QUANTUM FIELD THEORY IN CURVED SPACETIME

M.Sc. Thesis

By
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QUANTUM FIELD THEORY IN CURVED SPACETIME

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of
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Prateek Pant

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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **Quantum Field Theory in Curved Spacetime** in the partial fulfillment of the requirements for the award of the degree of **MASTER OF SCIENCE** and submitted in the **DISCIPLINE OF PHYSICS, Indian Institute of Technology Indore**, is an authentic record of my own work carried out during the time period from July 2019 to June 2021 under the supervision of **Dr. Manavendra Mahato**, Associate Professor, Indian Institute Of Technology Indore.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other institute.

Prateek 11/06/2021

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This is to certify that the above statement made by the candidate is correct to the best of my/our knowledge.

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Abstract

We will look at working of quantum fields in the curved spacetime which is a framework that includes all fundamental interactions while not requiring that gravity be quantized. String theory provides us the best possible tool to help in the unification of quantum field theory and gravity. The most celebrated idea of AdS/CFT duality, conjectured by Maldacena is a nice way to see it, which will also be the topic we will work on.

The constructions of the bulk fields in terms of the boundary fields for different settings have been carried out here. We look at the anti deSitter and deSitter cases and review some calculations in different coordinate systems for them. We employ the method of HKLL reconstruction. We carry on this construction over to the case of D1 branes and work in the 10 dimensional near horizon geometry dimensionally reduced to 3 dimensions. We look at a scalar field in this background and find its solution for different cases and develop a construction for it in terms of the boundary field theory, for the case of (horizon)$r_o \to 0$.

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Chapter 1

Introduction

Quantum field theory in curved spacetime is, as the name suggests the theory for quantization of fields propagating within the curved spacetime, while not requiring that gravity be quantized. The concept of a vacuum state in these theories is somewhat abstruse, which affects subsequent quantization procedure. The domain of applicability of such theories which requires a physical regime in which the influence of gravity on the propagation of quantum fields is significant but its own microscopic description is irrelevant would be in regions of extremely high spatial curvature such as in the vicinity of a black hole or in the very early stages of the universe. Such a regime would occur when the characteristic length-scale of a highly curved spacetime becomes comparable to the wavelength of quantum modes propagating thereon. Cosmological observations provide strong incentive for the study of QFT in curved spacetime, like the evidences for inflation. It has provided important physical insights like the Hawking's realization in 1976 that black holes aren't really black, but instead emit thermal radiation at a Hawking temperature proportional to the surface gravity κ, $T = \frac{\kappa}{2\pi}$, although not an observed phenomenon as the radiation temperature goes like $\frac{hc^3}{kGM}$, i.e., it is inversely proportional to the mass of the black hole, which makes it very small.
The primordial density perturbation spectrum emerging from cosmic inflation, that is the Bunch-Davies vacuum, can also be predicted using this formalism of QFT in curved spacetime, so this prediction is already verified if the inflation is correct. The theory of quantum field theory in curved spacetime can be considered as a first approximation to quantum gravity, but since gravity is not renormalizable in QFT, so merely formulating QFT in curved spacetime is not a theory of quantum gravity.

de Sitter space and anti-de Sitter space:
The main focus in this thesis will be on the de Sitter and anti-de Sitter spaces and the theories related to them. Named after Willem de Sitter, these spaces are maximally symmetric, like the Minkowski spacetime, which implies a constant curvature-de Sitter space being a constant positive curvature Lorentzian manifold while anti de sitter space having a constant negative curvature (zero in the Minkowski case). There metrics can be obtained by embedding a hyperboloid on the Minkowski space. The main application of de Sitter space is its use in general relativity, where it is one of the simplest mathematical models of the universe that is consistent with the observed accelerating expansion of the universe, while the AdS/CFT correspondence [11] supports a major role for the anti-de Sitter geometry in theoretical physics.

AdS/CFT is a conjecture that any complete theory of quantum gravity in an asymptotically AdS spacetime defines a CFT (Conformal field theory).It is claimed that this conformal field theory on the boundary is equivalent to the theory of gravitation on the bulk anti-de Sitter space. AdS/CFT perspective let us translate questions about quantum gravity into mathematically well posed questions about CFT. Although, it might not be possible to formulate all quantum gravity questions in CFT language. The AdS/CFT correspondence follows from the low-energy limit of open/closed duality of strings in D-brane systems.

The success of the AdS/CFT correspondence has motivated further proposal of holographic dualities between gravitational systems and conformal field theories. In particular, a similar correspondence has been proposed for gravity in de Sitter (dS) spacetime, **dS/CFT**. There has been an increasing interest for the theory of dS/CFT due to the recent astronomical observations which indicate that the cosmological constant in our universe is positive,i.e., it resembles de Sitter space[6-9].

Chapter 2

Theoretical Background

2.1. Quantum field theory in Curved Spacetime

Scalar field in curved spacetime
To generalize physical theories from flat to curved spacetime, we simply express the theories in a coordinate-invariant form and assert that they remain true when spacetime is curved. Here we have a look at the basic framework to work with a scalar field in a curved spacetime.

The curved space generalization of the action for a single scalar field ϕ would be

$$S_\phi = \int \left[-\frac{1}{2} g^{\mu\nu} \left(\nabla_\mu \phi\right) \left(\nabla_\nu \phi\right) - V(\phi) \right] \sqrt{-g} d^n x.$$

Lagrangian density of a scalar field in curved spacetime is

$$\mathcal{L} = \sqrt{-g} \left[-\frac{1}{2} g^{\mu\nu} \left(\nabla_\mu \phi\right) \left(\nabla_\nu \phi\right) - \frac{1}{2} m^2 \phi^2 - \xi R \phi^2 \right].$$

We have included a direct coupling to the curvature scalar R, parameterized by a constant ξ. Following two values of ξ are popularly used,

$$\xi = 0 \qquad \text{(minimal coupling)}$$

$$\xi = \frac{n-2}{4(n-1)} \qquad \text{(conformal coupling)}$$

The conjugate momentum can be written as

$$\Pi = \frac{\partial \mathcal{L}}{\partial (\nabla_o \phi)} = \sqrt{-g} \left(\nabla_o \phi\right).$$

We can impose the following canonical commutation relations.

$$\left[\phi(t,\vec{x}), \phi(t,\vec{x}')\right] = 0 \qquad \left[\Pi(t,\vec{x}), \Pi(t,\vec{x}')\right] = 0 \qquad \left[\phi(t,\vec{x}), \Pi(t,\vec{x}')\right] = \frac{i}{\sqrt{-g}} \delta^{(n-1)}(\vec{x}-\vec{x}')$$

The equation of motion for the scalar field obtained from its action is

$$\Box\phi - m^2\phi - \xi R\phi = 0. \tag{1}$$

The inner product for the solutions to this equation is generalized to the expression(For a spacelike hypersurface Σ)

$$(\phi_1, \phi_2) = -i \int_\Sigma (\phi_1 \nabla_\mu \phi_2^* - \phi_2^* \nabla_\mu \phi_1) n^\mu \sqrt{\gamma} d^{n-1}x$$

where γ_{ij} is the induced metric and n^μ is the normal vector. This is independent of the choice of Σ.
Expanding our field in terms of complete set of solutions, $f_i(x^\mu)$, to (1) that are orthonormal $(f_i, f_j) = \delta_{ij}$ (and corresponding conjugate modes with negative norm $(f_i^*, f_j^*) = -\delta_{ij}$)

$$\phi = \sum_i (\hat{a}_i f_i + \hat{a}_i{}^\dagger f_i^*)$$

where the coefficients $\hat{a}_i$ and $\hat{a}_i{}^\dagger$ are the annihilation and creation operators, following some commutation relations. The vacuum state $|0_f\rangle$ will be annihilated by all the annihilation operators as,

$$\hat{a}_i|0_f\rangle = 0 \text{ (for all i)}$$

and a state of n_i excitations will be defined as

$$|n_i\rangle = \frac{1}{\sqrt{n_i!}} (\hat{a}_i{}^\dagger)^{n_i} |0_f\rangle.$$

In Minkowski space,there is a natural set of modes that we can write, which are the eigenfunctions of $\frac{\partial}{\partial t}$, which is the Killing vector of the space. However, in general curved spacetime there is no canonical choice of a time variable w.r.t which we can classify the modes as being positive or negative frequency, like we could in flat space. So in the transition from the flat to curved spacetime we have lost any reason to prefer one set of modes over any other and we can consider an alternative set of modes $g_i(x^\mu)$ with all of the properties that our original modes possessed and expand the field with respect to it.

$$\phi = \sum_i (\hat{b}_i g_i + \hat{b}_i{}^\dagger g_i^*)$$

Bogolubov transformation is the transformation from one set of basis modes into another and is given by

$$g_i = \sum_j (\alpha_{ij} f_j + \beta_{ij} f_j^*)$$

$$f_i = \sum_j (\alpha_{ij}^* g_j - \beta_{ji} g_j^*)$$

where the matrices α_{ij} and β_{ij} are called the Bogolubov coefficients. Using the orthonormality of the mode functions, they can be expressed as

$$\alpha_{ij} = (g_i, f_j) \qquad \beta_{ij} = -(g_i, f_j^*).$$

They satisfy a set of normalization conditions given as

$$\sum_j (\alpha_{ik}\alpha^*_{jk} - \beta_{ik}\beta^*_{jk}) = \delta_{ij} \qquad\qquad \sum_j (\alpha_{ik}\beta_{jk} - \beta_{ik}\alpha_{jk}) = 0$$

and can be used to transform between the operators,

$$\hat{a}_i = \sum_j (\alpha_{ji}\hat{b}_j + \beta^*_{ji}\hat{b}^\dagger_j)$$

$$\hat{b}_i = \sum_j (\alpha^*_{ij}\hat{a}_j - \beta^*_{ij}\hat{a}^\dagger_j).$$

We can see from these relation that observers working with different set of modes will disagree on the number of particles observed or if observed at all. To know the number of particles observed in a system which, say, is in f-vacuum $|0_f\rangle$ and has no f-particles, by an observer using the g-modes we look at the expectation value of the g number operator in the f- vacuum.

$$\langle 0_f|\hat{n}_{gi}|0_f\rangle = \langle 0_f|b^\dagger_i b_i|0_f\rangle = \langle 0_f| \sum_{jk} (\alpha_{ij}\hat{a}^\dagger_j - \beta_{ij}\hat{a}_j)(\alpha^*_{ik}\hat{a}_k - \beta^*_{ik}\hat{a}^\dagger_k)|0_f\rangle$$

$$= \sum_j |\beta_{ij}|^2$$

There is no reason for this to vanish. So what appears like an empty vacuum from one perspective will be bubbling with particles according to another.

2.2. Maximally Symmetric Universes

Copernican principle is the idea that the universe is pretty much the same everywhere. It is related to two more mathematical properties that a manifold might have: isotropy and homogeneity. Isotropy applies at some specific point in the manifold and states that the space looks the same no matter in what direction you look. Homogeneity is the statement that the metric is the same throughout the manifold. In other words, given any two points p and q in M, there is an isometry that takes p into q.
If a space is isotropic everywhere, then it is homogeneous. Likewise if it is isotropic around one point and also homogeneous, it will be isotropic around every point.
The usefulness of homogeneity and isotropy is that they imply that a space is maximally symmetric, i.e., the space has its maximum possible number of killing vectors. Riemann tensor for a maximally symmetric n-dim manifold with metric $g_{\mu\nu}$ can be written

$$R_{\rho\sigma\mu\nu} = \kappa(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu})$$

where $\kappa = \frac{R}{n(n-1)}$ and the Ricci scalar R will be a constant over the manifold.
Since at any single point we can always put the metric into its canonical form, the kinds of maximally symmetric manifolds are characterized locally by the signature of the metric and the sign of the constant κ. For vanishing curvature ($\kappa = 0$) the maximally symmetric spacetime is the Minkowski space, with metric

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2.$$

The maximally symmetric spacetime with positive curvature ($\kappa > 0$) is called **de Sitter space** and the one with a negative curvature ($\kappa < 0$) is known as the **anti Sitter space**.To obtain the metric for these space we can embed a hyperboloid in flat space. So for de Sitter space if we consider a 5-dimensional Minkowski space and embed a hyperboloid($-u^2 + x^2 + y^2 + z^2 + w^2 = \ell^2$) on it with certain embedding coordinates defined in terms of a set of coordinates, say $\{t, \chi, \theta, \phi\}$ as,

$$u = \ell \sinh(t/\ell) \quad w = \ell \cosh(t/\ell) cos\chi \quad x = \ell \cosh(t/\ell) \sin \chi \cos \theta$$

$$y = \ell \cosh(t/\ell) \sin \chi \sin \theta \cos \phi \quad z = \ell \cosh(t/\ell) \sin \chi \sin \theta \sin \phi$$

we can get the metric to be

$$ds^2 = -dt^2 + \ell^2 \cosh^2(t/\ell) \left[d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2) \right]. \qquad (2)$$

ℓ is known as the de Sitter radius.
Let us consider a transformation from t to t' via

$$\cosh(t/\ell) = \frac{1}{\cos(t')}.$$

The metric (2) will be written as, under this change

$$ds^2 = \frac{\ell^2}{\cos^2(t')} d\bar{s}^2$$

where $d\bar{s}^2$ represents the metric on the Einstein static universe,

$$d\bar{s}^2 = -(dt')^2 + d\chi^2 + \sin^2 \chi d\Omega_2^2 \qquad (3)$$

where we have identified the expression in round parantheses in (2) as the metric on a 2-sphere, $d\Omega_2^2$.
The range of the new time coordinate is

$$-\pi/2 < t' < \pi/2$$

The conformal diagram of de Sitter space will simply be a representation of the patch of the Einstein static universe to which de Sitter is conformally related.

For the case of AdS we look at a fictitious 5-d flat manifold with metric $ds_5^2 = -du^2 - dv^2 + dx^2 + dy^2 + dz^2$, and embed a hyperboloid($-u^2 - v^2 + x^2 + y^2 + z^2 = -R^2$.) and induce coordinates $\{t', \rho, \theta, \phi\}$ on the hyperboloid via the embedding

$$u = R \sin t' \cosh \rho \quad v = R \cos t' \cosh \rho \quad x = R \sinh \rho \cos \theta$$

$$y = R \sinh \rho \sin \theta \cos \phi \quad z = R \sinh \rho \sin \theta \sin \phi$$

yielding a metric on this hyperboloid of the form

$$ds^2 = R^2(-\cosh^2 \rho dt'^2 + d\rho^2 + \sinh^2 \rho d\Omega_2^2)$$

where R is the anti desitter radius.
To obtain the conformal diagram for AdS, we do a coordinate transformation analogous to that used for de Sitter,but now on the radial coordinate.

$$\cosh \rho = \frac{1}{\cos \chi}$$

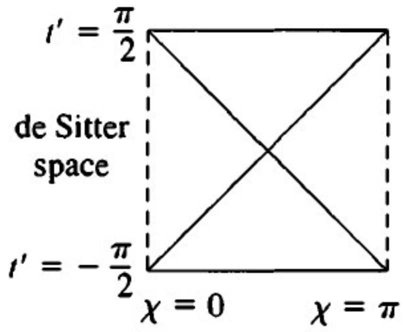

Figure 2.1: Conformal diagram for de Sitter spacetime.
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so that

$$ds^2 = \frac{R^2}{\cos^2 \chi} d\bar{s}^2$$

where $d\bar{s}^2$ is given by (3). The t' coordinate goes from $-\infty$ to ∞, while the range of the radial coordinate is

$$0 \leq \chi < \pi/2$$

Thus, anti-de Sitter space is conformally related to half of the Einstein static universe.

