Quantum entanglement and its applications in gravitational physics

M.Sc. Thesis

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Quantum entanglement and its applications in gravitational physics

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By

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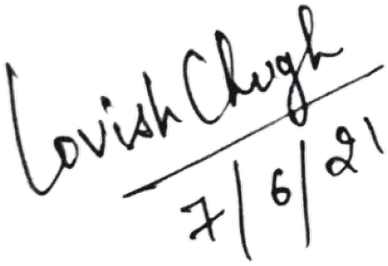
Declaration

I hereby certify that the work which is being presented in the thesis entitled **Quantum entanglement and its applications in gravitational physics** in the partial fulfillment of the requirements for the award of the degree of **Master of Science** and submitted in the **Department of Physics, Indian Institute of Technology, Indore**, is an authentic record of my own work carried out during the time period from **July, 2019** to **June, 2021** under supervision of **Dr. Debajyoti Sarkar, Assistant Professor, Indian Institute of Technology, Indore**.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other institute.



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This is to certify that the above statement made by the candidate is correct to the best of my/our knowledge.

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Abstract

In this work, the definitions and basics of quantum complexities are followed and various works which are done in regards of circuit and Kolmogorov complexity are studied. Based on these works, calculations with respect to fidelity susceptibility, capacity of entanglement, expected Kolmogorov complexity have been performed. First law of circuit complexity is calculated by using SYK model Hamiltonian which is also used in calculation of expected Kolmogorov complexity and infinitesimal change in Kolmogorov complexity. So in this way, we tried to build a relation between these two kinds of complexities. Nielsen's approach has been followed for the calculation of first law of circuit complexity. In this work we also tried to look for a relation between first law of thermodynamics and Kolmogorov complexity and hence tried to build first law of Kolmogorov complexity. Some open questions have been discussed at the end.

Table of Contents

Chapter 1: Introduction

In this work, we have studied various aspects of quantum information theory such as quantum complexity, fidelity susceptibility, capacity of entanglement etc.

Quantum computation and quantum information is the study of the information processing tasks that can be accomplished using quantum mechanical systems. For the classical system, the most basic piece of information is called a bit. A bit represents a two state system. We can also represent a bit by binary numbers 0 and 1. But if we work in a quantum mechanical system, the basic unit of information is quantum bit or qubit. While in classical system a bit can be in 0 state or 1 state, a qubit is somewhat more general. A qubit can exist in $|0\rangle$ or $|1\rangle$ or in a superposition state. This state is a linear combination of the states $|0\rangle$ and $|1\rangle$.

Apart from qubit another important parameter which needs to be introduced here is density operator. For that we need to first understand what is an ensemble. The definition of an ensemble provided by [1] suggests that an ensemble is a collection of identically prepared physical systems. To define density operator, firstly we need to define pure and mixed states. If all the objects are in the same state $|\psi_i\rangle$, the ensemble is represented by a pure state. In mixed states not all the N objects are in the same state i.e. N_i objects in the state $|\psi_i\rangle$ respectively, such that $\sum N_i = N$. Now, a density operator ρ is defined as

$$\rho = \sum_i \omega_i |\alpha^{(i)}\rangle\langle\alpha^{(i)}|. \tag{1}$$

For a pure ensemble we have $\omega_i = 1$ for some vector $|\alpha^{(i)}\rangle$ with $i = n$ for instance, and $\omega_i = 0$ for all other conceivable state kets so that the corresponding

density operator is written as

$$\rho = |\alpha^{(n)}\rangle\langle\alpha^{(n)}|. \tag{2}$$

There are few properties of density operators which are the following (1.) Density operator is hermitian, $\rho^\dagger = \rho$. (2.) Density operator satisfies the normalization condition, $Tr\rho = 1$. (3.) $\rho \geq 0$ positivity. (4.) for a pure ensemble $\rho^2 = \rho$. There is an important theorem related to density operators which is used in various calculations in this work. It states that the expectation value of an observable A in a state represented by a density matrix ρ, is given by

$$\langle A\rangle_\rho = Tr(\rho A). \tag{3}$$

It has a simple proof given as,

$$Tr(\rho A) = Tr(|\psi\rangle\langle\psi|A) = \sum_n \langle n|\psi\rangle\langle\psi|A|n\rangle \tag{4}$$

$$\implies \sum_n \langle\psi|A|n\rangle\langle n|\psi\rangle = \langle\psi|A|\psi\rangle = \langle A\rangle \tag{5}$$

After studying these preliminary parameters, let us now turn our attention to quantum complexity. Quantum complexity has come up as a very interesting approach for understanding various physical phenomena. In this work, firstly we have tried to study the basics of Quantum complexity, its classification. Then we tried to use its relation with other quantum information parameters which can be used to understand various physical processes. In this work we have tried to calculate quantities from various known parameters such as fidelity susceptibility, capacity of entanglement, quantum Fisher information in the field of quantum information.

In chapter 2, we will introduce the quantities required to understand quantum

complexities. Firstly we will define the basics of circuit and Kolmogorov complexity, then we will introduce fidelity susceptibility, capacity of entanglement and velocity coupling correspondence. In chapter 3, we will calculate the expected Kolmogorov complexity, first law of circuit complexity using Nielsen's approach and then we will try to search for first law of Kolmogorov complexity. In chapter 4, we will compile the calculations and in chapter 5, we will conclude our work and discuss some open questions.

Quantum complexity can be defined in various ways but here we are following the definitions of circuit and Kolmogorov complexity provided in [2], [3]. Kolmogorov complexity is also defined in [4] and [5]. These texts can be referred to get the greater insight of Kolmogorov complexity. [6] provides an overall understanding of quantum complexity theory in a broader sense. Defining field theory is rather difficult in quantum field theory. This is because it is difficult to work in infinite dimensional Hilbert space. Other important works which has been done in this regard are [7] and [8]. [7] discusses circuit complexity in quantum filed theory.

