Simplifying above equation to linear order we get,

$$R_{ab}(g) + \delta R_{ab}(h) = 0 \tag{3.7}$$

where $\delta R_{ab}(h)$ contain only the first order terms in h_{ab}. Since $R_{ab}(g) = 0$, the differential equations governing for the perturbations are obtained from the equation $\delta R_{ab}(h) = 0$. Now by Einsenhart formula [16], we get

$$\delta R_{ab} = -\delta\Gamma^d_{ab;d} + \delta\Gamma^d_{ad;b} \tag{3.8}$$

where,

$$\delta\Gamma^d_{ab} = \frac{g^{cd}}{2}(h_{ac;b} + h_{bc;a} - h_{ab;c}) \tag{3.9}$$

Now substituting from equation(3.9) to equation(3.8) we get Lichnerowicz operator

$$2\delta R_{ab} = \Delta^2 h_{ab} - 2R_{cadb}h^{cd} + 2R_{c(a}h^c_{b)} + \frac{1}{2}\Delta_b(2\Delta_c h^c_a - \Delta_a h) + \frac{1}{2}\Delta_a(2\Delta_c h^c_a - \Delta_b h) \tag{3.10}$$

by taking $\delta R_{ab} = 0$, we can get Lichnerowicz operator in the form of,

$$(\Delta_L h)_{ab} = 2R^c_{abd}h^d_c + R_{ca}h^c_b + R_{cb}h^c_a - \Delta^c\Delta_c h_{ab} \tag{3.11}$$

In summary;

$$g_{ab} \to g_{ab} + h_{ab}, \quad such \ that, \quad h^a_a = \Delta^a h_{ab} = 0 \quad and \quad R_{ab} \to R_{ab} + \frac{1}{2}(\Delta_L h)_{ab} \tag{3.12}$$

This does not eliminate all of the gauge freedom, but does simplify the perturbation equations,

$$\Delta_L h_{ab} = (\delta^c_a \delta^d_b \Box + 2R^{cd}_{ab}) h_{ab} \tag{3.13}$$

where Δ_L is Lichnerowicz operator.

In general relativity, physics is invariant under the general coordinate transformation (gct's), which are generated by vector fields ξ^a. The effect of an infinitesimal is to push the coordinates ϵ along the integral curves of ξ^a. Under such a gauge transformation, the metric transforms as

$$g_{ab} \to g_{ab} + 2\xi_{(a;b)} \tag{3.14}$$

here a pure gauge perturbation of the metric is in the form

$$h_{\xi ab} = 2\xi_{(a;b)} \tag{3.15}$$

But if ξ^a is divergence-free and harmonic then h_ξ satisfies both equations $(3.4) and (3.4)$. Therefore although there are $\frac{(N-1)(N+1)}{2}$ degrees of freedom in the solutions to the N-dimensional Lichnerowicz equation, $(N-1)$ of these are pure gauge, the remaining $\frac{N(N-3)}{2}$ being physical. It will turn out to be fairly straightforward to identify the gauge degrees of freedom.

3.1.1 What about boundary conditions?

There is the question of boundary conditions, which are the key to this problem. We want to place initial data on the Cauchy surface for the exterior spacetime, but such a surface necessarily touches the horizon, which is singular in Schwarzschild coordinates. Therefore there are two issues here: how to define "small" for the perturbation at the horizon, and secondly, which initial data surface to impose these constraints upon.

The first issue is straightforwardly dealt with. Although the horizon is singular in Schwarzschild coordinates, it is not a physical singularity merely a coordinate singularity. In four dimensions, non-singular coordinates have been known for some time as Kruskal coordinates. These require generalizing to higher dimensions, which is slightly more involved, but the transformation laws between Kruskal and Schwarzschild coordinates remain qualitatively the same at the horizon. Therefore, since Kruskal coordinates do not display their staticity in a straightforward manner, we perform a mode decomposition in Schwarzschild coordinates, transforming to Kruskal coordinates at the horizon to decide which modes are well behaved.