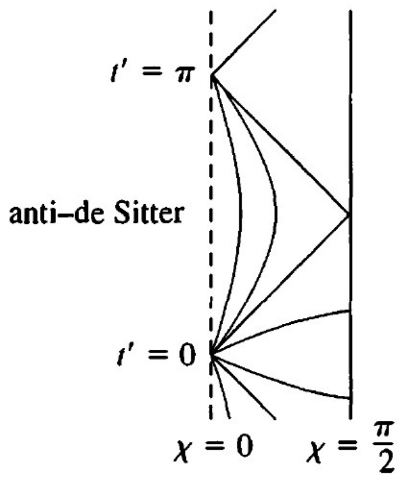

Figure 2.2: Conformal diagram for anti de Sitter spacetime.
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2.3. Coordinates in AdS

AdS in n+1 dimensions can be represented as a hyperboloid of radius R

$$X_o^2 + X_{n+1}^2 - \sum_{i=1}^{n} X_i^2 = R^2 \tag{4}$$

embedded in an n+2 dimensional flat space with metric $ds^2 = -dX_o^2 - dX_{n+1}^2 + \sum_{i=1}^{n} dX_i^2$ The coordinates X_a for $a = 0, .., n+1$ are known as embedding coordinates. The eq(4) can be solved by setting

$$\begin{aligned}
&X_o = R\sec\rho\cos t && && 0 \le \rho < \pi/2\\
&X_i = R\tan\rho\Omega_i && i = 1, ...n && -\pi < t \le \pi\\
&X_{n+1} = R\sec\rho\sin t && -1 \le \Omega_i \le 1 && \left[\sum_{i=1}^{n}\Omega_i^2 = 1\right]
\end{aligned}$$

$\big(\Omega_i$ parametrizes a S^{n-1} sphere$\big)$

The coordinates ρ, t and Ω_i cover the entire hyperboloid, thus are called the **global coordinates**. In these global coordinates the AdS boundary is the hypersurface $\rho = \pi/2$ This corresponds, in embedding coordinates, to the spatial infinity. So, we can compactify the AdS space by changing the range of the radial coordinate ρ to: $0 \le \rho \le \pi/2$ The AdS metric in these coordinates is given as:

$$ds^2 = R^2\sec^2\rho(-dt^2 + d\rho^2 + \sin^2\rho d\Omega_{n-1}^2). \tag{5}$$

There is another coordinate system for AdS, called the **Poincaré Patch**.To introduce the Poincaré coordinate system let us define the following light cone coordinates.

$$u = \frac{X_o - X_n}{R^2}\ ,$$

$$v = \frac{X_o + X_n}{R^2}.$$

Let

$$x^i = \frac{X_i}{Ru}\ ,\ t = \frac{X_{n+1}}{Ru}.$$

Then eq(4) becomes

$$R^4uv + R^2u^2(t^2 - \bar{x}^2) = R^2 \qquad \left(\bar{x}^2 = \sum_{i=1}^{n-1}(x^i)^2\right)$$

and which gives us the following expression for v.

$$v = \frac{1 - u^2(t^2 - \bar{x}^2)}{R^2u}$$

We therefore get the embedding coordinates as

$$\begin{aligned}
X_o &= \frac{1}{2u}(1 + u^2(R^2 + \bar{x}^2 - t^2))\\
X_n &= \frac{1}{2u}(1 + u^2(-R^2 + \bar{x}^2 - t^2))\\
X_i &= Rux^i \qquad i = 1, ..., n-1\\
X_{n+1} &= Rut.
\end{aligned}$$

It is useful to change to the coordinate $z = \frac{1}{u}$. So the Poincaré coordinates $z, \bar{x}, t$ are defined by the following relations

$$
\begin{aligned}
X_o &= \frac{1}{2z}(z^2 + R^2 + \bar{x}^2 - t^2)) \\
X_n &= \frac{1}{2z}(z^2 - R^2 + \bar{x}^2 - t^2)) \\
X_i &= \frac{Rx^i}{z} \qquad i = 1, ..., n-1 \\
X_{n+1} &= \frac{Rt}{z}
\end{aligned}
$$

The AdS metric in terms of these coordinates is given as

$$
ds^2 = \frac{R^2}{z^2}\left(-dt^2 + d\bar{x}^2 + dz^2\right) \tag{6}
$$

The coordinate z divides the AdS space in two regions($z > 0$ and $z < 0$) giving 2 Poincaré charts each corresponding to one half of the hyperboloid. The Poincaré AdS space is the region of the entire AdS corresponding to one of these two charts (the $z > 0$ is usually chosen).

The Poincaré coordinates can be written in terms of global coordinates as:

$$
t = \frac{X_{n+1}}{Ru} = \frac{X_{n+1}}{R\frac{(X_o - X_n)}{R^2}} = \frac{RX_{n+1}}{(X_o - X_n)} = \frac{R(R\sec\rho\sin\tau)}{(R\sec\rho\cos\tau - R\tan\rho\Omega_n)} = \frac{R\sin\tau}{\cos\tau - \Omega_n\sin\rho}
$$

$$
\begin{aligned}
z &= \frac{1}{u} = \frac{R^2}{(X_o - X_n)} = \frac{R\cos\rho}{\cos\tau - \Omega_n\sin\rho} \\
x_i &= \frac{X_i}{Ru} = \frac{\tan\rho\Omega_i}{u} = \frac{R\Omega_i\sin\rho}{\cos\tau - \Omega_n\sin\rho}
\end{aligned}
$$

Here the global time is denoted by τ to avoid confusion. Also even when $t \to \pm\infty$ we only cover a finite range in τ.

2.4. Quantum field theory on de Sitter space

Here we will look at the scalar field with the background of d-dimensional de Sitter space. dS_d may be realized as the hypersurface described by the equation

$$
-X_o^2 + X_1^2 + ...X_d^2 = \ell^2 \tag{7}
$$

in flat $d+1$-dimensional Minkowski space $M^{d,1}$, where ℓ is the de Sitter radius. This hypersurface in flat Minkowski space is a hyperboloid.
The action for the scalar field will be,

$$
S = -\frac{1}{2}\int d^dx\sqrt{-g}\left[(\nabla\phi)^2 + m^2\phi^2\right] \tag{8}
$$

Since this is a free field theory, all information is encoded in the two-point function of ϕ. We will study the Wightman function

$$
G(X, Y) = \langle 0|\phi(X)\phi(Y)|0\rangle \tag{9}
$$

which obeys the free field equation

$$(\nabla^2 - m^2)G(X,Y) = 0 \tag{10}$$

where ∇^2 is the Laplacian on dS_d.
Let us assume that the state $|0\rangle$ in (9) is invariant under the SO(d,1) de Sitter group. Then $G(X,Y)$ will be de Sitter invariant, and so at generic points can only depend on the de Sitter invariant length $\sigma(X,Y)$ between X and Y. Writing $G(X,Y) = G(\sigma(X,Y))$, (10) reduces to a differential equation in one variable σ

$$(1-\sigma^2)\partial_\sigma^2 G - d\sigma\partial_\sigma G - m^2 G = 0. \tag{11}$$

Let $z = \frac{1+\sigma}{2}$, then (11) becomes

$$z(1-z)G'' + (\frac{d}{2} - dz)G' - m^2 G = 0. \tag{12}$$

This is a hypergeometric equation whose solution is

$$G = c_{m,d}F(h_+, h_-, \frac{d}{2}, z) \tag{13}$$

where $c_{m,d}$ is a normalization constant and

$$h_\pm = \frac{1}{2}\left[(d-1) \pm \sqrt{(d-1)^2 - 4m^2}\right].$$

The hypergeometric function (13) has a singularity at z=1, or $\sigma = 1$, and a branch cut for $1 < \sigma < \infty$.The singularity occurs when the points X and Y are separated by a null geodesic. At short distances the scalar field is insensitive to the fact that it is in de Sitter space and the form of the singularity is precisely the same as that of the propagator in flat d-dim. Minkowski space. We can use this fact to fix $c_{m,d}$. Near $z = 1$ the hypergeometric function behaves as

$$F(h_+, h_-, \frac{d}{2}, \frac{1+\sigma}{2}) \sim \left(\frac{D^2}{4}\right)^{1-d/2} \frac{\Gamma(\frac{d}{2})\Gamma(\frac{d}{2}-1)}{\Gamma(h_+)\Gamma(h_-)}$$

where $D = \cos^{-1}\sigma$ is the geodesic separation between the the points. Comparing with the usual short-distance singularity $\frac{\Gamma(\frac{d}{2})}{2(d-2)\pi^{d/2}}(D^2)^{1-\frac{d}{2}}$ fixes the coefficient to be

$$c_{m,d} = 4^{1-d/2}\frac{\Gamma(h_+)\Gamma(h_-)}{\Gamma(\frac{d}{2})\Gamma(\frac{d}{2}-1)}\frac{\Gamma(\frac{d}{2})}{2(d-2)\pi^{d/2}} = \frac{\Gamma(h_+)\Gamma(h_-)}{(4\pi)^{d/2}\Gamma(d/2)}.$$

Since (11) has a $\sigma \to -\sigma$ symmetry, so if $G(\sigma)$ is a solution then $G(-\sigma)$ is also a solution. The second linearly independent solution to (5) is therefore

$$F(h_+, h_-, \frac{d}{2}, \frac{1-\sigma}{2}). \tag{14}$$

The singularity now at $\sigma = -1$, corresponds to X being null separated from the antipodal point to Y, which sounds rather unphysical at first, but since the antipodal points in de Sitter space are always separated by a horizon, the Green function (14) can be thought of as arising from an image source behind the horizon, and nonsingular everywhere within an observer's horizon. Hence the "unphysical" singularity can't be detected by any experiment.
The linear combination of (13) and (14) corresponds to a one parameter family of invariant Green functions G_α, in the de Sitter space, and thus a one-parameter family of de Sitter invariant vacuum states $|\alpha\rangle$ such that $G_\alpha(X,Y) = \langle\alpha|\phi(X)\phi(Y)|\alpha\rangle$.

2.5. String theory

It is the leading candidate for a theory that unifies the four basic forces of nature viz. Electromagnetic, Weak, Strong and Gravitation. In this theory each particle is identified as a specific vibrational mode of an elementary microscopic string. The elementary objects in string theory are one dimensional in contrast to usual zero-dimensional elementary particles. Due to the presence of only one sort of string and all particles arising from its vibrations, all of them are naturally incorporated into a single theory. So, suppose a decay process $\alpha \to \beta + \gamma$, will be imagined as a single string vibrating in such a way that it is identified as particle α that breaks into two strings that vibrate in ways that identify them as particles β and γ in this theory. Since strings may turn out to be extremely tiny, it may be difficult to observe directly the string-like nature of particles. The theory also contains **branes** that are objects which can have any no. of allowed dimensions. They are dynamical objects which can propagate through spacetime consistent with the principles of quantum mechanics, have mass and can have other attributes such as charge. A string can be regarded as a brane of one dimension. A p-brane (p-dim. Brane) sweeps out a (p+1) dimensional volume in spacetime. World sheets (2-dimensional surfaces) describe time evolution of strings and world volume for general higher dimensional branes (compared to world line for point particle). The absence of adjustable dimensionless parameters in the theory and the fact that the dimensionality of spacetime is fixed (emerges from a calculation-whereas in the standard model the information of a four-dimensional spacetime used to build the theory is not derived) is a sign of the uniqueness of string theory. It has one dimensionful parameter, the string length ℓ_s, whose value can be roughly imagined as the typical size of strings. Some of the dimensions in this theory may hide from plane view if they curl up into a space that is small enough to escape detection in experiments done with low energies. The lack of adjustable dimensionless parameters meant that the theory cannot be deformed or changes continuously by changing these parameters, so there could be other theories that cannot be reached by continuous deformations which leads us to the question that how many string theories are there? There are open strings, which have 2 endpoints and closed strings with no endpoints in the theory. One can consider theories with only closed strings, and theories with both open and closed strings. Since open strings generally can close to form closed strings, we don't consider theories with only open strings. Another subdivision in string theory is between bosonic and superstring theories. Bosonic strings live in 26 dimensions and their vibrations represent bosons, but the bosonic string theories are not realistic as they lack fermions. The superstrings live in ten dimensional spacetime, and their spectrum of states includes bosons and fermions related by supersymmetry. All realistic models of string theory are built from superstrings. There exists 5 possible 10 dimensional superstring theories (type IIA, type IIB, type I, SO(32) heterotic, and E8xE8 heterotic. The five different superstring theories are related to one another by duality symmetries viz. T-duality and S-duality. As all the five different string theories are related to one another by these duality maps, it strongly suggests presence of a single underlying theory known as "M-theory" or sometimes referred to as "U-theory" or universal theory. In special limits, M-theory is described by one of the (compactified) five different weakly coupled string theories. Besides these five theories there exists a sixth one, which is (10 + 1)-dimensional supergravity theory (SUGRA). M-theory in low energy limit reduces to this 11-dimensional supergravity theory and the term

M-theory sometimes refers to this low energy supergravity theory. Another way of defining M-theory is as the strong coupling limit of Type IIA string theory. The Type IIA string theory is taken equivalent to the M-theory compactified on a circle, where the radius of compactification is proportional to the string coupling.

2.6. AdS/CFT Duality

The AdS/CFT correspondence or the gauge/gravity duality was first proposed by Juan Maldacena in late 1997 [1]. Large N limits of certain conformal field theories in d dimensions can be described in terms of supergravity (and string theory) on the product of d+1-dimensional AdS space with a compact manifold. He conjectured that $\mathcal{N} = 4$ U(N) super-Yang-Mills theory in 3 + 1 dimensions is the same as (or dual to) type IIB superstring theory on $AdS_5 \times S_5$

AdS/CFT correspondence in simple terms states that the boundary of anti-de Sitter space can be regarded as the "spacetime" for a conformal field theory. It is claimed that this conformal field theory on the boundary is equivalent to the theory of gravitation on the bulk anti-de Sitter space in the sense that there is a "dictionary" for translating calculations within one theory to calculations in the other.
This correspondence is usually described as a "holographic duality" because this relationship between the two theories is analogous to the connection between a three-dimensional object and its image as a hologram. Although a hologram is two-dimensional, it encodes information about all three dimensions of the corresponding object it represents. In the same way, theories which are related by the AdS/CFT correspondence are conjectured to be exactly equivalent, despite living in several numbers of dimensions. So, the conformal field theory like a hologram captures information about the higher-dimensional quantum gravity theory.
More generally, the AdS/CFT perspective allows us to translate questions of quantum gravity into mathematically well posed questions on CFT. Although, it might not be possible to formulate all quantum gravity questions in CFT language.
String theory provides a “method” to seek out explicit examples of CFTs and their dual gravitational theories. The basic idea is to look at the low energy description of D-brane systems from the view point of open and closed strings. Let us illustrate the argument by quickly summarizing the prototypical example of AdS/CFT. Consider N coincident D3-branes of type IIB string theory in 10 dimensional Minkowski spacetime. Closed strings propagating in 10 dimensions can interact with the D3-branes. These interactions can be described in the following two equivalent ways:
(a) Defining D3-branes as a submanifold where open strings can end, which means that a closed string interacts with the D3-branes by breaking the string loop into an open string with endpoints attached to the D3-branes.

(b) Defining D3-branes as solitons of closed string theory,i.e., they create a non-trivial curved background where closed strings propagate.

