APPLICATIONS OF SEMICLASSICAL GRAVITY IN BLACK HOLE PHYSICS

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By

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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **"APPLICATIONS OF SEMICLASSICAL GRAVITY IN BLACK HOLE PHYSICS"** in the partial fulfilment of the requirements for the award of the degree of **MASTER OF SCIENCE** and submitted in the **DEPARTMENT OF PHYSICS, Indian Institute of Technology Indore**, is an authentic record of my own work carried out during the time period from July 2019 to June 2021 under the supervision of Dr. Debajyoti Sarkar, Assistant Professor, Department of Physics, IIT Indore.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other institute.

Mathew
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This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

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Mathew Joshy

Dedicated to my

mom and dad

Abstract

The work done in this note falls under the regime of semiclassical gravity. Semiclassical gravity involves the description of the way in which classical gravitational fields interacts with matter fields which are quantized. It is in fact a model, which is an attempt for finding a unification between general relativity and quantum mechanics, which is one of the ultimate goals of physics. The prediction of black holes and their characteristic horizons were one of the most incredible contribution of Einstein's general theory of relativity. But when we analyze their behaviour in the framework of semiclassical gravity, interesting scenarios might come up. For instance, classical solutions are no longer solutions and wormholes arise. In this thesis, I calculate the sub-leading terms in the classical black hole solution and check whether the classical solution with subleading terms can be considered as a solution to the semiclassical gravitational equations. I have checked 1-loop effective action does yield a solution. A numerical approach has been made to look for possible solutions that satisfy the semiclassical equations. Also, a numerical solution for the exact semiclassical equations has been found. I have discussed open questions at the end.

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Chapter 1

Introduction

1.1 The Schwarzschild solution

Einstein's General Theory of Relativity is considered as one of the greatest scientific accomplishments we have ever made. It completely reformulated our concept of gravity. According to Newton, gravity was nothing but a force. But through his field equations in General Relativity, Einstein proved that gravity is a property of the spacetime itself. However, in the weak-field limit and when the speeds of objects are very small compared to the speed of light, it will naturally reduce to Newtonian gravity. The Einstein field equation is a tensorial equation, and is given as

$$G_{\mu\nu} = \frac{8\pi G_N}{c^4} T_{\mu\nu} \tag{1.1}$$

where $G_{\mu\nu}$ is known as the Einstein tensor and is given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \tag{1.2}$$

where $R_{\mu\nu}$, R, $g_{\mu\nu}$ and $T_{\mu\nu}$ stands for Ricci tensor, Ricci scalar, metric tensor and energy-momentum tensor respectively. G_N is the Newtonian gravitational constant and c is speed of light. In this note, all the computations are done in units where $c = 1$.

A static and spherically symmetric vacuum solution of Einstein's field equations is given by the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2 \tag{1.3}$$

Here the last two terms give a 2-sphere S^2. M is the mass of the spherical body and r, θ, ϕ are the usual spherical coordinates. In fact, it is the only black hole solution which is spherically symmetric and static and are solutions to vacuum Einstein's equation. The Schwarzschild black hole is described by this metric. The black hole is characterized by a surrounding spherical boundary, called the event horizon, with horizon radius given by $r_h = \frac{2GM}{c^2}$. Horizons are typical signature of black holes. There are no causal connections between the inside and outside of the black hole horizon. Inside the horizon, the escape velocity for an object is greater than the speed of light. At the horizon, the time component g_{tt} in (1.3) goes to zero. A horizon must be there, to protect an otherwise naked sigularity [1].

1.2 Semiclassical gravity

In the semiclassical picture, we consider the gravitational field as classical, but the matter fields as quantized [2]. This treatment gives us the modified Einstein equations, which may be called semiclassical equations of motion. The existence of black holes is no longer evident once we take the quantum modifications into consideration. No-horizon scenarios might arise. The back reaction of matter fields could lead to new solutions that might possess significantly different properties compared to classical solution. Two important observations regarding semiclassical black holes in 4-D spacetime were made in [3]

1) A leading order static spherically symmetric classical solution is not a solution in semiclassical theory.

2) In semiclassical gravity, a minimal 2-sphere is not a horizon, rather it is a throat of a wormhole.
Some other results on semiclassical black holes can be found in [4–11]

In this note, I have done the analysis of the next to leading order terms in the semiclassical equations. It is in principle possible that addition of subleading terms somehow makes the classical solutions obey the semiclassical equations of motion. It's seen that for general field content of scalar, spin-$\frac{1}{2}$ and spin-1 fields, even the classical solution with sub-leading terms is not a solution in semiclassical gravity. So it is not an artifact of considering near-horizon expansion of classical spacetime. However, it can be observed that for a fraction of above fields, a black hole solution might exist. We look for possible solutions to the modified gravitational equations, and see that in fact there exists a solution, which is satisfied by all the equtions. To be precise, the solution should take the classical contribution also into account and we expect it to be of the form $\Omega = \Omega_{class} + \Omega_{semi-class}$. Here Ω^2 is the time component of the metric. Ω_{class} is the classical solution and $\Omega_{semi-class}$ is the semiclassical correction. This means that when $\Omega_{class} = 0$ (i.e, in the would be classical horizon case), g_{tt} is still $\neq 0$. In fact $\Omega_{semi-class}$ turns out to be a positive quantity. This implies a wormhole [1].

In chapter 2, I have done a review of the theoretical background of my project. In section 3.1, we study subleading terms in the classical solution and analyze the corresponding modifications to the semiclassical equations. In section 3.2, we make a numerical approach to study the existence of possible solutions to the semiclassical equations. An exact numerical solution for the equations has been found in section 3.3. Finally in 4, we conclude the results and discuss open questions. Some additional content including equations can be found in appendix A.

Chapter 2

Theoretical Background

2.1 Horizons in General relativity

There are two fundamental features of classical horizons. Firstly, they are surfaces in a static and spherically symmetric metric, on which the time component of the metric vanishes. Secondly, in GR, a minimal 2-sphere embedded in a 4-D static spacetime is a horizon.

Consider a static spherically symmetric metric with the general form

$$\begin{aligned} ds^2 &= \Omega^2(z) g_{\mu\nu} dx^\mu dx^\nu \\ &= \Omega^2(z) \left(dt^2 + N^2(z) dz^2 + R^2(z)(d\theta^2 + sin^2\theta d\phi^2) \right) \end{aligned} \tag{2.1}$$

We work in the Euclidean signatue so that the notion of black hole temperature and entropies are well defined. Euclideanization gives rise to black hole temperature [12]. Now, the geometric radius of a 2-sphere is given by $r(z) = R(z)\Omega(z)$. Consider the gauge $N(z) = 1$. Assuming there exists a horizon at $r = r_h$ with a finite temperature $T = 1/\beta$, we get the near horizon behavior, comparing it with the near horizon limit of (1.3)

$$\Omega(z) = e^{\frac{-2\pi z}{\beta}} + ..., \quad R(z) = r_h e^{\frac{2\pi z}{\beta}} + ... \tag{2.2}$$

where ... stands for the subleading terms. Here z is the new radial coordinate. The relation between z and the previous radial coordinate r is given by

$$z = -\frac{\beta}{4\pi} \ln(r - r_h) \tag{2.3}$$

In the z coordinate, at $r = r_h$, we have $z = \infty$. Also, we can see from (2.2) that $\Omega(z)$ goes to zero at $z = \infty$, which is nothing but the horizon condition.

By varying the Einstein-Hilbert action with respect to $\Omega(z)$ and $N(z)$, we obtain the equations of motion

$$2rr'' + r'^2 - 1 = 0$$

$$\Omega(r'^2 - 1) + 2rr'\Omega' = 0 \tag{2.4}$$

Here we have used the gauge $N(z) = 1/\Omega(z)$. Now, suppose that the 2-sphere at $z = \rho = \rho_h$ is a surface of minimal area, i.e. $r' = 0$ at $\rho = \rho_h$. It readily follows from (2.4) that $\Omega(\rho_h)$ vanishes and hence the surface is a horizon, since $\Omega = 0$ means that the time component of the metric $G_{\mu\nu} = \Omega^2 g_{\mu\nu}$ is zero. i.e. $G_{tt} = 0$

Now we may analyze the same aspects in the framework of semiclassical gravity.

