Some aspects A_4 flavor symmetry

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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **"Some aspects A_4 flavor symmetry"** in the partial fulfillment of the requirements for the award of the degree of **MASTER OF SCIENCE** and submitted in the **DISCIPLINE OF PHYSICS, Indian Institute of Technology Indore,** is an authentic record of my own work carried out during the time period from july 2020 to june, 2021 under the supervision of **Dr. Dipankar Das,** Assistant Professor.The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other institute.

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12/06/21

Signature of student with date

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This is to certify that the above statement made by the candidate is correct to the best of my/our knowledge.

19/06/21

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Abstract

We review pedagogically non-Abelian discrete groups, which play an important role in the particle physics. We show group-theoretical aspects for S_3 and A_4 concrete groups, such as representations, their tensor products. We explain how to derive, conjugacy classes, characters, representations, and tensor products for these groups (with a finite number).We also present typical flavor models by using A_4 group. Breaking patterns of discrete groups and decompositions of multiplets are important for applications of the non-Abelian discrete symmetry. We discuss these breaking patterns of the non-Abelian discrete group, which are a powerful tool for model buildings.

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Chapter 1

Introduction

We have demonstrated group-theoretical aspects for A_4 group explicitly, such as representation and their tensor products. We have shown them explicitly for non-Abelian discrete group A_4 , and discussed how to drive conjugacy classes, character tables, representations and tensor products for these groups (A_4).
A4 (discrete group of even permutations of four items) emerged as a unique discrete group that can duplicate the TBM (tribimaximal mixing) pattern in a very cost-effective manner among all the discrete groups explored in the literature. It's a smalles group with a three-dimensional representation that can accommodate three flavours of leptons and explain the textures of fermion mass matrices. [Karmakar, 2018] In the quark sector, models based on the A_4 symmetry as a possible family symmetry were first introduced in Refs. [Wyler, 1979; Branco, Nilles and Rittenberg, 1980]. After the impact of the symmetry on the Yukawa matrices is known, some structure for the vacuum expectation values (vev) has to be assumed before moving on to the mass matrices and respective phenomenological predictions.
Degee, Ivanov, and Keus[Degee, Ivanov and Keus, 2013] have introduced a geometrical procedure to minimize highly symmetric scalar potentials, and solved the problem for a three Higgs doublet model (3HDM) potential with an A_4 or an S_4 symmetry. In this thesis, we consider models with three Higgs doublets ϕ_i in a triplet representation of A_4 and The models contain only three generations of left-handed quark doublets Q_L , right-handed up-type quark singlets u_R , and right-handed down-type quark singlets d_R.[Felipe, Serodio and Silva, 2013]

Chapter 2

Group theory

Group Theory is the study of symmetry, whenever an object or a system's property is invariant under a transformation than we can analyze the object using group theoretic methods.

2.1 GROUP

2.1.1 Definition of a Group

A group G consists of a set of entities g_α called group elements. Which we could compose together. Composition or multiplication satisfies the following axioms :

1. Closure :- Given any two elements g_α and g_β , the product $g_\alpha.g_\beta$ is equal to another element g in G.

$$g_\alpha.g_\beta = g_\gamma$$

2. Associativity: Composition is associative

$$(g_\alpha.g_\beta).g_\gamma = g_\alpha.(g_\beta.g_\gamma)$$

3. Existence of the identity: There exists a group element, known as the identity and denoted by I, such that

$$I.g_\alpha = g_\alpha \text{and} \quad g_\alpha.I = g_\alpha.$$

4. Existence of the inverse: For every group element g,there exists a unique group element, known as the inverse of g_α and denoted by g_α^{-1} , such that

$$g_\alpha^{-1}.g_\alpha = I \text{and} \quad g_\alpha.g_\alpha^{-1} = I.$$

2.1.2 Abelian nonabelian groups

A group for which the composition rule is commutative is said to be abelian.

$$g_\alpha.g_\beta = g_\beta.g_\alpha$$

And a group for which the composition rule is not commutative is said to be nonabelian.

$$g_\alpha.g_\beta \neq g_\beta.g_\alpha$$

Examples of group :- 1. The two square roots of 1, $(1, -1)$, form the group Z_2 under ordinary multiplication.

2. Similarly, the three cube roots of 1 form the group $Z_3 = 1, \omega, \omega^2$ with $\omega = \exp(\frac{2\iota\pi}{3})$. and the four fourth roots of 1 form the group $Z_4 = 1, \iota, -1, -\iota$, where famously (or infamously) $\iota = \exp(\frac{\iota\pi}{2})$. More generally, the N, Nth roots of 1 form the group $Z_N = \exp(\frac{\iota 2\pi j}{N}) :$ $j = 0, ..., N-1$. The composition of group elements is defined by $\exp(\frac{\iota 2\pi j}{N}).\exp(\frac{\iota 2\pi k}{N}) = \exp(\frac{\iota 2\pi (j+k)}{N})$.

2.1.3 Multiplication table

A finite group with n elements can be characterized by its multiplication table . We construct a square $n \otimes n$ table, writing the product $g_i g_j$ in the square in the i^{th} row and the j^{th} column:

	$\cdots$	$\cdots$	g_j	$\cdots$	$\cdots$
$\vdots$	$\ddots$				
$\vdots$		$\ddots$			
g_i			$\mathrm{g}_i g_j$		
$\vdots$				$\ddots$	
$\vdots$					$\ddots$

Table 2.1: Multiplication table

2.1.4 Homomorphism and isomorphism

A map $f : G \to G'$ of a group G into the group G' Is called a homomorphism If it preserves the multiplicative structure of G,

$$1.\text{if} f(g_1)f(g_2) = f(g_1 g_2) \tag{2.1}$$

$$2.f(I) = I(\text{ the identity of G is mapped to the identity of G'}). \tag{2.2}$$

A homomorphism becomes an isomorphism if the map is one-to-one and onto.

2.2 Representation Theory

Given a group, the idea is to associate each element g with a $d \otimes d$ matrix $D(g)$such that

$$D(g_1)D(g_2) = D(g_1 g_2)$$

for any two group elements g_1 and g_2. The matrix $D(g)$ is said to represent the element g, and the set of matrices $D(g)$ for all $g \in G$ is said to furnish or provide a representation of G. The size of the matrices, d, is known as the dimension of the representation.

