INDIAN INSTITUTE OF TECHNOLOGY INDORE

Master Thesis

Model Building Using Dihedral Symmetries

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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **MODEL BUILDING USING DIHEDRAL SYMMETRIES** in the partial fulfillment of the requirements for the award of the degree of **MASTER OF SCIENCE** and submitted in the **DEPARTMENT OF PHYSICS**, **Indian Institute of Technology Indore**, is an authentic record of my own work carried out during the time period from July 2020 to June 2021 under the supervision of **Dr. Dipankar Das**, Assistant Professor, Department of Physics, IIT Indore.

The matter presented in this thesis by me has not been submitted for the award of any other degree of this or any other institute.

Ayushi 15/06/21

Signature of the student with date
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This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

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Dedicated to my Parents

Abstract

Department of Physics

Master of Science

Model Building Using Dihedral Symmetries

by Ayushi Srivastava

The standard model is tremendously successful in describing and predicting the outcome of several particle physics experiments. It provides a way to generate masses of quarks and leptons, but it does not explain the masses' hierarchy. It also does not explain the hierarchical pattern followed by quark mixings. This is known as the Flavor puzzle. In this thesis, we have constructed a four Higgs doublet model using dihedral symmetry D_4 to explain the hierarchy in masses and mixings. We have transformed our fields in such a way that the source of generation of mass for the third generation is different from the first two generations, which allows us to dilute the Yukawa coupling hierarchies. We have also connected quark mixings with the dynamics of the scalar sector, suggesting that the fact that the third generation of quarks is much heavier than the first two generations is intimately connected to the smallness of the off-Cabibbo elements.

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Abbreviations

SM	**S**tandard **M**odel
BSM	**B**eyond **S**tandard **M**odel
QCD	**Q**uantum **C**hromo-**D**ynamics
LH	**L**eft **H**anded
RH	**R**ight **H**anded
2HDM	Two **H**iggs-**D**oublet **M**odel
3HDM	Three **H**iggs-**D**oublet **M**odel
VEV	**V**acuum **E**xpectation **V**alue

The reward for work well done is the opportunity to do more ...

Jonas Salk

1

Standard Model And The Flavor Puzzle

The Standard Model of particle physics, a triumph of 20th-century physics, provides a remarkably successful framework for explaining three of the four known forces of nature. These forces or interactions are described by specifying the particles which mediate the interactions. And the number of mediators of an interaction is equal to the number of generators of the corresponding gauge group.

In the Standard Model, we have to following gauge group:

$$SU(3)_c \times SU(2)_L \times U(1)_Y \,, \tag{1.1}$$

where $SU(3)_c$ is the Quantum chromodynamics or QCD gauge group and $SU(2)_L \times U(1)_Y$ is the electroweak part. This symmetry is spontaneously broken into,

$$SU(3)_c \times SU(2)_L \times U(1)_Y \to SU(3)_c \times U(1)_Q. \tag{1.2}$$

Here Y and Q denote the weak hypercharge and the electric charge generators respectively. The QCD part of the gauge group describes the strong interactions. These interactions are mediated by eight gluons G_a. The $SU(2)_L \times U(1)_Y$ is called electroweak part of the Standard Model, because it describes both electromagnetic and weak interactions. These interactions are mediated by γ, $W^{\pm}$, Z_0 and the neutral Higgs boson H. This breaking is induced by the Vacuum expectation value(VEV) of Higgs field. It is an electroweak doublet given as

$$\phi = \begin{pmatrix} \phi^+ \\ \frac{v+H(x)+i\zeta}{2} \end{pmatrix} . \tag{1.3}$$

ϕ^+ and ζ are the unphysical modes which are eaten up by the W and Z in the process of symmetry breaking. H is the physical scalar mode called as the Higgs bosons. v is the VEV of H whose value is 246 GeV.

The flavor puzzle that we are dealing with comes from the Standard electroweak model. We will only focus on it from now on.

1.1 Standard Electroweak Model

Weak interactions are parity-violating. This means that the left chiral projection of the fermion field and the right chiral projection of the fermion field should have different transformation properties under an internal symmetry, that is, different interactions. These fermions are supposed to transform like the following gauge multiplets:

$$Q_{jL} \equiv \begin{pmatrix} p_{1L} \\ n_{1L} \end{pmatrix}, \quad \begin{pmatrix} p_{2L} \\ n_{2L} \end{pmatrix}, \quad \begin{pmatrix} p_{3L} \\ n_{3L} \end{pmatrix} : (2, 1/6)\,, \tag{1.4a}$$

$$p_{jR} \equiv \begin{pmatrix} p_{1R} \end{pmatrix}, \quad \begin{pmatrix} p_{2R} \end{pmatrix}, \quad \begin{pmatrix} p_{3R} \end{pmatrix} : (1, 2/3)\,, \tag{1.4b}$$

$$n_{jR} \equiv \begin{pmatrix} n_{1R} \end{pmatrix}, \quad \begin{pmatrix} n_{2R} \end{pmatrix}, \quad \begin{pmatrix} n_{3R} \end{pmatrix} : (1, -1/3)\,, \tag{1.4c}$$

$$L_{jL} \equiv \begin{pmatrix} \nu^E_{1L} \\ E_{1L} \end{pmatrix}, \quad \begin{pmatrix} \nu^E_{2L} \\ E_{2L} \end{pmatrix}, \quad \begin{pmatrix} \nu^E_{3L} \\ E_{3L} \end{pmatrix} : (2, -1/2)\,, \tag{1.4d}$$

$$E_{jR} \equiv \begin{pmatrix} E_{1R} \end{pmatrix}, \quad \begin{pmatrix} E_{2R} \end{pmatrix}, \quad \begin{pmatrix} E_{3R} \end{pmatrix} : (1, -1)\,. \tag{1.4e}$$

The quark and lepton fields are written in gauge basis here. We will see more about it in later section. On the right, we have shown the gauge transformation properties of the multiplets. The first term in the bracket indicates whether it is a doublet or a singlet, and the second term is the $U(1)$ hypercharge Y. The left-handed fields transform non-trivially under $SU(2)_L$, that's why the subscript L is written. No right-handed neutrinos were observed when the Standard Model was being constructed and also they have not been observed yet. And they were not needed because, at that time, neutrinos were believed to be massless. We can get the mass terms for the quarks and charged leptons when we consider the Yukawa interactions, but the left-handed neutrinos have nothing to get couple to in order to get the mass. So at the renormalizable level of SM Lagrangian, the neutrino's stay massless.

1.1.1 The Pure Gauge Lagrangian

The $SU(N)$ group has $N^2 - 1$ generators. Therefore, $SU(2)$ gauge group has three generators ($T_a = \sigma_a/2$) and $U(1)$ has one generator. Corresponding to the four generators, there are four gauge bosons in this theory. The pure gauge Lagrangian can be written as

$$\mathscr{L}_{\text{gauge}} = -\frac{1}{4} W^a_{\mu\nu} W_a^{\dagger\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu}\,, \tag{1.5}$$

where W^a_μ are the $SU(2)$ gauge bosons, and B_μ is the U(1) gauge boson. None of these gauge bosons are physical particles. The linear combinations of these bosons make up photon, $W^\pm$ and Z bosons.

1.1.2 Spontaneous Symmetry Breaking

The gauge bosons mediating weak interactions are massive but the above Lagrangian did not contain any mass term because such a term would not be gauge invariant. However, since we ultimately we want massive weak gauge bosons, we will have to break the $SU(2)_L \times U(1)_Y$ gauge group spontaneously, by introducing some type of Higgs scalar field. We take $SU(2)$ doublet of scalar, and write this as:

$$\phi \quad \equiv \quad \begin{pmatrix} \phi_+ \\ \phi_0 \end{pmatrix} : (2, 1/2)\,. \tag{1.6}$$

where the subscript denotes the hypercharge. The terms of Lagrangian which contain ϕ are

$$\mathscr{L} = (D_\mu(\phi))^\dagger (D^\mu(\phi)) - V(\phi)\,. \tag{1.7}$$

Here, D_μ is the covariant derivative given as

$$D_\mu = \partial_\mu + ig\frac{\sigma_a}{2}W_\mu^a + ig'YB_\mu, \tag{1.8}$$

where g and g' are gauge the coupling corresponding to $SU(2)_L$ and $U(1)_Y$, respectively.

The scalar potential is given as

$$V(\phi) = \mu^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2\,. \tag{1.9}$$

Choosing $\mu^2 < 0$ and $\lambda > 0$ leads to the required spontaneous symmetry breaking. The minimum of this potential is obtained for

$$|\langle\phi_1\rangle|^2 + |\langle\phi_2\rangle|^2 = \frac{v^2}{2}\,, \tag{1.10}$$

where ϕ_1 and ϕ_2 are the scalar fields which transform like a doublet of $SU(2)$ symmetry (which is denoted by ϕ_+ and ϕ_0 after assigning the hypercharge as done in Eq. (1.6)), the angular bracket denotes the value at the minimum, and

$$v = \sqrt{\frac{-\mu^2}{\lambda}}\,. \tag{1.11}$$

We will consider our system is in the minimum when

$$\langle\phi\rangle = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}, \tag{1.12}$$

i.e.,

$$\langle\phi_1\rangle = 0\,, \quad \langle\mathbb{R}\mathrm{e}\phi_2\rangle = v/\sqrt{2}\,, \quad \langle\mathbb{I}\mathrm{m}\phi_2\rangle = 0\,. \tag{1.13}$$

For an unbroken generator (Combination of $T^a + Y$) that leaves the vacuum invariant, there will be no goldstone boson and the corresponding gauge boson remains massless. If a broken generator non-trivially transforms the vacuum, then there exist a goldstone boson and the corresponding gauge boson is massive.

Claim: $Q \equiv T^3 + Y$ is the unbroken generator.

$$Q = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} + \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} . \tag{1.14}$$

Now, the field ϕ transforms as

$$\phi \rightarrow e^{i\alpha Q}\phi = \phi + i\alpha Q\phi + O(\alpha^2) , \tag{1.15a}$$

$$\langle\phi\rangle \rightarrow \langle\phi\rangle + i\alpha \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \langle\phi\rangle , \tag{1.15b}$$

$$\begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} + i\alpha \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} . \tag{1.15c}$$

Thus there is one diagonal generator which annihilates the vacuum or leaves the vacuum invariant. Thus, the original symmetry is therefore broken down to a $U(1)$ symmetry generated by Q. This is the $U(1)$ group of QED, which we can denote by writing $U(1)_{\text{em}}$. We can expand around the minimum and that expansion will give us the mass terms for $W^\pm$ and Z gauge bosons, photon remains massless.

1.1.3 Gauge Interaction Lagrangian

The kinetic energy terms for the fermions are given as:

$$\begin{aligned}\mathscr{L}_{\text{Fermion}} = \sum_{j=1}^{3} \bar{Q}_{L,j} i\gamma^\mu D_\mu Q_{L,j} + \bar{p}_{R,j} i\gamma^\mu D_\mu p_{R,j} + \bar{n}_{R,j} i\gamma^\mu D_\mu n_{R,j} + \\ \bar{L}_{L,j} i\gamma^\mu D_\mu L_{L,j} + \bar{E}_{R,j} i\gamma^\mu D_\mu E_{R,j} .\end{aligned} \tag{1.16}$$

The covariant derivatives acting on various fermion fields are given by

$$D_\mu Q_{L,j} = (\partial_\mu + \frac{i}{2} g\tau^a W^a_\mu + \frac{i}{6} g' B_\mu) Q_{L,j} , \tag{1.17a}$$

$$D_\mu u_{R,j} = (\partial_\mu + \frac{2i}{3} g' B_\mu) u_{R,j} , \tag{1.17b}$$

$$D_\mu d_{R,j} = (\partial_\mu - \frac{i}{3} g' B_\mu) d_{R,j} , \tag{1.17c}$$

$$D_\mu L_{L,j} = (\partial_\mu + \frac{i}{2} g\tau^a W^a_\mu - \frac{i}{2} g' B_\mu) L_{L,j} , \tag{1.17d}$$

$$D_\mu E_{R,j} = (\partial_\mu - i g' B_\mu) E_{R,j} . \tag{1.17e}$$

1.1.4 Yukawa Interaction Lagrangian

The Yukawa couplings are the couplings of fermions with scalar fields. It is known that there are three generations of quarks and leptons. The left-chiral components of quarks/leptons comes in $SU(2)$ doublet and the right-chiral components in singlets for any generation. Since all the generations have the same group transformation properties, the Yukawa couplings can connect any two generations. The Lagrangian is given as:

$$\mathscr{L}_{\text{Yuk}} = -\sum_{i,j=1}^{3} (Y_{ij}^{d}\overline{Q}_{iL}\phi n_{jR} + Y_{ij}^{u}\overline{Q}_{iL}\tilde{\phi} p_{jR} + Y_{ij}^{l}\overline{L}_{iL}\phi E_{jR} + h.c.)\,, \tag{1.18}$$

where $\tilde{\phi} = i\sigma_2\phi^*$. The terms in the Lagrangian are gauge invariant. Both ψ_L and ϕ are $SU(2)$ doublets, so the combination of them is $SU(2)$ singlet. The field E_R is a $SU(2)$ singlet, so the overall term transforms trivially under $SU(2)$. For the $U(1)$ part of the gauge group, the sum of hypercharges of the combination of the fields comes out to be zero. Hence, the interaction is invariant under the $SU(2)_L \times U(1)_Y$ gauge group.