Chapter 2: Quantities of quantum information in field theory

2.1 Introduction to Quantum complexity

In the broad sense complexity can be studied under two sections. One is circuit complexity and the other is algorithmic complexity or better known as Kolmogorov complexity. As our main aim is to understand complexity in field theory, one of the major works done in this direction is to relate complexity with entropy [2]. This work suggests that the phase space probability distribution for a classical non-relativistic gas often separates into two factors, one depending on the positions of the particles and the other on the momenta,

$$P(x,p) = F(x)G(p). \tag{6}$$

So, the total entropy is a sum of two terms, the positional entropy associated with the distribution $F(x)$, and the kinetic entropy associated with $G(p)$,

$$S = -\int F(x)logF(x)dx - \int G(p)logG(p)dp. \tag{7}$$

Where the positional entropy associated with the distribution $F(x)$; and the kinetic entropy associated with $G(p)$ [2]. According to [2], there is a conjecture which states that,

At any instant, the ensemble average of the computational complexity of the quantum system Q; is proportional to the classical positional entropy of the auxiliary system A.

This conjecture suggests a relation between circuit (or computational) complexity and classical entropy. We can relate the positional entropy of a classical particle in an auxiliary classical system A to the complexity of quantum system Q because the circuit complexity will only depend on the position of the

particle and not its velocity. To understand this we follow the definition of computational complexity. Briefly computational or circuit complexity defines the complexity of quantum circuit required to achieve a particular state from some reference state. So, the complexity will depend upon the position of the unitary operator in the state space that is used to achieve the final state. Because if we change the position of final state in state space then unitary operator will be changed and hence there will be change in complexity. For studying this we need to consider a quantum system. As a particular example we consider a quantum system Q consisting of K qubits interacting through a k-local Hamiltonian of the form

$$H = \sum_{i_1<i_2<...<i_3} \sum_{a_1=\{x,y,z\}} ... \sum_{a_k=\{x,y,z\}} J^{a_1,a_2,...,a_k}_{i_1,i_2,...,i_k} \sigma^{a_1}_{i_1} \sigma^{a_2}_{i_2} ... \sigma^{a_k}_{i_k}. \tag{8}$$

It can be simplified in schematic form as

$$H = \sum_I J_I \sigma_I. \tag{9}$$

where I runs over generalized $(4^K - 1)$ Pauli operators and J's are coupling constants. Here k-local means that the building blocks of a k-local Hamiltonian are Hermitian operators that involve at most k qubits. The classical system A represents the evolution of a quantum system as the motion of a non-relativistic particle moving on $SU(2^K)$ as K qubits are defined by 2^K variables. The space $SU(2^K)$ is a homogeneous group space generated by $(4^K - 1)$ generators. [2] identifies computational complexity with positional entropy as computational complexity has only to do with the distance of a point from the origin.

Now for the coupling constants J defined in (8), we have a velocity-coupling correspondence, or just V/J-correspondence. It suggests that J_I are the initial values of the velocity components V_I which we will define in later section

in this chapter. Through that the kinetic entropy of the classical auxiliary system A can be identified with the entropy of the probability distribution $P(J)$ defined by quantum system Q, since Kolmogorov complexity measures the length of the shortest algorithm that can prepare a string. Applied to the string of J's it would define a Kolmogorov complexity for each specific instance of a Hamiltonian. Here we need to consider velocity because as we discussed in (6) that phase space probability distribution can be divided in two parts. One part depends on position and other on momenta. We also know that momenta is just product of mass and velocity. So indirectly we can say that we divide the probability distribution into position and velocity part. Through this we can assume a relation between kinetic entropy and $P(J)$.

2.2 Relation of complexity with entropy with respect to time evolution of complexity

If we consider a time development operator $U(t) = e^{-\iota Ht}$, for a k-local Hamiltonian H. [2] suggests that the computational complexity of $U(t)$ evolve with time. We see that the complexity $C(t)$ grows linearly as

$$C(t) = Kt, \tag{10}$$

for a time exponential in K. At time $T \sim e^K$ the complexity reaches its maximum possible value C_{max} and flattens out for a very long time.

$$C_{max} = e^K. \tag{11}$$

On a much longer timescale of order $exp(e^k)$ quantum recurrences quasi-periodically return the complexity to sub exponential values. This pattern is equivalent to evolution of classical entropy. However, for classical case the linear growth of entropy will persist only a time polynomial (n the number

of degrees of freedom), the maximum entropy will also be of order of degrees of freedom and the recurrence time will be simply exponential not doubly exponential.

2.3 Circuit Complexity

Circuit complexity or the gate complexity or computational complexity defines how a final state is arrived from an initial state through application of quantum circuit. Quantum circuit consists of a number of quantum gates which are used in a manner to achieve the desired unitary operation. If U is a unitary operation which is carried out to give the final state, then the quantity $m_\zeta(U)$ is defined as minimum number of gates from ζ to achieve U, ζ be a set of unitary gates which is universal on K qubits. Quantum gates are analogous to classical logic gates except that they must be implemented reversibly. According to quantum postulate, quantum states evolve unitarily. The operator which can act on a single qubit state has to be unitary $U^\dagger U = 1$, which preserves norm of state. Quantum circuit contains logic gates connected by straight lines, which do not represent physical wires but indicates the direction of logic flow with time. [3] introduces a metric $d(.,.)$ on $SU(2^K)$ such that $d(I,U) \leq m_\zeta(U)$. Here I is the K-qubit identity operation. Thus the metric d provides a lower bound on the number of gates required to implement U. In chapter 3, we will use the approach of [3] to calculate first law of circuit complexity.