2.2.1 Equivalent representations

Two representations,$D(g)$ and $D'(g)$, are really the same representation (more formally, the two representations are equivalent) if they are related by a similarity transformation.

$$D'(g) = S^{-1}D(g)S$$

As explained in the review of linear algebra, $D(g)$ and $D'(g)$ are essentially the same matrix, merely written in two different bases, with the matrix S relating one set of basis vectors to the other set. Then given a representation $D(g)$, define $D'(g)$ by similarity transformation with some S whose inverse exists. Then $D(g)$ is also a representation, since

$$D'(g_1)D'(g_2) = (S^{-1}D(g_1)S)(S^{-1}D(g_2)S)$$

$$= S^{-1}D(g_1)D(g_2)S$$

$$= S^{-1}D(g_1g_2)S$$

$$= D'(g_1g_2). \tag{2.3}$$

2.2.2 Reducible or irreducible representation

The vector space v_j , on which representation matrices act, is called a representation space such as $D(g)_{ij}v_j \quad (j = 1, ..., n)$. The dimension n of the vector space $v_j \quad (j = 1, ..., n)$ is called as a dimension of the representation.A subspace in the representation space is called invariant subspace if $D(g)_{ij}v_j$ for any vector v_j in the subspace and any element $g \in G$ also corresponds to a vector in the same subspace. If a representation has an invariant subspace, such a representation is called reducible. A representation is irreducible if it has no invariant subspace. In particular, a representation is called completely reducible if $D(g)$ for $g \in G$ are written as the following block diagonal form,

$$\begin{pmatrix} D_1(g) & 0 & & & \\ 0 & D_2(g) & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & D_{r'}(g) \end{pmatrix}$$

where each $D_\alpha(g)$ for $\alpha = 1, ..., r$ is irreducible. This implies that a reducible represen- tation $D(g)$ is the direct sum of $D_\alpha(g)$,

$$\sum_{\alpha=1}^{r} \oplus D_\alpha(g). \tag{2.4}$$

Chapter 3

S_3 Group

S_N GROUP

All possible permutations among N objects x_i with $i = 1, ..., N$, form a group,

$$(x_1, ..., x_n) \to (x_{i1}, ..., x_{iN}) \tag{3.1}$$

This group is the so-called S_N with the order $N!$, and S_N is often called as the symmetric group. In the following we show concrete aspects on S_N for smaller N. The simplest one of S_N except the trivial S_1 is S_2,which consists of two permutations,

$$(x_1, x_2) \to (x_1, x_2), (x_1, x_2) \to (x_2, x_1)$$

This is nothing but Z_2, that is Abelian. Thus, we start with S_3.[Ishimori et al., 2009]

3.1 S_3 Group

S_3 consists of all permutations among three objects, (x_1, x_2, x_3) and its order is equal to $3! = 6$. All of six elements correspond to the following transformations,

$$e : (x_1, x_2, x_3) \to (x_1, x_2, x_3),$$

$$a_1 : (x_1, x_2, x_3) \to (x_2, x_1, x_3),$$

$$a_2 : (x_1, x_2, x_3) \to (x_3, x_2, x_1),$$

$$a_3 : (x_1, x_2, x_3) \to (x_1, x_3, x_2),$$

$$a_4 : (x_1, x_2, x_3) \to (x_3, x_1, x_2),$$

$$a_5 : (x_1, x_2, x_3) \to (x_2, x_3, x_1).$$

Their multiplication forms a closed algebra, e.g.

$$a_1 a_2 : (x_1, x_2, x_3) \to (x_1, x_2, x_3)$$

$$a_2 a_1 : (x_1, x_2, x_3) \to (x_3, x_1, x_2),$$

$$a_4 a_2 : (x_1, x_2, x_3) \to (x_1, x_3, x_2),$$

i.e.,

$$a_1 a_2 = a_5,$$

$$a_2 a_1 = a_4,$$

$$a_4 a_2 = a_2 a_1 a_2 = a_3,$$

Thus, by defining $a_1 = a, a_2 = b$, all of elements are written as

$$e, a, b, ab, ba, bab$$

.

Note that $aba = bab$. The S_3 group is a symmetry of an equilateral triangle as shown in Figure . The elements a and ab correspond to a reflection and the $2\pi/3$ rotation, respectively.

S_3 is the group of all permutations of three objects and it has 6 elements divided into three irreducible representation, namely, two singlets 1, 1' and one doublet 2.

The orthogonality relation

$$\sum_{\alpha}[\chi_\alpha(C_1]^2 = \sum_{\alpha} m_n n^2 = m_1 + 4m_2 + 9m_3 + \cdots = 6 \tag{3.2}$$

and

$$\sum_n m_n = 3$$

where $m_n \geq 0$, then $(m_1, m_2) = (2, 1)$.Thus, irreducible representation of S_3 include two singlets 1, 1' and a doublet 2.

Character Table of S_3 representations

Conjugacy class	Order	χ_1	$\chi_{1'}$	χ_2
C_1	1	1	1	2
C_2	3	1	1	-1
C_3	2	1	-1	0

Table 3.1: Character table of S_3

3.1.1 Matrix representation of S_3

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, b = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix},$$

$$ab = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, ba = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, bab = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

3.1.2 Tensor products of S_3

Tensor products of irreducible representation for S_3. Two one-dimensional representation denoted by 1 and 1', and one two-dimensional representation denoted by 2. Then,

Tensor products of two doublets $(x_1, x_2), (y_1, y_2)$

$$\Gamma_{l_\alpha \times l_\alpha}{}^{\alpha}(g) \otimes \Gamma_{l_\beta \times l_\beta}{}^{\beta}(g) = \Gamma_{l_\alpha l_\beta \times l_\alpha l_\beta}{}^{red.}(g) = \oplus a^\alpha \Gamma^\alpha \tag{3.3}$$

$$D_{2\times2}{}^{\alpha}(g)\otimes D_{2\times2}{}^{\beta}(g)=D_{4\times4}{}^{red.}(g) \tag{3.4}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{2\times1} \otimes \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_{2\times1} = \begin{pmatrix} x_1y_1 \\ x_1y_2 \\ x_2y_1 \\ x_2y_2 \end{pmatrix}_{4\times1} = reduciblevectorspace = \begin{pmatrix} A_{1\times1} \\ B_{1\times1} \\ \begin{pmatrix} C \\ D \end{pmatrix}_{2\times2} \end{pmatrix}$$

Direct product of S_3 group representation

$$\Gamma(e)=e\otimes e=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \Gamma(a)=a\otimes a=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\Gamma(b)=b\otimes b=\begin{pmatrix} \frac{1}{4} & \frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4} & \frac{3}{4} \\ \frac{\sqrt{3}}{4} & -\frac{1}{4} & \frac{3}{4} & -\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{3}{4} & -\frac{1}{4} & -\frac{\sqrt{3}}{4} \\ \frac{3}{4} & -\frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} & \frac{1}{4} \end{pmatrix}, \Gamma(ab)=ab\otimes ab=\begin{pmatrix} \frac{1}{4} & \frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4} & \frac{3}{4} \\ -\frac{\sqrt{3}}{4} & \frac{1}{4} & -\frac{3}{4} & \frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & -\frac{3}{4} & \frac{1}{4} & \frac{\sqrt{3}}{4} \\ \frac{3}{4} & -\frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} & \frac{1}{4} \end{pmatrix},$$

$$\Gamma(ba)=ba\otimes ba=\begin{pmatrix} \frac{1}{4} & -\frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} & \frac{3}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} & -\frac{3}{4} & -\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & -\frac{3}{4} & \frac{1}{4} & -\frac{\sqrt{3}}{4} \\ \frac{3}{4} & \frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4} & \frac{1}{4} \end{pmatrix}, \Gamma(bab)=bab\otimes bab=\begin{pmatrix} \frac{1}{4} & -\frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} & \frac{3}{4} \\ -\frac{\sqrt{3}}{4} & -\frac{1}{4} & \frac{3}{4} & \frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & \frac{3}{4} & -\frac{1}{4} & \frac{\sqrt{3}}{4} \\ \frac{3}{4} & \frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4} & \frac{1}{4} \end{pmatrix}$$