2.2 The semiclassical treatment

2.2.1 Semiclassical equations of motion

The semiclassical gravitational action is obtained by adding a quantum effective action $\Gamma(G)$ to the Einstein-Hilbert action $W_{EH}(G)$. Writing the metric (2.1) in the form $G_{\mu\nu} = e^{2\sigma} g_{\mu\nu}$, the semiclassical gravitational action $W_{grav} = W_{EH}(G) + \Gamma[G]$ is given by [2, 13, 14]

$$\begin{aligned} W_{grav} =& -\frac{1}{2\kappa}\int d^4x\sqrt{G}R(G) - \frac{a}{(4\pi)^2}\int d^4x\sqrt{g}\sigma C^2 \\ & -\frac{2b}{(4\pi)^2}\int d^4x\sqrt{g}\Big(2E^{\mu\nu}\nabla_\mu\sigma\nabla_\nu\sigma + 2\Box\sigma\nabla_\mu\sigma\nabla^\mu\sigma \\ & + (\nabla_\mu\sigma\nabla^\mu\sigma)^2\Big) + \frac{b}{(4\pi)^2}\int d^4x\sqrt{g}\sigma E + \Gamma_0[g_{\mu\nu}] \end{aligned} \tag{2.5}$$

Here $\kappa = 8\pi G_N$ is the classical gravitational coupling, $E_{\mu\nu}$ is the Einstein tensor corresponding to the metric g, E is the Euler density and C is the weyl tensor and these are also with respect to the metric g. Also a and b are given by

$$a = \frac{n_0}{120} + \frac{n_{1/2}}{20} + \frac{n_1}{10} \quad , \quad b = \frac{n_0}{360} + \frac{11n_{1/2}}{360} + \frac{31n_1}{180} \tag{2.6}$$

where n_s is the number of fields of spin s. $\Gamma_0[g_{\mu\nu}]$ is the quantum effective action computed on the optical metric $g_{\mu\nu}$. After calculating W_{grav} for our metric (2.1) and varying the resulting action with respect to $\sigma(z)$ and $N(z)$, we get the semiclassical equations of motion given by

$$\begin{aligned} 0 =& \frac{2e^{2\sigma}}{\kappa}\Bigg[\frac{2RR''}{N} + \frac{6RR'\sigma'}{N} - \frac{2RR'N'}{N^2} + \frac{R'^2}{N} + \frac{3R^2\sigma''}{N} - \frac{3R^2\sigma'N'}{N^2} \\ & + \frac{3R^2\sigma'^2}{N} - N\Bigg] + \frac{a}{6\pi^2}\Bigg[-\frac{R''}{RN} - \frac{R''^2}{2N^3} - \frac{R'^3N'}{RN^4} - \frac{R'^2N'^2}{2N^5} \\ & + \frac{R'N'}{RN^2} - \frac{R'^4}{2R^2N^3} + \frac{R'^2}{R^2N} + \frac{R'R''N'}{N^4} + \frac{R'^2R''}{RN^3} - \frac{N}{2R^2}\Bigg] \\ & + \frac{b}{\pi^2N^4}\Bigg[RNR''\sigma'^2 + \frac{1}{2}NR'^2\sigma'' - 3RR'\sigma'^2N' - \frac{3}{2}R'^2\sigma'N' \\ & + RNR'\sigma'^3 + NR'^2\sigma'^2 + 2RNR'\sigma'\sigma'' + NR'R''\sigma' - \frac{3}{2}R^2\sigma'^3N' \\ & + \frac{3}{2}R^2N\sigma'^2\sigma'' - \frac{1}{2}N^3\sigma'' + \frac{1}{2}N^2\sigma'N'\Bigg] \end{aligned} \tag{2.7}$$

$$
\begin{aligned}
0 =& \frac{e^{2\sigma}}{\kappa N^2}\Big[(R' + R\sigma')(R' + 3R\sigma') - N^2\Big] \\
&+ \frac{b\sigma'^2}{8\pi^2 N^4}\Big[-2N^2 + 8RR'\sigma' + 6R'^2 + 3R^2\sigma'^2\Big] \\
&+ \frac{a}{12\pi^2 R^2}\bigg[\frac{R^2\sigma R''^2}{N^4} + \frac{2R^2R'^2\sigma'N'}{N^5} + \frac{2RR'^3\sigma'}{N^4} - \frac{2RR'\sigma'}{N^2} \\
&+ \frac{2R^2\sigma R'^2N''}{N^5} - \frac{5R^2\sigma R'^2N'^2}{N^6} + \frac{\sigma R'^4}{N^4} - \frac{2R^2R'''\sigma R'}{N^4} \\
&- \frac{2R^2R'R''\sigma'}{N^4} + \frac{4R^2\sigma R'R''N'}{N^5} - \sigma\bigg] + \frac{1}{4\pi\beta}\delta_N\Gamma_0
\end{aligned}
\tag{2.8}
$$

The effective action Γ_0 is given by [14–17]

$$
(4\pi\beta)^{-1}\delta_N\Gamma_0 = -\frac{\pi^2 c_H}{90\beta^4}R^2(z) - \frac{\lambda_H}{72\beta^2}\left(R'^2N^{-2} - 1\right) \tag{2.9}
$$

where $c_H = n_0 + \frac{7}{2}n_{1/2} + 2n_1$ and $\lambda_H = n_{1/2} + 4n_1$. (2.9) should be used in equation of motion (2.8).

2.2.2 Leading order classical solution for semiclassical equations

Now, we proceed to check whether a static black hole solution satisfy the semiclassical equations. We use the gauge $N(z) = 1$. To leading order, we have the near horizon behaviour of $\sigma(z)$ and $R(z)$ as

$$
\sigma(z) = -\frac{2\pi z}{\beta} + ..., \qquad R(z) = r_h e^{\frac{2\pi z}{\beta}} + ... \tag{2.10}
$$

We use (2.10) in the equations of motion (2.7) and (2.8), and they respectively give

$$
\sigma \text{ variation:} \quad 0 = \mathcal{O}\left(e^{-\frac{4\pi z}{\beta}}\right) \tag{2.11}
$$

$$
N \text{ variation:} \quad 0 = -\left(n_0 + 6n_{1/2} - 18n_1\right)\frac{\pi^2}{180\beta^4}R^2(z) \tag{2.12}
$$

Here (2.11) is satisfied to leading order. But the RHS of (2.12) contains a divergent term. The divergent term may go to zero for a particular field content, e.g. if $n_0 = 6$, $n_{1/2} = 2$ and $n_1 = 1$. But in general, the divergent term is non-vanishing. Thus (2.12) is not satisfied in general. Therefore we can see that a static spherically symmetric metric with a finite temperature horizon is not a solution in semiclasssical gravity.

2.2.3 Horizons in semiclassical gravity

We proceed to check whether a minimal sphere is necessarily a horizon in the theory of semiclassical gravity. We use the gauge $N(z) = 1/\Omega(z)$. Two minimality conditions $r'(\rho_h) = 0$ and $\Omega'(\rho_h) = 0$ are imposed at the turning point $\rho = \rho_h$. Now (2.7) and (2.8) will respectively become

$$\frac{2\Omega}{\kappa}(1 - 2rr'' - r^2\frac{\Omega''}{\Omega}) + \frac{\bar{a}}{r^2\Omega}(\Omega + \Omega rr'' - r^2\Omega'')^2 + \bar{b}\Omega'' = 0 \quad (2.13)$$

$$-\frac{\Omega^2}{\kappa} - \frac{\bar{a}}{r^2}\ln\Omega^{-1}[(\Omega rr'' - r^2\Omega'')^2 - \Omega^2] - \frac{\gamma r^2}{\beta^4\Omega^2} + \frac{\lambda}{\beta^2} = 0 \quad (2.14)$$

where $\bar{a} = \frac{a}{12\pi^2}$, $\bar{b} = \frac{b}{2\pi^2}$, $\gamma = \frac{c_H\pi^2}{90}$, $\lambda = \frac{\lambda_H}{72}$. For simplicity, we consider the special case $\lambda = 0$ (only scalar fields are present). Further analysis of (2.13) and (2.14) gives a bound on Ω given by

$$\Omega < \Omega_0 = e^{-\frac{r^2}{\bar{a}\kappa}} \sim e^{-S_{BH}} \quad (2.15)$$

(2.15) shows that the value of Ω at the turning point is bounded by the exponential of negative of the Bekenstein-Hawking Entropy S_{BH}. The value of temperature is also bounded and is given by

$$T^4 < \frac{\bar{a}\Omega_0^4}{4\gamma r^4} \quad (2.16)$$

Then it follows from these results that the minimal 2-sphere is not a horizon as in the classical theory, but a wormhole.