Now we turn to the actual stability analysis: are there any unstable modes? Due to the symmetries of the spacetime, we can split up the perturbation into a purely transverse piece, a mixed transverse/D-Schwarzschild piece, and a purely Schwarzschild piece. This can be represented schematically as

$$\begin{pmatrix} h_{\mu\nu} & h_{\mu i} \\ h_{j\nu} & h_{ij} \end{pmatrix} \tag{3.16}$$

where μ runs from 1 to D and i is $10-D$. In a Kaluza-Klein spirit, we can interpret these perturbation as a scalar, vector, and tensor respectively concerning the D-dimensional Schwarzschild spacetime.

It is straightforward to show that there are no unstable modes with non-zero scalar or vector pieces meeting our criteria of being well behaved at both infinity and future event horizon. However for a D-dimensional s-wave of the form

$$h^{\mu i} = 0 = h^{ij} \tag{3.17}$$

$$h^{\mu\nu} = e^{\omega t} e^{\iota \mu_i x^i} \begin{pmatrix} H^{tt} & H^{tr} & 0 & 0 & ... \\ H^{tr} & H^{rr} & 0 & 0 & ... \\ 0 & K & 0 & 0 & ... \\ 0 & 0 & 0 & \frac{K}{sin^2} & ... \\ ... & ... & ... & ... & ... \end{pmatrix} \tag{3.18}$$

Using metric (3.2) and the perturbation equation(3.18) the Lichnerowicz equation reduces to

$$(\Delta_L + \Sigma_i \mu_i^2) h_{\mu\nu} = 0 \tag{3.19}$$

where Δ_L is the D-dimensional Lichnerowicz operator.

The notable point is that above equation $(3, 19)$ shows a pure D-dimensional gauge perturbations, i.e, $h_{\mu\nu} = \xi_{(\mu;\nu)}$, satisfies $\Delta_L \xi_{\mu;\nu} = 0$ which means that a pure gauge perturbation of the metric must be a zero mode of the D-dimensional Lichnerowicz operator equation for stability so as long as $\mu^2 = \Sigma_i \mu_i^2 \neq 0$ in equation (3.19), $h_{\mu\nu}$ will be a real physical perturbation.

Chapter 4

Instability of charged black holes in anti-de Sitter space

A strange paradox in black hole physics is that they are often thermodynamically unstable, with negative specific heat, as seen in the four-dimensional Schwarzschild solution of mass M has negative specific heat,

$$c = \frac{\partial T_H}{\partial M} = -\frac{1}{8\pi M} \tag{4.1}$$

i.e,

$$c < 0 \tag{4.2}$$

If we saw the classical treatment of perturbation of black hole in way of Regge and Wheeler[13] and Zerrilli and Vishweshwara, they are stable against small perturbations of the metric [16]. Using string theory concepts such as intersecting D-branes, tremendous progress has been made in providing a microscopic statistical mechanical account of black hole thermodynamics in recent years[10]. The black holes so described without exception have positive specific heat. Typically they are near extremal solutions to the four-or-five dimensional compactification of string theory with several electric and/or magnetic charges and a mass that almost saturates the BPS bound. The statistical mechanical account of their entropy relies on a low-energy field theory description of the D-branes from which they are constructed. It is no surprise, then, that the specific heat turns out to be positive: this is a criterion that is met by the statistical mechanics of almost any sensible field theory.

An obvious initial step in extending the success of string theory to more astrophysically significant black holes is to look for thermodynamically unstable variations of black holes for which string theory provides a dual description. In this Letter, we exhibit perhaps the simplest example of such a black hole (the anti-de Sitter space Reissner-Nordstrom solution), demonstrate its thermodynamic instability, and show via numerics that the solution is unstable in a linearized analysis. The instability should correspond to the onset of Bose condensation in the dual field theory [8].

4.1 $AdS_4 - RN$ solution and its Thermodynamics

Since $AdS4 - RN$ is a $N = 8$ gauged supergravity solution, the maximally symmetric $AdS4$ vacuum is the kaluza-klein reduction of the M-theory $AdS^4 \times S^7$ vacuum. Furthermore, gauged supergravity with $N = 8$ is a consistent truncation of eleven-dimensional supergravity [2]. This indicates that any four-dimensional classical solution lifts to an identical eleven-dimensional classical solution. As a result, an instability discovered in four dimensions is certain to exist in eleven.[7]

As we have seen metric form RN black hole in chapter1 there it was not in maximally symmetric space but here we will present metric particularly in anti-de sitter space. Then we will explore our knowledge of $N = 8$ supergravity Lagrangian and will prove that the N=8 supergravity Lagrangian can be converted back into AdS_4-RN solution.