2.4 Kolmogorov complexity

Kolmogorov complexity or algorithmic complexity is complexity of bit strings. Kolmogorov complexity measures the length of shortest algorithm that can prepare a string. [9] provides an effective definition for Kolmogorov complexity

for any classical system but it is even valid for a quantum system. It is given as the following: let U be any reference machine satisfying for all $i \in N$, $y \in \{0,1\}^*$, $U(\langle i, y\rangle) = \phi_i(y)$. The Kolmogorov complexity of x is

$$K(x) = min_z\{l(z) : U(z) = x, z \in \{0,1\}^*\} \tag{12}$$

$$= min_z\{l(\langle i, y\rangle) : \phi(y) = x, y \in \{0,1\}^*, i \in N\}, \tag{13}$$

where i defines a set of Turing machines, N is a set of natural numbers, ϕ_i is the enumeration of corresponding functions which are computed by the respective Turing machines. Turing Machine can be defined as a computer program written in a general-purpose language. We can think of z as a program that prints x. $\{0,1\}^*$ defines a set of finite strings or sequences. For example, $\{0,1\}^* = \{\epsilon, 0, 1, 00, 01, 10, 11, ...\}$ here ϵ denotes the empty word with no letters. $(\langle i, y\rangle)$ indicates a pair of i^{th} machine operating on y. (12), (13) suggest that $K(x)$ of a finite object x is the shortest effective binary description of x. Since z is a program which prints x, so (12) defines minimum length of z.
[2] suggests that average Kolmogorov complexity is related to entropy. As we discussed before that Kolmogorov complexity is related to kinetic entropy and through velocity coupling correspondence that we are going to discuss later, we can relate the entropy of probability distribution of J to Kolmogorov complexity. If we define entropy with that of Shannon's formula

$$S = -\int P(J) log P(J) dJ. \tag{14}$$

Then average kolmogorov complexity is given as

$$-\sum P(J) log P(J) = \sum P(J)\mathscr{C}_k(J) = \langle \mathscr{C}_k \rangle, \tag{15}$$

where $P(J)$ are probability distribution of coupling constants, here for an example this distribution follow SYK model defined in (8). Here $P(J)$ has

Gaussian distribution.

Now, in quantum information theory, there are various parameters which are needed to use complexity as a probe for studying gravitation such as fidelity susceptibility and capacity of entanglement. First we will study the basics of these quantities and later we will use these to study complexity.

2.5 Fidelity susceptibility

Fidelity is a measure of closeness of two quantum states. Following [10] we take a pure quantum state $|\psi(\lambda_1)\rangle$ in the Hilbert space. Here λ is a parameter. A neighboring state $|\psi(\lambda_2)\rangle$ can be achieved by infinitesimal change in the parameter λ. Now taking the inner product of two states for a small perturbation $\delta\lambda = \lambda_2 - \lambda_1$, using Taylor's expansion, we have

$$\langle\psi(\lambda_1)|\psi(\lambda_2)\rangle = 1 - G_\lambda \delta\lambda^2 + \mathscr{O}(\delta\lambda^3), \tag{16}$$

here the $\delta\lambda$ term is zero because we are looking for the minima of the fidelity, therefore its first derivative with respect to λ would be zero. This expression, considered as a metric can measure the distance between two pure states. Now if we consider density matrices [11], parameterized by a continuous parameter θ, for two density matrices ρ and σ, Fidelity is given as

$$F(\rho,\sigma) = Tr\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}. \tag{17}$$

Using Taylor series expansion for an infinitesimal flow $\theta + \epsilon$, we can define

$$F(\rho(\theta),\rho(\theta+\epsilon)) = 1 - \frac{1}{2}\chi_F(\theta)\epsilon^2 + ... \tag{18}$$

$$\chi_F(\theta) = -\frac{\partial^2 F(\rho(\theta),\rho(\theta+\epsilon))}{\partial\epsilon^2}\bigg|_{\epsilon=0}, \tag{19}$$

here χ_F is known as reduced fidelity susceptibility. $F(\rho(\theta),\rho(\theta+\epsilon))$ can be used as a measure of how much the density matrix changes along an infinitesimal

flow $\theta + \epsilon$.

Now we will study capacity of entanglement which is an important quantum information quantity. We will use this in chapter 3 to create a relation of expected Kolmogorov complexity and fidelity susceptibility.

2.6 Capacity of entanglement

For a density matrix ρ, the entanglement entropy is given as

$$S = -Tr(\rho \ \log \rho).$$

Introducing a modular Hamiltonian $K = -log\rho$, the entanglement entropy becomes

$$S = Tr(K\rho) = \langle K \rangle.$$

Now as given in [11], capacity of entanglement is defined as

$$C_E(\rho) = Tr(\rho(-\log \rho)^2) - (-Tr(\rho \ \log \rho))^2 \tag{20}$$

$$C_E(\rho) = \langle K^2 \rangle - \langle K \rangle^2. \tag{21}$$

For a thermodynamic system with inverse temperature β, one can define the capacity of entanglement to be equal to the heat capacity of the system

$$C_E(\rho) = \beta^2[\langle H^2 \rangle_\beta - \langle H \rangle_\beta^2], \tag{22}$$

here

$$\rho_{thermal} = e^{-\beta H}.$$

If we have a deformation in the density matrix ρ, with a deformation parameter θ, i.e.

$$\rho(\theta) = \frac{e^{-(1+\theta)K}}{Tr[e^{-(1+\theta)K}]} \tag{23}$$

$$\implies \log(\rho(\theta)) = -(1+\theta)K - log(Tr[e^{-(1+\theta)K}]), \tag{24}$$

then the capacity of entanglement is given as

$$C_E(\rho) = Tr[\rho(-\log\rho)^2] - [-Tr(\rho\ \log\rho)]^2 \tag{25}$$

$$C_E(\rho) = Tr[\rho\{(1+\theta)^2K^2 + 2log(Tr[e^{-(1+\theta)K}]) \tag{26}$$

$$+ 2(1+\theta)K\log\big(Tr[e^{-(1+\theta)K}]\big)\}]$$
$$- [Tr\{\rho\{(1+\theta)K + \log\big(Tr[e^{-(1+\theta)K}]\big)\}\}]^2,$$

$$C_E(\rho) = Tr[\rho(1+\theta)^2K^2 + 2\log\big(Tr[e^{-(1+\theta)K}]\big)\rho + \tag{27}$$

$$2(1+\theta)\log\big(Tr[e^{-(1+\theta)K}]\big)\rho K]$$
$$- [Tr\{\rho(1+\theta)K + \log\big(Tr[e^{-(1+\theta)K}]\big)\rho\}]^2$$

$$C_E(\rho) = 2\log\big(Tr[e^{-(1+\theta)K}]\big) + 2(1+\theta)\log\big(Tr[e^{-(1+\theta)K}]\big)\langle K\rangle + \tag{28}$$

$$(1+\theta)^2\langle K^2\rangle - (1+\theta)^2\langle K\rangle^2 - 2\log\big(Tr[e^{-(1+\theta)K}]\big) - 2(1+\theta)\log\big(Tr[e^{-(1+\theta)K}]\big)\langle K\rangle.$$