Chapter 3

Results and Discussions

3.1 Subleading terms in classical solution

In the results that are mentioned in 2 , the entire calculations are based on the leading order terms and all the subleading terms have been neglected (.... in (2.2) indicate that). It is a matter of great interest to check whether a classical solution can be considered as a solution to semiclassical equations, by also taking the subleading terms in the classical solution into account. Now, the semiclassical equations will contain additional subleading terms and we may look for any further modifications to the results that arise due to the presence of additional terms. Consider the general metric

$$ds^2 = g(r)dt^2 + \frac{dr^2}{g(r)} + R^2(r)d\Omega_2^2 \tag{3.1}$$

where $d\Omega_2^2$ is the metric of a 2-sphere S^2. In our analysis, we are interested in the near horizon behaviour of the metric. So we Taylor expand $g(r)$ around the horizon radius $r = r_h$ to get

$$g(r) = \frac{4\pi}{\beta}(r - r_h) + \frac{g''(r_h)}{2}(r - r_h)^2 \tag{3.2}$$

where we have used $g(r_h) = 0$ which is nothing but the horizon condition. Also, from [12] we have $g'(r_h) = \frac{4\pi}{\beta}$, where β is the inverse of horizon temperature. $g''(r_h)$ is an unknown variable which we want to compute here.

Our next aim is to write the metric (3.1) in the form

$$ds^2 = e^{2\sigma}\left(dt^2 + dz^2 + R^2(z)d\Omega_2^2\right) \tag{3.3}$$

so that it is of the form (2.1) in $N = 1$ gauge.

The new radial coordinate $z = -\int \frac{dr}{g(r)}$ can be found to be

$$z = -\frac{\beta}{4\pi}\ln\left[\frac{r - r_h}{\frac{8\pi}{\beta} + (r - r_h)g''(r_h)}\right] \tag{3.4}$$

At $r = r_h$, z goes to infinity. Therefore in our new coordinate, the horizon is at $z \to \infty$. (3.4) can be rearranged to get

$$r - r_h = \frac{8\pi}{\beta\left(e^{\frac{4\pi z}{\beta}} - g''(r_h)\right)} \tag{3.5}$$

Comparing (3.1) and (3.3), we have

$$\sigma(z) = \frac{1}{2}\ln g(r) \tag{3.6}$$

where $g(r)$ is given by (3.2). Now substituting for $(r - r_h)$ from (3.5), σ is found to be

$$\sigma(z) = -\frac{2\pi z}{\beta} - \ln\left(1 - g'' e^{-\frac{4\pi z}{\beta}}\right) \tag{3.7}$$

Also from (3.1) and (3.3) we have

$$R^2(r) = e^{2\sigma}R^2(z) \tag{3.8}$$

Calculating $R(z)$ we get

$$R(z) = r_h e^{\frac{2\pi z}{\beta}} - r_h g'' e^{-\frac{2\pi z}{\beta}} \tag{3.9}$$

where we have used $R^2(r) = r_h^2$. By taking $g'' = 0$, equations (3.7) and (3.9) gives us back (2.10). Now we proceed to check the validity of the semiclassical equations of motions, when $\sigma(z)$ and $R(z)$ are given by (3.7) and (3.9) respectively. We have chosen the gauge $N=1$. The first equation (2.7) now becomes

$$\begin{aligned}
0 = & \frac{e^{\frac{4\pi z}{\beta}}}{\left(e^{\frac{4\pi z}{\beta}}-g''\right)^2}\Bigg[\frac{1}{1440\beta^4\kappa}(46080g''r_h^2\pi^2\beta^2-2880\beta^4) \\
& -\frac{g''}{9\beta^2}(n_{1/2}+10n_1)-\frac{1}{1440r_h^2\pi^2\beta^4}\Big(256g''^2r_h^4\pi^4(n_0 \\
& +6n^{1/2}+12n_1)+\beta^4(n_0+6n_{1/2}+12n_1)\Big)\Bigg]
\end{aligned} \tag{3.10}$$

Analysis of the second equation of motion (2.8) gives

$$\begin{aligned}
0 = & -\frac{e^{\frac{12\pi z}{\beta}}}{\left(e^{\frac{4\pi z}{\beta}}-g''\right)^2}\frac{r_h^2\pi^2}{180\beta^4}\Big[n_0+6n_{1/2}-18n_1\Big] \\
& +\frac{e^{\frac{8\pi z}{\beta}}}{\left(e^{\frac{4\pi z}{\beta}}-g''\right)^2}\Bigg[\frac{1}{360\beta^2}(n_0+6n_{1/2}-18n_1)-\frac{g''r_h^2\pi^2}{45\beta^4}(n_0 \\
& +6n_{1/2}-18n_1)\Bigg]-\frac{g''^3r_h^2\pi^2}{\left(e^{\frac{4\pi z}{\beta}}-g''\right)^2\beta^4}\Big[n_0+6n_{1/2}-18n_1\Big] \\
& +\frac{g''^2}{360\beta^2\left(e^{\frac{4\pi z}{\beta}}-g''\right)^2}\Big[n_0+6n_{1/2}-18n_1\Big] \\
& -\frac{g''^4r_h^2\pi^2}{180\beta^4}\frac{e^{-\frac{4\pi z}{\beta}}}{\left(e^{\frac{4\pi z}{\beta}}-g''\right)^2}\Big[n_0+6n_{1/2}-18n_1\Big] \\
& +\frac{e^{\frac{4\pi z}{\beta}}}{\left(e^{\frac{4\pi z}{\beta}}-g''\right)^2}\Bigg[\frac{\left(n_0+6n_{1/2}+12n_1\right)}{1440r_h^2\pi^2}\ln\left(1-g''e^{-\frac{4\pi z}{\beta}}\right) \\
& +\frac{z\left(n_0+6n_{1/2}+12n_1\right)}{720r_h^2\pi\beta} \\
& +\frac{g''}{180\beta^2}\left(n_0-4n_{1/2}-58n_1\right)-\frac{g''^2r_h^2\pi^2}{1440\beta^4}\Bigg(16(19n_0+74n_{1/2} \\
& -22n_1)+256(n_0+6n_{1/2}+12n_1)\ln\left(1-g''e^{-\frac{4\pi z}{\beta}}\right)\Bigg) \\
& -\frac{16zg''^2r_h^2\pi^3}{45\beta^5}(n_0+6n_{1/2}+12n_1)\Bigg]
\end{aligned} \tag{3.11}$$

Now let's analyze these semiclassical equations of motion. As $z \to \infty$ (near horizon), (3.10) gives

$$0 = \mathcal{O}(e^{-\frac{4\pi z}{\beta}}) \tag{3.12}$$

Thus near the horizon, the equation of motion resulting from the variation with respect to σ is still satisfied, as shown in [3].

Looking at the near horizon behaviour of equation (3.11), the first term in the RHS of (3.11) is divergent ($\mathcal{O}(e^{\frac{4\pi z}{\beta}})$), while the second term will be a constant term. All the other terms will go to zero near the horizon. Clearly, since the RHS$\neq 0$, the equation is not satisfied. Just out of interest, we may consider the case when the divergent term and the constant term in the RHS of (3.11) cancel out. i.e, the RHS goes to zero so that (3.11) is satisfied. In this case, we see that $g''(r_h)$ is given by

$$g''(r_h) = \frac{45\beta^2}{r_h^2\pi^2} - \frac{e^{\frac{4\pi z}{\beta}}}{4} \tag{3.13}$$

which negatively diverges near the horizon, but we expect it to be a finite positive quantity. In conclusion, we can say that even if we take the subleading terms in (2.2) into account, it still remains that a static spherically symmetric metric with finite temperature horizon is not a solution in semi-classical gravity.