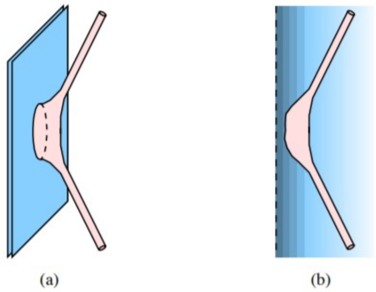

Figure 2.3: (a) Closed string scattering off branes in flat space. (b) Closed string propagating in a curved background.Reprinted from “TASI lectures on AdS/CFT” by J.Penedones,2016,p.33

Their equivalence is called open/closed duality. The AdS/CFT correspondence follows from the low-energy limit of this duality. We implement this low-energy limit by taking the string length $\ell_s \to 0$, keeping the string coupling g_s, the number of branes N and the energy fixed.

In description (a), the low energy excitations of the system form two decoupled sectors: massless closed strings propagating in 10 dimensional Minkowski spacetime and massless open strings attached to the D3-branes, which at low energies are well described by N = 4 Supersymmetric Yang-Mills (SYM) with gauge group SU(N).

The massless closed strings in description (b), propagate in the following geometry

$$ds^2 = \frac{1}{\sqrt{H(r)}}\eta_{\mu\nu}dx^\mu dx^\nu + \sqrt{H(r)}\left[dr^2 + r^2 d\Omega_5^2\right]$$

where $\eta_{\mu\nu}$ being the metric of the 4-d Minkowski space along the branes and

$$H(r) = 1 + \frac{R^4}{r^4}\ ,\ R^4 = 4\pi g_s N \ell_s^4.$$

Naively, the limit $\ell_s \to 0$ just produces 10 dimensional Minkowski spacetime. However, for the region close to the branes, at $r = 0$, one has to be careful. For $r \to 0$ there's an infinite red shift. So albeit near the stack of branes the energy E is arbitrarily large, for the observer at $r \to \infty$ it's finite thanks to the redshift. To work out the correct low-energy limit in the region around $r = 0$ we introduce a new coordinate $z = \frac{R^2}{r}$, which is kept fixed as $\ell_s \to 0$. This results in the following metric

$$ds^2 = R^2\frac{dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu}{z^2} + R^2 d\Omega_5^2$$

which is in turn the metric of $AdS_5 \times S_5$ both with radius R. Therefore, description (b) also leads to 2 decoupled sectors of low energy excitations: massless closed strings in 10D and full type IIB string theory on $AdS_5 \times S_5$. So we have the subsequent two pictures:

Picture 1: $\mathcal{N} = 4$ SYM in 4 dimensions $\bigoplus$ Super-gravity in 10 dimensions

Picture 2: Full type IIB string theory in $AdS_5 \times S_5$ $\bigoplus$ Super-gravity in 10 dimensions

Maldacena equated Picture I and Picture II, and thus conjectured the following duality:

$\mathcal{N} = 4$ SYM in 4 dimensions = Full type IIB string theory in $AdS_5 \times S_5$

2.7. D-branes

String theory contains D-branes which are solitonic objects and are an important class of branes that arise when one considers open strings. A Dp-brane is an extended object with p-spatial dimensions, where D stands for Dirichlet. In the presence of a D-brane, the endpoints of open strings must lie on the brane. This requirement imposes a number of Dirichlet boundary conditions on the motion of the open string endpoints. The worldsheet duality suggests that the D-brane is additionally a source of closed strings. When we consider the complete theory within the presence of these solitons we've modes that propagate within the bulk and modes that propagate on the solitons. The modes on the soliton interact with each other and with the bulk modes. It is possible, however, to define a limit of the full theory in which the bulk modes decouple from the modes living on the D-brane. This is typically a low energy limit, in which we tune the coupling constant so as to keep only the interactions among the modes living on the D-brane. The low energy effective theory of open strings on the Dp-brane is the U(N) gauge theory in (p + 1) dimensions with 16 supercharges[20]. When the open string endpoints have free boundary conditions along all spatial directions, we still have a D-brane, but now it is a space-filling D-brane. The D-brane extends all over space, and since open string endpoints can be anywhere on the D-brane, open string endpoints are completely free (like in bosonic string theory, where the number of spatial dimensions is 25, a D25-brane is a space-filling brane).
In this paper we will work with the D1 brane. The two-dimensional SU(N) Yang-Mills theory with sixteen supercharges at large N is conjectured to be is dual to the near horizon supergravity solution of D1-branes [21].

Near horizon geometry supergravity solution of the non-extremal D1-brane in the Einstein frame is given by

$$ds_{10}^2 = H^{-\frac{3}{4}}(r)\left(-f(r)dt^2 + dx_1^2\right) + H^{\frac{1}{4}}(r)\left(\frac{dr^2}{f(r)} + r^2 d\Omega_7^2\right)$$

$$e^{\phi(r)} = H(r)^{\frac{1}{2}}$$

$$F_7^{RR} = 6L^6\omega_{S_7}$$

where

$$H(r) = \left(\frac{L}{r}\right)^6, f = 1 - \frac{r_o^6}{r^6}, \text{ and } L^6 = g_{YM}^2 2^6 \pi^3 N \alpha'^4.$$

and $d\Omega_7^2$ refers to the metric on the unit 7-sphere and ω_{S_7} its volume form.

We now consistently truncate the 10 dimensional near horizon geometry of the D1-brane to 3 dimensions by dimensionally reducing on the 7-sphere using the following ansatz.

$$\begin{aligned} ds_{10}^2 &= e^{-14B(r)} g_{\mu\nu}(x) dx^\mu dx^\nu + e^{2B(r)} L^2 d\Omega_7^2 \\ &= e^{-14B(r)}\left(-c_T(r)^2 dt^2 + c_X(r)^2 dz^2 + c_R(r)^2 dr^2\right) + e^{2B(r)} L^2 dS_{X_7}^2 \end{aligned}$$

Using this ansatz in the 10-dimensional supergravity equations of motion, one obtains a set of coupled differential equations for the fields $c_T(r)$, $c_X(r)$, $c_R(r)$, $\phi(r)$ (dilaton) and $B(r)$ (breathing mode). It can be shown that on identifying

$$B(r) = -\frac{1}{24}\phi(r)$$

and keeping the Ramond-Ramond flux through the 7-sphere constant, one can obtain a consistent truncation of the 10-dimensional equations to effectively 3-dimensions[22][23]. The truncated set of equations of motion can be obtained from the following Einstein-dilaton system in 3 dimensions with action

$$S = \frac{1}{16\pi G_3}\int d^3x\sqrt{-g}\left[R - \frac{\beta}{2}\partial_\mu\partial^\mu\phi - \mathcal{P}(\phi)\right].$$

Here $\beta = \frac{16}{8}$ and G_3 is a three dimensional Newton's constant. $\mathcal{P}$ (dilaton potential) $= -\frac{24}{L^2}e^{\frac{4}{3}\phi}$
So the D1 brane in 10 dimensions reduces to

$$ds^2 = -c_T(r)^2dt^2 + c_X(r)^2dz^2 + c_R(r)^2dr^2 \tag{15}$$

$$\phi = -3\log\left(\frac{r}{L}\right)$$

where the background profiles are

$$c_T^2 = \left(\frac{r}{L}\right)^8 f,\ c_X^2 = \left(\frac{r}{L}\right)^8,\ c_R^2 = \frac{1}{f}\left(\frac{r}{L}\right)^2$$

with $f = 1 - \frac{r_o^6}{r^6}$.

Chapter 3

Bulk Reconstruction in Anti deSitter Space

Here we will look at the bulk field reconstructions in AdS_3, working in different coordinates and obtain the relation for the bulk scalar field in terms of a boundary field theory operator. The construction done here is known as the HKLL construction after Hamilton, Kabat, Lifschytz, and Lowe [3-5].

3.1. Poincare Coordinates

We will first look at the construction in Poincaré coordinates and obtain a smearing function using the Poincaré mode sum approach. An alternative approach is by Wick rotating to de Sitter space and using a retarded de Sitter Green's function[3]. The metric in these coordinates for AdS_3 can be written using (6) as

$$ds^2 = \frac{R^2}{z^2}\left(-dt^2 + dx^2 + dz^2\right) \tag{16}$$

From this we can write

$$g_{\mu\nu} = \frac{R^2}{z^2}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad g^{\mu\nu} = \frac{z^2}{R^2}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and we get

$$\sqrt{|g|} = \left[-\frac{R^2}{z^2} \times \frac{R^2}{z^2} \times \frac{R^2}{z^2}\right]^{1/2}.$$

The Laplacian Beltrami operator in these coordinates can be obtained as

$$\begin{aligned}
\Box &= \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^\alpha}\left(\sqrt{|g|}\, g^{\alpha\beta}\frac{\partial}{\partial x^\beta}\right) \\
&= \frac{z^3}{R^3}\left[\frac{\partial}{\partial t}\left(\frac{R^3}{z^3}\left(\frac{-z^2}{R^2}\right)\frac{\partial}{\partial t}\right) + \frac{\partial}{\partial x}\left(\frac{R^3}{z^3}\left(\frac{z^2}{R^2}\right)\frac{\partial}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{R^3}{z^3}\left(\frac{z^2}{R^2}\right)\frac{\partial}{\partial z}\right)\right] \\
&= \left(-\frac{z^2}{R^2}\frac{\partial}{\partial t^2} + \frac{z^2}{R^2}\frac{\partial}{\partial x^2} + \frac{z^2}{R^2}\frac{\partial}{\partial z^2} - \frac{z}{R^2}\frac{\partial}{\partial z}\right)
\end{aligned}$$

Equation of motion for a scalar field is given by

$$\left(-\Box + m^2\right)\phi(x,t,z). \tag{17}$$

Doing Fourier transform on $\phi(x,t,z)$, we can write

$$\phi(x,t,z) = \int d\omega dk a_{\omega k} e^{-i\omega t} e^{ikx} f_{\omega k}(z). \tag{18}$$

From (17) and (18), we get

$$-\omega^2 z^2 f(z) + k^2 z^2 f(z) + z\frac{\partial f}{\partial z} - z^2\frac{\partial f}{\partial z^2} + m^2 R^2 f(z) = 0.$$

This is a Bessel's differential equation whose solution will be of the form

$$f(z) = c_1 z J_\nu(\sqrt{\omega^2 - k^2} z) + c_2 z Y_\nu(\sqrt{\omega^2 - k^2} z)$$

where $\nu = \sqrt{1 + m^2 R^2}$.
For $\nu > 1$ only the Bessel function of first kind will be normalizable. So we get

$$\phi(x,t,z) = \int_{|\omega|>|k|} d\omega dk a_{\omega k} e^{-i\omega t} e^{ikx} z J_\nu(\sqrt{\omega^2 - k^2} z). \tag{19}$$

The boundary field in Poincaré coordinates is defined by

$$\phi_o(x,t) = \lim_{z\to 0} \frac{1}{z^\Delta}\phi(x,t,z) \qquad (\Delta = 1 + \nu).$$

Using the following relation

$$J_\nu(z) = \frac{\left(\frac{1}{2}z\right)^\nu}{\Gamma(\nu+1)}\ {}_0F_1\left(;\nu+1;\frac{-1}{4}z^2\right)$$

where

$${}_0F_1\left(;a;z\right) = \sum_{k=0}^{\infty}\frac{1}{a^k}\frac{z^k}{k!}$$

is the confluent Hypergeometric function of the first kind and ${}_0F_1\left(;\nu+1;\frac{-1}{4}z^2\right) \to 1$, when z$\to$0, we obtain

$$J_\nu(\sqrt{\omega^2 - k^2} z) = \frac{z^\nu}{2^\nu\Gamma(\nu+1)}\left(\omega^2 - k^2\right)^{\nu/2} F(;\nu+1;\frac{-1}{4}z^2).$$

The boundary field can thus be written as

$$\begin{aligned}\phi_o(x,t) &= \lim_{z\to 0}\frac{1}{z^\Delta}\int_{|\omega|>|k|} d\omega dk a_{\omega k} e^{-i\omega t} e^{ikx}\frac{z^{\nu+1}}{2^\nu\Gamma(\nu+1)}\left(\omega^2 - k^2\right)^{\nu/2} F(;\nu+1;\frac{-1}{4}z^2) \\ &= \frac{1}{2^\nu\Gamma(\nu+1)}\int_{|\omega|>|k|} d\omega dk a_{\omega k} e^{-i\omega t} e^{ikx}\left(\omega^2 - k^2\right)^{\nu/2}.\end{aligned}$$

We can invert this to obtain

$$a_{\omega k} = \frac{2^\nu\Gamma(\nu+1)}{(2\pi)^2\left(\omega^2 - k^2\right)^{\nu/2}}\int dt dx e^{i\omega t} e^{-ikx}\phi_o(x,t).$$

Therefore, we can write the bulk scalar field as

$$\phi(x,t,z) = \int dt'dx' K(t',x'|t,x,z)\phi_o(x',t') \tag{20}$$

where

$$K(t',x'|t,x,z) = \frac{2^\nu \Gamma(\nu+1)}{(2\pi)^2} \int_{|\omega|>|k|} d\omega dk a_{\omega k} e^{-i\omega(t-t')} e^{ik(x-x')} \frac{zJ_\nu(\sqrt{\omega^2-k^2}z)}{(\omega^2-k^2)^{\nu/2}}.$$

is the smearing function. We have obtained an expression of the bulk field in terms of the boundary field.

Poincaré mode sum

From eq(20) we have

$$\phi(x,t,z) = \frac{2^\nu \Gamma(\nu+1)}{(2\pi)^2} \int_{|\omega|>|k|} d\omega dk \frac{zJ_\nu(\sqrt{\omega^2-k^2}z)}{(\omega^2-k^2)^{\nu/2}} \left(\int dt'dx' e^{-i\omega(t-t')} e^{ik(x-x')} \phi_o(x',t') \right).$$

The second integral in this expression can be interpreted as the Fourier transform of the boundary field:

$$\phi(x,t,z) = 2^\nu \Gamma(\nu+1) \int_{|\omega|>|k|} d\omega dk \frac{zJ_\nu(\sqrt{\omega^2-k^2}z)}{(\omega^2-k^2)^{\nu/2}} e^{-i\omega t} e^{ikx} \phi_o(\omega,k). \tag{21}$$

Since,

$$J_\nu(b) = \frac{1}{2^{\nu-1}\Gamma(\nu) b^{-\nu}} \int_0^1 r(1-r^2)^{\nu-1} J_0(br) dr \tag{22}$$

we can write

$$J_\nu(\sqrt{\omega^2-k^2}z) = \frac{1}{2^{\nu-1}\Gamma(\nu)(\sqrt{\omega^2-k^2}z)^{-\nu}} \int_0^1 r(1-r^2)^{\nu-1} J_0(\sqrt{\omega^2-k^2}zr) dr.$$

Also, since

$$J_0(r\sqrt{\omega^2-k^2}) = \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{-ir\omega \sin\theta - kr\cos\theta} \tag{23}$$

we get

$$J_\nu(\sqrt{\omega^2-k^2}z) = \frac{1}{2^{\nu-1}\Gamma(\nu)(\sqrt{\omega^2-k^2}z)^{-\nu}} \int_0^1 r(1-r^2)^{\nu-1} dr \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{-izr\omega \sin\theta - kzr\cos\theta}.$$

Using the change of variables $r\sin\theta = \frac{t'}{z}$ and $r\cos\theta = \frac{y'}{z}$ we can write the above integral as

$$J_\nu(\sqrt{\omega^2-k^2}z) = \frac{1}{2^{\nu-1}\Gamma(\nu)(\sqrt{\omega^2-k^2}z)^{-\nu} 2\pi} \int_{t'^2+y'^2<z^2} dt'dy' \frac{1}{z^2} \left(\frac{z^2-t'^2-y'^2}{z^2} \right)^{\nu-1} \times e^{-i\omega t'} e^{-ky'} \tag{24}$$

where the integral is over a region in the form of a disk of radius z in the x-t plane. So from (21) and (24), we get

$$\phi(x,t,z) = \frac{\nu}{\pi}\int_{t'^2+y'^2<z^2} dt'dy' \left(\frac{z^2-t'^2-y'^2}{z}\right)^{\nu-1} \int d\omega dk e^{-i\omega(t+t')} e^{ik(x+iy')} \phi_o(\omega,k). \tag{25}$$

The second integral in (25) can be identified as $\phi_o(t+t', x+iy')$, so we have

$$\phi(x,t,z) = \frac{\Delta-1}{\pi}\int_{t'^2+y'^2<z^2} dt'dy' \left(\frac{z^2-t'^2-y'^2}{z}\right)^{\Delta-2} \phi_o(t+t', x+iy'). \tag{26}$$

Expressing the above result in terms of the AdS-invariant distance:

$$\sigma\left(t,x,z|t',x',z'\right) \frac{1}{2zz'}(z^2+z'^2+|x-x'|^2-(t-t')^2)$$

as

$$\phi(x,t,z) = \frac{\Delta-1}{\pi}\int_{t'^2+y'^2<z^2} dt'dy' \lim_{z'\to 0}\left(2z'\sigma(t,x,z|t+t',x+iy',z')\right)^{\Delta-2} \phi_o(t+t', x+iy'). \tag{27}$$

So we have expressed the bulk field in terms of an integral over a disk of radius z in the real t, imaginary x plane.