,

Projection matrices for S_3 irreducible representation

$$P_{\chi^{\alpha}}=\frac{l_{\chi^{\alpha}}}{|G|}\sum_{g\in G}\overline{\chi}^{\Gamma^{\alpha}}(g)\Gamma(g) \tag{3.5}$$

here, $l_{\chi^{\alpha}}$ and $|G|$is order 6.

$$P_{\chi^\alpha}\begin{pmatrix} x_1y_1 \\ x_1y_2 \\ x_2y_1 \\ x_2y_2 \end{pmatrix} = \begin{pmatrix} A_{1\times1} \\ B_{1\times1} \\ \begin{pmatrix} C \\ D \end{pmatrix}_{2\times2} \end{pmatrix}$$

$$P_{\chi_1} = \frac{1}{6}[\Gamma(e)+\Gamma(ab)+\Gamma(ba)+\Gamma(a)+\Gamma(b)+\Gamma(bab)] \tag{3.6}$$

$$P_{\chi_1} = \frac{1}{6}\begin{pmatrix} 3 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$P_{\chi_1}\begin{pmatrix} x_1y_1 \\ x_1y_2 \\ x_2y_1 \\ x_2y_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x_1y_1 \\ x_1y_2 \\ x_2y_1 \\ x_2y_2 \end{pmatrix} = \begin{pmatrix} \frac{x_1y_1}{2} + \frac{x_2y_2}{2} \\ 0 \\ 0 \\ \frac{x_1y_1}{2} + \frac{x_2y_2}{2} \end{pmatrix}$$

$1 = (x_1y_1 + x_2y_2)$ linear combination.

Similarly,

$$P_{\chi_{1'}} = \frac{1}{6}[\Gamma(e)+\Gamma(ab)+\Gamma(ba)-\Gamma(a)-\Gamma(b)-\Gamma(bab)] \tag{3.7}$$

$$P_{\chi_{1'}} = \frac{1}{6}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 3 & -3 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$P_{\chi_{1'}}\begin{pmatrix} x_1y_1 \\ x_1y_2 \\ x_2y_1 \\ x_2y_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1y_1 \\ x_1y_2 \\ x_2y_1 \\ x_2y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{x_1y_2}{2} - \frac{x_2y_1}{2} \\ -\frac{x_1y_2}{2} + \frac{x_2y_1}{2} \\ 0 \end{pmatrix}$$

$1'$= $(x_1y_2 - x_2y_1)$ linear combination.

and

$$P_{\chi_2} = \frac{1}{6}[2\Gamma(e) - \Gamma(ab) - \Gamma(ba)] \tag{3.8}$$

$$P_{\chi_2} = \frac{1}{6}\begin{pmatrix} 3 & 0 & 0 & -3 \\ 0 & 3 & 3 & 0 \\ 0 & 3 & 3 & 0 \\ -3 & 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$P_{\chi_2}\begin{pmatrix} x_1y_1 \\ x_1y_2 \\ x_2y_1 \\ x_2y_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x_1y_1 \\ x_1y_2 \\ x_2y_1 \\ x_2y_2 \end{pmatrix} = \begin{pmatrix} \frac{x_1y_1}{2} - \frac{x_2y_2}{2} \\ \frac{x_1y_2}{2} + \frac{x_2y_1}{2} \\ \frac{x_1y_2}{2} + \frac{x_2y_1}{2} \\ -\frac{x_1y_1}{2} + \frac{x_2y_2}{2} \end{pmatrix}$$

2= $\begin{pmatrix} x_2y_2 - x_1y_1 \\ x_1y_2 + x_2y_1 \end{pmatrix}$ linear combination.

1. Tensor products of two doublets (x_1, x_2) (y_1, y_2)

The linear combinations corresponding singlets and doublet

1: x$_1y_1 + x_2y_2$, $1'$: $x_1y_2 - x_2y_1$

and

$$2 : \begin{pmatrix} x_2y_2 - x_1y_1 \\ x_1y_2 + x_2y_1 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \otimes \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = (x_1y_1 + x_2y_2) \oplus (\mathrm{x}_1y_2 - x_2y_1) \oplus \begin{pmatrix} x_2y_2 - x_1y_1 \\ x_1y_2 + x_2y_1 \end{pmatrix}$$

$$2 \otimes 2 = 1 \oplus 1' \oplus 2$$

2. Tensor products of singlet (y) and doublet (x_1, x_2)

$$(y)_{1'} \otimes \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_2 = \begin{pmatrix} -yx_2 \\ yx_1 \end{pmatrix}$$

$$1' \otimes 2 = 2$$

3.Tensor products of singlet (x) and (y)

$$(x)_{1'} \otimes (y)_{1'} = (xy)_1$$

$$1' \otimes 1' = 1$$

, and

$$1 \otimes any = any$$

Chapter 4

A_4 Group

$A_4 Group$:

A_4 is the group of even permutations of four objects and it has 12 elements divided into four irreducible representation, namely, three singlets 1, 1', 1" and one triplet 3.

The orthogonality relation

$$\sum_{\alpha}[\chi_{\alpha}(C_1]^2 = \sum_{\alpha} m_n n^2 = m_1 + 4m_2 + 9m_3 + \cdots = 12 \tag{4.1}$$

and

$$\sum_{n} m_n = 4$$

where $m_n \geq 0$, then $(m_1, m_2, m_3) = (3, 0, 1)$.Thus, irreducible representation of A_4 include three singlets 1, 1' 1" and a triplet 3.

Character Table of S_3 representations

Conjugacy class	Order	χ_1	$\chi_{1'}$	$\chi_{1"}$	χ_3
C_1	1	1	1	1	3
C_3	2	1	1	1	-1
C_4	3	1	ω	ω^2	0
$C_{4'}$	3	1	ω^2	ω	0

Table 4.1: Character table of A_4

4.0.1 Tensor Products of A_4

Tensor products of irreducible representation for A_4. Three one-dimensional representation denoted by 1, 1' and, 1", and one three-dimensional representation denoted by 3. Then,

(1.) First, the alternating group A_4 can be defined in terms of two generators S, T satisfying the representation rules.

$$S^2 = T^3 = (ST)^3 = 1$$

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Conjugacy classes

$$C_1 : e$$

$$C_3 : S, TST^2, T^2ST$$

$$C_4 : T, TS, ST, STS$$

$$C_4' : T^2, ST^2, T^2S, TST$$

4.0.2 Matrix representation of A_4

$$e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, tst^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, t^2St = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$
$$, t = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, tS = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}, St = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}, StS = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$$
$$, t^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, St^2 = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, t^2S = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, tSt = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Tensor products of irreducible represention.

$$\Gamma_{l_\alpha \times l_\alpha}{}^{\alpha}(g) \otimes \Gamma_{l_\beta \times l_\beta}{}^{\beta}(g) = \Gamma_{l_\alpha l_\beta \times l_\alpha l_\beta}{}^{red.}(g) = \oplus a^\alpha \Gamma^\alpha \tag{4.2}$$