1.2 Fermion Masses From Yukawa Couplings

Gauge interactions are flavor universal, which means they do not care whether they are talking to the first, the second, or the third generation. These flavor universal couplings are broken in the Standard Model once we introduce the Yukawa couplings. And then, via the Higgs mechanism, inserting the VEV into the Yukawa coupling, we can get the fermion masses.
The Yukawa Lagrangian(from Eq. (1.18)) is given as

$$\mathscr{L}_{\text{Y}} = -\sum_{i,j=1}^{3} (Y_{ij}^{d}\overline{Q}_{iL}\phi n_{jR} + Y_{ij}^{u}\overline{Q}_{iL}\tilde{\phi} p_{jR} + Y_{ij}^{l}\overline{L}_{iL}\phi E_{jR} + h.c.)\,, \tag{1.19}$$

where $\tilde{\phi} = i\sigma_2\phi^* = \epsilon\phi^*$. Q_L, L_L are the left-handed $SU(2)$ quark and lepton doublets. n_R and p_R are right-handed quark singlets. And E_R is the right handed quark singlet. Now,

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \tilde{\phi} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \phi^+ \\ \phi^o \end{pmatrix}^* = \begin{pmatrix} \phi^{o*} \\ -\phi^- \end{pmatrix}. \tag{1.20}$$

For producing fermion mass terms,

$$\langle\phi\rangle = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} . \tag{1.21}$$

And the fermion mass matrix comes out to be

$$m_A = \frac{v}{\sqrt{2}} Y_A \, , \tag{1.22}$$

where A denotes u, d and l. These matrices are 3×3 square matrices. They do not have to be diagonal. And in reality, this is what happens.

1.2.1 Quark Sector

For the quark sector, the Yukawa Lagrangian is written as

$$\begin{aligned}
\mathscr{L}_{\text{Yuk}} = -(Y^d_{11}\bar{Q}_{1L}\phi n_{1R} + Y^d_{12}\bar{Q}_{1L}\phi n_{2R} + Y^d_{13}\bar{Q}_{1L}\phi n_{3R} + Y^d_{21}\bar{Q}_{2L}\phi n_{1R} + Y^d_{22}\bar{Q}_{2L}\phi n_{2R} \\
+Y^d_{23}\bar{Q}_{2L}\phi n_{3R} + Y^d_{31}\bar{Q}_{3L}\phi n_{1R} + Y^d_{32}\bar{Q}_{3L}\phi n_{2R} + Y^d_{33}\bar{Q}_{3L}\phi n_{3R} \\
+Y^u_{11}\bar{Q}_{1L}\tilde{\phi} p_{1R} + Y^u_{12}\bar{Q}_{1L}\tilde{\phi} p_{2R} + Y^u_{13}\bar{Q}_{1L}\tilde{\phi} p_{3R} + Y^u_{21}\bar{Q}_{2L}\tilde{\phi} p_{1R} + Y^u_{22}\bar{Q}_{2L}\tilde{\phi} p_{2R} \\
+Y^u_{23}\bar{Q}_{2L}\tilde{\phi} p_{3R} + Y^u_{31}\bar{Q}_{3L}\tilde{\phi} p_{1R} + Y^u_{32}\bar{Q}_{3L}\tilde{\phi} p_{2R} + Y^u_{33}\bar{Q}_{3L}\tilde{\phi} p_{3R} \\
+h.c.) \, ,
\end{aligned} \tag{1.23}$$

where Q is $SU(2)_L$ doublet given as

$$Q_{iL} = \begin{pmatrix} p_{iL} \\ n_{iL} \end{pmatrix} . \tag{1.24}$$

Using Eq. (1.21), writing the mass terms:

$$\begin{aligned}
\mathscr{L}_{\text{mass}} = -\frac{v}{\sqrt{2}}(Y^d_{11}\bar{n}_{1L}n_{1R} + Y^d_{12}\bar{n}_{1L}n_{2R} + Y^d_{13}\bar{n}_{1L}n_{3R} + Y^d_{21}\bar{n}_{2L}n_{1R} + Y^d_{22}\bar{n}_{2L}n_{2R} \\
+Y^d_{23}\bar{n}_{2L}n_{3R} + Y^d_{31}\bar{n}_{3L}n_{1R} + Y^d_{32}\bar{n}_{3L}n_{2R} + Y^d_{33}\bar{n}_{3L}n_{3R} \\
+Y^u_{11}\bar{p}_{1L}p_{1R} + Y^u_{12}\bar{p}_{1L}p_{2R} + Y^u_{13}\bar{p}_{1L}p_{3R} + Y^u_{21}\bar{p}_{2L}p_{1R} + Y^u_{22}\bar{p}_{2L}p_{2R} \\
+Y^u_{23}\bar{p}_{2L}p_{3R} + Y^u_{31}\bar{p}_{3L}p_{1R} + Y^u_{32}\bar{p}_{3L}p_{2R} + Y^u_{33}\bar{p}_{3L}p_{3R} \\
+h.c.) \, .
\end{aligned} \tag{1.25}$$

The above equation can also be written like this

$$\begin{aligned}\mathscr{L}_{\text{mass}} = \quad & -\frac{v}{\sqrt{2}}\begin{pmatrix}\bar{n}_{1L} & \bar{n}_{2L} & \bar{n}_{3L}\end{pmatrix}\begin{pmatrix}Y^d_{11} & Y^d_{12} & Y^d_{13}\\ Y^d_{21} & Y^d_{22} & Y^d_{23}\\ Y^d_{31} & Y^d_{32} & Y^d_{33}\end{pmatrix}\begin{pmatrix}n_{1L}\\ n_{2L}\\ n_{3L}\end{pmatrix}+\\ & -\frac{v}{\sqrt{2}}\begin{pmatrix}\bar{p}_{1L} & \bar{p}_{2L} & \bar{p}_{3L}\end{pmatrix}\begin{pmatrix}Y^u_{11} & Y^u_{12} & Y^u_{13}\\ Y^u_{21} & Y^u_{22} & Y^u_{23}\\ Y^u_{31} & Y^u_{32} & Y^u_{33}\end{pmatrix}\begin{pmatrix}p_{1L}\\ p_{2L}\\ p_{3L}\end{pmatrix}+h.c.\,. \qquad (1.26)\end{aligned}$$

Here,

$$M_d = \frac{v}{\sqrt{2}}\begin{pmatrix}Y^d_{11} & Y^d_{12} & Y^d_{13}\\ Y^d_{21} & Y^d_{22} & Y^d_{23}\\ Y^d_{31} & Y^d_{32} & Y^d_{33}\end{pmatrix}, \quad M_u = \frac{v}{\sqrt{2}}\begin{pmatrix}Y^u_{11} & Y^u_{12} & Y^u_{13}\\ Y^u_{21} & Y^u_{22} & Y^u_{23}\\ Y^u_{31} & Y^u_{32} & Y^u_{33}\end{pmatrix} \qquad (1.27)$$

are called the mass matrices for the down-type and up-type quark, respectively. These matrices are not diagonal. The fields p_i, n_i do not correspond to physical particles. We have to diagonalize the above matrices to get the physical fields.

Bi-unitary Transformation: For any matrix A, we can find two unitary matrices U_L and U_R such that $U_L A U_R^\dagger$ is diagonal, with real non-negative entries along the diagonal.

This means that to go from gauge eigenbasis to mass eigenbasis, we do the unitary transformation of fields, in which the left-chiral and right-chiral fields change by different amounts. Hence, the matrix sandwich between them is diagonal. Therefore,

$$m_A = U_L^\dagger \cdot m_A^{\text{diag}} \cdot U_R\,. \qquad (1.28)$$

The left-handed down-type fields transform as

$$\begin{pmatrix}n_{1L}\\ n_{2L}\\ n_{3L}\end{pmatrix} = U_L^\dagger\begin{pmatrix}d_L\\ s_L\\ b_L\end{pmatrix} \implies \begin{pmatrix}\bar{n}_{1L} & \bar{n}_{2L} & \bar{n}_{3L}\end{pmatrix} = \begin{pmatrix}\bar{d}_L & \bar{s}_L & \bar{b}_L\end{pmatrix} U_L\,. \qquad (1.29)$$

Similarly, for the right-handed down-type field

$$\begin{pmatrix} n_{1R} \\ n_{2R} \\ n_{3R} \end{pmatrix} = U_R^\dagger \begin{pmatrix} d_R \\ s_R \\ b_R \end{pmatrix} . \tag{1.30}$$

And for the left-chiral up-type fields,

$$\begin{pmatrix} p_{1L} \\ p_{2L} \\ p_{3L} \end{pmatrix} = V_L^\dagger \begin{pmatrix} u_L \\ c_L \\ t_L \end{pmatrix} \implies \begin{pmatrix} \bar{p}_{1L} & \bar{p}_{2L} & \bar{p}_{3L} \end{pmatrix} = \begin{pmatrix} \bar{u}_L & \bar{c}_L & \bar{t}_L \end{pmatrix} V_L . \tag{1.31}$$

Right-chiral feild transforms as

$$\begin{pmatrix} p_{1R} \\ p_{2R} \\ p_{3R} \end{pmatrix} = V_R^\dagger \begin{pmatrix} u_R \\ c_R \\ t_R \end{pmatrix} . \tag{1.32}$$

The mass terms for the down-type and up-type quarks can then be rewritten as

$$\begin{aligned} \mathscr{L}_{\text{mass}} = \quad & -\frac{v}{\sqrt{2}} \begin{pmatrix} \bar{d}_L & \bar{s}_L & \bar{b}_L \end{pmatrix} U_L \cdot M_d \cdot U_R^\dagger \begin{pmatrix} d_R \\ s_R \\ b_R \end{pmatrix} \\ & -\frac{v}{\sqrt{2}} \begin{pmatrix} \bar{u}_L & \bar{c}_L & \bar{t}_L \end{pmatrix} V_L \cdot M_u \cdot V_R^\dagger \begin{pmatrix} u_R \\ c_R \\ t_R \end{pmatrix} + h.c. . \end{aligned} \tag{1.33}$$

Here, $D_d = U_L \cdot M_d \cdot U_R^\dagger$ and $D_u = V_L \cdot M_u \cdot V_R^\dagger$ are diagonal matrices.

1.2.2 Lepton Sector

Writing explicitly the Yukawa interaction Lagrangian for the lepton sector:

$$\begin{aligned} \mathscr{L}_{\text{Yuk}} = -(&Y_{11}^l \bar{L}_{1L}\phi E_{1R} + Y_{12}^l \bar{L}_{1L}\phi E_{2R} + Y_{13}^l \bar{L}_{1L}\phi E_{3R} + Y_{21}^l \bar{L}_{2L}\phi E_{1R} + Y_{22}^l \bar{L}_{2L}\phi E_{2R} \\ &+ Y_{23}^l \bar{L}_{2L}\phi E_{3R} + Y_{31}^l \bar{L}_{3L}\phi E_{1R} + Y_{32}^l \bar{L}_{3L}\phi E_{2R} + Y_{33}^l \bar{L}_{3L}\phi E_{3R} + h.c) , \end{aligned} \tag{1.34}$$

where L is the $SU(2)_L$ doublet of the form

$$L_{iL} = \begin{pmatrix} \nu_{iL} \\ E_L \end{pmatrix} . \tag{1.35}$$

After spontaneous symmetry breaking, using Eq. (1.21), the mass terms are

$$\begin{aligned} \mathscr{L}_{\text{mass}} = -\frac{v}{\sqrt{2}}(Y^l_{11}\bar{E}_{1L}E_{1R} + Y^l_{12}\bar{E}_{1L}E_{2R} + Y^l_{13}\bar{E}_{1L}E_{3R} + Y^l_{21}\bar{E}_{2L}E_{1R} + Y^l_{22}\bar{E}_{2L}E_{2R} \\ +Y^l_{23}\bar{E}_{2L}E_{3R} + Y^l_{31}\bar{E}_{3L}E_{1R} + Y^l_{32}\bar{E}_{3L}E_{2R} + Y^l_{33}\bar{E}_{3L}E_{3R} + h.c.) \, , \end{aligned} \tag{1.36}$$

$$\mathscr{L} = -\frac{v}{\sqrt{2}} \sum_{i,j=1}^{3} (Y^l_{ij}\bar{E}_{iL}E_{jR} + h.c.) \, . \tag{1.37}$$

In the generation indices Y^l is a matrix. We can redefine our fields in a way such that the matrix Y^l becomes diagonal.

$$\begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}_L = E_L^\dagger \begin{pmatrix} e_L \\ \mu_L \\ \tau_L \end{pmatrix} , \quad \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}_R = E_R^\dagger \begin{pmatrix} e_R \\ \mu_R \\ \tau_R \end{pmatrix} . \tag{1.38}$$

Here, e, μ and τ represent physical fields. After changing the basis, the mass terms are written as

$$\mathscr{L}_{\text{mass}} = -\frac{v}{\sqrt{2}} \begin{pmatrix} \bar{e}_L & \bar{\mu}_L & \bar{\tau}_L \end{pmatrix} E_L \cdot M_d \cdot E_R^\dagger \begin{pmatrix} e_R \\ \mu_R \\ \tau_R \end{pmatrix} + h.c. \, . \tag{1.39}$$

As we can see, there are no mass terms for neutrinos because right-handed neutrinos do not exist in the Standard Model. There is only one kind of mass term, and those are for the charged leptons as given in Eq. (1.39). We could work in the basis of generations where these terms are diagonal.

There was no need to write the lepton fields as defined in Eq. (1.4a). We could have started with the physical fields. But we did it the same way for quarks and leptons to show the difference between the quark sector and the lepton sector.

1.3 Gauge Interactions

We have rotated our fermion fields with unitary transformation. Let's see what happens to the rest of the Standard Model lagrangian, in particular gauge interaction.