The anti-de sitter space Reissner Nordstrom solution is $(AdS_4 - RN)$;

$$ds^2 = -f dt^2 + \frac{1}{f} dr^2 + r^2 d\Omega^2 \tag{4.3}$$

where,

$$F_{0r} = \frac{Q}{\sqrt{8} r^2} \tag{4.4}$$

$$f = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \frac{r^2}{L^2} \tag{4.5}$$

We will work throughout in units where $G_4 = 1$.

The lagrangian of $N = 8$ gauged eleven dimensional supergravity contains the terms,

$$\mathcal{L} = \frac{\sqrt{g}}{16\pi}[R - \Sigma_{i=1}^{3}(\frac{1}{2}((\partial\phi)^2) + \frac{2}{L^2}Cosh\phi_i) - 2\Sigma_{A=1}^{4}e^{\alpha_i^{(A)}\phi_i}(F_{\mu\nu}^2)] \tag{4.6}$$

where

$$\alpha_i^A = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \tag{4.7}$$

and admits the black hole solution [5]

$$ds^2 = -\frac{F}{\sqrt{H}}dt^2 + \frac{\sqrt{H}}{F}dz^2 + \sqrt{H}z^2 d\Omega^2 \tag{4.8}$$

$e^{2\phi_1} = \frac{h_1 h_2}{h_3 h_4}$ $\qquad$ $e^{2\phi_2} = \frac{h_1 h_3}{h_2 h_4}$ $\qquad$ $e^{2\phi_3} = \frac{h_1 h_4}{h_2 h_3}$

and,

$H = \Pi_{A=1}^{4} h_A$ $\qquad$ $F = 1 - \frac{\mu}{z} + \frac{z^2}{L^2}H$ $\qquad$ $h_A = 1 + \frac{q_A}{z}$

$$F_{0z}^A = \pm\frac{1}{\sqrt{8}h_A^2}\frac{Q_A}{z^2} \tag{4.9}$$

Now from above, we can obtain the polynomial expression for mass M and entropy S in terms of non-extremality parameters (μ, q_A) ,

$$M = \frac{\mu}{2} + \Sigma_{A=1}^{4} q_A \quad ; \quad S = \pi z_H^2 \sqrt{H(z_H)} \tag{4.10}$$

where, z_H is the largest root of $F(z_H) = 0$. Only for certain range of the parameters (μ, q_A) do roots to this equation exist at all when they don't the solution is nakedly singular.

The conserved physical charges are thr Q_A, and they correspond to the four independent angular momenta of M-2 branes in eleven dimensions. If we take $Q_1 = Q_2 = Q_3 = Q_4$ and take transformation $r = z + q$ we can obtain our original metric in equation(4.3).[7]

It is straightforward to start with the Lagrangian in (4.6) and show that linearized perturbations to the equations of motion result in the following coupled equations which we will see in section(4.2).

4.2 Thermodynamic instability

We can calculate the entropy, mass, and conserved charges using the equations derived in earlier section. As previously stated, we can express entropy and mass in terms of the non-extremality parameter and find a polynomial that connects M, S, and, Q_A. This equation is simple to solve for M, but not for S in general.

Now we assume we know the function

$$M = M(S, Q_1, Q_2, ..., Q_n); \qquad or \quad M = M(S, Q_A) \tag{4.11}$$

where, we abbreviate Q_A is $Q_1, Q_2, ..., Q_n$.
We assume positive temperature (which is quite safe for our calculation in regular models of black holes since the Hawking temperature related to area of horizon, can never be negative), we can always invert the above expression $M = M(S, Q_A)$ to $S = S(M, Q_A)$.
A standard claim is that in classical thermodynamics is that the entropy for "sensible" matter must be concave down as a function of other extensive variables. Locally this means that the Hessian matrix [7],

$$H^S_{M.Q_A} = \begin{pmatrix} \frac{\partial^2 S}{\partial M^2} & \frac{\partial^2 S}{\partial M \partial Q_B} \\ \frac{\partial^2 S}{\partial Q_A \partial M} & \frac{\partial^2 S}{\partial Q_A \partial Q_B} \end{pmatrix} \tag{4.12}$$