Tensor products of two triplets $(a_1, a_2, a_3), (b_1, b_2, b_3)$

$$D_{3\times3}{}^{\alpha}(g) \otimes D_{3\times3}{}^{\beta}(g) = D_{9\times9}{}^{red.}(g) \tag{4.3}$$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_{3\times1} \otimes \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}_{3\times1} = \begin{pmatrix} a_1b_1 \\ a_1b_2 \\ a_1b_3 \\ a_2b_1 \\ a_2b_2 \\ a_2b_3 \\ a_3b_1 \\ a_3b_2 \\ a_3b_3 \end{pmatrix}_{9\times1} = \text{reducible vector space} = \begin{pmatrix} A_1 \\ B_{1'} \\ C_{1"} \\ \begin{pmatrix} D \\ E \\ F \end{pmatrix}_{3s} \\ \begin{pmatrix} G \\ H \\ I \end{pmatrix}_{3a} \end{pmatrix}_{9\times1}$$

Direct product of A_4 group representation

$$\Gamma(e) = e \otimes e = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}_{9\times 9}$$

$$\Gamma(s) = s \otimes s = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}_{9\times 9}$$

Similarly,

$$\Gamma(tst^2), \Gamma(t^2st), \Gamma(t), \Gamma(ts), \Gamma(st), \Gamma(t^2), \Gamma(st^2), \Gamma(t^2s), \Gamma(tst), \Gamma(sts)$$

,

Projection matrices for A_4 irreducible representation

$$P_{\chi^\alpha} = \frac{l_{\chi^\alpha}}{|G|} \sum_{g\in G} \overline{\chi}^{\Gamma^\alpha}(g)\Gamma(g) \tag{4.4}$$

here, l_{χ^α} is dim. and $|G|$is order 12.

$$P_{\chi_1} = \frac{1}{12}[\Gamma(e)+\Gamma(s)+\Gamma(tst^2)+\Gamma(t^2st)+\Gamma(t)+\Gamma(ts)+\Gamma(st)+\Gamma(sts)+\Gamma(t^2)+\Gamma(st^2)+\Gamma(t^2s)+\Gamma(tst)] \tag{4.5}$$

$$P_{\chi_1} = \frac{1}{12}\begin{pmatrix} 4&0&0&0&4&0&0&0&4\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 4&0&0&0&4&0&0&0&4\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 4&0&0&0&4&0&0&0&4 \end{pmatrix} = \begin{pmatrix} \frac{1}{3}&0&0&0&\frac{1}{3}&0&0&0&\frac{1}{3}\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ \frac{1}{3}&0&0&0&\frac{1}{3}&0&0&0&\frac{1}{3}\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ \frac{1}{3}&0&0&0&\frac{1}{3}&0&0&0&\frac{1}{3} \end{pmatrix}$$

$$P_{\chi_1}\begin{pmatrix} a_1b_1\\ a_1b_2\\ a_1b_3\\ a_2b_1\\ a_2b_2\\ a_2b_3\\ a_3b_1\\ a_3b_2\\ a_3b_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{3}&0&0&0&\frac{1}{3}&0&0&0&\frac{1}{3}\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ \frac{1}{3}&0&0&0&\frac{1}{3}&0&0&0&\frac{1}{3}\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ \frac{1}{3}&0&0&0&\frac{1}{3}&0&0&0&\frac{1}{3} \end{pmatrix}\begin{pmatrix} a_1b_1\\ a_1b_2\\ a_1b_3\\ a_2b_1\\ a_2b_2\\ a_2b_3\\ a_3b_1\\ a_3b_2\\ a_3b_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{3}a_1b_1+\frac{1}{3}a_2b_2+\frac{1}{3}a_3b_3\\ 0\\ 0\\ 0\\ \frac{1}{3}a_1b_1+\frac{1}{3}a_2b_2+\frac{1}{3}a_3b_3\\ 0\\ 0\\ 0\\ \frac{1}{3}a_1b_1+\frac{1}{3}a_2b_2+\frac{1}{3}a_3b_3 \end{pmatrix}$$

1= $(a_1b_1 + a_2b_2 + a_3b_3)$ linear combination.

$$P_{\chi_{1'}} = \frac{1}{12}[\Gamma(e)+\Gamma(s)+\Gamma(tst^2)+\Gamma(t^2st)+\omega\Gamma(t)+\omega\Gamma(ts) \\ +\omega\Gamma(st)+\omega\Gamma(sts)+\omega^2\Gamma(t^2)+\omega^2\Gamma(st^2)+\omega^2\Gamma(t^2s)+\omega^2\Gamma(tst)] \tag{4.6}$$

$$P_{\chi_{1'}} = \frac{1}{12}\begin{pmatrix} 4 & 0 & 0 & 0 & 4\omega & 0 & 0 & 0 & 4\omega^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4\omega^2 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 4\omega \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4\omega & 0 & 0 & 0 & 4\omega^2 & 0 & 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3}\omega & 0 & 0 & 0 & \frac{1}{3}\omega^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3}\omega^2 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3}\omega \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3}\omega & 0 & 0 & 0 & \frac{1}{3}\omega^2 & 0 & 0 & 0 & \frac{1}{3} \end{pmatrix}$$

$$P_{\chi_{1'}}\begin{pmatrix} a_1b_1 \\ a_1b_2 \\ a_1b_3 \\ a_2b_1 \\ a_2b_2 \\ a_2b_3 \\ a_3b_1 \\ a_3b_2 \\ a_3b_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3}\omega & 0 & 0 & 0 & \frac{1}{3}\omega^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3}\omega^2 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3}\omega \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3}\omega & 0 & 0 & 0 & \frac{1}{3}\omega^2 & 0 & 0 & 0 & \frac{1}{3} \end{pmatrix}\begin{pmatrix} a_1b_1 \\ a_1b_2 \\ a_1b_3 \\ a_2b_1 \\ a_2b_2 \\ a_2b_3 \\ a_3b_1 \\ a_3b_2 \\ a_3b_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{3}a_1b_1 + \frac{1}{3}\omega a_2b_2 + \frac{1}{3}\omega^2 a_3b_3 \\ 0 \\ 0 \\ 0 \\ \frac{1}{3}\omega^2 a_1b_1 + \frac{1}{3}a_2b_2 + \frac{1}{3}\omega a_3b_3 \\ 0 \\ 0 \\ 0 \\ \frac{1}{3}\omega a_1b_1 + \frac{1}{3}\omega^2 a_2b_2 + \frac{1}{3}a_3b_3 \end{pmatrix}$$

$1'$= $(a_1b_1 + \omega^2 a_2b_2 + \omega a_3b_3)$ linear combination.