3.2. Global coordinates

We will now look at the construction in Global coordinates. The metric in these coordinates for AdS_3 can be written using (5) as

$$ds^2 = R^2 \sec^2 \rho(-dt^2 + d\rho^2 + \sin^2 \rho d\varphi^2)$$

where $0 \le \rho < \pi/2$, $-\pi < t \le \pi$ and $0 \le \varphi < 2\pi$.

From this we can write the following expressions.

$$g_{\mu\nu} = R^2 \sec^2 \rho \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sin^2 \rho \end{bmatrix} \qquad g^{\mu\nu} = \frac{1}{R^2 \sec^2 \rho} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \csc^2 \rho \end{bmatrix}$$

$$\sqrt{|g|} = R^3 \sec^3 \rho \sin \rho$$

The Laplacian Beltrami operator in these coordinates can be obtained as

$$\begin{aligned}
\Box &= \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^\alpha} \left(\sqrt{|g|}\, g^{\alpha\beta} \frac{\partial}{\partial x^\beta} \right) \\
&= \frac{1}{R^3 \sec^3 \rho \sin \rho} \left[\frac{\partial}{\partial t} \left(R^3 \sec^3 \rho \sin \rho \frac{-1}{R^2 \sec^2 \rho} \frac{\partial}{\partial t} \right) + \frac{\partial}{\partial \rho} \left(R^3 \sec^3 \rho \sin \rho \frac{1}{R^2 \sec^2 \rho} \frac{\partial}{\partial \rho} \right) \right. \\
&\left. + \frac{\partial}{\partial \varphi} \left(R^3 \sec^3 \rho \sin \rho \frac{\csc^2 \rho}{R^2 \sec^2 \rho} \frac{\partial}{\partial \varphi} \right) \right] \\
&= \left(\frac{-1}{R^2} \cos^2 \rho \frac{\partial^2}{\partial t^2} + \frac{1}{R^2} \cot^2 \rho \frac{\partial^2}{\partial \varphi^2} + \frac{1}{R^2 \sec^2 \rho} \frac{\partial^2}{\partial \rho^2} + \frac{1}{R^2} \cot \rho \frac{\partial}{\partial \rho} \right)
\end{aligned}$$

The equation of motion for a scalar field in this background is

$$\left(-\Box + m^2\right) \phi = 0.$$

when we take ϕ of the form $e^{-i\omega t} \chi(\rho) e^{ik\varphi}$, we get

$$\begin{aligned}
\Box \phi =& \frac{\cos^2 \rho}{R^2} \omega^2 e^{-i\omega t} \chi(\rho) e^{ik\varphi} + \frac{\cot^2 \rho}{R^2} e^{-i\omega t} \chi(\rho)(-k^2) e^{ik\varphi} + \frac{\cos^2 \rho}{R^2} e^{-i\omega t} e^{ik\varphi} \frac{\partial^2 \chi}{\partial \rho^2} \\
&+ \frac{\cot \rho}{R^2} e^{-i\omega t} e^{ik\varphi} \frac{\partial \chi}{\partial \rho}.
\end{aligned}$$

$$\left(-\Box + m^2\right) \phi = -\frac{\partial^2 \chi}{\partial \rho^2} - \frac{1}{\cos \rho \sin \rho} \frac{\partial \chi}{\partial \rho} + \left(\frac{k^2}{\sin^2 \rho} + \frac{m^2 R^2}{\cos^2 \rho} - \omega^2 \right) \chi = 0 \qquad (28)$$

Let $\chi(\rho) = (\cos \rho)^{2h} (\sin \rho)^{2b} f(\rho)$

$$\begin{aligned}
\frac{\partial \chi}{\partial \rho} =& 2h(\cos \rho)^{2h-1}(-\sin \rho)(\sin \rho)^{2b} f(\rho) + 2b(\cos \rho)^{2h} (\sin \rho)^{2b-1} \cos \rho f(\rho) \\
&+ (\cos \rho)^{2h} (\sin \rho)^{2b} f'(\rho)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 \chi}{\partial \rho^2} =& 2h(2h-1)(\cos \rho)^{2h-2} (\sin \rho)^{2b+2} f(\rho) - 2h(\cos \rho)^{2h-1}(2b+1)(\sin \rho)^{2b} \cos \rho f(\rho) \\
&- 2h(\cos \rho)^{2h-1} (\sin \rho)^{2b+1} f'(\rho) - 2b(2h+1)(\cos \rho)^{2h} (\sin \rho)^{2b-1} f(\rho) \\
&+ 2b(2b-1)(\cos \rho)^{2h+2} (\sin \rho)^{2b-2} f(\rho) + 2b(\cos \rho)^{2h+1} (\sin \rho)^{2b-1} f'(\rho) \\
&- 2h(\cos \rho)^{2h-1} (\sin \rho)^{2b+1} f'(\rho) + 2b(\cos \rho)^{2h+1} (\sin \rho)^{2b-1} f'(\rho) \\
&+ (\cos \rho)^{2h} (\sin \rho)^{2b} f''(\rho)
\end{aligned}$$

Plugging these in (28) and putting $h(h-1) = \frac{m^2R^2}{4}$, $4b^2 = k^2$ ($h_\pm = \frac{1\pm\sqrt{1+m^2R^2}}{2}, b = \pm k/2$) gives

$$\begin{aligned}
&-2h(2h-1)(\cos\rho)^{2h-2}(\sin\rho)^{2b+2}f(\rho) + 2h(2b+1)(\cos\rho)^{2h}(\sin\rho)^{2b}f(\rho)\\
&\quad + 2h(\cos\rho)^{2h-1}(\sin\rho)^{2b+1}f'(\rho) + 2b(2h+1)(\cos\rho)^{2h}(\sin\rho)^{2b-1}f(\rho)\\
&\quad - 2b(2b-1)(\cos\rho)^{2h+2}(\sin\rho)^{2b-2}f(\rho) - 2b(\cos\rho)^{2h+1}(\sin\rho)^{2b-1}f'(\rho)\\
&\quad + 2h(\cos\rho)^{2h-1}(\sin\rho)^{2b+1}f'(\rho) - 2b(\cos\rho)^{2h+1}(\sin\rho)^{2b-1}f'(\rho)\\
&\quad - (\cos\rho)^{2h}(\sin\rho)^{2b}f''(\rho) + 2h(\cos\rho)^{2h-2}(\sin\rho)^{2b}f(\rho) - 2b(\cos\rho)^{2h}(\sin\rho)^{2b-2}f(\rho)\\
&\quad - (\cos\rho)^{2h-1}(\sin\rho)^{2b-1}f'(\rho)\\
&\qquad + \left(\frac{4b^2}{sin^2\rho} + \frac{4h(h-1)}{cos^2\rho} - \omega^2\right)(\cos\rho)^{2h}(\sin\rho)^{2b}f(\rho) = 0.
\end{aligned}$$

Further simplifying it gives

$$\begin{aligned}
&-2h(2h-1)(\sin\rho)^4 f(\rho) + 2h(2b+1)(\cos\rho)^2(\sin\rho)^2 f(\rho) + 2h(\sin\rho)^2 f(\rho)\\
&\quad + 2b(2h+1)(\cos\rho)^2\sin\rho f(\rho) - 2b(2b-1)(\cos\rho)^4 f(\rho) - 2b(\cos\rho)^2 f(\rho) - \cos\rho\sin\rho f'(\rho)\\
&\quad + 2h\cos\rho(\sin\rho)^3 f'(\rho) - 2b(\cos\rho)^3\sin\rho f'(\rho) + 2h\cos\rho(\sin\rho)^3 f'(\rho) - 2b(\cos\rho)^3\sin\rho f'(\rho)\\
&\quad - \cos\rho^2\sin\rho^2 f''(\rho) + \left(\frac{4b^2}{\sin^2\rho} + \frac{4h(h-1)}{\cos^2\rho} - \omega^2\right)\cos\rho^2\sin\rho^2 f(\rho) = 0.
\end{aligned}$$

The coefficient of $f(\rho)$ can be expanded and simplified to the following expression.

$$\begin{aligned}
&4\cos^2\rho\sin^2\rho\left(-h^2\tan^2\rho + \frac{h}{2}\tan^2\rho + 2hb + \frac{h}{2} + \frac{b}{2} - \frac{b\csc^2\rho}{2} + \frac{h}{2}\sec^2\rho - b^2\cot^2\rho\right.\\
&\quad\left. + \frac{b}{2}\cot^2\rho + b^2\csc^2\rho + h^2\sec^2\rho - h\sec^2\rho - \frac{\omega^2}{4}\right) f(\rho)\\
&= 4\cos^2\rho\sin^2\rho\left((h+b)^2 - \frac{\omega^2}{4}\right) f(\rho)
\end{aligned}$$

Let us carry out a change of varible to $y = \sin^2\rho$ for simplification. This will give us the following change of derivatives

$$\frac{dy}{d\rho} = 2\sin\rho\cos\rho \qquad \frac{d^2y}{d\rho^2} = 2\cos^2\rho - 2\sin^2\rho$$

$$\frac{df}{d\rho} = \frac{df}{dy}\frac{dy}{d\rho} = 2\sin\rho\cos\rho\frac{df}{dy}$$

$$\begin{aligned}
\frac{d^2f}{d\rho^2} &= \frac{d}{d\rho}\left(\frac{df}{dy}\right)\frac{dy}{d\rho} + \frac{df}{dy}\frac{d}{d\rho}\left(\frac{df}{dy}\right)\\
&= \frac{d^2f}{dy^2}\left(\frac{dy}{d\rho}\right)^2 + \frac{df}{dy}\frac{d^2y}{d\rho^2} = \frac{d^2f}{dy^2}\left(4\sin^2\rho\cos^2\rho\right)^2 + \frac{df}{dy}\left(2\cos^2\rho - 2\sin^2\rho\right).
\end{aligned}$$

The coefficient of $f'(\rho)$ and $f''(\rho)$ can be written in terms of the variable y as

$$4\cos^2\rho\sin^2\rho\left(-1 - 2b + (2h+2b+1)y\right) f'(y) - y(1-y)f''(y).$$

∴ Eq (28) can be written as

$$y(1-y)f''(y) + (2b+1-(2h+2b+1)y)f'(y) - \left[(h+b)^2 - \frac{\omega^2}{4}\right] f(y) = 0. \quad (29)$$

Choosing a different variable : $x = \cos^2\rho$ gives eq (52) as

$$x(1-x)f''(x) + (2h-(2h+2b+1)x)f'(x) - \left[(h+b)^2 - \frac{\omega^2}{4}\right] f(x) = 0. \quad (30)$$

→ **Behaviour at the origin**

To analyze the behaviour at the origin it is convenient to study the solutions of eq (29).Choosing without loss of generality, $h = h_+$, we obtain the scalar field as

$$\psi = e^{-i\omega t}e^{ik\varphi}(\cos\rho)^{2h_+}(\sin\rho)^k{}_2F_1\,(h_+ + \frac{1}{2}(k+\omega), h_+ + \frac{1}{2}(k-\omega), k+1; \sin^2\rho) \quad (31)$$

The other solution of (29) would not be valid as only those solutions are considered for which the boundary term of the classical action vanishes at the origin ($\rho = 0$):

$$S_{\text{origin}} = \lim_{\rho\to 0}\int_{\rho\text{ fixed}} dtd\varphi\sqrt{g}g^{\rho\rho}\Phi\partial_\rho\Phi \to 0$$

so that we do not have contributions to correlation functions coming from the interior, in the following relation between string theory in the bulk and field theory on the boundary:

$$Z_{\text{eff}}(\Phi_i) = e^{iS_{\text{eff}}(\Phi_i)} = \langle Te^{i\int_{\mathcal{B}}\Phi_{b,i}\mathcal{O}^i}\rangle$$

Here S_{eff} is the effective action in the bulk, $\Phi_{b,i}$ is the field Φ_i restricted to the boundary, and T is the time-ordering symbol in the field theory on the boundary $\mathcal{B}$. The expectation value on the right hand side is taken in the boundary field theory, with $\Phi_{b,i}$ treated as a source term. In the classical supergravity limit, given a boundary field we solve for the corresponding bulk field and use it to relate the bulk effective action to boundary correlation functions.