satisfies, $H^S_{M,Q_A} \leq 0$ i.e, it has no positive eigen values and similarly $H^M_{S,Q_A} \geq 0$. What is meant by this statement let's figure out, consider the simplest case for n=o, i.e, case without charge we have only mass M here which is equivalent to the energy of the system and take $\frac{\partial^2 S}{\partial M^2}$

$$H^S_{M,Q_a} = \frac{\partial^2 S}{\partial M^2} \tag{4.13}$$

and the specific heat of the system is given by

$$C_v = T(\frac{\partial^2 M}{\partial S^2})^{-1} \tag{4.14}$$

So from above, we get negative specific heat, which means the system is unstable. If we start at temperature T then it is possible to change the entropy without changing the total energy by having some regions at temperature $T + \delta T$ and others at $T - \delta T$. Since we are implicitly assuming a thermodynamics limit, it is relevant how big the domains of high and low temperature are. In a more refined description (e.g. Landau-Ginzburg theory), these domains might have preferred size or at least a minimal size.[6]

So local thermodynamical instability can now be expressed as convexity (as Hessian contains only second order partial derivatives, which leads to either convexity or concavity behavior of functions) of the function $M(S, Q_1, Q_2)$ (here we have considered only two kinds of charges). By forming the Hessian of $M(S, Q_1, Q_2)$, it is straight forward to verify the convexity along the line $Q_1 = Q_2 = Q$ when $\pi L Q > S$, or equivalently $M\sqrt{L} < Q^{\frac{3}{2}}$[7].

The associated eigen vector for the Hessian matrix has the form $(0, 1, -1)$. It looks like one charge wants to increase and the other charge decreases, here we got the signs of violation of the No-Hair theorem. Of course, this can happen on the account of global charge conservation[4]. It is worth noting that a black hole horizon exists in the large black hole limit if and only if $M\sqrt{L} \geq \frac{2}{3^{\frac{3}{4}} Q^{\frac{3}{2}}}$ (sign of violation of cosmic censorship conjecture). Thus, there is a narrow range of thermodynamically unstable $AdS_4 - RN$ black holes which border on nakedly singular.

4.3 Dynamical instability equations

Let's try to derive the equation of motion using the Euler Lagrange equation considering variable ϕ, we get

$$\frac{\partial \mathcal{L}}{\partial \phi_i} = \frac{\partial}{\partial \phi_i}[-\frac{\sqrt{g}}{16\pi}\Sigma_{i=1}^{3}\frac{1}{2}(\partial\phi)^2 + \frac{2}{L^2}cosh\phi_i - 2\Sigma_{A=1}^{4}e^{\alpha_i^{(A)}\phi_i}(F_{\mu\nu}^{(A)})^2] \tag{4.15}$$

$$\partial_\mu(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}) = \partial_\mu(\frac{\partial}{\partial(\partial_\mu \phi_i)}[-\frac{\sqrt{g}}{16\pi}\Sigma_{i=1}^{3}\frac{1}{2}(\partial\phi)^2 + \frac{2}{L^2}cosh\phi_i - 2\Sigma_{A=1}^{4}e^{\alpha_i^{(A)}\phi_i}(F_{\mu\nu}^{(A)})^2] \tag{4.16}$$

From equation (4.15) and equation (4.16) we get,

$$\Box\phi - \Sigma_{i=1}^{3}\frac{2}{L^2}sinh\phi_i - 2\Sigma_{A=1}^{4}e^{\alpha_i^{(A)}}\alpha_i^{(A)}(F_{\mu\nu}^{(A)})^2 = 0 \tag{4.17}$$

Now take linear perturbation, in ϕ and $F_{\mu\nu}$,

$$\phi \to \phi + \delta\phi \qquad\qquad F \to F + \alpha_i^{(A)}\delta F_{\mu\nu} \tag{4.18}$$

Now adopt this perturbation in equation (4.17),

$$\Box(\phi+\delta\phi) + \Sigma\frac{2}{L^2}sinh(\phi+\delta\phi) - 2\Sigma_{A=1}^{4}e^{\alpha_i^{(A)}(\phi+\delta\phi)}\alpha_i^{(A)}(F_{\mu\nu} + \alpha_i^{(A)}F_{\mu\nu}^{(A)})^2 = 0 \tag{4.19}$$

$$\Box\phi+\Box\delta\phi+\frac{2}{L^2}(sinh\phi_i+cosh\phi_i\delta\phi_i)-2\Sigma_{A=1}^{4}e^{\alpha_i^{(A)}i}(\alpha_i^{(A)})^2)(1+\delta\phi_i)(F_{\mu\nu}^{(A)}+\alpha_i^{(A)}F_{\mu\nu})^2 = 0 \tag{4.20}$$