$$\begin{aligned} P_{\chi_{1''}} = \frac{1}{12}[\Gamma(e) + \Gamma(s) + \Gamma(tst^2) + \Gamma(t^2st) + \omega^2\Gamma(t) \\ + \omega^2\Gamma(ts) + \omega^2\Gamma(st) + \omega^2\Gamma(sts) + \omega\Gamma(t^2) \\ + \omega\Gamma(st^2) + \omega\Gamma(t^2s) + \omega\Gamma(tst)] \quad (4.7) \end{aligned}$$

$$P_{\chi_{1''}} = \frac{1}{12}\begin{pmatrix} 4 & 0 & 0 & 0 & 4\omega^2 & 0 & 0 & 0 & 4\omega \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4\omega & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 4\omega^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4\omega^2 & 0 & 0 & 0 & 4\omega & 0 & 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3}\omega^2 & 0 & 0 & 0 & \frac{1}{3}\omega \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3}\omega & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3}\omega^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3}\omega^2 & 0 & 0 & 0 & \frac{1}{3}\omega & 0 & 0 & 0 & \frac{1}{3} \end{pmatrix}$$

$$P_{\chi_{1''}}\begin{pmatrix} a_1b_1 \\ a_1b_2 \\ a_1b_3 \\ a_2b_1 \\ a_2b_2 \\ a_2b_3 \\ a_3b_1 \\ a_3b_2 \\ a_3b_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3}\omega^2 & 0 & 0 & 0 & \frac{1}{3}\omega \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3}\omega & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3}\omega^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3}\omega^2 & 0 & 0 & 0 & \frac{1}{3}\omega & 0 & 0 & 0 & \frac{1}{3} \end{pmatrix}\begin{pmatrix} a_1b_1 \\ a_1b_2 \\ a_1b_3 \\ a_2b_1 \\ a_2b_2 \\ a_2b_3 \\ a_3b_1 \\ a_3b_2 \\ a_3b_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{3}a_1b_1 + \frac{1}{3}\omega^2 a_2b_2 + \frac{1}{3}\omega a_3b_3 \\ 0 \\ 0 \\ 0 \\ \frac{1}{3}\omega a_1b_1 + \frac{1}{3}a_2b_2 + \frac{1}{3}\omega^2 a_3b_3 \\ 0 \\ 0 \\ 0 \\ \frac{1}{3}\omega^2 a_1b_1 + \frac{1}{3}\omega a_2b_2 + \frac{1}{3}a_3b_3 \end{pmatrix}$$

1”= $(a_1b_1 + \omega a_2b_2 + \omega^2 a_3b_3)$ linear combination.

$$P_{\chi_3} = \frac{3}{12}[3\Gamma(e) - \Gamma(s) - \Gamma(tst^2) - \Gamma(t^2st)] \tag{4.8}$$

$$P_{\chi_3} = \frac{1}{4}\begin{pmatrix} 0&0&0&0&0&0&0&0&0\\ 0&4&0&0&0&0&0&0&0\\ 0&0&4&0&0&0&0&0&0\\ 0&0&0&4&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&4&0&0&0\\ 0&0&0&0&0&0&4&0&0\\ 0&0&0&0&0&0&0&4&0\\ 0&0&0&0&0&0&0&0&0 \end{pmatrix} = \begin{pmatrix} 0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0 \end{pmatrix}$$

$$P_{\chi_3}\begin{pmatrix} a_1b_1\\ a_1b_2\\ a_1b_3\\ a_2b_1\\ a_2b_2\\ a_2b_3\\ a_3b_1\\ a_3b_2\\ a_3b_3 \end{pmatrix} = \begin{pmatrix} 0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0 \end{pmatrix}\begin{pmatrix} a_1b_1\\ a_1b_2\\ a_1b_3\\ a_2b_1\\ a_2b_2\\ a_2b_3\\ a_3b_1\\ a_3b_2\\ a_3b_3 \end{pmatrix} = \begin{pmatrix} 0\\ a_1b_2\\ a_1b_3\\ a_2b_1\\ 0\\ a_2b_3\\ a_3b_1\\ a_3b_2\\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0\\ a_1b_2\\ a_1b_3\\ a_2b_1\\ 0\\ a_2b_3\\ a_3b_1\\ a_3b_2\\ 0 \end{pmatrix} = \begin{pmatrix} D\\ E\\ F \end{pmatrix}_{3s} = \begin{pmatrix} a_2b_3 + a_3b_2\\ a_3b_1 + a_1b_3\\ a_1b_2 + a_2b_1 \end{pmatrix}_{3s}$$

,

$$\begin{pmatrix} 0 \\ a_1b_2 \\ a_1b_3 \\ a_2b_1 \\ 0 \\ a_2b_3 \\ a_3b_1 \\ a_3b_2 \\ 0 \end{pmatrix} = \begin{pmatrix} G \\ H \\ I \end{pmatrix}_{3a} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}_{3a}$$

1. Tensor products of two triplets (a_1, a_2, a_3) (b_1, b_2, b_3)

The linear combinations corresponding singlets and triplet

1: $\mathrm{a}_1b_1 + a_2b_2 + a_3b_3$, $1'$: $a_1b_1 + \omega^2\mathrm{a}_2b_2 + \omega\mathrm{a}_3b_3$,1" : $a_1b_1 + \omega\mathrm{a}_2b_2 + \omega^2\mathrm{a}_3b_3$

and

$$3_s : \begin{pmatrix} a_2b_3 + a_3b_2 \\ a_3b_1 + a_1b_3 \\ a_1b_2 + a_2b_1 \end{pmatrix}, \; 3_a : \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \otimes \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = (a_1b_1{+}a_2b_2{+}a_3b_3) \oplus (\mathrm{a}_1b_1{+}\omega^2\mathrm{a}_2b_2{+}\omega\mathrm{a}_3b_3) \oplus (a_1b_1{+}\omega\mathrm{a}_2b_2{+}\omega^2\mathrm{a}_3b_3) \oplus$$

$$\begin{pmatrix} a_2b_3 + a_3b_2 \\ a_3b_1 + a_1b_3 \\ a_1b_2 + a_2b_1 \end{pmatrix} \oplus \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$$

$$3 \otimes 3 = 1 \oplus 1' \oplus 1" \oplus 3_s \oplus 3_a$$

2. Tensor products of singlet (a) and triplet (b_1, b_2, b_3)

$$1' \otimes 3 = 3$$

and

$$1" \otimes 3 = 3$$

3.Tensor products of singlet (a) and (b)

$$1 \otimes 1 = 1$$

,

$$1' \otimes 1" = 1$$

and

$$1 \otimes any = any$$

(2.) Another two generators s ,t satisfying the alternating group A_4 representation rules,

$$s^2 = t^3 = (st)^3 = 1$$

$s = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}$

Conjugacy classes

$$C_1 : e$$

$$C_3 : s, tst^2, t^2st$$

$$C_4 : t, ts, st, sts$$

$$C_4' : t^2, st^2, t^2s, tst$$

Then representation matrix A_4

$e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, s = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, tst^2 = \frac{1}{3}\begin{pmatrix} -1 & 2\omega & 2\omega^2 \\ 2\omega^2 & -1 & 2\omega \\ 2\omega & 2\omega^2 & -1 \end{pmatrix}, t^2st = \frac{1}{3}\begin{pmatrix} -1 & 2\omega^2 & 2\omega^2 \\ 2\omega & -1 & 2\omega^2 \\ 2\omega & 2\omega & -1 \end{pmatrix}, t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}, ts = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2\omega^2 & -\omega^2 & 2\omega^2 \\ 2\omega & 2\omega & -\omega \end{pmatrix}, st = \frac{1}{3}\begin{pmatrix} -1 & 2\omega^2 & 2\omega \\ 2 & -\omega^2 & 2\omega \\ 2 & 2\omega^2 & -\omega \end{pmatrix},$
$sts = \frac{1}{3}\begin{pmatrix} -1 & 2\omega & 2\omega^2 \\ 2\omega & -\omega^2 & 2 \\ 2\omega^2 & 2 & -\omega \end{pmatrix}, t_2 = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, st_2 = \frac{1}{3}\begin{pmatrix} -1 & 2\omega & 2\omega^2 \\ 2 & -\omega & 2\omega^2 \\ 2 & 2\omega & -\omega^2 \end{pmatrix}, t_2s = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2\omega & -\omega & 2\omega \\ 2\omega^2 & 2\omega^2 & -\omega^2 \end{pmatrix}, tst = \frac{1}{3}\begin{pmatrix} -1 & 2\omega^2 & 2\omega \\ 2\omega^2 & -\omega & 2 \\ 2\omega & 2 & -\omega^2 \end{pmatrix},$