From here, we can see that capacity of entanglement is given by variance of K, given by,

$$C_E(\rho) = (1+\theta)^2(\langle K^2\rangle - \langle K\rangle^2) \tag{29}$$

$$= (1+\theta)^2\langle(\Delta K)^2\rangle. \tag{30}$$

However, it is an interesting question as to how can (30) be generalized. For this purpose, we take the perturbation V with parameter θ (V doesn't commute with K).

$$\rho(\theta) = \frac{e^{-K-\theta V}}{Tr[e^{-K-\theta V}]} \tag{31}$$

$$\log(\rho(\theta)) = -K - \theta V - \log\big(Tr[e^{-K-\theta V}]\big). \tag{32}$$

Now, we know,

$$C_E(\rho) = Tr[\rho(-\log\rho)^2] - [-Tr(\rho\ \log\rho)]^2 \quad (33)$$

$$= Tr[\rho(K+\theta V+\log\big(Tr[e^{-K-\theta V}]\big))^2]- \quad (34)$$

$$\{Tr[\rho(K+\theta V+\log\big(Tr[e^{-K-\theta V}]\big))]\}^2$$

$$\begin{aligned} C_E(\rho) = Tr(\rho K^2)+\theta^2 Tr(\rho V^2)+2\log\big(Tr[e^{-K-\theta V}]\big)Tr(\rho)+2\theta Tr[\rho(K\cdot V)] \\ + 2\log\big(Tr[e^{-K-\theta V}]\big)Tr(\rho K) + 2\theta\log\big(Tr[e^{-K-\theta V}]\big)Tr(\rho V) \\ -[Tr(\rho K)]^2-\theta^2[Tr(\rho V)]^2-2\log\big(Tr[e^{-K-\theta V}]\big)-2Tr(\rho K)Tr(\rho V)\theta \\ - 2\log\big(Tr[e^{-K-\theta V}]\big)Tr(\rho K) - 2\theta Tr(\rho V)\log\big(Tr[e^{-K-\theta V}]\big). \end{aligned}$$

The above cumbersome equation can be solved in a very simple format as,

$$\begin{aligned} C_E(\rho) = \langle K^2\rangle+\theta^2\langle V^2\rangle+2log(Tr[e^{-K-\theta V}])+2\theta\langle K\cdot V\rangle+2log(Tr[e^{-K-\theta V}])\langle K\rangle \\ + 2\theta log(Tr[e^{-K-\theta V}])\langle V\rangle - \langle K\rangle^2 - \theta^2\langle V\rangle^2 - 2log(Tr[e^{-K-\theta V}]) \\ - 2log(Tr[e^{-K-\theta V}])\langle K\rangle - 2\theta log(Tr[e^{-K-\theta V}])\langle V\rangle - 2\theta\langle K\rangle\langle V\rangle, \end{aligned}$$

$$C_E(\rho) = \langle K^2\rangle - \langle K\rangle^2 + \theta^2[\langle V^2\rangle - \langle V\rangle^2] + 2\theta[\langle K\cdot V\rangle - \langle K\rangle\langle V\rangle] \quad (35)$$

$$= \langle(\Delta K)^2\rangle + \theta^2\langle(\Delta V)^2\rangle + 2\theta[\langle K\cdot V\rangle - \langle K\rangle\langle V\rangle]. \quad (36)$$

The results in (35), (29) can be verified by using V as K in (35). We have done these calculations by using the definition of capacity of entanglement provided in [11].

2.7 Velocity coupling correspondence

If we think of the time evolution of the Hamiltonian defined by (9), we can think it as the moving point $U(t)$, which we may think of as the motion of a fictitious particle moving on $SU(2^K)$. $SU(2^K)$ defines a state space. So, the motion of a point in the state space is equivalent to the motion of a unitary

operator which defines a series of quantum gates required to achieve the state. Using Schrodinger equation,

$$\iota \dot{U} = HU. \tag{37}$$

The particle starts at the point $U = 1$ i.e. the identity matrix. Using (9) in (37), we get,

$$\dot{U} = -\iota \sum_I J_I \sigma_I U. \tag{38}$$

If we solve for J_I,

$$\dot{U}U^\dagger = -\iota \sum_I J_I \sigma_I U U^\dagger. \tag{39}$$

Now, we know that for unitary operator, $UU^\dagger = 1$, this gives,

$$\iota \dot{U}U^\dagger = \sum_I J_I \sigma_I, \tag{40}$$

$$\iota \sigma_K \dot{U}U^\dagger = \sum_I J_I \sigma_K \sigma_I. \tag{41}$$

Applying trace on both sides of above equation,

$$\iota Tr(\sigma_K \dot{U}U^\dagger) = \sum_I J_I Tr \sigma_K \sigma_I. \tag{42}$$

Now,

$$Tr(\sigma_K \sigma_I) = \delta_{KI}. \tag{43}$$

So, we get,

$$J_I = \iota Tr(\sigma_I \dot{U}U^\dagger). \tag{44}$$

At the origin $U = 1$,

$$\iota Tr(\dot{U}\sigma_I) = J_I. \tag{45}$$

The left side of this equation is the projection of initial velocity onto the tangent space axes oriented along the Pauli basis. In the other words the J_I are initial values of the velocity component V_I.

$$J_I = V_I\Big|_{initial} \quad . \tag{46}$$

Above relation is known as velocity coupling correspondence.