By evaluation above two equation under very small perturbation we will get following set of equations which are

$$[\Box + \frac{2}{L^2} - 8F_{\mu\nu}^2]\delta\phi_1 - 16F_{\mu\nu}\delta F_{\mu\nu} = 0 \tag{4.21}$$

$$d\delta F = 0 \tag{4.22}$$

$$d * \delta F + d\delta\phi_i \Lambda * F = 0 \tag{4.23}$$

For equation (4.22) we used the Bianchi Identity in the form of one-form,

$$dF = 0 \tag{4.24}$$

In the above equations, we only took ϕ_1, since variation in ϕ_2 and ϕ_3 do not couple and can be consistently set to zero.

Now, Assume an ansatz form for variation in $\delta\phi_1$,

$$\delta\phi_1 = Re[e^{-i\omega t} Y_{lm} \delta\tilde{\phi_1}] \tag{4.25}$$

As we got the coupled equation(4.21), to decouple it we use the dyadic formalism as discussed in the appendix. In efforts to decouple equation(4.21), in the final result we get a fourth-order ordinary differential equation,

$$(\frac{\omega^2}{f} + \partial_r f \partial_r - l(l+1)r^2)r^3(^2 f + \partial_r f \partial_r - \frac{l(l+1)}{r^2} - \frac{2M}{r^3} + \frac{4Q^2}{r^4})r\delta\tilde{\phi_1}(r) \tag{4.26}$$

To carry out the numerical study we rewrite the above equation into dimensionless radial variable u, the dimensionless charge parameter χ, dimensionless mass parameter σ, and a dimensionless frequency $\tilde{\omega}$ such that ,

$$u = \frac{r}{M^{\frac{1}{3}} L^{\frac{2}{3}}} \qquad \chi = \frac{Q}{M^{\frac{2}{3}} L^{\frac{1}{3}}} \qquad \sigma = (\frac{L}{M})^{\frac{2}{3}} \qquad \tilde{\omega} = \omega \frac{L^{\frac{4}{3}}}{M^{\frac{1}{3}}} \tag{4.27}$$

put above variables in equation(4.25), we get

$$(\frac{\tilde{\omega}^2}{f} + \partial_u \tilde{f} \partial_u - \sigma\frac{l(l+1)}{u^2})u^3(\frac{\tilde{\omega}^2}{\tilde{f}} + \partial_u \tilde{f} \partial_u - \sigma\frac{l(l+1)}{u^2} - \frac{2}{u^3} + \frac{4\chi^2}{u^4})u\delta\tilde{\phi_1} = 4\chi^2(\frac{\tilde{\omega}^2}{\tilde{f}} + \partial_u \tilde{f} \partial_u)\delta\tilde{\phi_1} \tag{4.28}$$

where, $\tilde{f} = \sigma - \frac{2}{u} + \frac{\chi^2}{u^2} + u^2$

The above equation we obtained can be used to find the instability modes with variables partial wave number l, σ, and χ by using Mathematica.

The black brane limit, where the horizon is R^2 rather than S^2, is $\sigma = 0$. Only in this case should we place total faith in thermodynamic considerations. Aside from that, finite-size effects may relate to thermodynamic and dynamical instabilities.At $\sigma = 0$, a horizon exists only if $\chi \leq \frac{\sqrt{3}}{2^{2/3}} = 1.091$, and a thermodynamic instability appears for $\chi > 1$.[7].

Chapter 5

Conclusions

Newman-Penrose formalism is an elegant way to study different solutions of Einstein equation. They are also elegant to analyze gravitational perturbations, unstable modes and gravitational waves. N-P formalism can be utilized to explore other instabilities and hence to investigate the dynamics of the associated transitions.