Tensor product of A_4:

tensor product of two triplet $(\alpha_1, \alpha_2, \alpha_3)$ $(\beta_1, \beta_2, \beta_3)$

$$D^{\alpha}_{3\times3}(g) \otimes D^{\beta}_{3\times3}(g) = D^{red.}_{9\times9}(g) = \otimes m^{\alpha} D(g) \tag{4.9}$$

$$\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}_{3\times 1} \otimes \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}_{3\times 1} = \begin{pmatrix}\alpha_1\beta_1\\ \alpha_1\beta_2\\ \alpha_1\beta_3\\ \alpha_2\beta_1\\ \alpha_2\beta_2\\ \alpha_2\beta_3\\ \alpha_3\beta_1\\ \alpha_3\beta_2\\ \alpha_3\beta_3\end{pmatrix}_{9\times 1} = \begin{pmatrix} A_1\\ B_{1'}\\ C_{1"}\\ \begin{pmatrix} D\\ E\\ F\end{pmatrix}_{3s}\\ \begin{pmatrix} G\\ H\\ I\end{pmatrix}_{3a}\end{pmatrix}_{9\times 1}$$

$$\Gamma(e) = e\otimes e = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}_{9\times 9},$$

$$\Gamma(s) = s\otimes s = \frac{1}{9}\begin{pmatrix}
1 & -2 & -2 & -2 & 4 & 4 & -2 & 4 & 4\\
-2 & 1 & -2 & 4 & -2 & 4 & 4 & -2 & 4\\
-2 & -2 & 1 & 4 & 4 & -2 & 4 & 4 & -2\\
-2 & 4 & 4 & 1 & -2 & -2 & -2 & 4 & 4\\
4 & -2 & 4 & -2 & 1 & -2 & 4 & -2 & 4\\
4 & 4 & -2 & -2 & -2 & 1 & -2 & 4 & 4\\
-2 & 4 & 4 & -2 & 4 & 4 & 1 & -2 & -2\\
4 & -2 & 4 & 4 & -2 & 4 & -2 & 1 & -2\\
-2 & 4 & 4 & -2 & 4 & 4 & -2 & -2 & 1
\end{pmatrix}_{9\times 9}$$

Similarly

$$\Gamma(tst^2), \Gamma(t^2st), \Gamma(t), \Gamma(ts), \Gamma(st), \Gamma(t^2), \Gamma(st^2), \Gamma(t^2s), \Gamma(tst), \Gamma(sts)$$

,

$$P_{\chi^\alpha} = \frac{l_{\chi^\alpha}}{|G|} \sum_{g \in G} \overline{\chi}^{\Gamma^\alpha}(g)\Gamma(g) \tag{4.10}$$

here, $l_{\chi^\alpha} is dim.$ and $|G|$is order 12.

$$P_{\chi_1} = \frac{1}{12}[\Gamma(e) + \Gamma(s) + \Gamma(tst^2) + \Gamma(t^2st) + \Gamma(t) + \Gamma(ts) + \Gamma(st) + \Gamma(sts) + \Gamma(t^2) + \Gamma(st^2) + \Gamma(t^2s) + \Gamma(tst)] \tag{4.11}$$

$$P_{\chi_{1'}} = \frac{1}{12}[\Gamma(e) + \Gamma(s) + \Gamma(tst^2) + \Gamma(t^2st) + \omega\Gamma(t) + \omega\Gamma(ts) + \omega\Gamma(st) + \omega\Gamma(sts) + \omega^2\Gamma(t^2) + \omega^2\Gamma(st^2) + \omega^2\Gamma(t^2s) + \omega^2\Gamma(tst)] \tag{4.12}$$

$$P_{\chi_{1"}} = \frac{1}{12}[\Gamma(e) + \Gamma(s) + \Gamma(tst^2) + \Gamma(t^2st) + \omega^2\Gamma(t) + \omega^2\Gamma(ts) + \omega^2\Gamma(st) + \omega^2\Gamma(sts) + \omega\Gamma(t^2) + \omega\Gamma(st^2) + \omega\Gamma(t^2s) + \omega\Gamma(tst)] \tag{4.13}$$

$$P_{\chi_3} = \frac{1}{4}[3\Gamma(e) - \Gamma(s) - \Gamma(tst^2) - \Gamma(t^2st)] \tag{4.14}$$

1. Tensor products of two triplets $(\alpha_1, \alpha_2, \alpha_3)$ $(\beta_1, \beta_2, \beta_3)$

The linear combinations corresponding singlets and triplet

1: $\alpha_1\beta_1 + \alpha_2\beta_3 + \alpha_3\beta_2$, $1'$: $\alpha_3\beta_3 + \alpha_1\beta_2 + \alpha_2\beta_1$,1" : $\alpha_2\beta_2 + \alpha_3\beta_1 + \alpha_1\beta_3$

and

$$3_s : \begin{pmatrix} 2\alpha_1\beta_1 - \alpha_2\beta_3 - \alpha_3\beta_2 \\ 2\alpha_3\beta_3 - \alpha_1\beta_2 - \alpha_2\beta_1 \\ 2\alpha_2\beta_2 - \alpha_3\beta_1 - \alpha_1\beta_3 \end{pmatrix}$$

$$3_a : \begin{pmatrix} \alpha_2\beta_3 - \alpha_3\beta_2 \\ \alpha_1\beta_2 - \alpha_2\beta_1 \\ \alpha_3\beta_1 - \alpha_1\beta_3 \end{pmatrix}$$

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \otimes \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = (\alpha_1\beta_1 + \alpha_2\beta_3 + \alpha_3\beta_2)_1 \oplus (\alpha_3\beta_3 + \alpha_1\beta_2 + \alpha_2\beta_1)_{1'} \oplus (\alpha_2\beta_2 + \alpha_3\beta_1 + \alpha_1\beta_3)_{1"} \oplus \begin{pmatrix} 2\alpha_1\beta_1 - \alpha_2\beta_3 - \alpha_3\beta_2 \\ 2\alpha_3\beta_3 - \alpha_1\beta_2 - \alpha_2\beta_1 \\ 2\alpha_2\beta_2 - \alpha_3\beta_1 - \alpha_1\beta_3 \end{pmatrix} \oplus \begin{pmatrix} \alpha_2\beta_3 - \alpha_3\beta_2 \\ \alpha_1\beta_2 - \alpha_2\beta_1 \\ \alpha_3\beta_1 - \alpha_1\beta_3 \end{pmatrix}$$