Chapter 3: Calculations

3.1 Expected Kolmogorov complexity

In the introduction of complexity we defined the SYK model Hamiltonian (8), we also defined the coupling constants, for these coupling constants we have a probability distribution function given as

$$P(J) = \frac{\exp\left(-\frac{1}{2}B\sum_I J_I^2\right)}{Z}, \tag{47}$$

here B can be understood as inverse temperature β as used earlier. Now for this distribution we use the relation of expected Kolmogorov complexity with entropy (15). Since coupling constants are not binary bits but can take any real value, so can convert summation to integration. Partition function is given as,

$$\begin{aligned} Z &= \prod_I \int P(J)dJ && (48)\\ Z &= \prod_I \int_0^\infty exp\Big(-\frac{1}{2}BJ^2dJ\Big) && (49)\\ &= \prod_I \frac{1}{2}\sqrt{\frac{2\pi}{B}} && (50)\\ &= \Big(\sqrt{\frac{\pi}{2B}}\Big)^{4^K-1} && (51) \end{aligned}$$

Following (48), probability distribution is

$$P(J) = \Big(\sqrt{\frac{2B}{\pi}}\Big)^{4^K-1}\prod_I \exp\Big(-\frac{1}{2}BJ_I^2\Big). \tag{52}$$

Therefore, from (15), we have

$$\langle \mathscr{C}_k \rangle = -\int \Big(\sqrt{\frac{2B}{\pi}}\Big)^{4^K-1}\prod_I \exp\Big(-\frac{1}{2}BJ_I^2\Big) log\Big[\Big(\sqrt{\frac{2B}{\pi}}\Big)^{4^K-1}\prod_I \exp\Big(-\frac{1}{2}BJ_I^2\Big)\Big]dJ,$$

$$= -\int \Big(\sqrt{\frac{2B}{\pi}}\Big)^{4^K-1} \prod_I \exp\Big(-\frac{1}{2}BJ_I^2\Big) log\Big(\sqrt{\frac{2B}{\pi}}\Big)^{4^K-1} dJ \quad (53)$$

$$+ \int \Big(\sqrt{\frac{2B}{\pi}}\Big)^{4^K-1} \prod_I \exp\Big(-\frac{1}{2}BJ_I^2\Big) \Big(\sum_I \frac{1}{2}BJ_I^2\Big) dJ \quad (54)$$

Finally, we get,

$$\langle \mathscr{C}_k \rangle = -\frac{4^K-1}{2} log\Big(\frac{2B}{\pi}\Big) + \frac{4^K-1}{2}\Big(\sqrt{\frac{\pi}{2B}}\Big)^{4^K-2}. \quad (55)$$

This is the expected value of Kolmogorov complexity for the given probability distribution. [2] defines the variance of Hamiltonian for a infinite temperature thermofield-double state as

$$(\Delta H)^2 = TrH^2 \quad (56)$$

$$= Tr \sum_I \sum_L \sigma_I \sigma_L J_I J_L. \quad (57)$$

$$= \sum_I \sum_L J_I J_L Tr \sigma_I \sigma_L \quad (58)$$

Here, J_I and J_L are just numbers, not operators. Now,

$$Tr\sigma_I\sigma_L = \delta_{IL}. \quad (59)$$

Therefore,

$$\langle (\Delta H)^2 \rangle = \langle \sum_I J_I^2 \rangle. \quad (60)$$

Now from (52), we can write

$$log(P(J)) = log\Big(\sqrt{\frac{2B}{\pi}}\Big)^{4^K-1} - \Big(\sum_I \frac{1}{2}BJ_I^2\Big). \quad (61)$$

Therefore using (56)

$$\langle log(P(J)) \rangle = log\Big(\sqrt{\frac{2B}{\pi}}\Big)^{4^K-1} - \langle (\Delta H)^2 \rangle \frac{1}{2}B. \quad (62)$$

We know that,

$$\langle log(P(J))\rangle = \int P(J) log P(J) dJ.$$

As a result the expected Kolmogorov Complexity is given by,

$$-\int P(J) log P(J) dJ = \int P(J)\mathscr{C}_k(J) dJ = \langle \mathscr{C}_k \rangle. \tag{63}$$

Finally, from (62) and (63), we get

$$\langle \mathscr{C}_k \rangle = -log\Big(\sqrt{\frac{2B}{\pi}}\Big)^{4^K-1} + \langle (\Delta H)^2 \rangle \frac{1}{2} B. \tag{64}$$

Now for a thermal state, we can use (22)

$$\langle (\Delta H)^2 \rangle = \frac{C_E}{B^2}, \tag{65}$$

so, using (64) and (65),

$$\langle \mathscr{C}_k \rangle = -log\Big(\sqrt{\frac{2B}{\pi}}\Big)^{4^K-1} + \frac{C_E}{2B}. \tag{66}$$

This is relation of expected Kolmogorov complexity with the capacity of entanglement. Now, for calculating the change in Kolmogorov complexity we need to assume two values of expected Kolmogorov complexity at B_1 and B_2.

$$\begin{aligned} \mathscr{C}_k(B_1) &= -log\Big(\sqrt{\frac{2B_1}{\pi}}\Big)^{4^K-1} + \frac{C_E}{2B_1} &(67)\\ &= -log\Big(\sqrt{\frac{2B_1}{\pi}}\Big)^{4^K-1} + \frac{B_1}{2}\langle (\Delta H)^2 \rangle_{B_1}. &(68) \end{aligned}$$

Similarly for B_2

$$\mathscr{C}_k(B_2) = -log\Big(\sqrt{\frac{2B_2}{\pi}}\Big)^{4^K-1} + \frac{B_2}{2}\langle (\Delta H)^2 \rangle_{B_2}. \tag{69}$$

Here, we are changing state by changing B, as temperature is B^{-1}. From here, we can compute change in Kolmogorov complexity as

$$\delta\mathscr{C}_k(\Delta B) = log\Big(\sqrt{\frac{B_1}{B_2}}\Big)^{4^K-1} + \frac{1}{2}\Big[B_2\langle (\Delta H)^2 \rangle_{B_2} - B_1\langle (\Delta H)^2 \rangle_{B_1}\Big]. \tag{70}$$