In the context of the dynamical instability, Schwarzschild black holes are stable against smaller perturbation [16], but are thermodynamically unstable. It can be seen in equation(4.1) as their specific heat are negative. If we evaluate the higher dimensional solutions like brane we found that there exists a clumping instability in a certain range of parameters μ^2, called Classical instability or Greggory-Laflamme instability, these stabilities are in the form of "negative mass square" as shown by Reall via Euclidean path integral considering canonical ensemble [12], as an eigenvalue of Lichnerowicz operator called tachyonic modes technically the instabilities from the string theory perspective.
Thermodynamical calculations are done via constructing Hessian matrix and linearized perturbation of supergravity Lagrangian, the equation of motion obtained gives certain results a conclusive argument for Greggory-Laflamme conjecture which is
"For a black brane solution to be free of dynamical instabilities, it is necessary and sufficient for it to be locally thermodynamically stable."[7] Here, local thermodynamic stability is defined as having an entropy that is concave down as a function of the mass and the conserved charges. Or we can state the Greggory-Laflamme conjecture as

" for a black brane with translational symmetry, a Gregory-Laflamme instability exists precisely when the brane is thermodynamically unstable".[7]

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Chapter 6

Appendix

6.1 Tetrad Formalism

In the general theory of relativity, we use Einstein field equation in a local coordinate basis as in chapter one, but there are some methods which are advantageous, in some contexts, to proceed we choose a suitable tetrad basis of four linearly independent vectors-fields, projecting the relevant quantities on to the chosen basis, and considering the equation satisfied by them. This is tetrad formalism.

In this formalism, we choose our tetrad basis in such a way that it depends on the underlying symmetries of the spacetime which we wish to grasp and is to some extent a part of the problem.

Tetrad representation Set up at each point of spacetime a basis of four contravariant vectors.[3]

$$e^i_{(a)} \quad where,\ a = 1, 2, 3, 4 \tag{6.1}$$

here, (a) is tetrad indices and i is tensor indices. For tensor indices we have transformation property;

$$e_{(a)i} = g_{ik} e^i_{(a)} \tag{6.2}$$

$$e^i_{(a)} e^{(b)}_i = \delta^{(b)}_{(a)} \tag{6.3}$$

where g_{ik} denotes the metric tensor, and also take

$$e^i_{(a)} e_{(b)i} = \eta_{(a)(b)} \tag{6.4}$$

where,

$\eta_{(a)(b)}$ is a constant symmetric metric. Now for any quantity in the usual frame say A^j we can take the projection of it in tetrad frame by

$$A_{(a)} = e_{(a)j} A^j = e^j_{(a)} A_j \qquad A^{(a)} = \eta^{(a)(b)} A_b = e^{(a)}_j A^j = e^{(a)j} A_j \tag{6.5}$$

Now we will define covariant differentiation in terms of tetrads;

$$A_{j;i} = e^{(a)}_j e^{(b)}_i A_{(a),(b)} - \gamma_{(c)(a)(b)} e^{(a)} e^{(b)} A^{(c)} \tag{6.6}$$

where, $\gamma_{(c)(a)(b)}$ is Ricci Rotation Coefficient, which can be defined as,

$$\gamma_{(c)(a)(b)} = e^k_{(c)} e_{(a)k;i} e^i_{(b)} \tag{6.7}$$

6.2 Neumann-Penrose Formalism

This formalism is the tetrad formalism with a special choice of basis vectors, called null vectors (so-called as null tetrads), represented as

$$(l, n, m, \bar{m}) \tag{6.8}$$

The superiority of this formalism lies in the choice of null tetrads, Penrose's belief was that the essential element of spacetime is its light cone structure which makes possible the introduction of spinor basis, and it will appear that the light-cone structure which makes possible the introduction of a spinor basis. And it will appear that the light-cone structure of the spactime of the black hole solutions of the general relativity is exactly the kind that, makes the Neumann Penrose formalism most effective for grasping the inherent symmetries of the spacetime and revealing their analytical richness.