$$3 \otimes 3 = 1 \oplus 1' \oplus 1" \oplus 3_s \oplus 3_a$$

2. Tensor products of singlet (α) and triplet $(\beta_1, \beta_2, \beta_3)$

$$(\alpha)_{1'} \otimes \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} \alpha\beta_3 \\ \alpha\beta_1 \\ \alpha\beta_2 \end{pmatrix}$$

$$1' \otimes 3 = 3$$

and

$$(\alpha)_{1"} \otimes \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} \alpha\beta_2 \\ \alpha\beta_3 \\ \alpha\beta_1 \end{pmatrix}$$

$$1" \otimes 3 = 3$$

3.Tensor products of singlet (α) and (β)

$$(\alpha)_{1'} \otimes (\beta)_{1"} = (\alpha\beta)_1$$

$$1' \otimes 1" = 1$$

$$1 \otimes any = any$$

Chapter 5

Model with three Higgs doublets in the triplet representation of A_4

5.1 The potential of the three Higgs fields

The three Higgs doublets fields $\phi_i = (\phi_1, \phi_2, \phi_3)$ transform in a triplet representation of A_4 symmetry,

$$3 : \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}$$

Or products of the $\phi_i = (\phi_1, \phi_2, \phi_3)$ and $\phi_i^\dagger = (\phi_1^\dagger, \phi_2^\dagger, \phi_3^\dagger)$,

$$\begin{aligned} \phi^\dagger \otimes \phi = & \frac{1}{\sqrt{2}}(\phi^\dagger{}_3\phi_2 + \phi^\dagger{}_2\phi_3, \phi^\dagger{}_3\phi_1 + \phi^\dagger{}_1\phi_3, \phi^\dagger{}_1\phi_2 + \phi^\dagger{}_2\phi_1) : 3_s \\ & + \frac{1}{\sqrt{2}}(\phi^\dagger{}_3\phi_2 - \phi^\dagger{}_2\phi_3, \phi^\dagger{}_3\phi_1 - \phi^\dagger{}_1\phi_3, \phi^\dagger{}_1\phi_2 - \phi^\dagger{}_2\phi_1) : 3_a \\ & + \frac{1}{\sqrt{3}}(\phi_1^\dagger\phi_1 + \phi_2^\dagger\phi_2 + \phi_3^\dagger\phi_3) : 1 \\ & + \frac{1}{\sqrt{3}}(\phi_1^\dagger\phi_1 + \omega^2\phi_2^\dagger\phi_2 + \omega\phi_3^\dagger\phi_3) : 1^{'} \\ & + \frac{1}{\sqrt{3}}(\phi_1^\dagger\phi_1 + \omega\phi_2^\dagger\phi_2 + \omega^2\phi_3^\dagger\phi_3) : 1" \end{aligned} \tag{5.1}$$

Here $\omega = \exp(\frac{2\iota\pi}{3})$, and s , a stand for the symmetric and anti-symmetric triplet components.

The A_4-symmetric 3HDM can be represented by the following potential[Degee, Ivanov and Keus, 2013]

$$\begin{aligned} V = &-\frac{M_0}{\sqrt{3}}(\phi_1^\dagger\phi_1 + \phi_2^\dagger\phi_2 + \phi_3^\dagger\phi_3) + \frac{\Lambda_0}{3}(\phi_1^\dagger\phi_1 + \phi_2^\dagger\phi_2 + \phi_3^\dagger\phi_3)^2 \\ &+ \frac{\Lambda_3}{3}[(\phi_1^\dagger\phi_1)^2 + (\phi_2^\dagger\phi_2)^2 + (\phi_3^\dagger\phi_3)^2 - (\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2) - (\phi_2^\dagger\phi_2)(\phi_3^\dagger\phi_3) - (\phi_3^\dagger\phi_3)(\phi_1^\dagger\phi_1)] \\ &+ \Lambda_1[(Re\phi_1^\dagger\phi_2)^2 + (Re\phi_2^\dagger\phi_3)^2 + (Re\phi_3^\dagger\phi_1)^2] \\ &+ \Lambda_2[(Im\phi_1^\dagger\phi_2)^2 + (Im\phi_2^\dagger\phi_3)^2 + (Im\phi_3^\dagger\phi_1)^2] \\ &+ \Lambda_4[(Re\phi_1^\dagger\phi_2)(Im\phi_1^\dagger\phi_2) + (Re\phi_2^\dagger\phi_3)(Im\phi_2^\dagger\phi_3) + (Re\phi_3^\dagger\phi_1)(Im\phi_3^\dagger\phi_1)] \end{aligned} \tag{5.2}$$

Here parameters M_0 and Λ_i are assumed to take generic values.

Degee, Ivanov, and Keus have introduced a geometrical procedure to minimize highly symmetric scalar potentials, and solved the problem for a three Higgs doublet model (3HDM) potential with an A_4 or an S_4 symmetry. And it is found that the possible vev(vacuum expectation values) alignments for the A_4 symmetric potential which may correspond to a global minimum are[Degee, Ivanov and Keus, 2013]

$$\nu(1,0,0),$$

$$\nu(1,1,1)$$

$$\nu(1,\eta,\eta^*), \text{with} \quad \eta = \exp i\pi/3,$$

$$\nu(1, \exp i\alpha, 0), \text{ with any } \alpha. \tag{5.3}$$

Possible representation of the left-handed quark doublets (Q_L), the right-handed up quark singlets (u_R), and the right-handed down quark singlets (d_R), when the three Higgs doublets are in a triplet representation 3.

Q_L	u_R	d_R
3	3	3
3	3	three singlets
3	three singlets	3
3	three singlets	three singlets
three singlets	3	3

Table 5.1: Possible representation of the left-handed quark doublets (Q_L), the right-handed up quark singlets (u_R), and the right-handed down quark singlets (d_R), when the three Higgs doublets are in a triplet representation 3

The notation "three singlets" stands for the following independent possibilities for the fields in each of the three generations:

$$(1,1,1),(1,1',1"),(1,1,1'),(1',1',1'),(1,1',1'),$$

$$(1',1',1"),(1,1,1"),(1',1",1"),(1,1",1"),(1",1",1").$$

5.2 Yukawa Lagrangian terms and quark mass matrices

We are now ready to construct the Yukawa matrices for the various cases. As a first example, let us consider the case $\phi{\sim}3,(\bar{Q}_{L1},\bar{Q}_{L2},\bar{Q}_{L3})\sim(1,1,1'),d_R\sim 3$ and $u_R\sim 3$. We start with the down sector. Since $\bar{Q}_{L1}$ is in the 1 representation, it must couple to the $(\phi\otimes d_R)_1$ combination from Eq.(5.1).The same is true for $\bar{Q}_{L2}$, with an independent coefficient.

This leads to the Yukawa terms

$$\alpha_1\bar{Q}_{L1}[\phi_1 d_{R1}+\phi_2 d_{R2}+\phi_3 d_{R3}]+\alpha_2\bar{Q}_{L2}[\phi_1 d_{R1}+\phi_2 d_{R2}+\phi_3 d_{R3}] \tag{5.4}$$

Since $\bar{Q}_{L3}$ is in the $1'$ representation, we can only obtain a singlet with the 1" combination $(\phi\otimes d_R)_{1"}$ in Eq.(5.1) . This leads to the Yukawa term

$$\alpha_3 \bar{Q}_{L3}[\phi_1 d_{R1} + \omega\phi_2 d_{R2} + \omega^2\phi_3 d_{R3}] \tag{5.5}$$

Once the fields ϕ_i are substituted by the vevs ν_i, these terms give the down-type quark mass matrix, M_d

$M_d = \begin{pmatrix} \alpha_1\nu_1 & \alpha_1\nu_2 & \alpha_1\nu_3 \\ \alpha_2\nu_1 & \alpha_2\nu_2 & \alpha_2\nu_3 \\ \alpha_3\nu_1 & \omega\alpha_3\nu_2 & \omega^2\alpha_3\nu_3 \end{pmatrix}$, with arbitrary complex constants α_i.