For a small ϵ change around B_1, such that

$$B_2 - B_1 = \epsilon$$

we can take Taylor expansion of Kolmogorov complexity around B_1

$$\mathscr{C}_k(B_2) = \mathscr{C}_k(B_1) + \epsilon \left.\frac{\partial \mathscr{C}_k}{\partial B}\right|_{B_1}, \tag{71}$$

where

$$\delta\mathscr{C}_k = \left.\frac{\partial \mathscr{C}_k}{\partial B}\right|_{B_1}$$

$$\mathscr{C}_k(B) = -\frac{(4^K-1)}{2} log\Big(\frac{2B}{\pi}\Big) + \frac{B}{2}\langle(\Delta H)^2\rangle_B \tag{72}$$

$$\left.\frac{\partial \mathscr{C}_k}{\partial B}\right|_{B_1} = -\frac{(4^K-1)}{2B_1} + \frac{\langle(\Delta H)^2\rangle_{B_1}}{2} + \frac{B_1}{2}\frac{\partial}{\partial B_1}\langle(\Delta H)^2\rangle_{B_1}. \tag{73}$$

This is change in Kolmogorov complexity for a small ϵ change.

Now [11] provides a relation between reduced fidelity susceptibility and capacity of entanglement given as

$$\chi_F = \frac{1}{4} C_E. \tag{74}$$

This relation can be understood in a very simple way. Using (17), if we take $\sigma = \rho + \epsilon\delta\rho$, and letting ρ being diagonal, we have

$$F = Tr\sqrt{\sqrt{\rho}(\rho + \epsilon\delta\rho)\sqrt{\rho}} \tag{75}$$

$$\implies F = \sum_{i,j}\sqrt{\lambda_i^2 + \epsilon\delta\rho_{ij}\lambda_i}, \tag{76}$$

here λ_i are eigenvalues of ρ. Now, differentiating Fidelity with respect to ϵ,

$$\frac{\partial F}{\partial \epsilon} = \sum_{i,j}\frac{1}{2\sqrt{\lambda_i^2 + \epsilon\delta\rho_{ij}\lambda_i}}\delta\rho_{ij}\lambda_i \tag{77}$$

$$\frac{\partial^2 F}{\partial \epsilon^2} = -\sum_{i,j} \frac{1}{(4\lambda_i^2 + \epsilon \delta\rho_{ij}\lambda_i)^{3/2}} (\delta\rho_{ij})^2 (\lambda_i)^2. \tag{78}$$

Putting $\epsilon = 0$ gives,

$$\left.\frac{\partial^2 F}{\partial \epsilon^2}\right|_{\epsilon=0} = -\sum_{i,j} \frac{1}{4(\lambda_i)^3} (\partial\rho_{ij})^2 (\lambda_i)^2. \tag{79}$$

Using (18), we have

$$\chi_F = \frac{1}{2(\lambda_i + \lambda_j)} (\delta\rho)_{ij} (\delta\rho)_{ji}. \tag{80}$$

Now, here we define a symmetric logarithmic derivative $L(\theta)$ of $\rho(\theta)$ [11] as

$$L(\theta) = \frac{1}{\rho(\theta)} \frac{\partial \rho(\theta)}{\partial \theta} \tag{81}$$

$$\frac{\partial \rho(\theta)}{\partial \theta} = \frac{1}{2}(L\rho + \rho L). \tag{82}$$

Now, quantum Fisher information $g(\theta)$ [11] is defined as,

$$g(\theta) = Tr(\rho(\theta) L^2(\theta)) \tag{83}$$

$$\implies g(\theta) = Tr\Big(\frac{\partial \rho(\theta)}{\partial \theta} L(\theta)\Big). \tag{84}$$

Now, using Taylor expansion,

$$\rho + \epsilon\delta\rho = \rho + \epsilon \frac{\partial \rho(\theta)}{\partial \theta}. \tag{85}$$

Using (81) and (85), we get,

$$\delta\rho = \frac{1}{2}(\rho L + L\rho). \tag{86}$$

In component form,

$$\delta\rho_{ij} = \frac{1}{2}(L_{ij}\lambda_j + \lambda_i L_{ij}) \tag{87}$$

$$\implies L_{ij} = \frac{2}{(\lambda_i + \lambda_j)} \delta\rho_{ij}. \tag{88}$$

This gives us,

$$L^2(\theta) = \frac{4}{(\lambda_i + \lambda_j)^2}\delta\rho_{ij}\delta\rho_{ji}. \tag{89}$$

Using (89), we get,

$$Tr(\rho L^2(\theta)) = \sum_{i,j} \frac{2}{(\lambda_i + \lambda_j)}\delta\rho_{ij}\delta\rho_{ji}. \tag{90}$$

From (90) and (80), we have,

$$g = 4\chi_F. \tag{91}$$

Now, [11] provides us the relation of capacity of entanglement and Quantum Fisher information such that $C_E = g$, so following (91), we have,

$$C_E = 4\chi_F. \tag{92}$$

Using this we can relate expected Kolmogorov complexity with reduced fidelity susceptibility as

$$\langle \mathscr{C}_k \rangle = -log\Big(\sqrt{\frac{2B}{\pi}}\Big)^{4^K-1} + \frac{2\chi_F}{B}. \tag{93}$$

3.2 Nielsen's approach for circuit complexity and 1st law of circuit complexity

Nielsen provides a geometric approach towards computational complexity [2],[12]. For achieving a target state $|\psi_T\rangle$ from a reference state $|\psi_R\rangle$, $C(|\psi_T\rangle)$ is minimum number of gates required for the unitary transformation. The unitary transformation is given as

$$U(\sigma) = \overleftarrow{P} exp\Big[-i\int_0^\sigma ds\ H(s)\Big], \quad H(s) = \sum_I Y^I(s)O_I. \tag{94}$$

Here, s parameterize the position (or distance) along the trajectory, while $\overleftarrow{P}$ indicates right-to-left path ordering in interpreting the exponential operator. The Hamiltonian $H(s)$ is a linear combination of hermitian operators O_I. The coefficients $Y^I(s)$ are control functions specifying which gates are being applied at a particular point s along the trajectory.

Nielsen's approach to identifying the optimal circuit is to minimize the cost defined as

$$\mathscr{D}(U(\sigma)) = \int_0^1 ds\ F(U(s), Y^I(s)), \tag{95}$$

where F is a local cost function assumed to depend only on the position $U(s)$ and the $Y^I(s)$.