6.2.1 Null Basis and the Spin Coefficients

Null tetrad:

$$(l, n, m, \bar{m}) \tag{6.9}$$

Orthogonality condition of null tetrads:

$$l \cdot m = l \cdot \bar{m} = n \cdot m = n \cdot \bar{m} = 0 \tag{6.10}$$

As from the names these are null vectors so another property will be,

$$l \cdot l = n \cdot n = m \cdot m = \bar{m} \cdot \bar{m} = 0 \tag{6.11}$$

Now the normalization condition will be,

$$l \cdot n = 1 \qquad and \qquad m \cdot \bar{m} = -1 \tag{6.12}$$

From the above conditions, we can define a metric for null bases,

$$\eta_{(a)(b)} = \eta^{(a)(b)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{6.13}$$

with correspondent,

$$e_1 = l \qquad e_2 = n \qquad e_3 = m \qquad e_4 = \bar{m} \tag{6.14}$$

6.2.2 Maxwell's Equation in Neumann-Penrose Formalism

In the Neumann-Penrose formalism, the antisymmetric Maxwell tensor, F_{ij}, is replaced by three complex scalars,

$$\phi_0 = F_{13} = F_{ij}l^i m^j \quad \phi_1 = \frac{1}{2}(F_{12}+F_{43}) = \frac{1}{2}F_{ij}(l^i n^j + \bar{m}^i m^j) \quad \phi_2 = F_{42} = F_{ij}(\bar{m}^i m^j) \tag{6.15}$$

Now, Maxwell's equation can be written in the terms of ϕ_0, ϕ_1 and, ϕ_2.[16]

$$D\phi_1 - \bar{\delta}\phi_0 = (\pi - 2\alpha)\phi_0 \tag{6.16}$$

$$D\phi_2 - \bar{\delta}\phi_1 = -\lambda\phi_0 + 2\pi\phi_1 + (\rho - 2\epsilon)\phi_2 \tag{6.17}$$

$$\delta\phi_1 - \Delta\phi_0 = (\mu - 2\gamma)\phi_0 + 2\tau\phi_1 - \sigma\phi_2 \tag{6.18}$$

$$\delta\phi_2 - \Delta\phi_1 = -\nu\phi_0 + 2\mu\phi_1 + (\tau - 2\beta)\phi_2 \tag{6.19}$$

Now we can represent the null tetrad in a matrix form.[7]

$$\sigma^{\mu}_{\Delta\dot{\Delta}} = \begin{pmatrix} l^\mu & m^\mu \\ \bar{m}^\mu & n^\mu \end{pmatrix} \tag{6.20}$$

and set

$$D = l^\mu \partial_\mu \qquad \Delta = n^\mu \partial_\mu \qquad \delta = m^\mu \partial_\mu \qquad \bar{\delta} = \bar{m}^\mu \partial_\mu \tag{6.21}$$

Vector indices can be converted into dyadic indices by setting

$$v_{\Delta\dot{\Delta}} = \sigma^{\mu}_{\Delta\dot{\Delta}} v_\mu \tag{6.22}$$

Now we will define unique covariant derivative D_μ, by its action on spinor ψ_Γ is

$$D_\mu \psi_\Gamma = \partial_\mu \psi_\Gamma - \psi_\Sigma \gamma_{\mu}{}^{\Sigma}{}_{\Gamma} \tag{6.23}$$

where, $\gamma_{\mu}{}^{\Sigma}{}_{\Gamma}$ called spin-coefficient, which can be written as,

$$\gamma_{0\dot{0}\Sigma\Gamma} = \begin{pmatrix} \kappa & \epsilon \\ \epsilon & \pi \end{pmatrix} \qquad \gamma_{0\dot{1}\Sigma\Gamma} = \begin{pmatrix} \sigma & \beta \\ \beta & \mu \end{pmatrix}$$
$$\gamma_{10\Sigma\Gamma} = \begin{pmatrix} \rho & \alpha \\ \alpha & \lambda \end{pmatrix} \qquad \gamma_{1\dot{1}\Sigma\Gamma} = \begin{pmatrix} \tau & \gamma \\ \gamma & \nu \end{pmatrix} \tag{6.24}$$

The above matrix combinations are formed by 12 non-vanishing Ricci spin coefficient [3]. Some Ricci spin coefficients calculation can be found in reference [15].