The up-quark Yukawa terms

$$\beta_1\bar{Q}_{L1}[\phi_1 u_{R1}+\phi_2 u_{R2}+\phi_3 u_{R3}]+\beta_2\bar{Q}_{L2}[\phi_1 u_{R1}+\phi_2 u_{R2}+\phi_3 u_{R3}]+\beta_3\bar{Q}_{L3}[\phi_1 u_{R1}+\omega\phi_2 u_{R2}+\omega^2\phi_3 u_{R3}] \tag{5.6}$$

A similar analysis of the up-type quark mass matrix, M_u

$M_u = \begin{pmatrix} \beta_1\nu_1^* & \beta_1\nu_2^* & \beta_1\nu_3^* \\ \beta_2\nu_1^* & \beta_2\nu_2^* & \beta_2\nu_3^* \\ \beta_3\nu_1^* & \omega\beta_3\nu_2^* & \omega^2\beta_3\nu_3^* \end{pmatrix}$, where β_i are arbitrary complex constants.

We define the Hermitian matrices

$$H_d = M_d {M_d}^\dagger, H_u = M_u {M_u}^\dagger, \tag{5.7}$$

As a second example, let us consider the case $\phi \sim 3$, $(\bar{Q}_{L1}, \bar{Q}_{L2}, \bar{Q}_{L3}) \sim (1, 1', 1")$, $d_R \sim 3$, and $u_R \sim 3$. We find Yukawa terms and quark mass matrix for down sector:

$$\alpha_1\bar{Q}_{L1}[\phi_1 d_{R1}+\phi_2 d_{R2}+\phi_3 d_{R3}]+\alpha_2\bar{Q}_{L2}[\phi_1 d_{R1}+\omega\phi_2 d_{R2}+\omega^2\phi_3 d_{R3}] \\ +\alpha_3\bar{Q}_{L3}[\phi_1 d_{R1}+\omega^2\phi_2 d_{R2}+\omega\phi_3 d_{R3}] \quad (5.8)$$

$$M_d=\begin{pmatrix}\alpha_1\nu_1 & \alpha_1\nu_2 & \alpha_1\nu_3\\ \alpha_2\nu_1 & \omega\alpha_2\nu_2 & \omega^2\alpha_2\nu_3\\ \alpha_3\nu_1 & \omega^2\alpha_3\nu_2 & \omega\alpha_3\nu_3\end{pmatrix}, \text{ with arbitrary complex constants } \alpha_i.$$

And for up-type quark sector yields:

$$\beta_1\bar{Q}_{L1}[\phi_1 u_{R1}+\phi_2 u_{R2}+\phi_3 u_{R3}]+\beta_2\bar{Q}_{L2}[\phi_1 u_{R1}+\omega\phi_2 u_{R2}+\omega^2\phi_3 u_{R3}] \\ +\beta_3\bar{Q}_{L3}[\phi_1 u_{R1}+\omega^2\phi_2 u_{R2}+\omega\phi_3 u_{R3}] \quad (5.9)$$

$$M_u=\begin{pmatrix}\beta_1\nu_1^* & \beta_1\nu_2^* & \beta_1\nu_3^*\\ \beta_2\nu_1^* & \omega\beta_2\nu_2^* & \omega^2\beta_2\nu_3^*\\ \beta_3\nu_1^* & \omega^2\beta_3\nu_2^* & \omega\beta_3\nu_3^*\end{pmatrix}, \text{ where } \beta_i \text{ are arbitrary complex constants.}$$

If the Higgs potential is invariant under A_4, it has been shown of Eq.(5.3) the possible vev alignment $\nu(1,1,1)$ and $\nu(1,\eta,\eta^*)$. In that case M_d and M_u are digonalized,

$$M_d.\frac{1}{\sqrt{3}}\begin{pmatrix}1 & 1 & 1\\ 1 & \omega & \omega^2\\ 1 & \omega^2 & \omega\end{pmatrix}=\begin{pmatrix}\alpha_1 & \alpha_1 & \alpha_1\\ \alpha_2 & \omega\alpha_2 & \omega^2\alpha_2\\ \alpha_3 & \omega^2\alpha_3 & \omega\alpha_3\end{pmatrix}.\frac{1}{\sqrt{3}}\begin{pmatrix}1 & 1 & 1\\ 1 & \omega & \omega^2\\ 1 & \omega^2 & \omega\end{pmatrix} \\ =\frac{1}{\sqrt{3}}\begin{pmatrix}3\alpha_1 & 0 & 0\\ 0 & 3\alpha_2 & 0\\ 0 & 0 & 3\alpha_3\end{pmatrix} \quad (5.10)$$

and

$$M_u.\frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} = \begin{pmatrix} \beta_1 & \beta_1 & \beta_1 \\ \beta_2 & \omega\beta_2 & \omega^2\beta_2 \\ \beta_3 & \omega^2\beta_3 & \omega\beta_3 \end{pmatrix}.\frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}$$

$$= \frac{1}{\sqrt{3}}\begin{pmatrix} 3\beta_1 & 0 & 0 \\ 0 & 3\beta_2 & 0 \\ 0 & 0 & 3\beta_3 \end{pmatrix} \quad (5.11)$$

$$H_d = M_d {M_d}^\dagger = 3\nu^2 \begin{pmatrix} |\alpha_1|^2 & 0 & 0 \\ 0 & |\alpha_2|^2 & 0 \\ 0 & 0 & |\alpha_3|^2 \end{pmatrix}, \quad (5.12)$$

$$H_u = H_u = M_u {M_u}^\dagger = 3\nu^2 \begin{pmatrix} |\beta_1|^2 & 0 & 0 \\ 0 & |\beta_2|^2 & 0 \\ 0 & 0 & |\beta_3|^2 \end{pmatrix}, \quad (5.13)$$

meaning that, in these cases, all quark masses are non-vanishing and non-degenerate.

Requiring non-vanishing quark by itself, restricts the representations of $Q_L; u_R; d_R$ to the five possibilities $s; 3; 3$, $3; s; s$, $3; s; 3$, $3; 3; s$, and $3; 3; 3$, where s stands for $(1, 1', 1")$, with the vevs restricted to $\nu(1, 1, 1)$ or $\nu(1, \eta, \eta^*)$.

Chapter 6

Conclusions

We have studied the possibility of generating the quark masses in the context of three Higgs doublet models extended by discrete A_4 symmetry. Assuming that the Higgs fields are in the triplet representation of the discrete group. We show that none of the feasible vev alignments that correspond to a global minimum of the scalar potential leads to phenomenologically feasible mass matrices for the Standard Model's three generations of quarks.

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