Rotation of right-handed quark fields: Rotation of right-handed quark fields leaves the Lagrangian invariant. To see this, consider the gauge interaction term for p_R:

$$\begin{aligned} i\bar{p}_R\gamma^\mu D_\mu p_R = i\bar{p}_R\gamma^\mu(\partial_\mu + i\frac{2}{3}g'B_\mu)p_R = & \quad i\bar{p}_R\gamma^\mu(\partial_\mu p_R + i\frac{2}{3}g'B_\mu p_R) \\ = & \quad i\bar{p}_R\gamma^\mu\partial_\mu p_R - \frac{2}{3}g'\bar{p}_R\gamma^\mu B_\mu p_R\,. \end{aligned} \tag{1.40}$$

p_R transforms as

$$p_R = V_R^\dagger u_R \implies \bar{p}_R = \bar{u}_R V_R\,. \tag{1.41}$$

Therefore,

$$i\bar{p}_R\gamma^\mu\partial_\mu p_R - \frac{2}{3}g'\bar{p}_R\gamma^\mu B_\mu p_R = i\bar{u}_R\gamma^\mu\partial_\mu u_R - \frac{2}{3}g'\bar{u}_R\gamma^\mu B_\mu u_R\,. \tag{1.42}$$

Whether we choose the gauge eigenbasis or mass eigenbasis, the gauge interaction term looks the same for the right-handed quark field.

Gauge interaction term for left-handed quark field: The matrix M_u and M_d is not necessarily be diagonalized by the same matrix. This mismatch between the left-handed up-type and down-type quark sector is accounted by the Cabibbo–Kobayashi–Maskawa(CKM) matrix [1, 2]. It enters the charged current interaction term mediated by W_μ^+ gauge bosons as shown below

$$\mathscr{L}_{\rm int} = \quad \frac{g}{\sqrt{2}}\bar{p_L}\gamma^\mu W_\mu^+ n_L\,. \tag{1.43}$$

After rotating the fields, Eq. (1.43) is written as

$$\begin{aligned} \mathscr{L}_{\rm int} \quad &= -\frac{g}{\sqrt{2}}\bar{u_L}V_L\gamma^\mu W_\mu^+ U_L^\dagger d_L\,, \\ &= -\frac{g}{\sqrt{2}}\bar{u_L}V_L U_L^\dagger\gamma^\mu W_\mu^+ d_L\,, \end{aligned}$$

$$= -\frac{g}{\sqrt{2}}\bar{u_L} V_{\mathrm{CKM}} \gamma^\mu W_\mu^+ d_L\,. \tag{1.44}$$

CKM matrix is sometimes also referred to as the quark mixing matrix because the charged current gauge interactions couples any up-type quark to a down-type quark of any generation. Note that the mixing does not appear in neutral currents involving the Z boson. The neutral current interactions do not change quark flavor. There are no flavor-changing interactions in the SM in the lepton sector because we don't have any rotation matrices that don't drop out from the rest of the Lagrangian.

1.4 CKM Matrix

The quark mixing matrix is given as

$$V_{\mathrm{CKM}} = V_L \cdot U_L^\dagger = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}. \tag{1.45}$$

1.4.1 Counting The Number Of Parameters Of V_{CKM}

As we can see, CKM Matrix is unitary matrix. Number of parameters of $N \times N$ unitary matrix is N^2. Here, N corresponds to number of generations. If we allow only the real values of mixing matrix, then this matrix would represent a rotation matrix in $N-$dimensional space. The rotation matrix has $\frac{1}{2}N(N-1)$ parameters. Then the remaining parameters of the unitary matrix, $N^2 - \frac{1}{2}N(N-1) = \frac{1}{2}N(N+1)$, represent the phases.

A 3×3 CKM matrix therefore have three real parameters and six phases. But not all of these phases are physical and we can just rotate them away. Our interaction Lagrangian is insensitive to the phases of left-handed field. Possible field redefinition:

$$u_{jL} \to e^{i\phi(u_{jL})} u_{jL}\,, \quad d_{jL} \to e^{i\phi(d_{jL})} d_{jL}\,. \tag{1.46}$$

It therefore seems that we can absorb 6 phases from the CKM matrix. But this is not really correct, because one of the phases will be fixed by redefining the other fields. Counting this one exception out, we can now write the number of physically observable phases in the CKM matrix to be $6-5=1$. Hence, the CKM matrix has four parameters, three quark mixing angles and one

CP-violating complex phase. It, therefore, seems that we can absorb 6 phases from the CKM matrix. But this is not correct because one of the phases will be fixed by redefining the other fields. Counting this one exception out, we can now write the number of physically observable phases in the CKM matrix to be $6 - 5 = 1$. Hence, the CKM matrix has four parameters, three quark mixing angles, and one CP-violating complex phase.

1.4.2 Parametrization Of CKM Matrix

Many parametrizations of the CKM matrix have been proposed in the literature. We will discuss here two parametrizations: the standard parametrization [3, 4] recommended by particle data group [5] and Wolfenstein Parametrization [6].

1.4.2.1 Standard Parametrization

The standard parametrization is given by:

$$V_{\text{CKM}} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}, \tag{1.47}$$

where $c_{ij} = \cos(\theta_{ij})$ and $s_{ij} = \sin(\theta_{ij})$ with $(i, j = 1, 2, 3)$. And δ is the phase necessary for CP violation. The elements of CKM matrix modifies the strength of charged-current interaction and by studying various processes, it is possible to estimate the magnitude of the elements. The angles θ_{ij} can be chosen to lie in the first quadrant, so $c_{ij}, s_{ij} \geq 0$. The estimates of magnitudes are given as [5]:

$$V_{\text{CKM}} = \begin{pmatrix} 0.97370 \pm 0.00014 & 0.2245 \pm 0.0008 & 0.00382 \pm 0.00024 \\ 0.221 \pm 0.004 & 0.987 \pm 0.011 & 0.0410 \pm 0.0014 \\ 0.0080 \pm 0.0003 & 0.0388 \pm 0.0011 & 1.013 \pm 0.030 \end{pmatrix}. \tag{1.48}$$

The matrix is almost unit matrix, with diagonal element close to unity and small off-diagonal elements. Also, the off-diagonal elements involving the first two generations are greater than all other off-diagonal elements. More explicitly, we can write $s_{13} << s_{23} << s_{12} << 1$. This hierarchy is conveniently described by the Wolfenstein parameterization.

1.4.2.2 Wolfenstein Parameterization

Wolfenstein introduced a parameter $\lambda = s_{12}$ to denote the smallness of all elements of the mixing matrix. s_{23} is then defined by $A\lambda^2$, which is another order of smallness down from s_{12}. The element V_{ub} is called $A\lambda^3(\rho - i\eta)$, cubic in λ. The factor $\rho - i\eta$ ensures that this element is complex. The rest of the elements are then found using the unitarity of the matrix. Traditionally, the form of the matrix is

$$V_{\text{CKM}} = \begin{pmatrix} 1 - \frac{1}{2}\lambda^2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1 - \frac{1}{2}\lambda^2 & A\lambda^2 \\ A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 & 1 \end{pmatrix}. \tag{1.49}$$

There are four parameters in this parametrization, as is there in Eq. (1.47). The Wolfenstein parametrization is certainly more transparent than the standard parametrization. But Eq. (1.49) is only an approximation. To achieve sufficient level of accuracy, one must include higher order terms in λ.

1.5 Why Would One Go Beyond SM?

The Standard Model is a tremendously successful theory. It's probably the most predictive and precise theoretical explanation for observations across a wide range of energies. So why would one go beyond the Standard Model?

While we have learned a great deal in the 20th century, there are still many things that are not known. The model provides a way to generate masses of quarks and leptons but it does not explain the hierarchical masses of quarks and leptons (Flavor puzzle). The first generation quark masses are of few MeVs and the third generation's are in Gevs. The approximate masses [5] are shown in the table below:

$m_u \approx 2.16$	$m_c \approx 1279$	$m_t \approx 172760$
$m_d \approx 4.67$	$m_s \approx 93$	$m_b \approx 4180$
$m_e \approx 0.511$	$m_\mu \approx 105.66$	$m_\tau \approx 1776.86$

TABLE 1.1: Quarks and leptons masses in MeVs

As we can see from the table that there is about five orders of magnitude between the various quark masses and similarly in the lepton sector. Therefore, the Yukawa couplings needed to generate the fermion masses spans six order of magnitude($h_e \sim 10^{-6}$ to $h_t \sim 1$), whereas the gauge couplings do not exhibit such an apparent hierarchy. This is something that we do not understand. These are just some free parameters in the standard model. Aside from the fermion masses, CKM matrix also has a very hierarchical structure. This becomes clear if we look at the Wolfenstein Parametrization(Eq. (1.49)).

These are the problems that we will explain by providing a model in which sources of generation of masses for the third generation is different from the first two generation. In the next chapter, we will go through the basics of group theory. And then we will discuss some properties of Dihedral groups, necessary for constructing models beyond Standard Model(BSM). In the last chapter, we will discuss our model and its implications.

"Symmetry is a vast subject, significant in art and nature. Mathematics lies at its root, and it would be hard to find a better one on which to demonstrate the working of mathematics intellect."

Hermann Weyl

2

The Group Of Dihedral Symmetries

Groups arise everywhere in nature, science, and mathematics, usually as collections of transformations of some set that preserve some exciting structure. These transformations are the symmetry operations that can be classified as either continuous or discrete. In each case, these operations are represented by the group elements. For continuous symmetries, we have continuous groups (Lie groups), and for discrete symmetries, we have discrete groups that can be both finite and infinite. We can create a group from any geometric shape by looking at the symmetries of the shape. The symmetries are the transformations where we flip and rotate the shape to look the same before and after. The symmetries form a group called "The group of symmetries". When the shape is a regular polygon, the group of symmetries is called the Dihedral Group. The knowledge of group theory can serve as a powerful tool for simplifying the complex system and has often been used in elementary particle physics to construct new theoretical models.

2.1 Introduction To Group Theory

A group $\mathbb{G}$ is a set of elements under operation "*" that have the following properties:

1. *CLOSURE*: If a,b are the members of $\mathbb{G}$ then there exist c in $\mathbb{G}$ such that

$$c = a * b \,. \tag{2.1}$$

2. *ASSOCIATIVITY*: If a,b,c are the members of $\mathbb{G}$ then

$$a * (b * c) = (a * b) * c \,. \tag{2.2}$$

3. *EXISTENCE OF IDENTITY*: There exist e in $\mathbb{G}$ such that

$$e * a = a * e = a \,. \tag{2.3}$$

4. *INVERSE*: For every element a in $\mathbb{G}$ there is a corresponding inverse element a^{-1} such that

$$a * a^{-1} = a^{-1} * a = e \,. \tag{2.4}$$

2.1.1 Abelian Group

If a group has a further property $a * b = b * a$ for all a,b in the group $\mathbb{G}$, the group is called Abelian.

2.1.2 Subgroup

A subgroup is a set of elements in group $\mathbb{G}$ under operation * which satisfy all the properties of the group under the same operation. Identity Element 'e' itself forms a group.

2.1.3 Types Of Group

Groups can be infinite or finite :-

Infinite Groups contain infinite number of elements (Refer to example 1).

Finite Groups contain finite number element in which every element a has a finite order n such that $a^n = e$ (Example 2).

2.1.4 Multiplication Table

The group multiplication table is a square grid with one row and one column for each element in the set, the grid is filled in so that the element in the row belonging to a and belonging to column b is $a * b$. Construction of multiplication table is done in example 2.

2.1.5 Examples

1. **Group of integers $\{\mathbb{Z}, +\}$**: This is an infinite group under addition operation.
   (a) Integers are closed under addition.
   (b) Addition is associative.
   (c) Identity element: e=0.
   (d) Inverse exist: For every element a in $\mathbb{Z}$ there exist -a in $\mathbb{Z}$ such that $a + (-a) = 0 = e$.
2. **Fourth root of unity under multiplication**: This is a finite group whose elements are $1, -1, i$ and $-i$.

$a * b$	1	-1	i	-i
1	1	-1	i	-i
-1	-1	1	-i	i
i	i	-i	-1	1
-i	-i	i	1	-1

TABLE 2.1: Multiplication table for fourth root of unity

   (a) Elements are closed under multiplication.
   (b) Multiplication of numbers is associative.
   (c) Identity Element: e=1.
   (d) Inverse Exist: For every element there exist its corresponding inverse which can be found using multiplication table.
   To find the inverse of -1:
   1. Locate -1 in row 1.

2. Check the corresponding column.

Here, the column corresponding to -1 is

$$\begin{bmatrix} -1 \\ 1 \\ -i \\ i \end{bmatrix}$$

3. Stop at the row at which the identity element appears. The identity element in this example is 1, which appears in the third row.

4. Check the corresponding element in the first column. The first column and third row give the inverse of -1, which is -1.

Similarly, the inverse of i is $-i$ and $-i$ is i. Identity is its inverse.

3. **Set of non singular matrices**

   Set of all matrices whose determinant is not zero forms a non-abelian group under matrix multiplication, where the identity element is given as

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

   for a $n \times n$ matrix.

2.1.6 Order

The order of a group is defined as the number of elements in the group.

Lagrange's theorem: If a subset H of the group G is also a group, H is called the subgroup of G. Then according to this theorem order of the subgroup H must be a divisor of the order of G.

The order of an element a is the number h for which $a^h = e$.

2.1.7 Conjugacy Class

The elements $g^{-1}ag$ for $g \in G$ are called elements conjugate to the element a. The set including all elements to conjugate to an element a of G, $\{g^{-1}ag, \forall g \in G\}$, is called a conjugacy class. The order of all the elements in a conjugacy class is same since

$$(gag^{-1})^h = ga(g^{-1}g)a(g^{-1}g)\cdots ag^{-1} = ga^hg^{-1}geg^{-1} = e\,. \tag{2.5}$$

The conjugacy class of the identity element e consist of a single element e.