Now we will apply the above machinery to our $AdS_4 - RN$ case. For $AdS_4 - RN$, a convenient choice of the null tetrad and the corresponding non-zero coefficient are as follows;

$$l^\mu = (\frac{1}{f}, 1, 0, 0) \qquad n^\mu = \frac{1}{2}(1, -f, 0, 0)$$
$$m^\mu = \frac{1}{r\sqrt{2}}(0, 0, 1, \iota cosec\theta) \qquad \bar{m}^\mu = \frac{1}{r\sqrt{2}}(0, 0, 1, -\iota cosec\theta) \tag{6.25}$$

$$\rho = -\frac{1}{r}, \quad \mu = -\frac{f}{2r}, \quad \gamma = \frac{f'}{4}, \quad \alpha = -\beta = -\frac{cot\theta}{\sqrt{8}r} \tag{6.26}$$

If we take black brane limit, then we should replace cosecθ by 1 in (6.25) and $\alpha = 0 = \beta$ in (6.26). If we go without the black brane limit, we trade the real antisymmetric tensor $F_{\mu\nu}$ for a complex symmetric tensor,

$$\phi^{(0)}_{\Delta\Gamma} = \begin{pmatrix} \phi^{(0)}_o & \phi^{(0)}_1 \\ \phi^{)(0)}_1 & \phi^{(0)}_2 \end{pmatrix} \tag{6.27}$$

through the formula,

$$4\sqrt{2}F_{\mu\nu}\sigma^\mu_{\Delta\dot{\Delta}}\sigma^\nu_{\Gamma\dot{\Gamma}} = \phi^{(0)}_{\Delta\Gamma}\epsilon_{\dot{\Delta}\dot{\Gamma}} + \phi^{\bar{(0)}}_{\dot{\Delta}\dot{\Gamma}}\epsilon_{\dot{\Delta}\dot{\Gamma}} \tag{6.28}$$

The $4\sqrt{2}$ factor is for convenience, the $AdS_4 - RN$ background has $\phi^{(0)}_1 = \frac{Q}{r^2}$, the $AdS_4 - RN$ and all other components zero. In the same way we are trading in $\delta F_{\mu\nu}$ for $\phi_{\Delta\Gamma}$, whose components are ϕ_0, ϕ_1 and ϕ_2 with familiar factor of $4\sqrt{2}$. Finally we write ϕ in place of $\delta\phi_1$ to avoid ambiguity in the meaning to δ.

Now we can get the pair of coupled differential equation in components from the above expression of Maxwell's anti-symmetric field tensor straightforwardly and obtain the equation similar to the set of equations from (6.16) to (6.19) [16].

$$\begin{aligned}
(D-2\rho)\phi_1-(\bar{\delta}-2\alpha)\phi_0 &= -\phi_1^{(0)}D\phi \\
(\Delta+\mu-2\gamma)\phi_0-\delta\phi_1 &= 0 \\
(D-2\rho)\phi_2-\delta\phi_1 &= 0 \\
(\delta+2\beta)\phi_2-(\Delta+2\mu)\phi_1 &= \phi_1^{(0)}\Delta\phi
\end{aligned} \tag{6.29}$$

These all equations are coupled in ϕ_0, ϕ_1, ϕ_2, we can decouple these equations as separated equations where only single ϕ_i appears.

$$\begin{aligned}
[(D-3\rho)(\Delta+\mu-2\gamma)-\delta(\bar{\delta}-2\alpha)]\phi_0 &= -\phi_1^{(0)}\delta D\phi \\
[(\Delta+3\mu)(D-\rho)-\bar{\delta}(\delta+2\beta)]\phi_2 &= -\phi_1^{(0)}\bar{\delta}\Delta\phi \\
[(D-2\rho)(\Delta+2\mu)-(\delta+\beta-\alpha)\bar{\delta}]\phi_1 &= -\phi_1^{(0)}D\Delta\phi \\
[\Box+\frac{2}{L^2}+2(\phi_1^{(0)})^2]\phi &= -4\phi_1^{(0)}Re\phi_1
\end{aligned} \tag{6.30}$$

where we have considered the fact that spin coefficients are all real for AdS_4-RN.

Now take the last two equations to replace $Re\phi_1$ algebraically, we get

$$[(D-2\rho)(\Delta+2\mu)-(\delta+\beta-\alpha)\delta]\frac{1}{4\phi_{1^{(0)}}}[\Box+\frac{2}{L^2}+2(\phi_1^{(0)})^2]=\phi_1^{(0)}D\Delta\phi \tag{6.31}$$

Now by putting $Re[e^{-\iota\omega t}Y_{lm}\delta\tilde{\phi}_1(r)]$, we get equation (4.26).

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