→ **Behaviour at the boundary**

To study the behaviour at the boundary it is most convenient to work with solutions of (30), thus giving the expressions for the scalar field as:

$$\Phi^{(+)} = e^{-i\omega t}e^{ik\varphi}(\cos\rho)^{2h_+}(\sin\rho)^k{}_2F_1\,(h_+ + \frac{1}{2}(k+\omega), h_+ + \frac{1}{2}(k-\omega), 2h_+; \cos^2\rho) \quad (32)$$

and

$$\Phi^{(-)} = e^{-i\omega t}e^{ik\varphi}(\cos\rho)^{2h_-}(\sin\rho)^k{}_2F_1\,(h_- + \frac{1}{2}(k+\omega), h_- + \frac{1}{2}(k-\omega), 2h_-; \cos^2\rho). \quad (33)$$

Using

$$\begin{aligned} F(a,b;c;z) =& \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F(a,b;a+b-c+1;1-z) \\ &+ \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-z)^{c-a-b}F(c-a,c-b;c-a-b+1;1-z) \end{aligned}$$

we get

$$F(h_+ + \frac{1}{2}(k+\omega), h_+ + \frac{1}{2}(k-\omega), k+1; \sin^2\rho) = C^{(+)} F\Big(h_+ + \frac{1}{2}(k+\omega), h_+ + \frac{1}{2}(k-\omega),$$
$$2h_+; \cos^2\rho\Big) + C^{(-)}(\cos^2\rho)^{-\nu} F(h_- + \frac{1}{2}(k-\omega), h_- + \frac{1}{2}(k+\omega), 2h_-; \cos^2\rho)$$

where the constants $C^{(+)}$ and $C^{(-)}$ are

$$C^{(+)} = \frac{\Gamma(k+1)\Gamma(-\nu)}{\Gamma(h_- + \frac{1}{2}(k-\omega))\Gamma(h_- + \frac{1}{2}(k+\omega))} \qquad C^{(-)} = \frac{\Gamma(k+1)\Gamma(\nu)}{\Gamma(h_+ + \frac{1}{2}(k+\omega))\Gamma(h_+ + \frac{1}{2}(k-\omega))}.$$

$$\begin{aligned}
\implies \psi &= e^{-i\omega t} e^{ik\varphi} (\cos\rho)^{2h_+} (\sin\rho)^k {}_2F_1\,(h_+ + \frac{1}{2}(k+\omega), h_+ + \frac{1}{2}(k-\omega), k+1; \sin^2\rho) \\
&= C^{(+)} e^{-i\omega t} e^{ik\varphi} (\cos\rho)^{2h_+} (\sin\rho)^k F(h_+ + \frac{1}{2}(k+\omega), h_+ + \frac{1}{2}(k-\omega), 2h_+; \cos^2\rho) \\
&+ C^{(-)} e^{-i\omega t} e^{ik\varphi} (\cos\rho)^{2h_+ - 2\nu} (\sin\rho)^k F(h_- + \frac{1}{2}(k-\omega), h_- + \frac{1}{2}(k+\omega), 2h_-; \cos^2\rho) \\
&= C^{(+)}\Phi^{(+)} + C^{(-)}\Phi^{(-)} \qquad\qquad (2h_+ - 2\nu = 2h_-)
\end{aligned}$$

For $\nu > 1$, $C^{(-)}$ must vanish for a fluctuating solution because the norm of $\Phi^{(-)}$ diverges at the boundary. This will happen if one of the gamma functions in the denominator has zero or a negative integer as its argument,i.e.,if

$$\omega = \pm(\Delta + k + 2n) \qquad n = 0, 1, 2, ... \qquad (\Delta = 2h_+)$$

Quantizing the bulk field in terms of the modes obtained above, we get

$$\phi(\rho, t, \varphi) = \sum_{n,k} a_{\omega k}\psi_{\omega k} + \psi^*_{\omega k} a^\dagger_{\omega k}$$

where $a_{\omega k}$, $a^\dagger_{\omega k}$ are the annihilation and creation operators.They create normalizable particle excitations in the bulk.
We define the (right) boundary in global coordinates as

$$\phi^R_{\text{boundary}} = \lim_{\rho\to\pi/2} \frac{1}{\cos^\Delta\rho}\phi = \mathcal{O} \qquad \text{(Let)}.$$

$$\lim_{\rho\to\pi/2} \frac{1}{\cos^\Delta\rho}\psi_{\omega k} = e^{-i\omega_{nk}t} e^{ik\varphi} F(h_+ + \frac{1}{2}(k+\omega), h_+ + \frac{1}{2}(k-\omega), k+1; 1) := \Psi_{\omega k}(t, \varphi)$$

So we can write

$$\sum_{n,k} \Psi_{\omega k} a_{\omega k} + \Psi^*_{\omega k} a^\dagger_{\omega k} = \mathcal{O}(t, \varphi).$$

For simplicity, we will consider the case where Δ is an integer. Then the solution becomes periodic in time and we can limit the range of t to $-\pi$ to π. When Δ is an integer $\Psi_{\omega k}$ are orthogonal to all $\Psi^*_{\omega k}$.

$$\text{Let } \tilde{\Psi}_{\omega k}(t, \varphi) = \frac{1}{F(h_+ + \frac{1}{2}(k+\omega), h_+ + \frac{1}{2}(k-\omega), k+1; 1)} e^{-i\omega_{nk}t} e^{ik\varphi}.$$

Using the orthonormality and completeness of the functions $e^{-i\omega_{nk}t}$ and $e^{ik\varphi}$, we can solve for $a_{\omega k}$:

$$a_{\omega k} = \int_{-\pi}^{\pi} dt \int_0^{2\pi} d\varphi \tilde{\Psi}^*_{\omega k}(t,\varphi)\mathcal{O}(t,\varphi).$$

Thus we have

$$\begin{aligned}\phi(\rho,t,\varphi) &= \sum_{n,k} \psi_{\omega k} \int_{-\pi}^{\pi} dt' \int_0^{2\pi} d\varphi' \tilde{\Psi}^*_{\omega k}(t',\varphi')\mathcal{O}(t',\varphi') \\ &\qquad + \psi^*_{\omega k} \int_{-\pi}^{\pi} dt'' \int_0^{2\pi} d\varphi'' \tilde{\Psi}_{\omega k}(t'',\varphi'')\mathcal{O}(t'',\varphi'') \\ &= \sum_{n,k} \int_{-\pi}^{\pi} dt' \int_0^{2\pi} d\varphi' \left(\sum_{n,k} \psi_{\omega k}(\rho,t,\varphi)\tilde{\Psi}^*_{\omega k}(t',\varphi') + c.c \right) \mathcal{O}(t',\varphi').\end{aligned}$$

Since

$$\sum_{n,k} \psi_{\omega k}(\rho,t,\varphi)\tilde{\Psi}^*_{\omega k}(t',\varphi') = \frac{(\cos\rho)^{\Delta}(\sin\rho)^k F(h_+ + \frac{1}{2}(k+\omega), h_+ + \frac{1}{2}(k-\omega), k+1; \sin^2\rho)}{F(h_+ + \frac{1}{2}(k+\omega), h_+ + \frac{1}{2}(k-\omega), k+1; 1)}$$

is real and is therefore equal to its complex conjugate, we get the final form for the expression:

$$\phi(\rho,t,\varphi) = \int_{-\pi}^{\pi} dt' \int_0^{2\pi} d\varphi' K(\rho,t,\omega;t',\varphi')\mathcal{O}(t',\varphi')$$

where

$$K(\rho,t,\omega;t',\varphi') = \sum_{n,k} \frac{2(\cos\rho)^{\Delta}(\sin\rho)^k F(h_+ + \frac{1}{2}(k+\omega), h_+ + \frac{1}{2}(k-\omega), k+1; \sin^2\rho)}{F(h_+ + \frac{1}{2}(k+\omega), h_+ + \frac{1}{2}(k-\omega), k+1; 1)}.$$

This is known as the smearing function. So we have encoded the information of the bulk in the boundary through this smearing function.

Chapter 4

De Sitter Construction

Here we look at the smearing construction done for the de Sitter space for the dS/CFT conjecture which is a natural extension in terms of the AdS/CFT correspondence in the context of global holography [12]. Due to the CFT obtained from the asymptotic behavior of de Sitter space being non-unitary, both the normalizable and non normalizable boundary values can be considered unlike the AdS case [13]. Due to the resemblance of AdS and dS spaces via analytic continuation, one would think to do the same for the smearing prescription from AdS to dS, but we face the issue of not obtaining causal correlation functions. With the analytically continued smearing construction for AdS, a field operator in de Sitter is then expressed as an integral defined on the past or future boundary. The domain of integration for smearing is in the past/future light cone of the bulk point, which reduces to a standard Cauchy problem to express the bulk point in terms of boundary operators as evolving the initial conditions using the retarded Green's function in de Sitter space.
Analytically continuing the AdS Poincaré patch$\left(ds^2 = \frac{R_{\mathrm{AdS}}^2}{z^2}\left(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2\right)\right)$ to de Sitter flat slicing via

$$z \to \eta, t \to t, x^i \to ix^i, R_{\mathrm{AdS}} = iR_{\mathrm{dS}}$$

we get

$$ds^2 = \frac{-d\eta^2 + d\mathbf{x}^2}{\eta^2} \quad (R_{\mathrm{dS}} = 1).$$

(with t treated as one of the spatial coordinates in de Sitter space)
After the analytic continuation, the spatial lightcone in AdS becomes timelike, and the bulk operator will now commute with the operators inside its own timelike lightcone and fails to commute with the ones outside, which is not the right behavior for being causal. In AdS we go from the bulk to the boundary in a spatial direction, and we have both positive frequency and negative frequency modes in the time direction while sticking to just normalizable modes(in the local operator) in z direction. As we continue to dS, the z direction becomes the time direction and keeping one set of modes in this direction turns into keeping either positive or negative frequency mode, which spoils microcausality.

As has been pointed out by several authors [14][15][16], the analytic continuation of AdS correlators would not give the correlation function in any de Sitter invariant vacuum. The reason why this happens is the definitions of correlation functions in dS and AdS are not related to each other via analytic continuation.
The analytically continued correlation function from AdS to de Sitter space requires

us to not only know the early stage evolution of the wavefunction but the later stage as well, which is what one would not do in cosmology. Also, fixing a certain boundary condition at the future infinity is manifestly acausal.This acausal behavior will manifest itself as the breakdown of microcausality:operators on a single spatial slice fail to commute.
The way to define the correlation function in de Sitter and in more generic cosmology should just involve the Hartle–Hawking wavefunction and its complex conjugate, and corresponds to an in–in path integral[17][14]:

$$\langle\Psi|\tilde{\phi}(x_1,\eta)\ldots\tilde{\phi}(x_n,\eta)|\Psi\rangle_{\mathrm{dS,FRW}} = \int_\eta \mathcal{D}\tilde{\phi}\Psi_E^*[\tilde{\phi}]\tilde{\phi}(x_1,\eta)\ldots\tilde{\phi}(x_n,\eta)\Psi_E[\tilde{\phi}].$$

where η is a certain spatial slice on which we compute correlation functions.Ψ_E refers to both "a wavefunction at early time" and "a wavefunction of the universe in the Euclidean (Hartle–Hawking) vacuum". Here one no longer specifies the boundary condition at the future boundary. This definition obeys microcausality—the spacelike separated operators commute inside the correlation functions and timelike separated ones do not commute. The simplest one of this type of correlation functions is the Wightman function for a free scalar field in de Sitter space.

Thus a construction of a de Sitter bulk operator that computes de Sitter cosmology should reproduce the Wightman function, and it should also contain both positive and negative frequency modes in de Sitter space in order to ensure causality.

We look at how a local scalar operator with mass $m^2 > \left(\frac{d}{2}\right)^2$ in de Sitter space is constructed with a CFT located at the boundary.

4.1. Flat Slicing

Here we work in the flat patch of de Sitter space(dS_{d+1}), which covers only half of the global geometry. One can either choose the past wedge to work on, or the future wedge, and the boundary CFT will live on $\mathcal{I}^-$ or $\mathcal{I}^+$ respectively. Here for the moment we choose the past wedge.
Here we define

$$\Delta = \frac{d}{2} + i\sqrt{m^2 - \left(\frac{d}{2}\right)^2}$$

and near the boundary a positive/negative frequency mode has behavior

$$\Phi_+(\eta \to 0) \sim \eta^\Delta \mathcal{O}_+ \ , \ \Phi_-(\eta \to 0) \sim \eta^{d-\Delta}\mathcal{O}_-$$

where $\mathcal{O}_\pm$ are single–trace operators in the boundary CFT, with scaling dimensions Δ and $d - \Delta$ respectively.
As stated before, a causal operator should have both the components:

$$\Phi(\eta \to 0) \sim A\eta^\Delta \mathcal{O}_+ + B\eta^{d-\Delta}\mathcal{O}_-$$

To construct the bulk operator, we evolve the initial data at $\mathcal{I}^-$ with the retarded Green's function, which is given by

$$G_{ret}(x, x') = G_E(x, x') - G_E(x', x)$$

with G_E being the Wightman function in Euclidean vacuum, which was calculated in (13)(with d→d+1, for dS$_d+1$ space) and also [18]

$$G_E(x,x') = \frac{\Gamma(\Delta)\Gamma(d-\Delta)}{(4\pi)^{\frac{d+1}{2}}\Gamma\left(\frac{d+1}{2}\right)} F(\Delta, d-\Delta, \frac{d+1}{2}, \frac{1+\sigma}{2})$$

where

$$\sigma = \frac{\eta^2 + \eta'^2 - (\mathbf{x}-\mathbf{x}')^2}{2\eta\eta'}$$

is the de Sitter invariant distance function.
When one of the bulk points x' approaches the boundary $\eta' \to 0$, the fourth argument of the hypergeometric function grows large and is dominated by σ

$$\frac{1+\sigma}{2} \sim \frac{\sigma}{2} \sim \frac{\eta^2 - (\mathbf{x}-\mathbf{x}')^2}{4\eta\eta'}$$

$$\begin{aligned} \because F(\alpha,\beta,\gamma;z) =& \frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\gamma-\alpha)\Gamma(\beta)}(-z)^{-\alpha}F(\alpha,\alpha-\gamma+1,\alpha-\beta+1;\frac{1}{z}) \\ &+ \frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\beta)}(-z)^{-\beta}F(\beta,\beta-\gamma+1,\beta-\alpha+1;\frac{1}{z}) \end{aligned} \tag{34}$$

(when neither $\alpha-\beta$ nor $\gamma-\alpha-\beta$ is an integer and is thus applicable to the case when a de Sitter scalar has mass parameter $m^2 > \left(\frac{d}{2}\right)^2$ as well as to light particles with non-integer dimensions)
Therefore we can write

$$\begin{aligned} F(\Delta,d-\Delta,\frac{d+1}{2},\frac{\sigma}{2}) =& \frac{\Gamma(\frac{d+1}{2})\Gamma(d-2\Delta)}{\Gamma(\frac{d+1}{2}-\Delta)\Gamma(d-\Delta)}\left(-\frac{\sigma}{2}\right)^{-\Delta} F(\Delta,\Delta-\frac{d+1}{2}+1,\Delta-(d-\Delta)+1;\frac{2}{\sigma}) \\ &+ \frac{\Gamma(\frac{d+1}{2})\Gamma(\Delta-(d-\Delta))}{\Gamma(\Delta)\Gamma(\frac{d+1}{2}-(d-\Delta))}\left(-\frac{\sigma}{2}\right)^{-(d-\Delta)} F(d-\Delta,d-\Delta-\frac{d+1}{2}+1,d-\Delta-\Delta+1;\frac{2}{\sigma}). \end{aligned} \tag{35}$$

$$\begin{aligned} F(\Delta,d-\Delta,\frac{d+1}{2},\frac{\sigma}{2}) =& \frac{\Gamma(\frac{d+1}{2})\Gamma(d-2\Delta)}{\Gamma(\frac{d+1}{2}-\Delta)\Gamma(d-\Delta)}\left(-\frac{4\eta\eta'}{\eta^2-(\mathbf{x}-\mathbf{x}')^2}\right)^{\Delta} \\ & F(\Delta,\Delta-\frac{d+1}{2}+1,\Delta-(d-\Delta)+1;\frac{2}{\sigma}) \\ &+ \frac{\Gamma(\frac{d+1}{2})\Gamma(2\Delta-d)}{\Gamma(\Delta)\Gamma(\Delta-(\frac{d-1}{2}))}\left(-\frac{4\eta\eta'}{\eta^2-(\mathbf{x}-\mathbf{x}')^2}\right)^{d-\Delta} F(d-\Delta,\frac{d+1}{2}-\Delta,d-2\Delta+1;\frac{2}{\sigma}) \end{aligned} \tag{36}$$