It was suggested in [3] that for a particular set of fields, the divergent terms in the RHS of $\mathcal{N}$ variation equation will vanish. For instance, if $n_0 = 6$, $n_{1/2} = 2$ and $n_1 = 1$, the equation was found to be satisfied. Surprisingly, this is the multiplet of $\mathcal{N} = 4$ super-Yang-Mills theory in 4 dimensions. It could mean that in this case, a black hole solution might exist. But the demonstration of it needed the analysis of subleading terms in the semi-classical equations, which is precisely what has been done in the above calculations. Looking at the RHS of (3.11), it is clear that for the particular case of $n_0 = 6$,

$n_{1/2} = 2$ and $n_1 = 1$, the divergent term will vanish. In addition, the second term which is a constant term also goes to zero, for this particular field content. All the remaining terms obviously go to zero in the near horizon limit. Therefore, it is clear that there could be a black hole solution for this particular interesting case and that the divergences will keep on vanishing for this field content for higher and higher order terms.

3.2 A numerical approach to study the existence of solution

Here I use a numerical approach to look for possible solutions to the semiclassical equations of motions. We find certain bounds from the equations and later plot them to analyze the nature of possible solutions.

Equations (2.13) and (2.14) can be rewritten respectively as

$$\frac{a'}{3}\left(1 + rr'' - r^2\frac{\Omega''}{\Omega}\right)^2 + b'\frac{r^2\Omega''}{\Omega} = -\frac{2r^2}{\kappa}\left(1 - 2rr'' - r^2\frac{\Omega''}{\Omega}\right) \tag{3.14}$$

and

$$\frac{c'_H}{\Omega^2 r^2} + \frac{\Omega^2}{\kappa} = -\frac{a'}{3r^2}\log\Omega^{-1}\left[(\Omega rr'' - r^2\Omega'')^2 - \Omega^2\right] \tag{3.15}$$

where we have assumed $\lambda = 0$ (sclar fields only, i.e only $n_0 \neq 0$) and $\beta^2 = \gamma r^2$. Also $a' = \frac{a}{4\pi^2}$, $b' = \frac{b}{2\pi^2}$ and $c'_H = \frac{c_H\pi^2}{90\gamma^2}$.

We expect $r'' > 0$ and $0 < \Omega < 1$. Now, irrespective of whether Ω'' is positive or negative, the LHS of (3.14) is always is positive. So, the RHS of (3.14) also has to be +ve. Now we can consider two cases. First we assume $\Omega'' < 0$ and get

$$1 - 2rr'' + r^2\frac{|\Omega''|}{\Omega} < 0 \tag{3.16}$$

or

$$1 - 2\tilde{P} + \tilde{f} < 0 \tag{3.17}$$

where $\tilde{P} = rr''$ and $\tilde{f} = r^2 \frac{|\Omega''|}{\Omega}$. Also (3.15) gives

$$\left(\tilde{P} + \tilde{f}\right)^2 < 1 \tag{3.18}$$

Below we plot above inequalities (3.17) and (3.18).

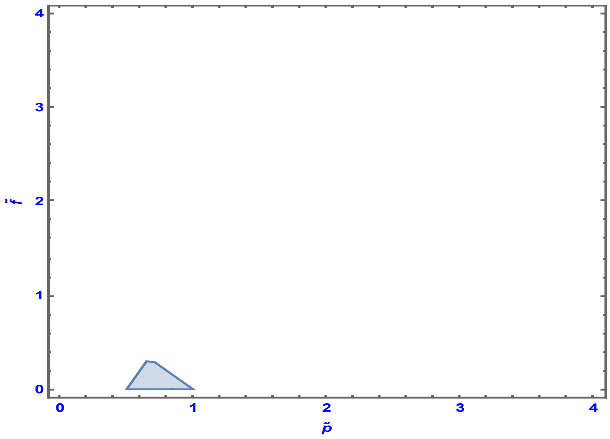

Figure 3.1: Plot of (3.17) together with the condition (3.18)

We can see that the allowed values of $\tilde{P}$ and $\tilde{f}$ appear to be really constrained. Now suppose we consider the ansatz $\Omega = e^{-r^2}$, for which $\Omega'' < 0$. This ansatz is inspired from the bound that we found in (2.15). In this case, we get 2 relations

$$1 - 2\tilde{P} + 2r^2\tilde{P} < 0 \quad \text{and} \quad \left(\tilde{P} + 2r^2\tilde{P}\right)^2 < 1 \tag{3.19}$$

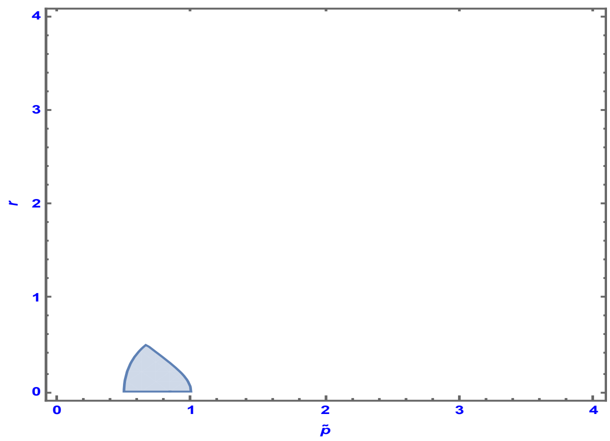

Figure 3.2: Plot of inequalities (3.19)

From figure 3.2, we can see that the allowed values of r are really small. But in reality, r could be arbitrarily large. So we can say that the semiclassical correction alone is not a good ansatz that could possibly serve as a solution to the semiclassical equations. Rather, we also have to take the classical part into account. Also, we can rule out the possibility of Ω'' being negative. Now lets consider the case when $\Omega'' > 0$. Here from (3.14) and (3.15) we have

$$1 - 2\tilde{P} - \tilde{f} < 0 \quad \text{and} \quad \left(\tilde{P} - \tilde{f}\right)^2 < 1 \tag{3.20}$$

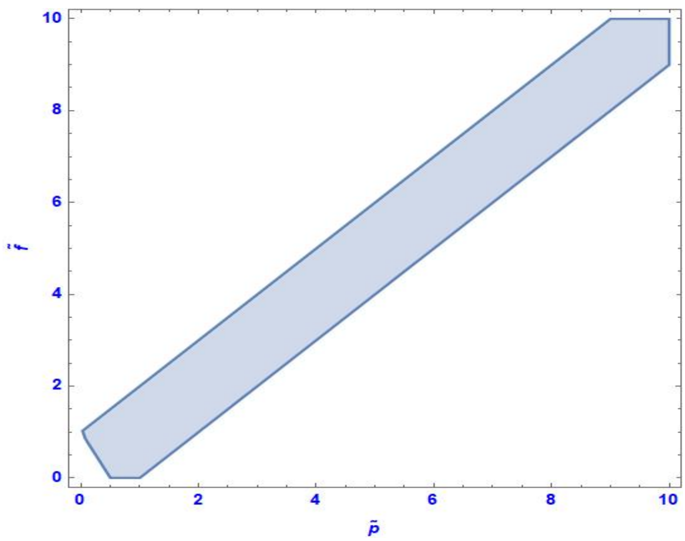

Figure 3.3: Plot of inequalities (3.20)

Figure 3.3 shows that the values of $\tilde{P}$ and $\tilde{f}$ are not limited, as it was observed in the $\Omega'' < 0$ case. So we come to the solid conclusion that Ω'' has to be a positive quantity.

We have seen that the semiclassical correction for Ω alone doesn't serve as a good ansatz. Then a reasonable expectation for a possible ansatz for Ω is of the form

$$\Omega = \Omega_{class} + \Omega_{semi-class} \tag{3.21}$$

We consider the ansatz

$$\Omega = \sqrt{\frac{4\pi}{\beta}(r - r_h)} + e^{-kr^2} \tag{3.22}$$

where the first term is the classical ansatz. For simplicity, we are using $k = 1$ unit. Now for this Ω, we have

$$\Omega'' = -2e^{-r^2} r r'' + \frac{\sqrt{\pi} r''}{\beta \sqrt{\frac{r-r_h}{\beta}}} \tag{3.23}$$

Ω'' is found to be positive for large r. Now in place of (3.20), we have the relations

$$1 - 2x''(x + r_h) - \frac{x''(x + r_h)^2 \left(-2e^{-(x+r_h)^2}(x + r_h) + \frac{1}{2\sqrt{x}}\right)}{e^{-(x+r_h)^2} + \sqrt{x}} < 0 \tag{3.24}$$

$$\left((x + r_h)x'' - \frac{x''(x + r_h)^2 \left(-2e^{-(x+r_h)^2}(x + r_h) + \frac{1}{2\sqrt{x}}\right)}{e^{-(x+r_h)^2} + \sqrt{x}}\right)^2 < 1 \tag{3.25}$$

where we have used $\beta = 4\pi$. Also we introduced a new variable $x = r - r_h$, as we want to study near-horizon behavior. Clearly, for a particular r_h, x is a measure of the radial distance from horizon. Also note that $x'' = r''$. Below we plot above relations between x and x'', for different values of r_h.