2.1.8 Characters And Orthogonality Relations

The character χ of the a representation $D(g)$ is given as the trace of that representation and they follow the following orthogonality relations:-

$$\sum_{g\in G} \chi_{D_\alpha(g)} * \chi_{D_\beta(g)} = N_G\delta_{\alpha\beta}\,, \tag{2.6a}$$

$$\sum_{\alpha} \chi_{D_\alpha(g_i)} * \chi_{D_\alpha(g_j)} = \frac{N_G}{n_i}\delta_{C_iC_j}\,, \tag{2.6b}$$

where D_α, D_β denotes the irreducible representation, N_G is the number of elements in the group, C_i represents the conjugacy class and n_i is the number of the elements in the conjugacy class.

2.1.9 Homomorphism And Isomorphism

Homomorphism: Homomorphism is a map between two groups such that the group operation is preserved. A function $f : G \to H$ between two groups is a homomorphism when it f satisfies the following property for all $x, y \in G$

$$f(x * y) = f(x) \diamond f(y). \tag{2.7}$$

Let's see an example of homomorphism. Suppose we have a map $f : (\mathbb{R}, +) \to (\mathbb{R}^+, *)$. For it to be a homomorphism, it should satisfy eqn(2.7).

$$f(x) = e^x\,, \tag{2.8}$$

$$f(y) = e^y\,, \tag{2.9}$$

$$f(x+y) \quad = \quad e^{x+y} = e^x \cdot e^y = f(x) * f(y)\,. \tag{2.10}$$

Therefore, from above equations we can see that f is a homomorphism.

Isomorphism: Two groups are said to be isomorphic if they have the same multiplication table. If G is isomorphic to H then there is one-to-one correspondence between the elements of G and H.

For $g_i \to h_i$ to be an isomorphism, the elements of H should satisfy $h_i \diamond h_j = h_k$ if $g_i * g_j = g_k$.
Example of isomorphism: The permutation group S_3 is isomorphic to symmetry operations that take an equilateral triangle into itself that is group D_3.

2.2 Representations

A set of square, non-singular matrices T(g) associated with the elements of a group $g \in G$ such that if $g_1 g_2 = g_3$ then $T(g_1)T(g_2) = T(g_3)$. That is, T is a homomorphism.

2.2.1 Identity Representation Matrix

If e is the identity element of the group, then $T(e) = 1$ (Identity matrix).

2.2.2 Identity Representation

$T(g) = 1$ for all g in G, also known as trivial representation.

2.2.3 Faithful Representation

All $T(g)$ are distinct. That is, T is isomorphism.

2.2.4 Example

Group of integers under addition modulo 4 : $(\mathbb{Z}_4 = \{0, 1, 2, 3\}, +_4)$.

Matrix representation: We will represent each of the above mentioned element as matrices.

$a +_4 b$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Table 2.2: Multiplication table for Z_4

Element 0 is represented by the matrix $\phi(0)$, element 1 is represented by the matrix $\phi(1)$, element 2 is represented by the matrix $\phi(2)$ and element 3 is represented by the matrix $\phi(3)$. We can think of these matrices as rotation matrices and they are given as following:
$\phi(1)$: Rotates $\mathbb{R}^2$ by 90 degree.

$$\phi(1) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{2.11}$$

$\phi(2)$: Rotates $\mathbb{R}^2$ by 180 degree.

$$\phi(2) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} * \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \tag{2.12}$$

$\phi(3)$: Rotates $\mathbb{R}^2$ by 270 degree.

$$\phi(3) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} * \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{2.13}$$

$\phi(0)$: Rotates $\mathbb{R}^2$ by 360 degree or identity.

$$\phi(0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} * \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{2.14}$$

$\phi(x) * \phi(y)$	$\phi(0)$	$\phi(1)$	$\phi(2)$	$\phi(3)$
$\phi(0)$	$\phi(0)$	$\phi(1)$	$\phi(2)$	$\phi(3)$
$\phi(1)$	$\phi(1)$	$\phi(2)$	$\phi(3)$	$\phi(0)$
$\phi(2)$	$\phi(2)$	$\phi(3)$	$\phi(0)$	$\phi(1)$
$\phi(3)$	$\phi(3)$	$\phi(0)$	$\phi(1)$	$\phi(2)$

Table 2.3: Multiplication table of the rotation matrices

As we can see that table 2 and 3 are similar. Therefore we can conclude that the above-given example is an example of faithful representation or one-to-one representation.

2.2.5 Equivalent Representation

If T is a representation then

$$T' = S^{-1}TS \tag{2.15}$$

is also a representation where S can be any arbitrary non-singular matrix because

$$T'(A)T'(B) = S^{-1}T(A)SS^{-1}T(B)S = S^{-1}T(A)T(B)S = S^{-1}T(AB)S = T'(AB) \tag{2.16}$$

T and T' are equivalent. By choosing different S we can get different representations of the same matrix.

Example: Consider T and S as given below. S^{-1} is also given for convenience.

$$T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad S^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \tag{2.17}$$

Then one can easily find the similarity transformation of T as

$$T' = S^{-1}TS = \begin{bmatrix} -3 & -5 \\ 2 & 3 \end{bmatrix} \tag{2.18}$$

2.2.6 Inequivalent Representations

Representations T and T' for which it is impossible to find a similarity transform S relating them.

2.2.7 Reducible And Irreducible Representation

A representation T is *Reducible Representation* if it's equivalent representation T' has the form

$$T' = \begin{bmatrix} T_1 & 0 & \cdots & 0 \\ 0 & T_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & T_n \end{bmatrix} \tag{2.19}$$

where T_n are square matrices. If a representation cannot be reduced in the above form then it is called *Irreducible Representation.* A reducible representation can be reduced into a number of irreducible representations.

2.3 Dihedral Groups

The dihedral group D_n $(n \geq 3)$ is the group of symmetries of a regular polygon with n vertices. The order of dihedral group is $2n$.

Dihedral group notation : $D_n = \{a, b | a^n = b^2 = (ab)^2 = 1\}$ where a, b represents rotation(by a multiple of 2π radians around the center) and reflection(about an axis), respectively, which takes the regular polygon back to itself.

2.3.1 Group Structure Of D_4

This group is the symmetry group of square. The eight symmetries of a square:

1. $R_0 = e = 0°$ **Rotation(Identity)**

2. $R_{90} = 90°$ **Rotation(Counter Clockwise)**

3. $R_{180} = 180°$ **Rotation(Counter Clockwise)**

4. $R_{270} =$ **270° Rotation(Counter Clockwise)**

5. F_X = **Reflection about horizontal axis**

6. F_Y = **Reflection about vertical axis**

7. F_{D1} = **Reflection about diagonal**

8. F_{D2} = **Reflection about other diagonal**

Characteristic Table for the above group of symmetries $\mathbb{G} = \{$e,R_{90},R_{180},R_{270},F_X,F_Y,F_{D1},F_{D2},$\}$

ab	e	R_{90}	R_{180}	R_{270}	F_X	F_Y	F_{D1}	F_{D2}
e	e	R_{90}	R_{180}	R_{270}	F_X	F_Y	F_{D1}	F_{D2}
R_{90}	R_{90}	R_{180}	R_{270}	e	F_{D2}	F_{D1}	F_X	F_Y
R_{180}	R_{180}	R_{270}	e	R_{90}	F_Y	F_X	F_{D2}	F_{D1}
R_{270}	R_{270}	e	R_{90}	R_{180}	F_{D1}	F_{D2}	F_Y	F_X
F_X	F_X	F_{D1}	F_Y	F_{D2}	e	R_{180}	R_{90}	R_{270}
F_Y	F_Y	F_{D2}	F_X	F_{D1}	R_{180}	e	R_{270}	R_{90}
F_{D1}	F_{D1}	F_Y	F_{D2}	F_X	R_{270}	R_{90}	e	R_{180}
F_{D2}	F_{D2}	F_X	F_{D1}	F_Y	R_{90}	R_{270}	R_{180}	e

TABLE 2.4: Multiplication table for D_4

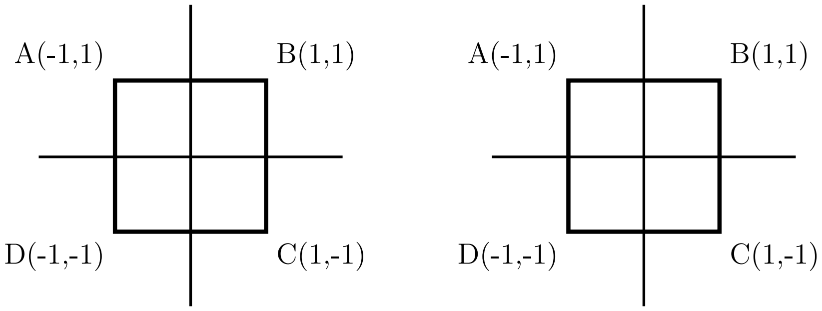
2.3.2 Finding Representation Matrices : Method 1

Consider a square on x-y plane with vertices A,B,C,D and corresponding position vectors r_A, r_B, r_C, r_D.

$$r_A = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, r_B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, r_C = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, r_D = \begin{bmatrix} -1 \\ -1 \end{bmatrix}. \tag{2.20}$$

1. **Matrix for identity transformation**

   $r_A \to r_A$, $r_B \to r_B$, $r_C \to r_C$, $r_D \to r_D$



The position coordinates are transformed by the matrix T_1 as following

$$\begin{bmatrix} u & v \\ w & x \end{bmatrix} * \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \tag{2.21a}$$

$$\begin{bmatrix} u & v \\ w & x \end{bmatrix} * \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{2.21b}$$

$$\begin{bmatrix} u & v \\ w & x \end{bmatrix} * \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \tag{2.21c}$$

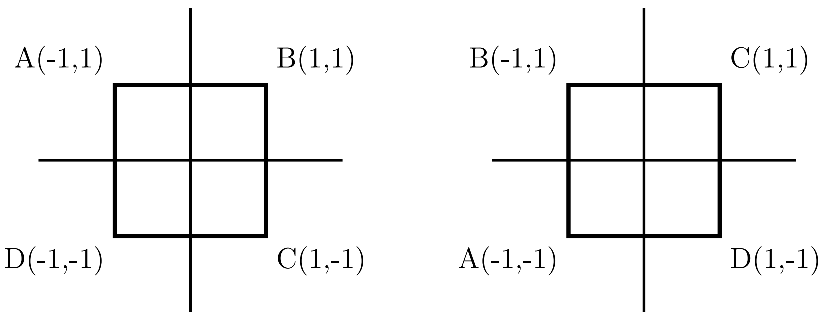
$$\begin{bmatrix} u & v \\ w & x \end{bmatrix} * \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}. \tag{2.21d}$$

Solving eqn 2.21, we can find the matrix T_1 as

$$T_1 = \begin{bmatrix} u & v \\ w & x \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{2.22}$$

2. **Matrix for 90° rotation**

   $r_A \to r_D,\ \ r_B \to r_A,\ \ r_C \to r_B,\ \ r_D \to r_C$



The position coordinates are transformed by the matrix T_2 as following

$$\begin{bmatrix} u & v \\ w & x \end{bmatrix} * \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \tag{2.23a}$$

$$\begin{bmatrix} u & v \\ w & x \end{bmatrix} * \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \tag{2.23b}$$

$$\begin{bmatrix} u & v \\ w & x \end{bmatrix} * \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{2.23c}$$

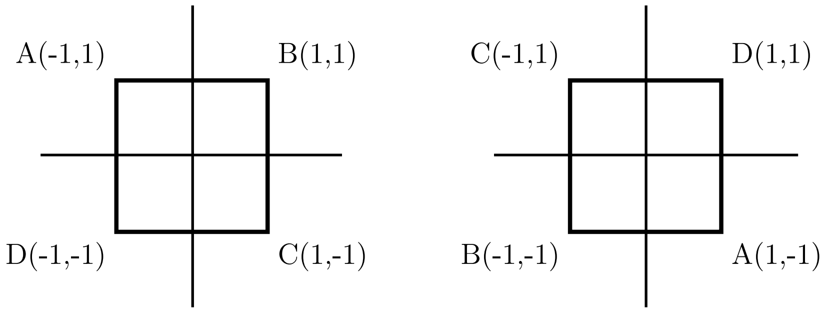
$$\begin{bmatrix} u & v \\ w & x \end{bmatrix} * \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \tag{2.23d}$$

Solving eqn 2.23, we can find the matrix T_2 as

$$T_2 = \begin{bmatrix} u & v \\ w & x \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{2.24}$$

3. **Matrix for 180° transformation**

$r_A \to r_C, \;\; r_B \to r_D, \;\; r_C \to r_A, \;\; r_D \to r_B$



The position coordinates are transformed by the matrix T_3 as following

$$\begin{bmatrix} u & v \\ w & x \end{bmatrix} * \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \tag{2.25a}$$

$$\begin{bmatrix} u & v \\ w & x \end{bmatrix} * \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \tag{2.25b}$$

$$\begin{bmatrix} u & v \\ w & x \end{bmatrix} * \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \tag{2.25c}$$