The circuit complexity is then the cost evaluated for the optimal trajectory.

$$C(|\psi_T\rangle) = Min\ \mathscr{D}$$

If we choose x^a as coordinates for the space of unitary operators $U(x^a)$.

$$U(x(\sigma)) = \overleftarrow{P} exp\Big[-i \int_0^\sigma ds H(x(s))\Big], \qquad H(s) = \sum_a \dot{x}^a(s) O_a(x). \tag{96}$$

So the cost becomes, $\mathscr{D} = \int_0^1 ds\ F(x^a(s), \dot{x}^a(s))$, here F is only a function of coordinates x^a and the velocities $\dot{x}^a$. So if we equate this cost function to variational principal of classical mechanics, then we can write Euler-Lagrange equation for this function

$$\frac{\partial F}{\partial x^a} - \frac{\partial}{\partial s}\Big(\frac{\partial F}{\partial \dot{x}^a}\Big) = 0, \tag{97}$$

and

$$C(|\psi_T\rangle) = Min \int_0^1 ds\ F(x^a(s), \dot{x}^a(s)). \tag{98}$$

As suggested by [12], from Nielsen's approach of circuit complexity we can proceed toward the first law of complexity. For this we need to examine

circuit complexity under a small perturbation. For a fixed reference state, if the target state is perturbed from $|\psi_T\rangle$ to $|\psi_T + \delta\psi\rangle$. Then,

$$\delta C = C(|\psi_T + \delta\psi\rangle) - C(|\psi_T\rangle). \tag{99}$$

Let us take $x^a(s,z)$ as geodesic solutions which satisfy the boundary conditions given as

$$x^a(s=0,z) = x_0^a(z), \qquad x^a(s=1,z) = x_1^a(z), \tag{100}$$

here z is a parameter of geodesic family. For a small variation δz around $z=0$ we can write,

$$x^a(s,z) = x^a(s) + \delta x^a, \qquad \delta x^a = v^a(s)\delta z,$$

here, $x^a(s) = x^a(s,z=0)$ and $v^a(s) = \partial_z x^a(s,z)|_{z=0}$. So, change in complexity is written as

$$\delta C = \int_0^1 ds\ [F(x^a(s) + v^a(s)\delta z, \dot{x}^a(s) + \dot{v}^a(s)\delta z) - F(x^a(s), \dot{x}^a(s))]. \tag{101}$$

First order change in complexity can be written as

$$\delta C = \int_0^1 ds\ \Big[\frac{\partial F}{\partial x^a}\delta x^a - \partial_s\Big(\frac{\partial F}{\partial \dot{x}^a}\Big)\delta\dot{x}^a + \partial_s\Big(\frac{\partial F}{\partial \dot{x}^a}\Big)\delta x^a\Big] \tag{102}$$

$$\delta C = \int_0^1 ds\ \Big[\frac{\partial F}{\partial x^a} - \frac{\partial}{\partial s}\Big(\frac{\partial F}{\partial \dot{x}^a}\Big)\Big]v^a(s) + \frac{\partial F}{\partial \dot{x}^a}v^a\Big|_{s=0}^{s=1}. \tag{103}$$

Following (97), change in complexity can be written as

$$\delta C = \frac{\partial F}{\partial \dot{x}^a}v^a\Big|_{s=0}^{s=1} \tag{104}$$

$$= p_a v^a\Big|_{s=1} - p_a v^a\Big|_{s=0}, \tag{105}$$

here $v^a = \delta x^a$ and $p_a = \frac{\partial F}{\partial \dot{x}^a}$. By observing the results here, we can draw an analogy that in the state space, F can be identified as a Lagrangian and p_a

can be identified as its conjugate momentum.

For a fixed reference state we can change in complexity as

$$\delta C = p_a \delta x^a \Big|_{s=1}. \tag{106}$$

According to [12], (106) is referred as first law of circuit complexity.

From the first law of circuit complexity as described above, we can calculate variation in complexity for the Hamiltonian provided in [2]. In [12], the Hamiltonian used for unitary transformation given as

$$H(s) = \sum_I Y^I(s) O_I,$$

and the Hamiltonian used in [2] has the form

$$H = \sum_I J_I \sigma_I$$

Here we observe that the structure of two Hamiltonian is similar for several reasons. Firstly O_I are hermitian operators whereas σ_I are Pauli matrices. J's are coupling constants whereas Y's are control function which have similar role as coupling constants. Second, from [2] we have coupling constant velocity correspondence and in [12], Y's are related to the velocity coordinates of state space. So, we can use (9) for unitary transformation in (94).

Now for calculating variation in circuit complexity, we need a local cost function F, [12] gives several properties of this cost function which are (1) Smoothness, (2) Positivity, (3) Triangle inequality and (4) Positive homogeneity, using them we can take cost function as

$$F(U, Y) = \sum_I |Y^I| \tag{107}$$

$$or\ F(U, J) = \sum_I |J_I|. \tag{108}$$

Since J has a Gaussian distribution, we can take its average value as

$$\langle \sum_I J_I \rangle = \sum_I \int J_I P(J_I) dJ_I \tag{109}$$

$$= (\sqrt{\frac{2B}{\pi}})^{4^K-1} \sum_I \Big(\prod_I \int_0^\infty J_I \; exp(-\frac{1}{2} B J_I^2) dJ_I \Big), \tag{110}$$

solving this integral, we get the local cost function as

$$F = \left(\sqrt{\frac{2}{B\pi}} \right)^{4^K-1} (4^K - 1), \tag{111}$$

$$p_a = \frac{\partial F}{\partial \dot{x}^a} = \Big(\sqrt{\frac{2}{\pi}} \Big)^{4^K-1} (4^K - 1) \frac{(1 - 4^K)}{2B^{\frac{4^K-1}{2}}} \frac{\partial B}{\partial \dot{x}^a} \tag{112}$$

$$\delta C = p_a \frac{\partial x^a(s,z)}{\partial z}\Big|_{s=1} \tag{113}$$

$$\delta C = \Big(\sqrt{\frac{2}{\pi}} \Big)^{4^K-1} (4^K - 1) \frac{(1 - 4^K)}{2B^{\frac{4^K-1}{2}}} \frac{\partial B}{\partial \dot{x}^a} \frac{\partial x^a(s,z)}{\partial z}\Big|_{s=1}. \tag{114}$$

So here we get a relation of variation of circuit complexity for a fixed reference state.