So the Wightman function can be wriiten as:

$$\begin{aligned} G_E(x,x') =& \frac{\Gamma(\Delta)\Gamma(d-2\Delta)}{(4\pi)^{\frac{d+1}{2}}\Gamma(\frac{d+1}{2}-\Delta)}\left(-\frac{4\eta\eta'}{\eta^2-(\mathbf{x}-\mathbf{x}')^2}\right)^{\Delta} \\ & F(\Delta,\Delta-\frac{d-1}{2},2\Delta-d+1;\frac{2}{\sigma}) \\ &+ \frac{\Gamma(d-\Delta)\Gamma(2\Delta-d)}{(4\pi)^{\frac{d+1}{2}}\Gamma(\Delta-(\frac{d-1}{2}))}\left(-\frac{4\eta\eta'}{\eta^2-(\mathbf{x}-\mathbf{x}')^2}\right)^{d-\Delta} F(d-\Delta,\frac{d+1}{2}-\Delta,d-2\Delta+1;\frac{2}{\sigma}) \end{aligned} \tag{37}$$

$$G_E(x,x') = c_{\Delta,d}\,(-\sigma)^{\Delta-d}\,F(d-\Delta,\frac{d+1}{2}-\Delta,d-2\Delta+1;\frac{2}{\sigma}) + c^*_{\Delta,d}\,(-\sigma)^{-\Delta}\,F(\Delta,\Delta-\frac{d-1}{2},2\Delta-d+1;\frac{2}{\sigma}) \quad (38)$$

where

$$c_{\Delta,d} = \frac{\Gamma(d-\Delta)\Gamma(2\Delta-d)}{2^{\Delta-d}(4\pi)^{\frac{d+1}{2}}\Gamma(\Delta-(\frac{d-1}{2}))}$$

$$c^*_{\Delta,d} = \frac{\Gamma(d-\Delta^*)\Gamma(2\Delta^*-d)}{2^{\Delta^*-d}(4\pi)^{\frac{d+1}{2}}\Gamma(\Delta^*-(\frac{d-1}{2}))} = \frac{\Gamma(\Delta)\Gamma(d-2\Delta)}{2^{-\Delta}(4\pi)^{\frac{d+1}{2}}\Gamma(\frac{d+1}{2}-\Delta)}.$$

Similarly we can also write the following Wightman function,

$$G_E(x',x) = c_{\Delta,d}\,(-\sigma')^{\Delta-d}\,F(d-\Delta,\frac{d+1}{2}-\Delta,d-2\Delta+1;\frac{2}{\sigma'}) + c^*_{\Delta,d}\,(-\sigma')^{-\Delta}\,F(\Delta,\Delta-\frac{d-1}{2},2\Delta-d+1;\frac{2}{\sigma'}). \quad (39)$$

where

$$\sigma' = \frac{\eta'^2+\eta^2-(\mathbf{x}'-\mathbf{x})^2}{2\eta'\eta}.$$

As $\eta'\to 0$

$$G_E(x,x') = c_{\Delta,d}\,(-\sigma-i\epsilon)^{\Delta-d} + c^*_{\Delta,d}\,(-\sigma-i\epsilon)^{-\Delta} \quad (40)$$

and

$$G_E(x',x) = c_{\Delta,d}\,(-\sigma+i\epsilon)^{\Delta-d} + c^*_{\Delta,d}\,(-\sigma+i\epsilon)^{-\Delta}. \quad (41)$$

$\Big[\because F(a,b,c,z) = 1 + \frac{abz}{c} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \ldots\ldots \implies F(a,b,c,0) = 1$

So for $\eta'\sim 0$ $\left(\frac{2}{\sigma}\sim 0\right)$ we get

$F(\Delta,\Delta-\frac{d-1}{2},2\Delta-d+1;\frac{2}{\sigma})$ and $F(d-\Delta,\frac{d+1}{2}-\Delta,d-2\Delta+1;\frac{2}{\sigma})\to 0\Big]$

So we can write the retarded Green's function as

$$G_{ret}|_{\eta'\to 0} = c_{\Delta,d}\,(-\sigma-i\epsilon)^{\Delta-d} + c^*_{\Delta,d}\,(-\sigma-i\epsilon)^{-\Delta} - c.c \quad (42)$$

Using which we can obtain the following expression.

$$\partial_{\eta'}G_{ret}\sim\frac{1}{\eta'}\Big(c(\Delta-d)\,(-\sigma-i\epsilon)^{\Delta-d} - c^*\Delta\,(-\sigma-i\epsilon)^{-\Delta} + c^*\Delta\,(-\sigma+i\epsilon)^{-\Delta} - c(\Delta-d)\,(-\sigma+i\epsilon)^{\Delta-d}\Big) \quad (43)$$

The bulk operator is constructed by evolving an operator near the boundary:

$$\Phi(\eta,\mathbf{x}) = \int_{|\mathbf{x}'|<\eta} d^dx' \left(\frac{1}{\eta'}\right)^{d-1} \left(G_{ret}(\eta,\mathbf{x};\eta',\mathbf{x}')\,\partial_{\eta'}\Phi(\eta',\mathbf{x}') - \Phi(\eta',\mathbf{x}')\,\partial_{\eta'}G_{ret}(\eta,\mathbf{x};\eta',\mathbf{x}')\right)$$

where

$$\Phi(\eta',\mathbf{x}') \sim A(\eta')^{\Delta}\mathcal{O}_+(\mathbf{x}') + B(\eta')^{d-\Delta}\mathcal{O}_-(\mathbf{x}')$$

and

$$\partial_{\eta'}\Phi \sim A\Delta(\eta')^{\Delta-1}\mathcal{O}_+(\mathbf{x}') + B(d-\Delta)(\eta')^{d-\Delta-1}\mathcal{O}_-(\mathbf{x}').$$

$$\begin{aligned}\implies \Phi(\eta,\mathbf{x}) = \int_{|\mathbf{x}'|<\eta} d^dx' \Big[(\eta')^{\Delta-d}A\Big(cd\left((-\sigma-i\epsilon)^{\Delta-d}-(-\sigma+i\epsilon)^{\Delta-d}\right)+\\ 2c^*\Delta\left((-\sigma-i\epsilon)^{-\Delta}-(-\sigma+i\epsilon)^{-\Delta}\right)\Big)\mathcal{O}_+(\mathbf{x}')\\ -(\eta')^{-\Delta}B\Big(2c(\Delta-d)\left((-\sigma-i\epsilon)^{\Delta-d}-(-\sigma+i\epsilon)^{\Delta-d}\right)+\\ c^*d\left((-\sigma-i\epsilon)^{-\Delta}-(-\sigma+i\epsilon)^{-\Delta}\right)\Big)\mathcal{O}_-(\mathbf{x}')\Big] \end{aligned} \tag{44}$$

In the coefficient of $\mathcal{O}_+(\mathbf{x}')$ the factor $(\eta')^{\Delta-d}$ cancels with the factor of η' with the inverse power from $\sigma^{\Delta-d}$ and gives a well defined limit when $\eta'\to 0$, but it doesn't cancel with the factor in $\sigma^{-\Delta}$, leading to a fast oscillation when $\eta'\to 0$ so the term proportional to $\sigma^{-\Delta}$ vanishes.
Similarly we only have the contribution from the terms proportional to $\sigma^{-\Delta}$ for the $\mathcal{O}_-(\mathbf{x}')$ term.

$$\begin{aligned}\Phi(\eta,\mathbf{x}) = \int_{|\mathbf{x}'|<\eta} d^dx' \Big[(\eta')^{\Delta-d}A\Big(cd\left((-\sigma-i\epsilon)^{\Delta-d}-(-\sigma+i\epsilon)^{\Delta-d}\right)\Big)\mathcal{O}_+(\mathbf{x}')\\ -(\eta')^{-\Delta}B\Big(c^*d\left((-\sigma-i\epsilon)^{-\Delta}-(-\sigma+i\epsilon)^{-\Delta}\right)\Big)\mathcal{O}_-(\mathbf{x}')\Big]\end{aligned}$$

Notice that outside the bulk lightcone $\sigma\propto\eta^2-(\mathbf{x}-\mathbf{x}')^2<0$ so the ϵ prescription can be dropped and the integral gives a vanishing result. When we analytically continue the result into the bulk lightcone, the ϵ prescription will give a phase shift proportional to $\mathrm{Im}(\Delta-d)$ and $\mathrm{Im}(-\Delta)$ respectively.
So for $\mathcal{O}_+$ we have a smearing function proportional to $(\sigma\eta')^{\Delta-d}$ and for $\mathcal{O}_-$ we have a smearing function proportional to $(\sigma\eta')^{-\Delta}$. Also carrying out the transformation $\mathbf{x}'\to\mathbf{x}'+\mathbf{x}$ gives

$$\Phi(\eta,\mathbf{x}) = A_{\Delta,d}\int_{|\mathbf{x}'|<\eta} d^dx'\left(\frac{\eta^2-\mathbf{x}'^2}{\eta}\right)^{\Delta-d}\mathcal{O}_+(\mathbf{x}+\mathbf{x}')+B_{\Delta,d}\int_{|\mathbf{x}'|<\eta} d^dx'\left(\frac{\eta^2-\mathbf{x}'^2}{\eta}\right)^{-\Delta}\mathcal{O}_-(\mathbf{x}+\mathbf{x}'). \tag{45}$$

From (37) and for convenience setting $\mathbf{x}'$ to zero, we get

$$\begin{aligned}G_E(\eta,\mathbf{x};\eta'{\sim}0,\mathbf{x}'=0) \to \frac{\Gamma(\Delta)\Gamma(d-2\Delta)}{(4\pi)^{\frac{d+1}{2}}\Gamma(\frac{d+1}{2}-\Delta)}\left(-\frac{4\eta\eta'}{\eta^2-\mathbf{x}^2}\right)^{\Delta}\\ +\frac{\Gamma(d-\Delta)\Gamma(2\Delta-d)}{(4\pi)^{\frac{d+1}{2}}\Gamma(\Delta-(\frac{d-1}{2}))}\left(-\frac{4\eta\eta'}{\eta^2-\mathbf{x}^2}\right)^{d-\Delta}.\end{aligned} \tag{46}$$

Here we would like to normalize the boundary two-point functions so that we have

$$G_E(\eta\to 0,\mathbf{x};\eta'\to 0,\mathbf{x}'=0)\to(\eta\eta')^{\Delta}D_+(\mathbf{x})+(\eta\eta')^{d-\Delta}D_-(\mathbf{x})$$

where $D_\pm$ are the boundary CFT correlation functions which we take to be

$$D_+(\mathbf{x}) = \frac{2^{2\Delta}\Gamma(\Delta)\Gamma(d-2\Delta)}{(4\pi)^{\frac{d+1}{2}}\Gamma(\frac{d+1}{2}-\Delta)}\left(\frac{1}{\mathbf{x}^2}\right)^{\Delta} \qquad D_-(\mathbf{x}) = \frac{2^{2(d-\Delta)}\Gamma(d-\Delta)\Gamma(2\Delta-d)}{(4\pi)^{\frac{d+1}{2}}\Gamma(\Delta-(\frac{d-1}{2}))}\left(\frac{1}{\mathbf{x}^2}\right)^{d-\Delta}.$$

Computing correlation function between the bulk operator and an operator near the boundary

$$\langle \Phi(\eta, \mathbf{x}) \Phi(\eta' \to 0, 0) \rangle = A_{\Delta,d} \int_{|\mathbf{x}'|<\eta} d^d x' \left(\frac{\eta^2 - \mathbf{x}'^2}{\eta} \right)^{\Delta - d} \eta'^{\Delta} \langle \mathcal{O}_+(\mathbf{x}+\mathbf{x}') \mathcal{O}_+(0) \rangle$$
$$+ B_{\Delta,d} \int_{|\mathbf{x}'|<\eta} d^d x' \left(\frac{\eta^2 - \mathbf{x}'^2}{\eta} \right)^{-\Delta} \eta'^{d-\Delta} \langle \mathcal{O}_-(\mathbf{x}+\mathbf{x}') \mathcal{O}_-(0) \rangle$$

With the boundary correlator of the operators $\mathcal{O}_\pm$:

$\langle \mathcal{O}_+(\mathbf{x}) \mathcal{O}_+(0) \rangle = D_+(\mathbf{x}) \qquad \langle \mathcal{O}_-(\mathbf{x}) \mathcal{O}_-(0) \rangle = D_-(\mathbf{x}) \qquad \langle \mathcal{O}_+(\mathbf{x}) \mathcal{O}_-(0) \rangle = 0)$

we obtain

$$\langle \Phi(\eta, \mathbf{x}) \Phi(\eta' \to 0, 0) \rangle = A_{\Delta,d} \frac{2^{2\Delta} \Gamma(\Delta) \Gamma(d - 2\Delta)}{(4\pi)^{\frac{d+1}{2}} \Gamma(\frac{d+1}{2} - \Delta)} \int_{|\mathbf{x}'|<\eta} d^d x' \left(\frac{\eta^2 - \mathbf{x}'^2}{\eta} \right)^{\Delta - d} \frac{\eta'^{\Delta}}{(\mathbf{x}+\mathbf{x}')^{2\Delta}}$$
$$+ B_{\Delta,d} \frac{2^{2(d-\Delta)} \Gamma(d - \Delta) \Gamma(2\Delta - d)}{(4\pi)^{\frac{d+1}{2}} \Gamma(\Delta - (\frac{d-1}{2})))} \int_{|\mathbf{x}'|<\eta} d^d x' \left(\frac{\eta^2 - \mathbf{x}'^2}{\eta} \right)^{-\Delta} \frac{\eta'^{d-\Delta}}{(\mathbf{x}+\mathbf{x}')^{2(d-\Delta)}}. \quad (47)$$

$$\text{Let } f(\alpha, \beta) = \int_{|\mathbf{x}'|<\eta} d^d x' \left(\frac{\eta^2 - \mathbf{x}'^2}{\eta} \right)^{\alpha} \frac{1}{(\mathbf{x}+\mathbf{x}')^{2\beta}}. \quad (48)$$

For convenience we make the choice $x^1 = |\mathbf{x}| = R$, $x^2 = \cdots = x^d = 0$, thus

$$f(\alpha, \beta) = \mathrm{Vol}(\mathrm{S}^{d-2}) \int_0^\eta dr r^{d-1} \left(\frac{\eta^2 - r^2}{\eta} \right)^{\alpha} \int_0^\pi \frac{\sin^{d-2}\theta d\theta}{(R^2 + 2Rr\cos\theta + r^2)^\beta},$$
$$\mathrm{Vol}(\mathrm{S}^{d-2}) = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)}. \quad (49)$$