$\underline{r_h = 10}$

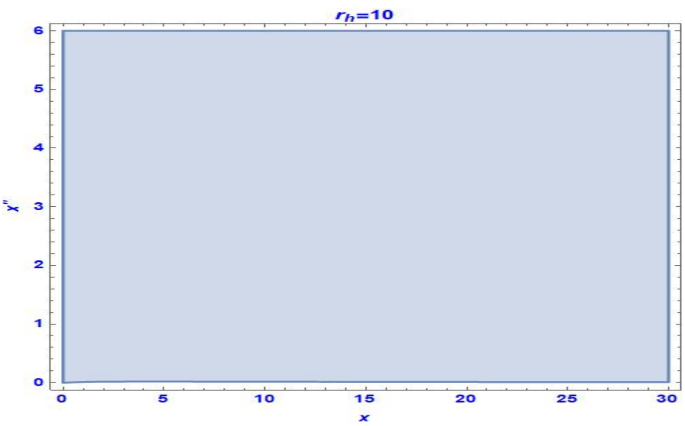

Figure 3.4: Plot of condition (3.24) for $r_h = 10$

Figure 3.4 shows relation (3.24) holds for entire range of x and x''.

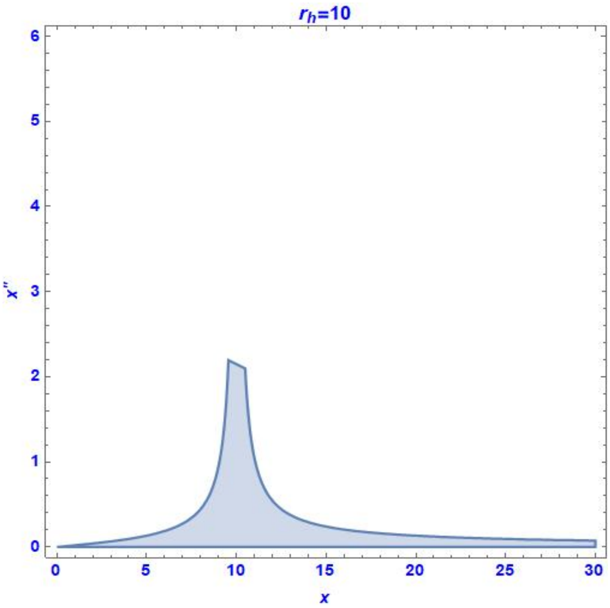

Figure 3.5: Plot of condition (3.25) for $r_h = 10$

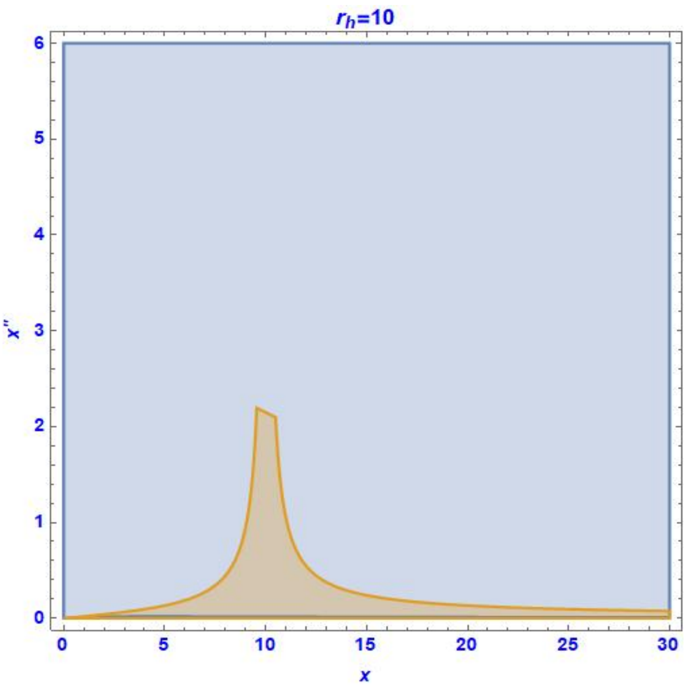

Figure 3.6: Overlap between inequalities (3.24) and (3.25) for $r_h = 10$

$\underline{r_h = 20}$

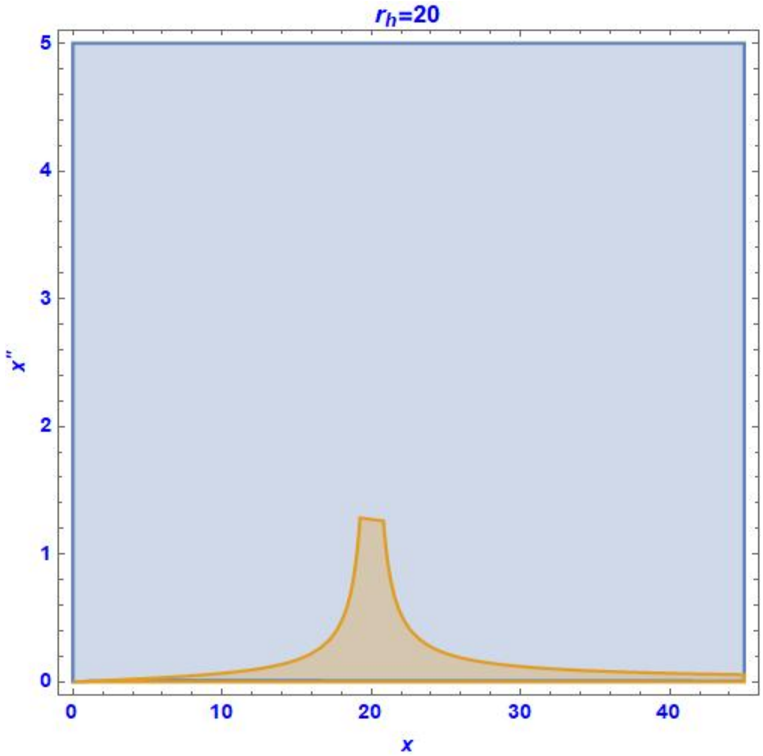

Figure 3.7: Overlap between inequalities (3.24) and (3.25) for $r_h = 20$

$\underline{r_h = 30}$

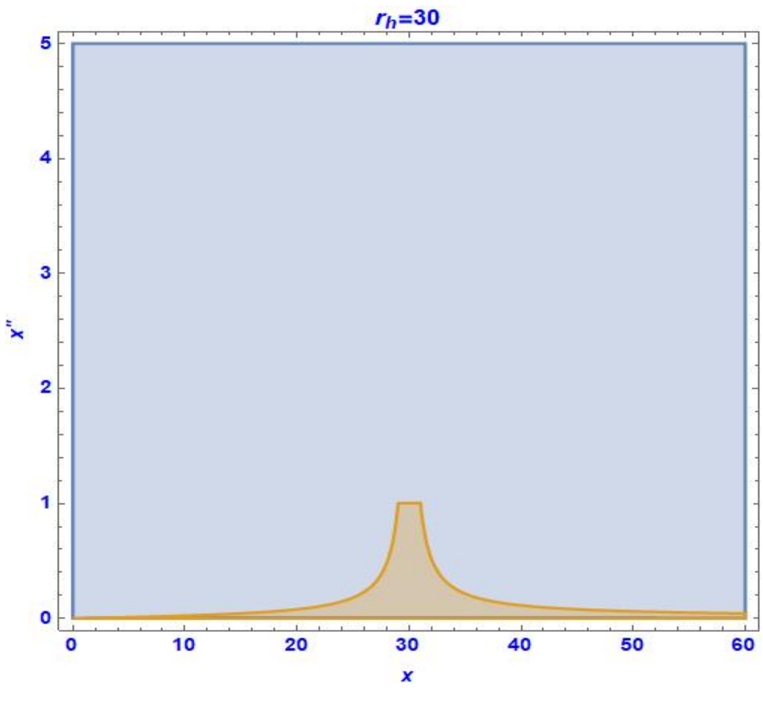

Figure 3.8: Overlap between inequalities (3.24) and (3.25) for $r_h = 30$

In the above figures, the overlapping regions represent a possible solution. A common feature of the plots is a spike at $x \approx r_h$ or at $r \approx 2r_h$. It can be seen that around $x = r_h$, the LHS of (3.25) goes to zero, which is the reason why we have a spike at this point. Understanding the physical implication of the spike is something that needs further study.