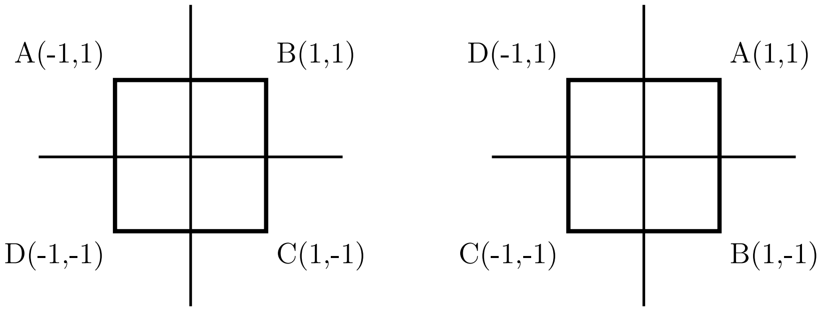
$$\begin{bmatrix} u & v \\ w & x \end{bmatrix} * \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{2.25d}$$

Solving eqn 2.25, we can find the matrix T_3 as

$$T_3 = \begin{bmatrix} u & v \\ w & x \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{2.26}$$

4. **Matrix for 270° transformation**

   $r_A \to r_B,\ \ r_B \to r_C,\ \ r_C \to r_D,\ \ r_D \to r_A$



The position coordinates are transformed by the matrix T_4 as following

$$\begin{bmatrix} u & v \\ w & x \end{bmatrix} * \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{2.27a}$$

$$\begin{bmatrix} u & v \\ w & x \end{bmatrix} * \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \tag{2.27b}$$

$$\begin{bmatrix} u & v \\ w & x \end{bmatrix} * \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \tag{2.27c}$$

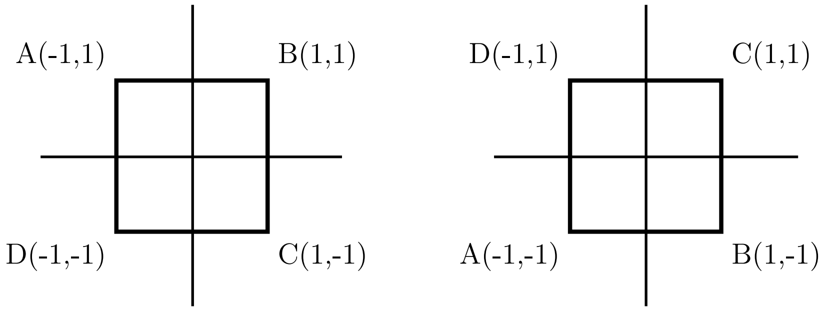
$$\begin{bmatrix} u & v \\ w & x \end{bmatrix} * \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}. \tag{2.27d}$$

Solving eqn 2.27, we can find the matrix T_4 as

$$T_4 = \begin{bmatrix} u & v \\ w & x \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{2.28}$$

5. **Matrix for reflection about horizontal axis**

   $r_A \to r_D, r_B \to r_C,\ \ r_C \to r_B,\ \ r_D \to r_A$



The position coordinates are transformed by the matrix T_X as following

$$\begin{bmatrix} u & v \\ w & x \end{bmatrix} * \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \tag{2.29a}$$

$$\begin{bmatrix} u & v \\ w & x \end{bmatrix} * \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \tag{2.29b}$$

$$\begin{bmatrix} u & v \\ w & x \end{bmatrix} * \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{2.29c}$$

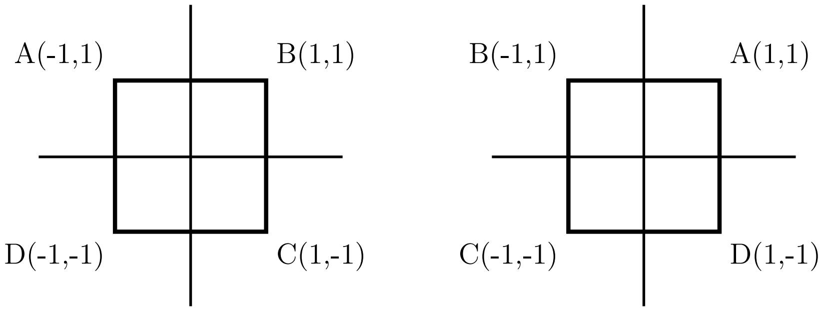
$$\begin{bmatrix} u & v \\ w & x \end{bmatrix} * \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}. \tag{2.29d}$$

Solving eqn 2.29, we can find the matrix T_X as

$$T_X = \begin{bmatrix} u & v \\ w & x \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{2.30}$$

6. **Matrix for reflection about vertical axis**

   $r_A \to r_B$, $r_B \to r_A$, $r_C \to r_D$, $r_D \to r_C$



The position coordinates are transformed by the matrix T_Y as following

$$\begin{bmatrix} u & v \\ w & x \end{bmatrix} * \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{2.31a}$$

$$\begin{bmatrix} u & v \\ w & x \end{bmatrix} * \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \tag{2.31b}$$

$$\begin{bmatrix} u & v \\ w & x \end{bmatrix} * \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \tag{2.31c}$$

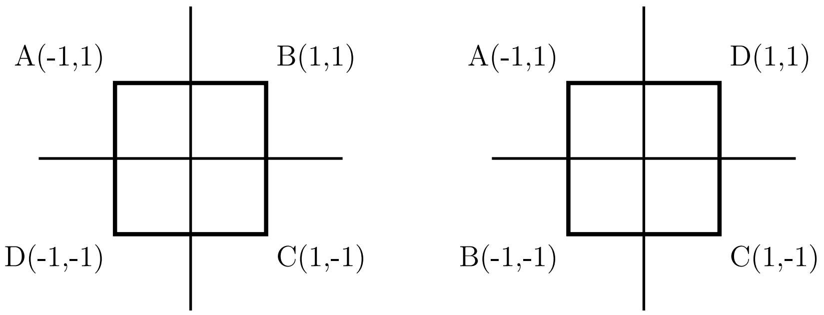
$$\begin{bmatrix} u & v \\ w & x \end{bmatrix} * \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \tag{2.31d}$$

Solving eqn 2.31, we can find the matrix T_Y as

$$T_Y = \begin{bmatrix} u & v \\ w & x \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{2.32}$$

7. **Matrix for reflection about diagonal**

$r_A \rightarrow r_A \;\; r_B \rightarrow r_D, \;\; r_C \rightarrow r_C, \;\; r_D \rightarrow r_B$



The position coordinates are transformed by the matrix T_{D1} as following

$$\begin{bmatrix} u & v \\ w & x \end{bmatrix} * \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \tag{2.33a}$$

$$\begin{bmatrix} u & v \\ w & x \end{bmatrix} * \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \tag{2.33b}$$

$$\begin{bmatrix} u & v \\ w & x \end{bmatrix} * \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \tag{2.33c}$$

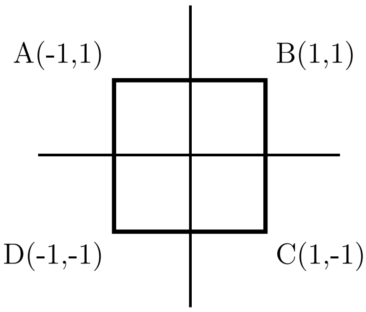
$$\begin{bmatrix} u & v \\ w & x \end{bmatrix} * \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{2.33d}$$

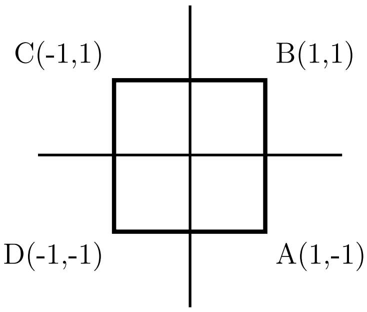
Solving eqn 2.33, we can find the matrix T_{D1} as

$$T_{D1} = \begin{bmatrix} u & v \\ w & x \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}. \tag{2.34}$$

8. **Matrix for reflection about other diagonal**

$r_A \to r_C \;\; r_B \to r_B, \;\; r_C \to r_A, \;\; r_D \to r_D$





The position coordinates are transformed by the matrix T_{D2} as following

$$\begin{bmatrix} u & v \\ w & x \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \tag{2.35a}$$

$$\begin{bmatrix} u & v \\ w & x \end{bmatrix} * \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{2.35b}$$

$$\begin{bmatrix} u & v \\ w & x \end{bmatrix} * \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \tag{2.35c}$$

$$\begin{bmatrix} u & v \\ w & x \end{bmatrix} * \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}. \tag{2.35d}$$

Solving eqn 2.35a, we can find the matrix T_{D2} as

$$T_{D2} = \begin{bmatrix} u & v \\ w & x \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} . \tag{2.36}$$

Matrix representation of the elements of D_4 :

$$\begin{aligned} &T_1 = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; T_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \; T_3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \; T_4 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \\ &T_X = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \; T_Y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \; T_{D1} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \; T_{D2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} . \end{aligned} \tag{2.37}$$

2.3.3 Finding 4×4 Matrix Representation : Method 2

Here we are going to discuss another method for finding the matrix representation of the elements. We will use the figures in subsection 4.1 and use the vertex numbering to find the permutations. And corresponding to each permutation, we will put the entries as $0s$ and $1s$ in the matrix.

1. **Identity Transformation** $\begin{matrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{matrix}$

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.38}$$

2. $90°$ **Rotation** $\begin{matrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{matrix}$

$$T_{90} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{2.39}$$

3. 180° **Rotation** $\begin{matrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{matrix}$

$$T_{180} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{2.40}$$

4. 270° **Rotation** $\begin{matrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{matrix}$

$$T_{270} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{2.41}$$

5. **Reflection about x-axis** $\begin{matrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{matrix}$

$$T_X = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{2.42}$$

6. **Reflection about y-axis** $\begin{matrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{matrix}$

$$T_Y = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{2.43}$$

7. **Reflection about Diagonal** D_1 $\begin{matrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{matrix}$

$$T_{D1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{2.44}$$

8. **Reflection about Diagonal** D_2 $\begin{matrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{matrix}$

$$T_{D2} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.45}$$

2.3.4 Reducing 4×4 Matrices

As we have seen in subsection 4.2, there is a 2×2 representation of the elements of D_4. So the 4×4 matrix representation found out in subsection 4.3 must be reducible. Now our task is to construct an orthogonal invertible matrix **S** which can decompose the above reducible matrices into block-diagonal form.

Why are we constructing a orthogonal matrix?

Symmetries of square is composed of rotations and reflections and the class of orthogonal matrices was defined in such a way so that they would represent rotations and reflections; this property is what makes this class of matrices so useful in the first place.

What is an orthogonal matrix? A matrix is said to be orthogonal if its columns $\{q^{(1)}, q^{(2)}...., q^{(n)}\}$ form a orthonormal set in $\mathbb{R}^n$. The below given matrix is an orthogonal matrix as it satisfies $S^T \cdot S = S \cdot S^T = I$.

$$S = \begin{bmatrix} -0.5 & 0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix}, \quad S^T = \begin{bmatrix} -0.5 & -0.5 & -0.5 & -0.5 \\ 0.5 & -0.5 & 0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix} \tag{2.46}$$

The block diagonal matrices can be easily find out using the similarity transformation $S^{-1} \cdot I \cdot S = S^T \cdot I \cdot S$ as shown below

1. **I**

$$I' = S^{-1} \cdot I \cdot S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.47}$$

2. $\mathbf{T}_{90}$

$$T'_{90} = S^{-1} \cdot T_{90} \cdot S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{2.48}$$

3. $\mathbf{T}_{180}$

$$T'_{90} = S^{-1} \cdot T_{90} \cdot S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{2.49}$$

4. $\mathbf{T}_{270}$

$$T'_{270} = S^{-1} \cdot T_{270} \cdot S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \tag{2.50}$$

5. $\mathbf{T}_X$

$$T'_X = S^{-1} \cdot T_X \cdot S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \tag{2.51}$$

6. $\mathbf{T}_Y$

$$T'_Y = S^{-1} \cdot T_Y \cdot S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.52}$$

7. $\mathbf{T}_{D1}$

$$T'_{D1} = S^{-1} \cdot T_{D1} \cdot S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \tag{2.53}$$

8. $\mathbf{T}_{D2}$

$$T'_{D2} = S^{-1} T_{D2} S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{2.54}$$

All the above matrices are in block diagonal form that means it can be further reduced into smaller 1X1 and 2X2 matrices.

2.3.5 Conjugacy Classes

The elements are classified into five conjugacy classes,

$$C_1 : \{e\}\,,\ \ C_2 : \{a, a^3\}\,,\ \ C'_1 : \{a^2\}\,,\ \ C'_2 : \{b, a^2 b\}\,,\ \ C''_2 : \{ab, a^3 b\} \tag{2.55}$$

Here, the subscript of C_n, n, denotes the number of elements in the conjugacy class C_n. Their order are found as

$$e = e\,, \quad h \ = \ 1\,, \tag{2.56a}$$

$$a^4 = (a^3)^4 = e\,, \quad h \ = \ 4\,, \tag{2.56b}$$

$$(a^2)^2 = e\,, \quad h \ = \ 2\,, \tag{2.56c}$$

$$(b^2) = (a^2 b)^2 = e\,, \quad h \;=\; 2\,, \tag{2.56d}$$

$$(ab)^2 = (a^3 b)^2 = e\,, \quad h \;=\; 2\,. \tag{2.56e}$$

2.3.6 Characters And Representations Of D_4

Let us study the irreps(irreducible representation) of D_4. The number of irreps of D_4 must be equal to the number of conjugacy classes, that is, five. We assume that there are m_n $n-$dimensional representations. m_n must satisfy

$$\sum_n m_n = 5\,. \tag{2.57}$$

The orthogonality relation 2.6a requires,

$$\sum_\alpha [\chi_{D_\alpha(C_1)}]^2 = \sum_n m_n n^2 = m_1 + 4m_2 + 9m_3 + \cdots = 8\,, \tag{2.58}$$

where $m_n \geq 0$. This equation has two possible solutions, $(m_1, m_2) = (4,1)$ and $(0,2)$, but only the former $(m_1, m_2) = (4,1)$ satisfies Eq. (2.57).