3.3 Search for first law of Kolmogorov complexity

Since variance calculates the deviation of a random variable around some central value or mean so, from (56), we can take change in energy around the mean value to be equal to square root of variance of Hamiltonian or the standard deviation.

$$dE = \sqrt{\langle (\Delta H)^2 \rangle} \tag{115}$$

$$= \sqrt{\langle \sum_I J_I^2 \rangle}. \tag{116}$$

Using,

$$\langle \sum_I J_I^2 \rangle = \sum_I \int_0^\infty J_I^2 P(J) dJ \tag{117}$$

$$= \sum_I \Big(\Big(\frac{2B}{\pi} \Big)^{\frac{4^K-1}{2}} \prod_I \int_0^\infty J_I^2 \; e^{-\frac{1}{2}BJ^2} dJ \Big). \tag{118}$$

Evaluating,

$$\int_0^\infty J_I^2 \; e^{-\frac{1}{2}BJ^2} dJ = \frac{1}{B}\sqrt{\frac{2\pi}{B}}, \tag{119}$$

we have,

$$\langle \sum_I J_I^2 \rangle = (4^K - 1)\Big(\frac{2}{B}\Big)^{4^K-1}. \tag{120}$$

So, from here we can calculate change in energy as square root of (120).

$$dE = \sqrt{4^K - 1}\sqrt{\Big(\frac{2}{B}\Big)^{4^K-1}} \tag{121}$$

Now, if here we use first law of thermodynamics, we can think of getting a relation,

$$dE = TdS - PdV. \tag{122}$$

Shannon's formula for entropy is given as,

$$S = -\int P(J) \; log \; P(J) \; dJ. \tag{123}$$

If we take continuous J function then by definition of expected Kolmogorov complexity, and taking B as inverse temperature,

$$TdS = \frac{1}{B}\delta \mathscr{C}_k \tag{124}$$

$$= -\frac{(4^K - 1)}{2B^2} + \frac{\langle (\Delta H)^2 \rangle_B}{2B} + \frac{1}{2}\frac{\partial}{\partial B}\langle (\Delta H)^2 \rangle_B. \tag{125}$$

So, from (124) and (121), we can calculate PdV as

$$\begin{aligned} PdV &= TdS - dE \qquad (126) \\ &= -\frac{(4^K-1)}{2B^2} + \frac{\langle(\Delta H)^2\rangle_B}{2B} + \frac{1}{2}\frac{\partial}{\partial B}\langle(\Delta H)^2\rangle_B - \sqrt{4^K-1}\sqrt{\left(\frac{2}{B}\right)^{4^K-1}} \qquad (127) \end{aligned}$$

Here, we have tried to look for a relation of first law of Kolmogorov complexity by using first law of thermodynamics. We can further work on this by understanding the physical significance of each term in (127).

Chapter 4: Result and discussion

In this section we will try to compile all the results which we obtained during our whole work process. These results broadly describe the change in complexity (Kolmogorov as well as computational). An important result which we got is expression of expected Kolmogorov complexity (55) which we can think of as absolute value of Kolmogorov complexity, using this we also tried to calculate infinitesimal change around this absolute value with the help of Taylor expansion (73). Another important calculation which revolve around expected Kolmogorov complexity is establishing its relation with capacity of entanglement and fidelity susceptibility which is a important result in the point of view of application of complexity in the field of gravitational physics as these quantities have established role in bulk theory. Another thing in regard of Kolmogorov complexity is the search of its first law on the lines of first law of thermodynamics which we think is a very interesting agenda that we tried to touch. This is a very new approach regarding first law of circuit complexity. In this approach, first we defined change in energy around the mean value to be equal to square root of variance of Hamiltonian. Then we used Hamiltonian (9) and the definition of expected Kolmogorov complexity (15) to relate first law of thermodynamics and Kolmogorov complexity. Further progress can be made in this regard as it is a very interesting and new approach. In the regime of circuit complexity first we studied Nielsen's idea of circuit complexity and then calculated its first law by taking SYK model Hamiltonian (114). In between these, we also tried to calculate capacity of entanglement for a modular Hamiltonian K for two different perturbations (29), (35).

As we mentioned earlier that Quantum complexity is an interesting and a new

field. Much work has been done and a lot more needs to be done in understanding this emerging field. We hope that this work will also play a small role in this regard.

Chapter 5: Conclusion and Scope for future work

So again we want to emphasis that till now we have calculated expected Kolmogorov Complexity, its relation with Fidelity susceptibility and capacity of entanglement and also change in Kolmogorov complexity for an infinitesimal change using Taylor expansion. In the domain of circuit complexity we calculated variation in circuit complexity for a particular Quantum Hamiltonian and we also tried to find a relation kolmogorov complexity with first law of thermodynamics. From here one thing which can be done is to relate these two complexities. There are a few works which are done in this direction like [13] from a different point of view. As we have used same SYK model Hamiltonian for calculating change in circuit complexity as well as Kolmogorov complexity. So an attempt can be made to compare these two and understand the relation between these two complexities. Another important aspect which is visible from the present calculation is to calculate variation of Kolmogorov complexity on the lines of first law of circuit complexity. This aspect can also be used to draw a relation between these two complexities. Use of complexity for understanding gravity is also an important area of research. In AdS/CFT, there are certain field theories which are dual to gravity in AdS. A lot of recent work have been done by assuming field theory complexity is volume in AdS. Another important aspect is time evolution of complexity. As we studied earlier that using a k-local Hamiltonian for a time development operator, we have a time evolution of computational complexity (10). We can study time evolution of complexity for a time dependent Hamiltonian. For a time dependent Hamiltonian we can calculate the explicit expression of time evolution of complexity.

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