The analysis done above is not a proof that (3.22) is the correct form of ansatz. In fact, any ansatz of the form $\Omega = \sqrt{\frac{4\pi}{\beta}(r - r_h)} + r^n e^{-r^2}$ will yield similiar results, where n can be an integer or fraction. But n has to be less than 7, beacuse of our requirement that the semiclassical correction has to be really small (between 0 and 1). The above analysis helps us to understand the overall nature of the solution.

3.3 An exact numerical solution for the wormholes

In this section, we find a numeric solution for the exact semiclassical equations of motion given by (2.7) and (2.8). We use the gauge $N(z) = 1/\Omega(z)$. Also, σ and its derivatives are rewritten in terms of Ω, using the relation $\Omega = e^{\sigma}$. We know the geometric radius of a 2-sphere is given by $r(z) = R(z)\Omega(z)$. We also use this relation in the exact semiclassical equations so that the final equations are in terms of $r(z)$ and $\Omega(z)$. For simplicity, we consider the case when there are only scalar fields. Choosing a trial value $n_0 = 360$ (only scalar fields), one gets

$$a = 3, \quad b = 1, \quad c_H = 360, \quad \lambda_H = 0 \tag{3.26}$$

Additionally, we take $\kappa = 1$ and $\beta = 4\pi$.

So far, we have been working in z coordinate and the horizon is at $z \to \infty$. Just because of technical difficulty to implement the initial conditions at ∞, we make a change of coordinate $y = e^{-z}$. Now, the would be horizon will be at $y = 0$. The value of y will keep on increasing as we move further and further away from the horizon. So we may say that y behaves like r. The final semiclassical equations in terms of y are given in appendix A. We tried to numerically solve both equations simultaneously. The plots of the numerical solutions are shown below.

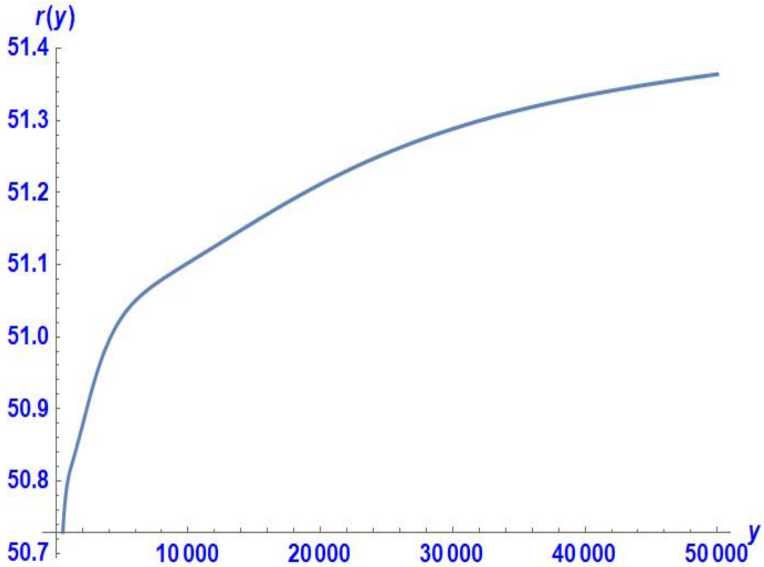

Figure 3.9: Plot of $r(y)$ vs y. Initial conditions were specified at $y = 0.5$ (at $y = 0.5 \implies r = 50$, $\Omega = 1.2$, $r' = 0.5$, $\Omega' = 0.5$, $r'' = 0.1$).

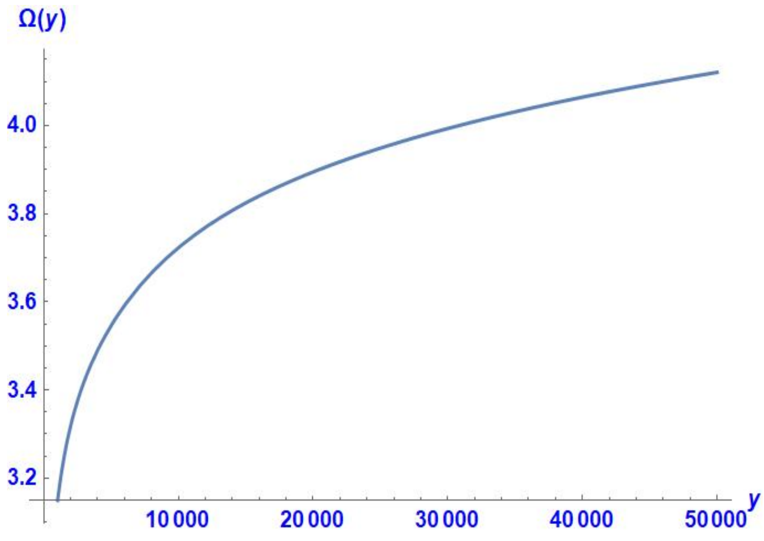

Figure 3.10: Plot of $\Omega(y)$ vs y.

In the above solutions, we start at a point slightly away from the horizon ($y = 0$), because getting too close to the horizon leads to diverging results. That is why we start at $y = 0.5$. The plot of $r(y)$ given in figure 3.9 is a sensible solution. The horizon, which is at $y = 0$, is expected to have a radius slightly less than the value $r(0.5) = 50$. As we move further and further away, i.e, as the value of y increases, $r(y)$ keeps on increasing in an exponentially small manner. The value of $r(y)$ is not getting saturated for higher and higher y values, which is in agreement with our expectation. Since we are solving the exact semiclassical equations, the solution must be satisfied irrespective of how far away we are looking at. It can be seen that the behaviour 3.9 will remain the same for even arbitrarily higher values of y.

In case of Ω, its value at $y = 0$ will be exponentially small ($0 < \Omega < 1$), as can be seen from (2.15). Since we start at a slightly far away point, we are using an initial value $\Omega = 1.2$. It can be observed from figure 3.10 that as the value of y increases, $\Omega(y)$ continues to behave like $r(y)$ as seen in figure 3.9. This is reminiscent of the result we saw in section 3.2. As y increases, the classical part of Ω given by $\Omega = \sqrt{\frac{4\pi}{\beta}(r - r_h)}$ will dominate. This is the reason why $\Omega(y)$ has a similar behaviour as $r(y)$.

Below we give some solution plots, where we have also taken $n_{1/2}$ and n_1 fields into account in our computation. We vary n_s value for a particular field, keeping the n_s value of other fields as a non-zero constant.

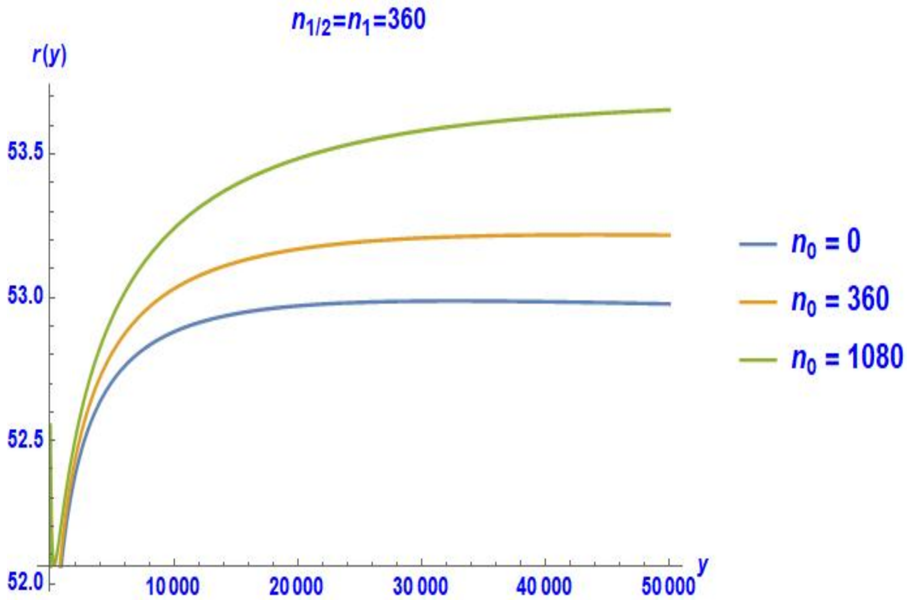

Figure 3.11: Plot of $r(y)$ vs y for different n_0 values, keeping $n_{1/2}$ and n_1 constant.