Using the formulae

$$\int_0^\pi \frac{\sin^{2\mu-1}\theta d\theta}{(1 + 2a\cos\theta + a^2)^\nu} = \frac{\Gamma(\mu)\Gamma(\frac{1}{2})}{\Gamma(\mu + \frac{1}{2})} F(\nu, \nu - \mu + \frac{1}{2}, \mu + \frac{1}{2}, a^2) \quad (50)$$

$$\int_0^1 (1-x)^{\mu-1} x^{\gamma-1} F(\alpha, \beta, \gamma, ax) dx = \frac{\Gamma(\mu)\Gamma(\gamma)}{\Gamma(\mu+\gamma)} F(\alpha, \beta, \gamma+\mu, a) \quad (51)$$

to obtain

$$\int_0^\pi \frac{\sin^{\frac{2(d-1)}{2}-1}\theta d\theta}{R^{2\beta}\left(1 + \frac{2r}{R}\cos\theta + \left(\frac{r}{R}\right)^2\right)^\beta} = \frac{\Gamma(\frac{(d-1)}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{d}{2})R^{2\beta}} F\left(\beta, \beta - \left(\frac{d-1}{2}\right) + \frac{1}{2}, \frac{d}{2}, \frac{r^2}{R^2}\right). \quad (52)$$

$$\Longrightarrow f(\alpha, \beta) = \frac{2\pi^{\frac{d-1}{2}}\Gamma(\frac{1}{2})}{\Gamma(\frac{d}{2})R^{2\beta}} \int_0^1 d(\eta r)(\eta r)^{d-1} \left(\frac{\eta^2 - (\eta r)^2}{\eta} \right)^\alpha F\left(\beta, \beta - \left(\frac{d-1}{2}\right) + \frac{1}{2}, \frac{d}{2}, \frac{(\eta r)^2}{R^2}\right)$$
$$= \frac{2\pi^{\frac{d-1}{2}}\Gamma(\frac{1}{2})}{\Gamma(\frac{d}{2})R^{2\beta}} \frac{\eta^{\alpha+d}}{2} \int_0^1 dy y^{\frac{d-2}{2}} (1-y)^\alpha F\left(\beta, \beta - \frac{d}{2} + 1, \frac{d}{2}, \frac{\eta^2 y}{R^2}\right) (\text{Let } r^2 = y)$$
$$= \frac{2\pi^{\frac{d-1}{2}}\Gamma(\frac{1}{2})}{\Gamma(\frac{d}{2})R^{2\beta}} \frac{\eta^{\alpha+d}}{2} \frac{\Gamma(\alpha+1)\Gamma(\frac{d}{2})}{\Gamma(\alpha + \frac{d}{2} + 1)} F\left(\beta, \beta - \frac{d}{2} + 1, \alpha + 1 + \frac{d}{2}, \frac{\eta^2}{R^2}\right)$$

$$\implies f(\alpha,\beta) = \frac{\pi^{\frac{d}{2}}\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{d}{2}+1)}\frac{\eta^{\alpha+d}}{|\mathbf{x}|^{2\beta}}F\left(\beta,\beta-\frac{d}{2}+1,\alpha+1+\frac{d}{2},\frac{\eta^2}{|\mathbf{x}|^2}\right) \tag{53}$$

$$\therefore \int_{|\mathbf{x}'|<\eta} d^dx' \left(\frac{\eta^2-\mathbf{x}'^2}{\eta}\right)^{\Delta-d} \frac{\eta'^{\Delta}}{(\mathbf{x}+\mathbf{x}')^{2\Delta}} = \frac{\eta'^{\Delta}\pi^{\frac{d}{2}}\Gamma(\Delta-d+1)}{\Gamma(\Delta-\frac{d}{2}+1)}\frac{\eta^{\Delta}}{|\mathbf{x}|^{2\Delta}} F\left(\Delta,\Delta-\frac{d}{2}+1,\Delta-\frac{d}{2}+1,\frac{\eta^2}{|\mathbf{x}|^2}\right)$$

$$\begin{aligned} &= \frac{(\eta\eta')^{\Delta}\pi^{\frac{d}{2}}\Gamma(\Delta-d+1)}{\Gamma(\Delta-\frac{d}{2}+1)}\frac{1}{|\mathbf{x}|^{2\Delta}}\left(1-\frac{\eta^2}{\mathbf{x}^2}\right)^{-\Delta} \quad (\because F(\alpha,\beta,\beta,z)=(1-z)^{-\alpha}) \\ &= \frac{\pi^{\frac{d}{2}}\Gamma(\Delta-d+1)}{\Gamma(\Delta-\frac{d}{2}+1)}(-1)^{\Delta}\left(\frac{\eta\eta'}{\eta^2-\mathbf{x}^2}\right)^{\Delta} \end{aligned} \tag{54}$$

Similarly,

$$\int_{|\mathbf{x}'|<\eta} d^dx' \left(\frac{\eta^2-\mathbf{x}'^2}{\eta}\right)^{-\Delta} \frac{\eta'^{d-\Delta}}{(\mathbf{x}+\mathbf{x}')^{2(d-\Delta)}} = \frac{\pi^{\frac{d}{2}}\Gamma(1-\Delta)}{\Gamma(\frac{d}{2}-\Delta+1)}(-1)^{d-\Delta}\left(\frac{\eta\eta'}{\eta^2-\mathbf{x}^2}\right)^{d-\Delta}. \tag{55}$$

From (47), (54), (55) we can write

$$\begin{aligned} \langle\Phi(\eta,\mathbf{x})\,\Phi(\eta'\to 0,0)\rangle &= A_{\Delta,d}\frac{2^{2\Delta}\Gamma(\Delta)\Gamma(d-2\Delta)}{(4\pi)^{\frac{d+1}{2}}\Gamma(\frac{d+1}{2}-\Delta)}\frac{\pi^{\frac{d}{2}}\Gamma(\Delta-d+1)}{\Gamma(\Delta-\frac{d}{2}+1)}(-1)^{\Delta}\left(\frac{\eta\eta'}{\eta^2-\mathbf{x}^2}\right)^{\Delta} \\ &+ B_{\Delta,d}\frac{2^{2(d-\Delta)}\Gamma(d-\Delta)\Gamma(2\Delta-d)}{(4\pi)^{\frac{d+1}{2}}\Gamma(\Delta-(\frac{d-1}{2}))}\frac{\pi^{\frac{d}{2}}\Gamma(1-\Delta)}{\Gamma(\frac{d}{2}-\Delta+1)}(-1)^{d-\Delta}\left(\frac{\eta\eta'}{\eta^2-\mathbf{x}^2}\right)^{d-\Delta}. \end{aligned} \tag{56}$$

$A_{\Delta,d}$ and $B_{\Delta,d}$ are fixed by demanding that the correlation function of Φ (56) recover the Wightman function in the Euclidean vacuum (46)

$$A_{\Delta,d} = \frac{\Gamma(\Delta-\frac{d}{2}+1)}{\pi^{\frac{d}{2}}\Gamma(\Delta-d+1)} \qquad B_{\Delta,d} = A_{d-\Delta,d} = \frac{\Gamma(\frac{d}{2}-\Delta+1)}{\pi^{\frac{d}{2}}\Gamma(1-\Delta)}. \tag{57}$$

$$\boxed{\begin{aligned} \Phi(\eta,\mathbf{x}) = &\frac{\Gamma(\Delta-\frac{d}{2}+1)}{\pi^{\frac{d}{2}}\Gamma(\Delta-d+1)}\int_{|\mathbf{x}'|<\eta} d^dx'\left(\frac{\eta^2-\mathbf{x}'^2}{\eta}\right)^{\Delta-d}\mathcal{O}_+(\mathbf{x}+\mathbf{x}') \\ &+\frac{\Gamma(\frac{d}{2}-\Delta+1)}{\pi^{\frac{d}{2}}\Gamma(1-\Delta)}\int_{|\mathbf{x}'|<\eta} d^dx'\left(\frac{\eta^2-\mathbf{x}'^2}{\eta}\right)^{-\Delta}\mathcal{O}_-(\mathbf{x}+\mathbf{x}'). \end{aligned}} \tag{58}$$

So, we expressed the local operators in de Sitter space that probe and create particle in the Euclidean vacuum state, in terms of the boundary CFT operators.

Chapter 5

Construction in D1-brane

Now we will look at the case of D1 branes. The 10 dimensional near horizon geometry of the D1 brane dimensionally reduced on the 7-sphere to 3 dimensions is given by eq(15)

$$ds^2 = -c_T(r)^2 dt^2 + c_X(r)^2 dz^2 + c_R(r)^2 dr^2$$

where

$$c_T^2 = \left(\frac{r}{L}\right)^8 f,\ c_X^2 = \left(\frac{r}{L}\right)^8,\ c_R^2 = \frac{1}{f}\left(\frac{r}{L}\right)^2,\ f = 1 - \frac{r_o^6}{r^6}.$$

From this we can write

$$g_{\mu\nu} = \begin{bmatrix} -c_T(r)^2 & 0 & 0 \\ 0 & c_X(r)^2 & 0 \\ 0 & 0 & c_R(r)^2 \end{bmatrix} \qquad g^{\mu\nu} = \begin{bmatrix} -1/c_T(r)^2 & 0 & 0 \\ 0 & 1/c_X(r)^2 & 0 \\ 0 & 0 & 1/c_R(r)^2 \end{bmatrix}$$

and we get

$$\sqrt{|g|} = c_T c_X c_R.$$

The Laplace-Beltrami operator in this background is given as:

$$\begin{aligned}
\Box &= \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^\alpha}\left(\sqrt{|g|}\, g^{\alpha\beta}\frac{\partial}{\partial x^\beta}\right) \\
&= \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial t}\left(\sqrt{|g|}\left(\frac{-1}{c_T^2}\right)\frac{\partial}{\partial t}\right) + \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial z}\left(\sqrt{|g|}\left(\frac{1}{c_X^2}\right)\frac{\partial}{\partial z}\right) + \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial r}\left(\sqrt{|g|}\left(\frac{1}{c_R^2}\right)\frac{\partial}{\partial r}\right) \\
&= \frac{-1}{c_T^2}\frac{\partial^2}{\partial t^2} + \frac{1}{c_X^2}\frac{\partial^2}{\partial z^2} + \frac{1}{c_T c_X c_R}\frac{\partial}{\partial r}\left(\frac{c_T c_X c_R}{c_R^2}\frac{\partial}{\partial r}\right) \\
&= \frac{-1}{c_T^2}\frac{\partial^2}{\partial t^2} + \frac{1}{c_X^2}\frac{\partial^2}{\partial z^2} + \frac{1}{c_T c_X c_R}\frac{\partial}{\partial r}\left(\frac{c_T c_X c_R}{c_R^2}\frac{\partial}{\partial r}\right).
\end{aligned}$$

where

$$\frac{\partial}{\partial r}\left(\frac{c_T c_X c_R}{c_R^2}\frac{\partial}{\partial r}\right) = \frac{1}{L^7}\frac{\partial}{\partial r}\left(r^7 f\frac{\partial}{\partial r}\right) = \frac{7}{L^7}r^6 f\frac{\partial}{\partial r} + \frac{r^7}{L^7}f\frac{\partial^2}{\partial r^2} + \frac{r^7}{L^7}\frac{\partial f}{\partial r}\frac{\partial}{\partial r}.$$

The equation of motion for a massless scalar field in this background is

$$\Box\phi = 0. \tag{59}$$

where ϕ is of the form $e^{-i\omega t}e^{ikz}\varphi(r)$
Fourier mode expanding the field will give us

$$\phi(r,t,z) = \int d\omega dk a_{\omega k} e^{-i\omega t} e^{ikz} \varphi_{\omega k}(r).$$

Plugging this into (59), we will get

$$\frac{-1}{c_T^2}(-i\omega)^2 e^{-i\omega t}e^{ikz}\varphi(r) + \frac{1}{c_X^2}(ik)^2 e^{-i\omega t}e^{ikz}\varphi(r) + \frac{e^{-i\omega t}e^{ikz}}{c_T c_X c_R}\left[\frac{7}{L^7}r^6 f\frac{d\varphi}{dr} + \frac{r^7}{L^7}f\frac{d^2\varphi}{dr^2} + \frac{r^7}{L^7}\frac{df}{\partial r}\frac{d\varphi}{dr}\right] = 0.$$

$$\implies \frac{\omega^2}{c_T^2}\varphi(r) - \frac{k^2}{c_X^2}\varphi(r) + \frac{1}{c_T c_X c_R}\left[\frac{7}{L^7}r^6 f\frac{d\varphi}{dr} + \frac{r^7}{L^7}f\frac{d^2\varphi}{dr^2} + \frac{r^7}{L^7}\frac{6r_0^6}{r^7}\frac{d\varphi}{dr}\right] = 0.$$

$$\implies \left(\frac{\omega^2 L^8 f}{r^8} - \frac{k^2 L^8}{r^8}\right)\varphi(r) + \left(\frac{7L^2}{r^3}f + \frac{6L^2 r_0^6}{r^9}\right)\frac{d\varphi}{dr} + \frac{L^2}{r^2}f\frac{d^2\varphi}{dr^2} = 0. \quad (60)$$

Let us look at the solutions for this equation in different cases. As we dont have some exact solution for this equation we will first look at the series solutions for it.

Let us consider a change of variables to $u = \frac{r_o^2}{r^2}$ for convenience.

$$\frac{d\varphi}{dr} = \frac{d\varphi}{du}\left(\frac{-2u^{3/2}}{r_o}\right) \qquad \frac{d^2\varphi}{dr^2} = \frac{d^2\varphi}{du^2}\left(\frac{4u^3}{r_o^2}\right) + \frac{d\varphi}{du}\left(\frac{6u^2}{r_o^2}\right)$$

So under this change eq(61) becomes

$$\left(\frac{\omega^2 L^8(1-u^3)u^4}{r_o^8} - \frac{k^2 L^8 u^4}{r_o^8}\right)\varphi(u) + \left(\frac{7L^2(1-u^3)u^{3/2}}{r_o^3} + \frac{6L^2 u^{9/2}}{r_o^3}\right)\left(-\frac{2u^{3/2}}{r_o}\right)\frac{d\varphi}{du} + \frac{6L^2u^3}{r_o^4}(1-u^3)\left[\frac{d^2\varphi}{du^2}\left(\frac{4u^3}{r_o^2}\right) + \frac{d\varphi}{du}\left(\frac{6u^2}{r_o^2}\right)\right] = 0.$$

and further simplifying it gives

$$\frac{24L^2u^6}{r_o^6}(1-u^3)\frac{d^2\varphi}{du^2} + \left(\frac{36L^2u^5}{r_o^6}(1-u^3) - \frac{14L^2}{r_o^4}(1-u^3)u^3 - \frac{12L^2u^6}{r_o^4}\right)\frac{d\varphi}{du} + \left(\frac{\omega^2L^8(1-u^3)u^4}{r_o^8} - \frac{k^2L^8u^4}{r_o^8}\right)\varphi(u) = 0. \quad (61)$$

Here $u = 1$ is a regular singular point of the differential equation. So we will find a Frobenius solution around this point.
We assume the following series solution for φ

$$\varphi(u) = (u-1)^{\hbar}\sum_{n=0}^{\infty} a_n(u-1)^n.$$

From this we obtain the following relations

$$\varphi'(u) = \sum_{n=0}^{\infty} a_n(n+\hbar)(u-1)^{n+\hbar-1} \qquad \varphi''(u) = \sum_{n=0}^{\infty} a_n(n+\hbar)(n+\hbar-1)(u-1)^{n+\hbar-2}$$

and plugging these relations in (61) gives

$$
\begin{aligned}
&-\frac{24L^2}{r_o^6}u^6(u^2+u+1)\sum_{n=0}^{\infty}a_n(n+\hbar)(n+\hbar-1)(u-1)^{n+\hbar-1}\\
&-\frac{36L^2}{r_o^6}u^5(u^2+u+1)\sum_{n=0}^{\infty}a_n(n+\hbar)(u-1)^{n+\hbar}+\frac{14L^2}{r_o^4}u^3(u^2+u+1)\sum_{n=0}^{\infty}a_n(n+\hbar)(u-1)^{n+\hbar}\\
&-\frac{12L^2}{r_o^4}u^6\sum_{n=0}^{\infty}a_n(n+\hbar)(u-1)^{n+\hbar-1}+\frac{\omega^2L^8}{r_o^8}u^4(u^2+u+1)\sum_{n=0}^{\infty}a_n(u-1)^{n+\hbar+1}\\
&-\frac{k^2L^8}{r_o^8}u^4\sum_{n=0}^{\infty}a_n(u-1)^{n+\hbar}=0.
\end{aligned}
$$