Thus, the irreps of D_4 include four singlets and a doublet. Let us denote the singlets by $\mathbf{1}_{++}$, $\mathbf{1}_{+-}$, $\mathbf{1}_{-+}$ and $\mathbf{1}_{--}$, and the doublet by $\mathbf{2}$. Their characters are denoted by $\chi_{\mathbf{1}_{++}}(g)$, $\chi_{\mathbf{1}_{+-}}(g)$, $\chi_{\mathbf{1}_{-+}}(g)$, $\chi_{\mathbf{1}_{--}}(g)$ and $\chi_{\mathbf{2}}(g)$, respectively.

First, we consider the characters of four singlets. Because $b^2{=}(a^2) = e$, $(ab)^2 = e$, the characters of C_2, C_2' and C_2'' have two possibilities,

$$\chi_{D_\alpha}(C_2) = \pm 1\,, \quad \chi_{D_\alpha}(C_2') = \pm 1\,, \quad \chi_{D_\alpha}(C_2'') = \pm 1\,, \tag{2.59}$$

where D_α corresponds to four singlets $\mathbf{1}_{\pm\pm}$. Obviously, $\chi_{D_\alpha}(C_1) = 1$ and $\chi_{\mathbf{2}}(C_1) = 2$. Furthermore, one singlet corresponds to a trivial singlet, that is $\chi_{\mathbf{1}_{++}}(C_2) = \chi_{\mathbf{1}_{++}}(C_1') = \chi_{\mathbf{1}_{++}}(C_2') = \chi_{\mathbf{1}_{++}}(C_2'') = 1$.

2×2 matrix representation for D_4 can be generated by,

$$a = \begin{pmatrix} \cos(\pi/2) & -\sin(\pi/2) \\ \sin(\pi/2) & \cos(\pi/2) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\,, \quad b = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,. \tag{2.60}$$

From these generators, we can find out the characters of the representation of the elements as

$$\chi_{\mathbf{2}}(C_2) = 0\,, \quad \chi_{\mathbf{2}}(C_1') = -2\,, \quad \chi_{\mathbf{2}}(C_2') = 0\,, \quad \chi_{\mathbf{2}}(C_2'') = 0\,. \tag{2.61}$$

We will now use the orthogonality relations, Eqs. (2.6a) and (2.6b), to find out the remaining characters. Solving for $\chi_{\mathbf{1}_{+-}(C_i)}$, we get

$$\sum_g \chi_{\mathbf{1}_{++}}(g)\chi_{\mathbf{1}_{+-}}(g) = 1 + 2\chi_{\mathbf{1}_{+-}}(C_2) + \chi_{\mathbf{1}_{+-}}(C_1') + 2\chi_{\mathbf{1}_{+-}}(C_2') + 2\chi_{\mathbf{1}_{+-}}(C_2'') = 0\,, \tag{2.62}$$

$$\sum_g [\chi_{\mathbf{1}_{+-}}(g)]^2 = 1 + 2[\chi_{\mathbf{1}_{+-}}(C_2)]^2 + [\chi_{\mathbf{1}_{+-}}(C_1')]^2 + 2[\chi_{\mathbf{1}_{+-}}(C_2')]^2 + 2[\chi_{\mathbf{1}_{+-}}(C_2'')]^2 = 8\,, \tag{2.63}$$

$$\chi_{\mathbf{2}}(g)\chi_{\mathbf{1}_{+-}}(g) = 1\cdot 2 + 2\cdot 0\cdot \chi_{\mathbf{1}_{+-}}(C_2) + (-2)\cdot 0\cdot \chi_{\mathbf{1}_{+-}}(C_1') + 2\cdot 0\cdot \chi_{\mathbf{1}_{+-}}(C_2') + \\ 2\cdot 0\cdot \chi_{\mathbf{1}_{+-}}(C_2'') = 0\,. \tag{2.64}$$

Eq. (2.63) implies $\chi_{\mathbf{1}_{+-}}(C_i) = \pm 1$ and Eq. (2.64) implies $\chi_{\mathbf{1}_{+-}}(C_1') = 1$. Following the same procedure for $\chi_{\mathbf{1}_{-+}}(C_i)$, we get

$$\sum_g \chi_{\mathbf{1}_{++}}(g)\chi_{\mathbf{1}_{-+}}(g) = 1 + 2\chi_{\mathbf{1}_{-+}}(C_2) + \chi_{\mathbf{1}_{-+}}(C_1') + 2\chi_{\mathbf{1}_{-+}}(C_2') + 2\chi_{\mathbf{1}_{-+}}(C_2'') = 0\,, \tag{2.65}$$

$$\chi_{\mathbf{1}_{-+}}(C_i) = \pm 1\,, \tag{2.66}$$

$$\chi_{\mathbf{1}_{-+}}(C_1') = 1\,. \tag{2.67}$$

For $\chi_{\mathbf{1}_{--}}(C_i)$, we get

$$\sum_g \chi_{\mathbf{1}_{++}}(g)\chi_{\mathbf{1}_{--}}(g) = 1 + 2\chi_{\mathbf{1}_{--}}(C_2) + \chi_{\mathbf{1}_{--}}(C_1') + 2\chi_{\mathbf{1}_{--}}(C_2') + 2\chi_{\mathbf{1}_{--}}(C_2'') = 0\,, \tag{2.68}$$

$$\chi_{\mathbf{1}_{--}}(C_i) = \pm 1\,, \tag{2.69}$$

$$\chi_{\mathbf{1}_{--}}(C_1') = 1\,. \tag{2.70}$$

And we can also write the following relations :-

$$1 + 2\chi_{\mathbf{1}_{+-}}(C_2)\chi_{\mathbf{1}_{-+}}(C_2) + 1 + 2\chi_{\mathbf{1}_{+-}}(C_2')\chi_{\mathbf{1}_{-+}}(C_2') + 2\chi_{\mathbf{1}_{+-}}(C_2'')\chi_{\mathbf{1}_{-+}}(C_2'') = 0\,, \tag{2.71}$$

$$1 + 2\chi_{\mathbf{1}_{+-}}(C_2)\chi_{\mathbf{1}_{--}}(C_2) + 1 + 2\chi_{\mathbf{1}_{+-}}(C_2')\chi_{\mathbf{1}_{--}}(C_2') + 2\chi_{\mathbf{1}_{+-}}(C_2'')\chi_{\mathbf{1}_{--}}(C_2'') = 0\,, \tag{2.72}$$

$$1 + 2\chi_{\mathbf{1}_{-+}}(C_2)\chi_{\mathbf{1}_{--}}(C_2) + 1 + 2\chi_{\mathbf{1}_{-+}}(C_2')\chi_{\mathbf{1}_{--}}(C_2') + 2\chi_{\mathbf{1}_{-+}}(C_2'')\chi_{\mathbf{1}_{--}}(C_2'') = 0\,, \tag{2.73}$$

Solving the above equations, we can construct the character table as given below:

	h	$\chi_{\mathbf{1}_{++}}$	$\chi_{\mathbf{1}_{+-}}$	$\chi_{\mathbf{1}_{-+}}$	$\chi_{\mathbf{1}_{--}}$	$\chi_{\mathbf{2}}$
C_1	1	1	1	1	1	2
C_2	4	1	-1	-1	1	0
C_1'	2	1	1	1	1	-2
C_2'	2	1	1	-1	-1	0
C_2''	2	1	-1	1	-1	0

TABLE 2.5: Character table for D_4

2.3.7 Tensor Products

Now that we have constructed the character table and we know the irreducible representations of d4, our next task is to find out the tensor product of the irreps. All the irreps of D_4 are given as follows:

$$\mathbf{1}_{++}\,, \quad \mathbf{1}_{+-}\,, \quad \mathbf{1}_{-+}\,, \quad \mathbf{1}_{--}\,, \quad \mathbf{2} \tag{2.74}$$

The singlet representation can be easily found out using the character table and the doublet representation can be found out using the generator in Eq. (2.60).
Consider a doublet

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{2.75}$$

It transforms under D_4 as following:

$$e: \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{2.76a}$$

$$a: \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}, \tag{2.76b}$$

$$a^2: \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix}, \tag{2.76c}$$

$$a^3: \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}, \tag{2.76d}$$

$$b: \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}, \tag{2.76e}$$

$$ab: \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}, \tag{2.76f}$$

$$a^2b: \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 \\ x_2 \end{bmatrix}, \tag{2.76g}$$

$$a^3b: \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ -x_1 \end{bmatrix}. \tag{2.76h}$$

Singlet $\begin{bmatrix} w \end{bmatrix}_{\mathbf{1}_{++}}$ transforms under D_4 as follows

$$e: \begin{bmatrix} w \end{bmatrix} \to D(e) \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} w \end{bmatrix},$$

$$\begin{aligned}
a &: \begin{bmatrix} w \end{bmatrix} \to D(a) \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} w \end{bmatrix} , \\
a^2 &: \begin{bmatrix} w \end{bmatrix} \to D(a^2) \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} w \end{bmatrix} , \\
a^3 &: \begin{bmatrix} w \end{bmatrix} \to D(a^3) \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} w \end{bmatrix} , \\
b &: \begin{bmatrix} w \end{bmatrix} \to D(b) \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} w \end{bmatrix} , \\
ab &: \begin{bmatrix} w \end{bmatrix} \to D(ab) \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} w \end{bmatrix} , \\
a^2b &: \begin{bmatrix} w \end{bmatrix} \to D(a^2b) \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} w \end{bmatrix} , \\
a^3b &: \begin{bmatrix} w \end{bmatrix} \to D(a^3b) \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} w \end{bmatrix} .
\end{aligned} \tag{2.77}$$

Singlet $\begin{bmatrix} w \end{bmatrix}_{\mathbf{1}_{+-}}$ transforms under D_4 as follows

$$\begin{aligned}
e &: \begin{bmatrix} w \end{bmatrix} \to D(e) \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} w \end{bmatrix} , \\
a &: \begin{bmatrix} w \end{bmatrix} \to D(a) \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} -1 \end{bmatrix} \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} -w \end{bmatrix} , \\
a^2 &: \begin{bmatrix} w \end{bmatrix} \to D(a^2) \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} w \end{bmatrix} , \\
a^3 &: \begin{bmatrix} w \end{bmatrix} \to D(a^3) \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} -1 \end{bmatrix} \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} -w \end{bmatrix} , \\
b &: \begin{bmatrix} w \end{bmatrix} \to D(b) \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} w \end{bmatrix} , \\
ab &: \begin{bmatrix} w \end{bmatrix} \to D(ab) \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} -1 \end{bmatrix} \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} -w \end{bmatrix} , \\
a^2b &: \begin{bmatrix} w \end{bmatrix} \to D(a^2b) \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} w \end{bmatrix} , \\
a^3b &: \begin{bmatrix} w \end{bmatrix} \to D(a^3b) \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} -1 \end{bmatrix} \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} -w \end{bmatrix} .
\end{aligned} \tag{2.78}$$

Singlet $\begin{bmatrix} w \end{bmatrix}_{\mathbf{1}_{-+}}$ transforms under D_4 as follows

$$\begin{aligned}
e &: \begin{bmatrix} w \end{bmatrix} \to D(e) \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} w \end{bmatrix} , \\
a &: \begin{bmatrix} w \end{bmatrix} \to D(a) \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} -1 \end{bmatrix} \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} -w \end{bmatrix} , \\
a^2 &: \begin{bmatrix} w \end{bmatrix} \to D(a^2) \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} w \end{bmatrix} , \\
a^3 &: \begin{bmatrix} w \end{bmatrix} \to D(a^3) \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} -1 \end{bmatrix} \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} -w \end{bmatrix} , \\
b &: \begin{bmatrix} w \end{bmatrix} \to D(b) \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} -1 \end{bmatrix} \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} -w \end{bmatrix} , \\
ab &: \begin{bmatrix} w \end{bmatrix} \to D(ab) \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} w \end{bmatrix} , \\
a^2b &: \begin{bmatrix} w \end{bmatrix} \to D(a^2b) \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} -1 \end{bmatrix} \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} -w \end{bmatrix} , \\
a^3b &: \begin{bmatrix} w \end{bmatrix} \to D(a^3b) \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} w \end{bmatrix} .
\end{aligned} \tag{2.79}$$

Singlet $\begin{bmatrix} w \end{bmatrix}_{\mathbf{1}_{--}}$ transforms under D_4 as follows

$$\begin{aligned}
e &: \begin{bmatrix} w \end{bmatrix} \to D(e) \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} w \end{bmatrix} \,, \\
a &: \begin{bmatrix} w \end{bmatrix} \to D(a) \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} w \end{bmatrix} \,, \\
a^2 &: \begin{bmatrix} w \end{bmatrix} \to D(a^2) \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} w \end{bmatrix} \,, \\
a^3 &: \begin{bmatrix} w \end{bmatrix} \to D(a^3) \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} w \end{bmatrix} \,, \\
b &: \begin{bmatrix} w \end{bmatrix} \to D(b) \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} -1 \end{bmatrix} \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} -w \end{bmatrix} \,, \\
ab &: \begin{bmatrix} w \end{bmatrix} \to D(ab) \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} -1 \end{bmatrix} \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} -w \end{bmatrix} \,, \\
a^2b &: \begin{bmatrix} w \end{bmatrix} \to D(a^2b) \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} -1 \end{bmatrix} \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} -w \end{bmatrix} \,, \\
a^3b &: \begin{bmatrix} w \end{bmatrix} \to D(a^3b) \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} -1 \end{bmatrix} \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} -w \end{bmatrix} \,.
\end{aligned} \tag{2.80}$$

2.3.7.1 Tensor Product Of Doublets

Take first doublet whose components are x_1 and x_2, and second doublet whose components are y_1 and y_2. If we take the tensor product of these two, we get

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \otimes \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 y_1 \\ x_1 y_2 \\ x_2 y_1 \\ x_2 y_2 \end{bmatrix} . \tag{2.81}$$

Now we know how a doublet transforms under different elements of D_4, Eq. (2.76). From that we can easily find out how each element $x_i y_j$ transforms.