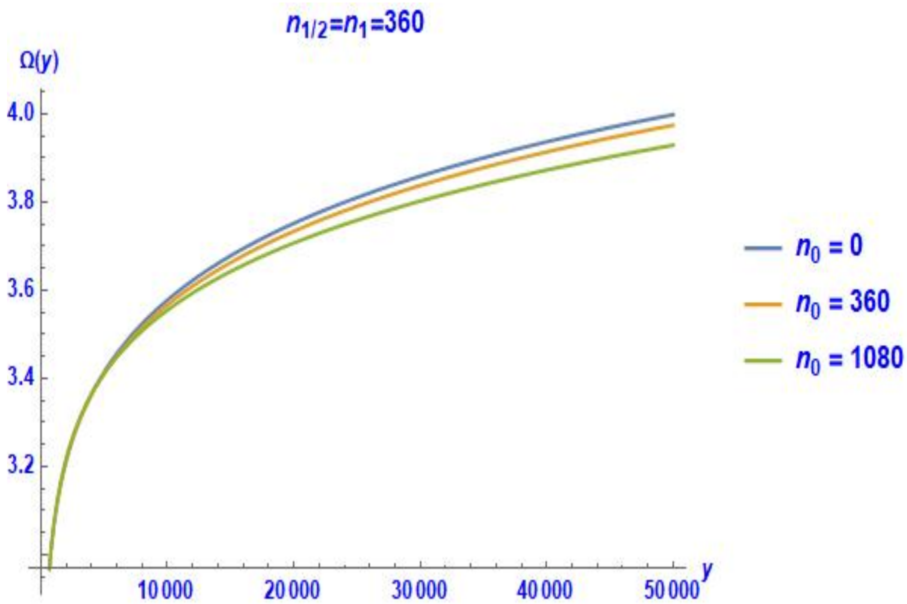

Figure 3.12: Plot of $\Omega(y)$ vs y for different n_0 values, keeping $n_{1/2}$ and n_1 constant.

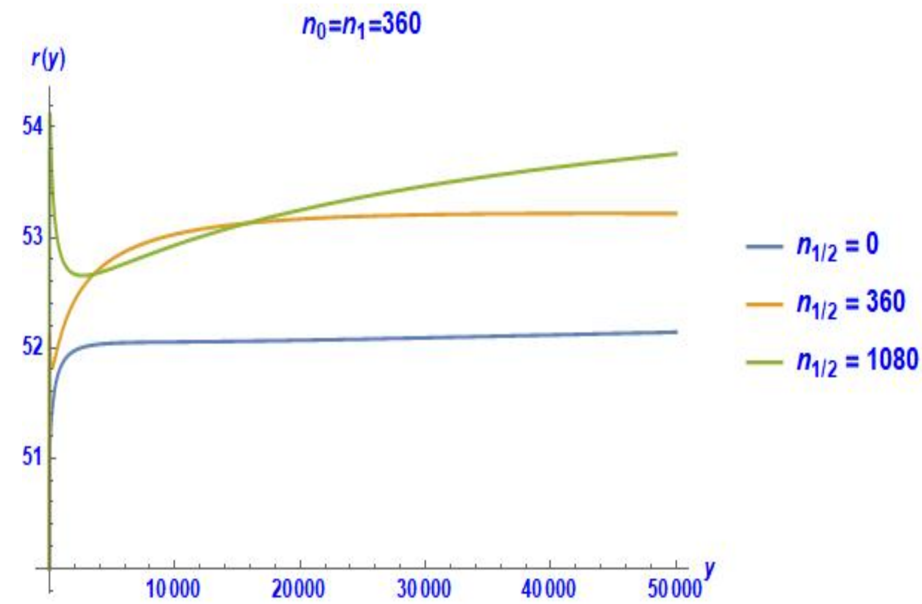

Figure 3.13: Plot of $r(y)$ vs y for different $n_{1/2}$ values, keeping n_0 and n_1 constant.

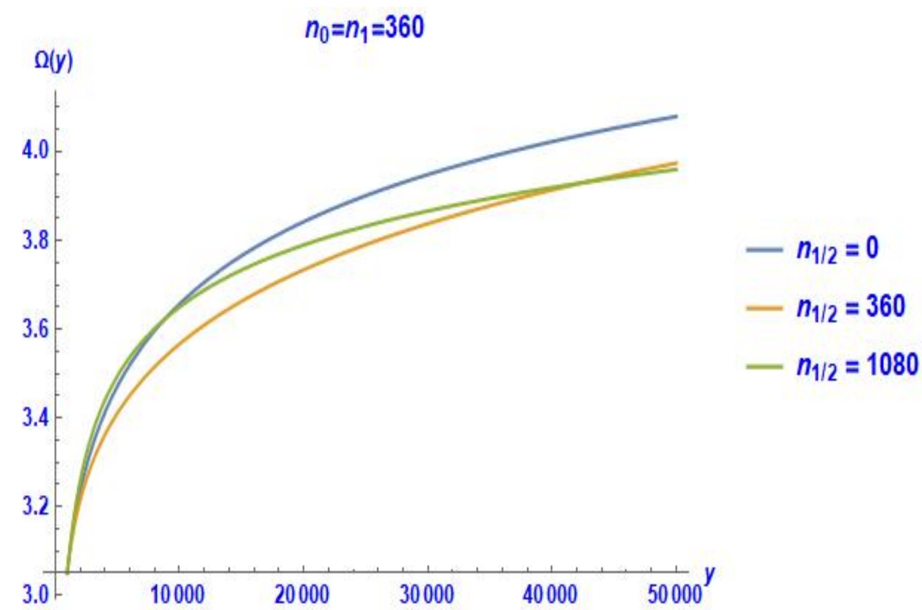

Figure 3.14: Plot of $\Omega(y)$ vs y for different $n_{1/2}$ values, keeping n_0 and n_1 constant.

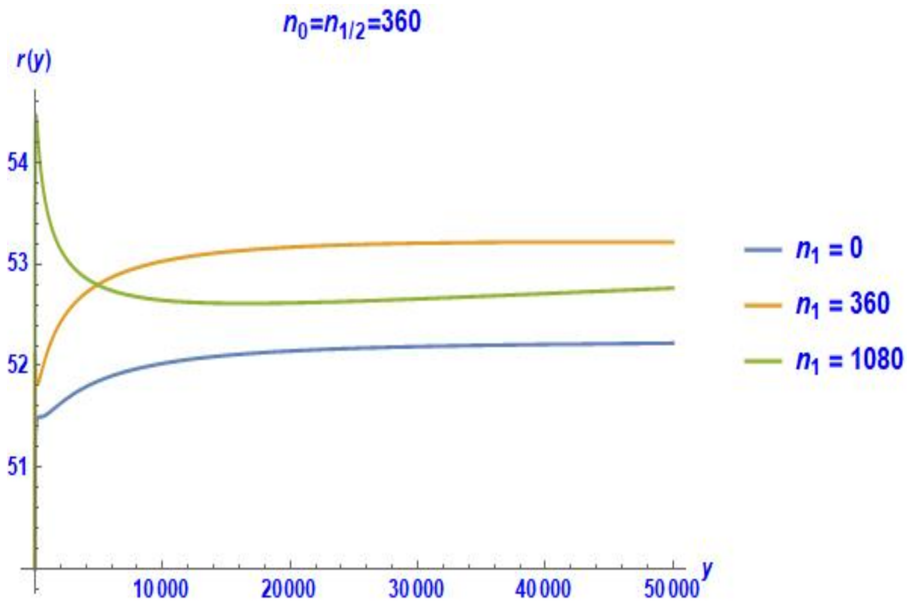

Figure 3.15: Plot of $r(y)$ vs y for different n_1 values, keeping n_0 and $n_{1/2}$ constant.