Under the change $(u-1)=x$ the above equation can be written as

$$
\begin{aligned}
&\frac{-24}{r_o^2}\sum_{n=0}^{\infty}a_n(n+\hbar)(n+\hbar-1)\left[x^{n+\hbar+4}+3x^{n+\hbar+3}+3x^{n+\hbar+2}+x^{n+\hbar+1}+3x^{n+\hbar}+3x^{n+\hbar-1}\right.\\
&\qquad\left.+3x^{n+\hbar+3}+9x^{n+\hbar+2}+9x^{n+\hbar+1}+3x^{n+\hbar+2}+9x^{n+\hbar+1}+9x^{n+\hbar}\right]\\
&\frac{-36}{r_o^2}\sum_{n=0}^{\infty}a_n(n+\hbar)\left[x^{n+\hbar+4}+3x^{n+\hbar+3}+3x^{n+\hbar+2}+2x^{n+\hbar+3}+6x^{n+\hbar+2}+6x^{n+\hbar+1}\right.\\
&\qquad\left.+x^{n+\hbar+2}+3x^{n+\hbar+1}+3x^{n+\hbar}\right]+14\sum_{n=0}^{\infty}a_n(n+\hbar)\left[x^{n+\hbar+2}+3x^{n+\hbar+1}+3x^{n+\hbar}\right]\\
&-12\sum_{n=0}^{\infty}a_n(n+\hbar)\left[x^{n+\hbar+2}+x^{n+\hbar-1}+3x^{n+\hbar+1}+3x^{n+\hbar}\right]+\frac{\omega^2L^6}{r_o^4}\sum_{n=0}^{\infty}a_n\times\\
&\left[x^{n+\hbar+5}+x^{n+\hbar+2}+3x^{n+\hbar+4}+3x^{n+\hbar+3}+x^{n+\hbar+4}+x^{n+\hbar+1}+3x^{n+\hbar+3}+3x^{n+\hbar+2}\right]\\
&-\frac{k^2L^6}{r_o^4}\sum_{n=0}^{\infty}a_n\left[x^{n+\hbar+1}+x^{n+\hbar}\right]=0.
\end{aligned}
$$

From this we can obtain the recursion relations for a_n and the indicial equation giving $\hbar=0$ and $\hbar=-\frac{r_o^2}{6}+1$.
The normalizable series solution will then be:

$$
\begin{aligned}
\varphi(r)=a_o\Bigg(&1-\frac{k^2L^6}{12r_o^4}\left(\left(\frac{r_o^2}{r^2}\right)-1\right)-\frac{\left(\frac{\omega^2L^6}{r_o^4}+\frac{3k^2L^6}{2r_o^4}-\frac{9k^2L^6}{r_o^6}-\frac{1}{12}\left(\frac{k^2L^6}{r_o^4}\right)^2\right)}{24\left(\frac{6}{r_o^2}+1\right)}\left(\left(\frac{r_o^2}{r^2}\right)-1\right)^2\\
&+\frac{1}{36\left(\frac{12}{r_o^2}+1\right)}\left(\frac{27k^2L^6}{r_o^6}+\frac{\omega^2k^2L^{12}}{12r_o^8}+\frac{1}{12}\left(\frac{k^2L^6}{r_o^4}\right)^2-\frac{k^2L^6}{2r_o^4}-\frac{4\omega^2L^6}{r_o^4}\right.\\
&\left.+\left(\frac{792}{r_o^2}+44+\frac{k^2L^6}{r_o^4}\right)\frac{\left(\frac{\omega^2L^6}{r_o^4}+\frac{3k^2L^6}{2r_o^4}-\frac{9k^2L^6}{r_o^6}-\frac{1}{12}\left(\frac{k^2L^6}{r_o^4}\right)^2\right)}{24\left(\frac{6}{r_o^2}+1\right)}\right)\left(\left(\frac{r_o^2}{r^2}\right)-1\right)^3-\ldots\Bigg).
\end{aligned}
$$

The large r solution (far away from the brane) will be given as:

$$\varphi(r\to\infty) = a_o\left(1+\frac{k^2L^6}{12r_o^4}-\frac{\left(\frac{\omega^2L^6}{r_o^4}+\frac{3k^2L^6}{2r_o^4}-\frac{9k^2L^6}{r_o^6}-\frac{1}{12}\left(\frac{k^2L^6}{r_o^4}\right)^2\right)}{24\left(\frac{6}{r_o^2}+1\right)}\right.$$
$$-\frac{1}{36\left(\frac{12}{r_o^2}+1\right)}\left(\frac{27k^2L^6}{r_o^6}+\frac{\omega^2k^2L^{12}}{12r_o^8}+\frac{1}{12}\left(\frac{k^2L^6}{r_o^4}\right)^2-\frac{k^2L^6}{2r_o^4}-\frac{4\omega^2L^6}{r_o^4}\right.$$
$$\left.\left.+\left(\frac{792}{r_o^2}+44+\frac{k^2L^6}{r_o^4}\right)\frac{\left(\frac{\omega^2L^6}{r_o^4}+\frac{3k^2L^6}{2r_o^4}-\frac{9k^2L^6}{r_o^6}-\frac{1}{12}\left(\frac{k^2L^6}{r_o^4}\right)^2\right)}{24\left(\frac{6}{r_o^2}+1\right)}\right)-...\right).$$

The solution close to the brane, where r_o would be dominant in u, is given as:

$$\varphi(r) = a_o\left(1-\frac{k^2L^6}{12r_o^2r^2}-\frac{\left(\omega^2L^6+3k^2L^6/2-\frac{9k^2L^6}{r_o^2}-\frac{1}{12}\left(\frac{k^2L^6}{r_o^2}\right)^2\right)}{24r^4\left(\frac{6}{r_o^2}+1\right)}\right.$$
$$+\frac{1}{36\left(\frac{12}{r_o^2}+1\right)}\left(\frac{27k^2L^6}{r_o^6}+\frac{\omega^2k^2L^{12}}{12r_o^8}+\frac{1}{12}\left(\frac{k^2L^6}{r_o^4}\right)^2-\frac{k^2L^6}{2r_o^4}-\frac{4\omega^2L^6}{r_o^4}\right.$$
$$\left.\left.+\left(792+44r_o^2+\frac{k^2L^6}{r_o^2}\right)\frac{\left(\omega^2L^6+\frac{3k^2L^6}{2}-\frac{9k^2L^6}{r_o^2}-\frac{1}{12}\left(\frac{k^2L^6}{r_o^2}\right)^2\right)}{24\left(\frac{6}{r_o^2}+1\right)}\right)-...\right).$$

To get a more manageable solution which is in terms of some exact known function we will consider the case of $r_o\to 0$, i.e., $f\to 1$. Then (60) reduces to:

$$\left(\frac{\omega^2L^8}{r^8}-\frac{k^2L^8}{r^8}\right)\varphi(r)+\frac{7L^2}{r^3}\frac{d\varphi}{dr}+\frac{L^2}{r^2}\frac{d^2\varphi}{dr^2}=0. \tag{62}$$

Defining a new variable $u=\frac{c^2}{r^4}$, where $c^2=\omega^2L^6-k^2L^6$ and under which the differentials change as

$$\frac{d\varphi}{dr}=\frac{d\varphi}{du}\left(\frac{-4u}{r}\right) \qquad \frac{d^2\varphi}{dr^2}=\frac{d^2\varphi}{du^2}\left(\frac{16u^2}{r^2}\right)+\frac{d\varphi}{du}\left(\frac{20u}{r^2}\right).$$

we can write the equation (62) as

$$cu^{-1/2}\varphi-8\frac{d\varphi}{du}+16u\frac{d^2\varphi}{du^2}=0.$$

This equation is of the form of Bessel's equation so we will get the solutions for the field in terms of Bessel's function. So we have

$$\varphi(r)=\frac{Ac^3}{r}J_3\left(\frac{c}{r}\right)-\frac{1}{4}i\frac{Bc^3}{r}Y_3\left(\frac{c}{r}\right).$$

Taking the normalizable part gives us:

$$\phi(r,t,z) = \int_{|\omega|>|k|} d\omega dk a_{\omega k} e^{-i\omega t} e^{ikz} \frac{Ac^3}{r} J_3\left(\frac{c}{r}\right).$$

We define the boundary field for this case as

$$\phi_o(t,z) = \lim_{r\to\infty} r^{\Delta}\phi(r,t,z) \qquad .$$

where $\Delta = 4$, which can be known by looking at the asymptotic solutions of the field. Using the following relation.

$$J_\nu(z) = \frac{\left(\frac{1}{2}z\right)^\nu}{\Gamma(\nu+1)}\ {}_0F_1\,(;\nu+1;\frac{-1}{4}z^2)$$

we can write the boundary field as

$$\phi_o(t,z) = \int_{|\omega|>|k|} \frac{A\left(c^2/2\right)^3}{\Gamma(4)} a_{\omega k} e^{-i\omega t} e^{ikz} d\omega dk.$$

Inverting this relation gives us:

$$a_{\omega k} = \frac{\Gamma(4)}{A(2\pi)^2\left(c^2/2\right)^3}\int dt dz e^{i\omega t} e^{-ikz}\phi_o(z,t).$$

Therefore, we can write the bulk field as

$$\phi(r,t,z) = \int dt' dz' K(t',z'|r,t,z)\phi_o(t',z') \tag{63}$$

where

$$K(t',z'|r,t,z) = \int_{|\omega|>|k|} d\omega dk a_{\omega k} e^{-i\omega(t-t')} e^{ik(z-z')} \frac{A}{c^3 r} J_3\left(\frac{c}{r}\right).$$

is the smearing function. So this expression provides a relation of the bulk field in terms of the boundary field.
Here we can see that in the $r_o \to 0$ limit we obtain a AdS type solution as we saw in the Poincaré coordinates construction. So using the similar process of mode sum we will try to obtain the relation of the bulk field in terms of a complexified boundary operator.
From eq (63) we get

$$\begin{aligned}\phi(r,t,z) &= \frac{48}{(2\pi)^2}\int_{|\omega|>|k|} d\omega dk \frac{1}{c^3 r} J_3\left(\frac{c}{r}\right)\int dt' dz' e^{-i\omega(t-t')} e^{ik(z-z')}\phi_o(t',z') \\ &= 48\int_{|\omega|>|k|} d\omega dk e^{-i\omega t} e^{ikz}\frac{1}{c^3 r} J_3\left(\frac{c}{r}\right)\phi_o(\omega,k).\end{aligned}$$

Using (22) and (23) we have

$$J_3\left(\frac{c}{r}\right) = \frac{1}{2^{3-1}\Gamma(3)\left(\frac{c}{r}\right)^{-3}}\int_0^1 x dx(1-x^2)^2 J_o\left(\frac{cx}{r}\right)$$

and

$$J_o\left(\frac{cx}{r}\right) = J_o\left(q\sqrt{\omega^2-k^2}\right) \qquad\qquad \left(q = \frac{cx}{r\sqrt{\omega^2-k^2}}\right)$$
$$= \frac{1}{2\pi}\int_0^{2\pi} d\theta e^{-iq\omega\sin\theta - kq\cos\theta}.$$

So we have the following expression.

$$J_3\left(\frac{c}{r}\right) = \frac{1}{8\times 2\pi}\left(\frac{r}{c}\right)^3\int_0^1 xdx(1-x^2)^2\int_0^{2\pi} d\theta e^{-iq\omega\sin\theta-kq\cos\theta}$$

Let $q\sin\theta = t'$, $q\cos\theta = y'$

$\left[\int\int \frac{c^2}{r^2(\omega^2-k^2)} xdxd\theta = \int\int dt'dy'\right]$

This will give us,

$$J_3\left(\frac{c}{r}\right) = \frac{1}{16\pi}\left(\frac{r}{c}\right)^3\int_{t'^2+y'^2<\frac{\omega^2-k^2}{r^2}} dt'dy'\frac{r^2(\omega^2-k^2)}{c^2}\left(1-(t'^2+y'^2)\frac{r^2(\omega^2-k^2)}{c^2}\right)^2 e^{-i\omega t'}e^{-ky'}$$
$$= \frac{1}{16\pi}\left(\frac{r^5}{c^3L^6}\right)\int_{t'^2+y'^2<\frac{\omega^2-k^2}{r^2}} dt'dy'\left(1-(t'^2+y'^2)\frac{r^2}{L^6}\right)^2 e^{-i\omega t'}e^{-ky'}$$

where the integral is over a disk like region of radius $\frac{\omega^2-k^2}{r^2}$ in the t-z plane.

Thus we obtain the following expression for the field

$$\phi(r,t,z) = \frac{3}{\pi}\left(\frac{r^5}{c^3}\right)\int_{t'^2+y'^2<\frac{\omega^2-k^2}{r^2}} dt'dy'\left(1-(t'^2+y'^2)\frac{r^2}{L^6}\right)^2\frac{1}{c^3r}\int d\omega dk e^{-i\omega(t+t')}e^{ik(z+iy')}\phi_o(\omega,k)$$
$$= \frac{\Delta-1}{\pi}\left(\frac{r^4}{c^6}\right)\int_{t'^2+y'^2<\frac{\omega^2-k^2}{r^2}} dt'dy'\left(1-(t'^2+y'^2)\frac{r^2}{L^6}\right)^{\Delta-2}\phi_o(t+t',z+iy').$$

So we have obtained a relation for a massless scalar field in D1 brane background in terms of the boundary field. So here we have the bulk field in terms of an integral over a disk of radius $\frac{\omega^2-k^2}{r^2}$ in the real t, imaginary z plane.
The information about the bulk has been encoded in the boundary operator through the smearing function and one should be able to use it to try to learn about bulk physics from the boundary in the D1 brane background.

Conclusion

In summary, we saw in this paper on how quantum field theory works in curved spacetime, especially in de Sitter and anti de Sitter spaces. We went through some bulk reconstruction calculations, so that we have a bulk field represented in terms of the boundary field operator. We reviewed some already carried out constructions in these spaces and tried to use them to work in the case of D1 brane. The calculations covered here were done in Poincaré and global coordinates of AdS, working in three dimensions and the flat patch of de Sitter space.

The bulk operator in the flat patch of de Sitter space was constructed in terms of the single-trace CFT operators and the construction was done in terms of CFT operators at past boundary $\mathcal{I}^-$. In the future wedge the bulk operators are constructed with CFT operators on $\mathcal{I}^+$, where the "retarded propagator" is now propagating the boundary operators back in time. Everything formulated in that case can be redone in the future wedge to get a local operator in the future wedge in terms of operators at $\mathcal{I}^+$, as the flat FRW slicing of de Sitter space defined on the future wedge is more relevant in cosmology.

We then looked at a massless scalar field in the case of the D1 brane and worked in the dimensionally reduced near horizon geometry to 3 dimensions. We found the solution for the scalar field in this background and tried to get a smearing function for the bulk field to relate it with the boundary field operator, wherein we worked in $r_0 \to 0$ limit which gave us a solution resembling that of AdS space. We also looked at the series solution for the field in this background.

In the future we could try to find the solutions for a massive scalar field or in certain limits such as (low frequency) and construct a smearing function for these cases. This set up is a part of well studied D1-D5 brane solution and its dual CFT is also well studied. Being a setup in lower dimensions, it is more amenable to calculations. The exercise of finiding Bulk reconstructions for various fields in terms of operators on the boundary will provide more insights and better understanding of the AdS/CFT conjecture.

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