Element **e**:

$$x'_1 y'_1 = x_1 y_1 \,, \tag{2.82a}$$
$$x'_1 y'_2 = x_1 y_2 \,, \tag{2.82b}$$
$$x'_2 y'_1 = x_2 y_1 \,, \tag{2.82c}$$
$$x'_2 y'_2 = x_2 y_2 \,. \tag{2.82d}$$

Element **a**:

$$x'_1 y'_1 = x_2 y_2 \,, \tag{2.83a}$$

$$x'_1y'_2 = -x_2y_1\,, \tag{2.83b}$$
$$x'_2y'_1 = -x_1y_2\,, \tag{2.83c}$$
$$x'_2y'_2 = x_1y_1\,. \tag{2.83d}$$

It is found that following linear combinations transforms like different singlets for element a.

$$x'_1y'_1 + x'_2y'_2 = [D(a)]_{\mathbf{1}_{++}}[x_1y_1 + x_2y_2]\,, \tag{2.84a}$$
$$x'_1y'_1 - x'_2y'_2 = [D(a)]_{\mathbf{1}_{+-}}[x_1y_1 - x_2y_2]\,, \tag{2.84b}$$
$$x'_1y'_2 + x'_2y'_1 = [D(a)]_{\mathbf{1}_{-+}}[x_1y_2 + x_2y_1]\,, \tag{2.84c}$$
$$x'_1y'_2 - x'_2y'_1 = [D(a)]_{\mathbf{1}_{--}}[x_1y_2 - x_2y_1]\,. \tag{2.84d}$$

Element $\mathbf{a}^2$:

$$x'_1y'_1 = x_1y_1\,, \tag{2.85a}$$
$$x'_1y'_2 = x_1y_2\,, \tag{2.85b}$$
$$x'_2y'_1 = x_2y_1\,, \tag{2.85c}$$
$$x'_2y'_2 = x_2y_2\,. \tag{2.85d}$$

It is found that following linear combinations transforms like different singlets for element a^2.

$$x'_1y'_1 + x'_2y'_2 = [D(a^2)]_{\mathbf{1}_{++}}[x_1y_1 + x_2y_2]\,, \tag{2.86a}$$
$$x'_1y'_1 - x'_2y'_2 = [D(a^2)]_{\mathbf{1}_{+-}}[x_1y_1 - x_2y_2]\,, \tag{2.86b}$$
$$x'_1y'_2 + x'_2y'_1 = [D(a^2)]_{\mathbf{1}_{-+}}[x_1y_2 + x_2y_1]\,, \tag{2.86c}$$
$$x'_1y'_2 - x'_2y'_1 = [D(a^2)]_{\mathbf{1}_{--}}[x_1y_2 - x_2y_1]\,. \tag{2.86d}$$

Element $\mathbf{a}^3$:

$$x'_1y'_1 = x_2y_2\,, \tag{2.87a}$$
$$x'_1y'_2 = -x_2y_1\,, \tag{2.87b}$$
$$x'_2y'_1 = -x_1y_2\,, \tag{2.87c}$$
$$x'_2y'_2 = x_1y_1\,. \tag{2.87d}$$

It is found that following linear combinations transforms like different singlets for element a^3.

$$x_1'y_1' + x_2'y_2' = [D(a^3)]_{\mathbf{1}_{++}}[x_1y_1 + x_2y_2]\,, \tag{2.88a}$$
$$x_1'y_1' - x_2'y_2' = [D(a^3)]_{\mathbf{1}_{+-}}[x_1y_1 - x_2y_2]\,, \tag{2.88b}$$
$$x_1'y_2' + x_2'y_1' = [D(a^3)]_{\mathbf{1}_{-+}}[x_1y_2 + x_2y_1]\,, \tag{2.88c}$$
$$x_1'y_2' - x_2'y_1' = [D(a^3)]_{\mathbf{1}_{--}}[x_1y_2 - x_2y_1]\,. \tag{2.88d}$$

Element **b**:

$$x_1'y_1' = x_1y_1\,, \tag{2.89a}$$
$$x_1'y_2' = -x_1y_2\,, \tag{2.89b}$$
$$x_2'y_1' = -x_2y_1\,, \tag{2.89c}$$
$$x_2'y_2' = x_2y_2\,. \tag{2.89d}$$

It is found that following linear combinations transforms like different singlets for element b.

$$x_1'y_1' + x_2'y_2' = [D(b)]_{\mathbf{1}_{++}}[x_1y_1 + x_2y_2]\,, \tag{2.90a}$$
$$x_1'y_1' - x_2'y_2' = [D(b)]_{\mathbf{1}_{+-}}[x_1y_1 - x_2y_2]\,, \tag{2.90b}$$
$$x_1'y_2' + x_2'y_1' = [D(b)]_{\mathbf{1}_{-+}}[x_1y_2 + x_2y_1]\,, \tag{2.90c}$$
$$x_1'y_2' - x_2'y_1' = [D(b)]_{\mathbf{1}_{--}}[x_1y_2 - x_2y_1]\,. \tag{2.90d}$$

Element **ab**:

$$x_1'y_1' = x_2y_2\,, \tag{2.91a}$$
$$x_1'y_2' = x_2y_1\,, \tag{2.91b}$$
$$x_2'y_1' = x_1y_2\,, \tag{2.91c}$$
$$x_2'y_2' = x_1y_1\,. \tag{2.91d}$$

It is found that following linear combinations transforms like different singlets for element ab.

$$x_1'y_1' + x_2'y_2' = [D(ab)]_{\mathbf{1}_{++}}[x_1y_1 + x_2y_2]\,, \tag{2.92a}$$
$$x_1'y_1' - x_2'y_2' = [D(ab)]_{\mathbf{1}_{+-}}[x_1y_1 - x_2y_2]\,, \tag{2.92b}$$
$$x_1'y_2' + x_2'y_1' = [D(ab)]_{\mathbf{1}_{-+}}[x_1y_2 + x_2y_1]\,, \tag{2.92c}$$

$$x'_1y'_2 - x'_2y'_1 \quad = \quad [D(ab)]_{\mathbf{1}_{--}}[x_1y_2 - x_2y_1]\,. \tag{2.92d}$$

Element $\mathbf{a}^2\mathbf{b}$:

$$x'_1y'_1 \quad = \quad x_1y_1\,, \tag{2.93a}$$
$$x'_1y'_2 \quad = \quad -x_1y_2\,, \tag{2.93b}$$
$$x'_2y'_1 \quad = \quad -x_2y_1\,, \tag{2.93c}$$
$$x'_2y'_2 \quad = \quad x_2y_2\,. \tag{2.93d}$$

It is found that following linear combinations transforms like different singlets for element a^2b.

$$x'_1y'_1 + x'_2y'_2 \quad = \quad [D(a^2b)]_{\mathbf{1}_{++}}[x_1y_1 + x_2y_2]\,, \tag{2.94a}$$
$$x'_1y'_1 - x'_2y'_2 \quad = \quad [D(a^2b)]_{\mathbf{1}_{+-}}[x_1y_1 - x_2y_2]\,, \tag{2.94b}$$
$$x'_1y'_2 + x'_2y'_1 \quad = \quad [D(a^2b)]_{\mathbf{1}_{-+}}[x_1y_2 + x_2y_1]\,, \tag{2.94c}$$
$$x'_1y'_2 - x'_2y'_1 \quad = \quad [D(a^2b)]_{\mathbf{1}_{--}}[x_1y_2 - x_2y_1]\,. \tag{2.94d}$$

Element $\mathbf{a}^3\mathbf{b}$:

$$x'_1y'_1 \quad = \quad x_2y_2\,, \tag{2.95a}$$
$$x'_1y'_2 \quad = \quad x_2y_1\,, \tag{2.95b}$$
$$x'_2y'_1 \quad = \quad x_1y_2\,, \tag{2.95c}$$
$$x'_2y'_2 \quad = \quad x_1y_1\,. \tag{2.95d}$$

It is found that following linear combinations transforms like different singlets for element a^3b.

$$x'_1y'_1 + x'_2y'_2 \quad = \quad [D(a^3b)]_{\mathbf{1}_{++}}[x_1y_1 + x_2y_2]\,, \tag{2.96a}$$
$$x'_1y'_1 - x'_2y'_2 \quad = \quad [D(a^3b)]_{\mathbf{1}_{+-}}[x_1y_1 - x_2y_2]\,, \tag{2.96b}$$
$$x'_1y'_2 + x'_2y'_1 \quad = \quad [D(a^3b)]_{\mathbf{1}_{-+}}[x_1y_2 + x_2y_1]\,, \tag{2.96c}$$
$$x'_1y'_2 - x'_2y'_1 \quad = \quad [D(a^3b)]_{\mathbf{1}_{--}}[x_1y_2 - x_2y_1]\,. \tag{2.96d}$$

From the above equations we can conclude that $[x_1y_1 + x_2y_2]$ transforms as $\mathbf{1}_{++}$ singlet, $[x_1y_1 - x_2y_2]$ transforms as $\mathbf{1}_{+-}$ singlet, $[x_1y_2 + x_2y_1]$ transforms as $\mathbf{1}_{-+}$ and $[x_1y_2 - x_2y_1]$ transforms as $\mathbf{1}_{--}$ singlet under each element of D_4.

Hence, the tensor product of two doublets can be written as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \otimes \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = [x_1y_1 + x_2y_2]_{\mathbf{1}_{++}} + [x_1y_1 - x_2y_2]_{\mathbf{1}_{+-}} + [x_1y_2 + x_2y_1]_{\mathbf{1}_{-+}} + [x_1y_2 - x_2y_1]_{\mathbf{1}_{--}} \tag{2.97}$$

2.3.7.2 Tensor Product Of Doublet And Singlet

$$\begin{bmatrix} w \end{bmatrix} \otimes \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} wx \\ wy \end{bmatrix} \tag{2.98}$$

Case 1: Consider the following singlet and doublet:

$$\begin{bmatrix} w \end{bmatrix}_{\mathbf{1}_{++}}, \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{\mathbf{2}}. \tag{2.99}$$

The tensor product is trivial and is given as

$$\begin{bmatrix} w \end{bmatrix}_{\mathbf{1}_{++}} \otimes \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{\mathbf{2}} = \begin{bmatrix} wx_1 \\ wx_2 \end{bmatrix}_{\mathbf{2}} \tag{2.100}$$

Case 2: The singlet and doublet for this case are

$$\begin{bmatrix} w \end{bmatrix}_{\mathbf{1}_{+-}}, \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{\mathbf{2}}. \tag{2.101}$$

Element $\mathbf{a}$

$$w'x'_1 = wx_2\,, \tag{2.102}$$

$$w'x'_2 = wx_1\,. \tag{2.103}$$

It is found that

$$\begin{bmatrix} wx_1 \\ -wx_2 \end{bmatrix} \to \begin{bmatrix} w'x'_1 \\ -w'x'_2 \end{bmatrix} = \begin{bmatrix} wx_2 \\ -wx_1 \end{bmatrix} = \begin{bmatrix} D(a) \end{bmatrix}_{\mathbf{2}} \begin{bmatrix} wx_1 \\ -wx_2 \end{bmatrix}. \tag{2.104}$$

Element $\mathbf{a}^2$

$$w'x'_1 = -wx_1\,, \tag{2.105}$$

$$w'x'_2 = -wx_2\,. \tag{2.106}$$

It is found that

$$\begin{bmatrix} wx_1 \\ -wx_2 \end{bmatrix} \to \begin{bmatrix} w'x'_1 \\ -w'x'_2 \end{bmatrix} = \begin{bmatrix} -wx_1 \\ wx_2 \end{bmatrix} = \Big[D(a^2)\Big]_{\mathbf{2}} \begin{bmatrix} wx_1 \\ -wx_2 \end{bmatrix} . \tag{2.107}$$

Element $\mathbf{a}^3$

$$w'x'_1 = -wx_2\,, \tag{2.108}$$

$$w'x'_2 = wx_1\,. \tag{2.109}$$

It is found that

$$\begin{bmatrix} wx_1 \\ -wx_2 \end{bmatrix} \to \begin{bmatrix} w'x'_1 \\ -w'x'_2 \end{bmatrix} = \begin{bmatrix} -wx_2 \\ -wx_1 \end{bmatrix} = \Big[D(a^3)\Big]_{\mathbf{2}} \begin{bmatrix} wx_1 \\ -wx_2 \end{bmatrix} . \tag{2.110}$$

Element $\mathbf{b}$

$$w'x'_1 = wx_1\,, \tag{2.111}$$

$$w'x'_2 = -wx_2\,. \tag{2.112}$$

It is found that

$$\begin{bmatrix} wx_1 \\ -wx_2 \end{bmatrix} \to \begin{bmatrix} w'x'_1 \\ -w'x'_2 \end{bmatrix} = \begin{bmatrix} wx_1 \\ wx_2 \end{bmatrix} = \Big[D(b)\Big]_{\mathbf{2}} \begin{bmatrix} wx_1 \\ -wx_2 \end{bmatrix} . \tag{2.113}$$