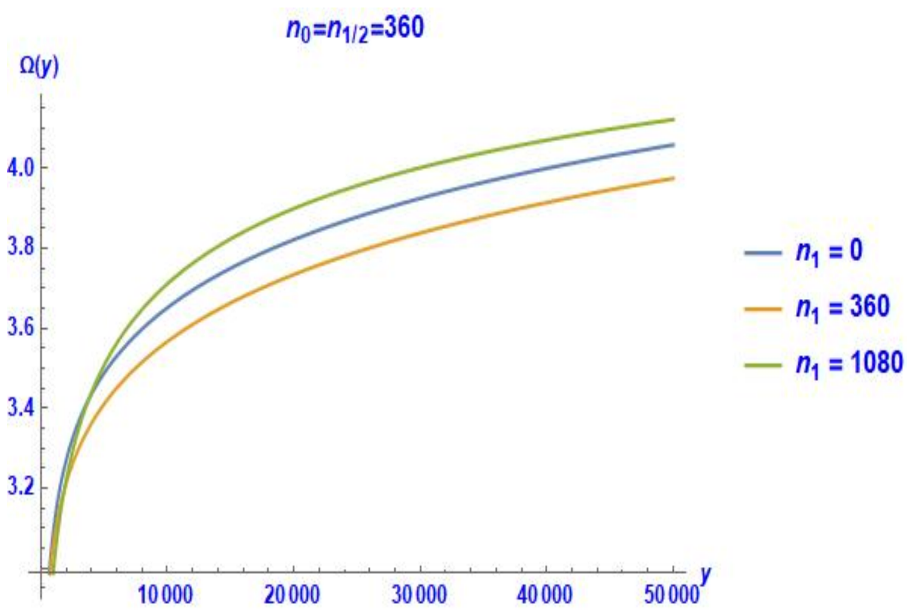

Figure 3.16: Plot of $\Omega(y)$ vs y for different n_1 values, keeping n_0 and $n_{1/2}$ constant.

Chapter 4

Conclusions and Scope for Future Work

4.1 Conclusions and future work

It has been shown that the classical solution with next to leading order terms taken into consideration cannot solve the semiclassical equations of motion. In other words, a static spherically symmetric metric with a finite temperature horizon still cannot exist in semiclassical gravity. However, we saw that for a particular set of fields, i.e for $n_0 = 6$, $n_{1/2} = 2$ and $n_1 = 1$, it is possible that a black hole solution might exist. And surprisingly, this happens to be the multiplet of $\mathcal{N} = 4$ super-Yang-Mills theory in 4 dimensions.

We saw that the semiclassical correction for Ω alone is not enough to solve the equations of motion. Rather, we also have to take the classical part into consideration. Through some plots, we found certain regions where a solution can exist. Also, we found an exact numerical solution for the wormholes by solving equations (2.13) and (2.14). To our knowledge, this numerical solution is the first evidence of a wormhole in 4 dimensions, which we obtained by rigorously solving the semiclassical equations. Although we couldn't find a unique ansatz as analytical solution, the analysis done helps to provide a good understanding of the overall nature of the solution. Pointing out again, the equations that need to be analytically solved are

$$\frac{2\Omega}{\kappa}(1-2rr^{''}-r^2\frac{\Omega^{''}}{\Omega})+\frac{\bar{a}}{r^2\Omega}(\Omega+\Omega rr^{''}-r^2\Omega^{''})^2+\bar{b}\Omega^{''}=0 \quad (4.1)$$

$$-\frac{\Omega^2}{\kappa}-\frac{\bar{a}}{r^2}\ln\Omega^{-1}[(\Omega rr^{''}-r^2\Omega^{''})^2-\Omega^2]-\frac{\gamma r^2}{\beta^4\Omega^2}+\frac{\lambda}{\beta^2}=0 \quad (4.2)$$

4.2 Outlook

Black holes are one of the most mysterious objects in physics. The physics of black holes is an extremely active field of research all around the world. Recently, the 2020 Nobel Prize in Physics honored the pioneering studies about the nature of black holes. Semiclassical gravitational theory, on which my project is based on, can have profound implications on the nature of black holes. The semiclassical gravitational equations are complicated non-linear differential equations. By analytically solving them, one will be providing exact solutions of four dimensional Einstein equations with quantum matter corrections, which is, as of yet, completely unknown. We know it in 2 dimensional case, but not in four dimensions [18]. So it will be an important addition to black hole literature. Works on semiclassical black holes can have great significance in the study of Hawking's information loss problem. In addition, semiclassical gravitational study could be a foundation stone for an ultimate theory of quantum gravity.

Appendix A

Appendix 1

The exact semiclassical equations of motion in terms of $y = e^{-z}$ are given below (for $n_0 = 360$ so that $a = 3$, $b = 1$, $c_H = 360$, $\lambda_H = 0$. Also we take $\kappa = 1$ and $\beta = 4\pi$). Here $r = r(y)$ and $\Omega = \Omega(y)$.

The N variation equation is,

$$
\begin{aligned}
0 =& \frac{1}{64\pi^2 r^2 \Omega^2}\Bigg[- r^4 - 64\pi^2 r^2 \Omega^4 + 64\pi^2 y^2 r^2 \Omega^4 r'^2 - 32 y^2 r \Omega^3 r' \Omega' \\
&+ 128\pi^2 y^2 r^3 \Omega^3 r' \Omega' - 32 y^3 r^2 \Omega^3 r'^2 \Omega' + 32 y^4 r \Omega^3 r'^3 \Omega' \\
&+ 16 y^2 r^2 \Omega^2 \Omega'^2 + 64 y^3 r^3 \Omega^2 r' \Omega'^2 - 16 y^4 r^2 \Omega^2 r'^2 \Omega'^2 + 8 y^4 r^4 \Omega'^4 \\
&- 32 y^4 r^2 \Omega^3 r' \Omega' r'' + 32 y^4 r^3 \Omega^2 \Omega'^2 r'' + 32 y^4 r^3 \Omega^2 r' \Omega' \Omega'' \\
&- 32 y^3 r^4 \Omega \Omega'^2 (\Omega' + y\Omega'')\Bigg] + \frac{\ln \Omega}{64\pi^2 r^2 \Omega^2}\Bigg[- 16 y^2 r^2 \Omega^4 r'^2 \\
&+ 16\Omega^4(-1 + y^4 r'^4) + 32 y^2 r^3 \Omega^3 r' \Omega' + 32 y^3 r^2 \Omega^3 r'^2 \Omega' \\
&- 64 y^4 r \Omega^3 r'^3 \Omega' - 64 y^3 r^3 \Omega^2 r' \Omega'^2 + 80 y^4 r^2 \Omega^2 r'^2 \Omega'^2 - 32 y^4 r^3 \Omega r' \Omega'^3 \\
&- 32 y^4 r^2 \Omega^3 r' \Omega' r'' + 32 y^4 r^3 \Omega^2 \Omega'^2 r'' + 16 y^4 r^2 \Omega^4 r''^2 + 64 y^4 r^2 \Omega^3 r'^2 \Omega'' \\
&- 96 y^4 r^3 \Omega^2 r' \Omega' \Omega'' + 32 y^3 r^4 \Omega \Omega'^2 (\Omega' + y\Omega'') - 32 y^3 r^2 \Omega^4 r' (2r'' \\
&+ y r''') + 32 y^3 r^3 \Omega^3 \Big(- y r'' \Omega'' + \Omega'(2r'' + y r''')\Big) + 32 y^3 r^3 \Omega^3 r' (2\Omega'' \\
&+ y\Omega''') - 16 y^2 r^4 \Omega^2 \Big(\Omega'^2 - y^2 \Omega''^2 + 2y\Omega'(2\Omega'' + y\Omega''')\Big)\Bigg]
\end{aligned} \tag{A.1}
$$

The σ variation equation is,

$$
\begin{aligned}
0 =& \frac{1}{4\pi^2 r^2 \Omega}\Bigg[\Omega^2\Bigg((1+8\pi^2 r^2 - y^2 r'^2)(-1+y^2 r'^2) + 2yr(-1+8\pi^2 r^2 \\
&+ y^2 r'^2)(r' + yr'') - y^2 r^2 (r' + yr'')^2\Bigg) - y^2 r^2\Bigg(yr'\Omega' - r\Omega' \\
&- yr\Omega''\Bigg)^2 + 2yr\Omega\Bigg(yr'\Omega'(-1+8\pi^2 r^2 + y^2 r'^2 + yrr' + y^2 rr'') \\
&+ r^2(4\pi^2 r + yr' + y^2 r'')(\Omega' + y\Omega'')\Bigg)\Bigg]
\end{aligned} \tag{A.2}
$$

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