Element $\mathbf{ab}$

$$w'x'_1 = -wx_2\,, \tag{2.114}$$

$$w'x'_2 = wx_1\,. \tag{2.115}$$

It is found that

$$\begin{bmatrix} wx_1 \\ -wx_2 \end{bmatrix} \to \begin{bmatrix} w'x'_1 \\ -w'x'_2 \end{bmatrix} = \begin{bmatrix} wx_2 \\ -wx_1 \end{bmatrix} = \Big[D(ab)\Big]_{\mathbf{2}} \begin{bmatrix} wx_1 \\ -wx_2 \end{bmatrix} . \tag{2.116}$$

Element $\mathbf{a^2b}$

$$w'x_1' = -wx_1\,, \tag{2.117}$$

$$w'x_2' = wx_2\,. \tag{2.118}$$

It is found that

$$\begin{bmatrix} wx_1 \\ -wx_2 \end{bmatrix} \rightarrow \begin{bmatrix} w'x_1' \\ -w'x_2' \end{bmatrix} = \begin{bmatrix} -wx_1 \\ -wx_2 \end{bmatrix} = \Big[D(a^2b)\Big]_{\mathbf{2}} \begin{bmatrix} wx_1 \\ -wx_2 \end{bmatrix}. \tag{2.119}$$

Element $\mathbf{a^3b}$

$$w'x_1' = wx_2\,, \tag{2.120}$$

$$w'x_2' = wx_1\,. \tag{2.121}$$

It is found that

$$\begin{bmatrix} wx_1 \\ -wx_2 \end{bmatrix} \rightarrow \begin{bmatrix} w'x_1' \\ -w'x_2' \end{bmatrix} = \begin{bmatrix} wx_2 \\ -wx_1 \end{bmatrix} = \Big[D(a^3b)\Big]_{\mathbf{2}} \begin{bmatrix} wx_1 \\ -wx_2 \end{bmatrix}. \tag{2.122}$$

The tensor product is given as

$$\Big[w\Big]_{\mathbf{1}_{+-}} \otimes \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{\mathbf{2}} = \begin{bmatrix} wx_1 \\ -wx_2 \end{bmatrix}_{\mathbf{2}} \tag{2.123}$$

Case 3: The singlet and doublet for this case are

$$\Big[w\Big]_{\mathbf{1}_{-+}}\,, \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{\mathbf{2}}\,. \tag{2.124}$$

Element $\mathbf{a}$

$$w'x_1' = wx_2\,, \tag{2.125}$$

$$w'x_2' = -wx_1\,. \tag{2.126}$$

It is found that

$$\begin{bmatrix} wx_2 \\ wx_1 \end{bmatrix} \to \begin{bmatrix} w'x'_2 \\ w'x'_1 \end{bmatrix} = \begin{bmatrix} -wx_1 \\ wx_2 \end{bmatrix} = \Big[D(a)\Big]_{\mathbf{2}} \begin{bmatrix} wx_2 \\ wx_1 \end{bmatrix} . \tag{2.127}$$

Element $\mathbf{a}^2$

$$w'x'_1 = -wx_1 \,, \tag{2.128}$$

$$w'x'_2 = -wx_2 \,. \tag{2.129}$$

It is found that

$$\begin{bmatrix} wx_2 \\ wx_1 \end{bmatrix} \to \begin{bmatrix} w'x'_2 \\ w'x'_1 \end{bmatrix} = \begin{bmatrix} -wx_2 \\ -wx_1 \end{bmatrix} = \Big[D(a^2)\Big]_{\mathbf{2}} \begin{bmatrix} wx_2 \\ wx_1 \end{bmatrix} . \tag{2.130}$$

Element $\mathbf{a}^3$

$$w'x'_1 = -wx_2 \,, \tag{2.131}$$

$$w'x'_2 = wx_1 \,. \tag{2.132}$$

It is found that

$$\begin{bmatrix} wx_2 \\ wx_1 \end{bmatrix} \to \begin{bmatrix} w'x'_2 \\ w'x'_1 \end{bmatrix} = \begin{bmatrix} wx_1 \\ -wx_2 \end{bmatrix} = \Big[D(a^3)\Big]_{\mathbf{2}} \begin{bmatrix} wx_2 \\ wx_1 \end{bmatrix} . \tag{2.133}$$

Element **b**

$$w'x'_1 = -wx_1 \,, \tag{2.134}$$

$$w'x'_2 = wx_2 \,. \tag{2.135}$$

It is found that

$$\begin{bmatrix} wx_2 \\ wx_1 \end{bmatrix} \to \begin{bmatrix} w'x'_2 \\ w'x'_1 \end{bmatrix} = \begin{bmatrix} wx_2 \\ -wx_1 \end{bmatrix} = \Big[D(b)\Big]_{\mathbf{2}} \begin{bmatrix} wx_2 \\ wx_1 \end{bmatrix} . \tag{2.136}$$

Element **ab**

$$w'x'_1 = wx_2 \,, \tag{2.137}$$

$$w'x'_2 = wx_1\,. \tag{2.138}$$

It is found that

$$\begin{bmatrix} wx_2 \\ wx_1 \end{bmatrix} \to \begin{bmatrix} w'x'_2 \\ w'x'_1 \end{bmatrix} = \begin{bmatrix} wx_1 \\ wx_2 \end{bmatrix} = \Big[D(ab)\Big]_{\mathbf{2}} \begin{bmatrix} wx_2 \\ wx_1 \end{bmatrix}. \tag{2.139}$$

Element $\mathbf{a^2b}$

$$w'x'_1 = wx_1\,, \tag{2.140}$$

$$w'x'_2 = -wx_2\,. \tag{2.141}$$

It is found that

$$\begin{bmatrix} wx_2 \\ wx_1 \end{bmatrix} \to \begin{bmatrix} w'x'_2 \\ w'x'_1 \end{bmatrix} = \begin{bmatrix} -wx_2 \\ wx_1 \end{bmatrix} = \Big[D(a^2b)\Big]_{\mathbf{2}} \begin{bmatrix} wx_2 \\ wx_1 \end{bmatrix}. \tag{2.142}$$

Element $\mathbf{a^3b}$

$$w'x'_1 = -wx_2\,, \tag{2.143}$$

$$w'x'_2 = -wx_1\,. \tag{2.144}$$

It is found that

$$\begin{bmatrix} wx_2 \\ wx_1 \end{bmatrix} \to \begin{bmatrix} w'x'_2 \\ w'x'_1 \end{bmatrix} = \begin{bmatrix} -wx_1 \\ -wx_2 \end{bmatrix} = \Big[D(a^3b)\Big]_{\mathbf{2}} \begin{bmatrix} wx_2 \\ wx_1 \end{bmatrix}. \tag{2.145}$$

The tensor product is given as

$$\Big[w\Big]_{\mathbf{1}_{-+}} \otimes \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{\mathbf{2}} = \begin{bmatrix} wx_2 \\ wx_1 \end{bmatrix}_{\mathbf{2}} \tag{2.146}$$

Case 4: The singlet and doublet for this case are

$$\Big[w\Big]_{\mathbf{1}_{--}}\,, \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{\mathbf{2}}. \tag{2.147}$$

Element **a**

$$w'x_1' = -wx_2\,, \tag{2.148}$$

$$w'x_2' = wx_1\,. \tag{2.149}$$

It is found that

$$\begin{bmatrix} wx_2 \\ -wx_1 \end{bmatrix} \to \begin{bmatrix} w'x_2' \\ -w'x_1' \end{bmatrix} = \begin{bmatrix} wx_1 \\ wx_2 \end{bmatrix} = \Big[D(a)\Big]_{\mathbf{2}} \begin{bmatrix} wx_2 \\ -wx_1 \end{bmatrix}\,. \tag{2.150}$$

Element $\mathbf{a}^2$

$$w'x_1' = -wx_1\,, \tag{2.151}$$

$$w'x_2' = -wx_2\,. \tag{2.152}$$

It is found that

$$\begin{bmatrix} wx_2 \\ -wx_1 \end{bmatrix} \to \begin{bmatrix} w'x_2' \\ -w'x_1' \end{bmatrix} = \begin{bmatrix} -wx_2 \\ wx_1 \end{bmatrix} = \Big[D(a^2)\Big]_{\mathbf{2}} \begin{bmatrix} wx_2 \\ -wx_1 \end{bmatrix}\,. \tag{2.153}$$

Element $\mathbf{a}^3$

$$w'x_1' = wx_2\,, \tag{2.154}$$

$$w'x_2' = -wx_1\,. \tag{2.155}$$

It is found that

$$\begin{bmatrix} wx_2 \\ -wx_1 \end{bmatrix} \to \begin{bmatrix} w'x_2' \\ -w'x_1' \end{bmatrix} = \begin{bmatrix} -wx_1 \\ -wx_2 \end{bmatrix} = \Big[D(a^3)\Big]_{\mathbf{2}} \begin{bmatrix} wx_2 \\ -wx_1 \end{bmatrix}\,. \tag{2.156}$$

Element **b**

$$w'x_1' = -wx_1\,, \tag{2.157}$$

$$w'x_2' = wx_2\,. \tag{2.158}$$

It is found that

$$\begin{bmatrix} wx_2 \\ -wx_1 \end{bmatrix} \to \begin{bmatrix} w'x'_2 \\ -w'x'_1 \end{bmatrix} = \begin{bmatrix} wx_2 \\ wx_1 \end{bmatrix} = \Big[D(b)\Big]_{\mathbf{2}} \begin{bmatrix} wx_2 \\ -wx_1 \end{bmatrix} . \tag{2.159}$$

Element **ab**

$$w'x'_1 = -wx_2 \, , \tag{2.160}$$

$$w'x'_2 = wx_1 \, . \tag{2.161}$$

It is found that

$$\begin{bmatrix} wx_2 \\ -wx_1 \end{bmatrix} \to \begin{bmatrix} w'x'_2 \\ -w'x'_1 \end{bmatrix} = \begin{bmatrix} wx_1 \\ wx_2 \end{bmatrix} = \Big[D(ab)\Big]_{\mathbf{2}} \begin{bmatrix} wx_2 \\ -wx_1 \end{bmatrix} . \tag{2.162}$$

Element $\mathbf{a}^2\mathbf{b}$

$$w'x'_1 = wx_1 \, , \tag{2.163}$$

$$w'x'_2 = -wx_2 \, . \tag{2.164}$$

It is found that

$$\begin{bmatrix} wx_2 \\ -wx_1 \end{bmatrix} \to \begin{bmatrix} w'x'_2 \\ -w'x'_1 \end{bmatrix} = \begin{bmatrix} -wx_2 \\ -wx_1 \end{bmatrix} = \Big[D(a^2b)\Big]_{\mathbf{2}} \begin{bmatrix} wx_2 \\ -wx_1 \end{bmatrix} . \tag{2.165}$$

Element $\mathbf{a}^3\mathbf{b}$

$$w'x'_1 = wx_2 \, , \tag{2.166}$$

$$w'x'_2 = wx_1 \, . \tag{2.167}$$

It is found that

$$\begin{bmatrix} wx_2 \\ -wx_1 \end{bmatrix} \to \begin{bmatrix} w'x'_2 \\ -w'x'_1 \end{bmatrix} = \begin{bmatrix} wx_1 \\ -wx_2 \end{bmatrix} = \Big[D(a^3b)\Big]_{\mathbf{2}} \begin{bmatrix} wx_2 \\ -wx_1 \end{bmatrix} . \tag{2.168}$$

The tensor product is given as

$$\begin{bmatrix} w \end{bmatrix}_{\mathbf{1}_{--}} \otimes \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{\mathbf{2}} = \begin{bmatrix} wx_2 \\ -wx_1 \end{bmatrix}_{\mathbf{2}} \tag{2.169}$$

2.3.7.3 Tensor Product Of Singlets

Similarly, we can find out the tensor product of singlets.

$$[w_1]_{\mathbf{1}_{s_1 s_1}} \otimes [w_2]_{\mathbf{1}_{s_1' s_2'}} = [w_1 w_2]_{\mathbf{1}_{s_1'' s_2''}} \tag{2.170}$$

where $s_1'' = s_1.s_1'$ and $s_2'' = s_2.'s_2'$.

2.3.8 Group Structure Of D_5

D_5 is the symmetry group of pentagon of order ten. It is isomorphic to $\mathbb{Z}_5 \rtimes \mathbb{Z}_2$. It consist of cyclic rotation, $\mathbb{Z}_5$ and reflection. That is, it is generated by two generators a and b, which act on the 5 edges of the pentagon. All of the elements can be written as $a^m b^k$ with $m = 0, 1, 2, 3, 4$ and $k = 0, 1$, given $a^5 = e, b^2 = e, bab = a^{-1}$. Using these relations, we can find the four conjugacy classes[1] as

$$C_1 = \{e\} \qquad C_2^{(1)} = \{a, a^4\} \qquad C_2^{(2)} = \{a^2, a^3\} \qquad C_5 = \{b, ab, a^2b, a^3b, a^4b\} \tag{2.171}$$

The character χ of the a representation $D(g)$ is given as the trace of that representation. The characters satisfy the orthogonality relations given as

$$\sum_{g \in G} \chi_{D_\alpha(g)} * \chi_{D_\beta(g)} = N_G \delta_{\alpha\beta} \qquad\qquad \sum_{\alpha} \chi_{D_\alpha(g_i)} * \chi_{D_\alpha(g_j)} = \frac{N_G}{n_i} \delta_{C_i C_j} \tag{2.172}$$

Here N_G is the order of the group, C_i denotes the conjugacy class of g_i and n_i denotes the number of elements in that conjugacy class. Let m_n represents m n-dimensional irreducible representations. Now, the statement "number of irreducible representations is equal to the

[1] The elements $g^{-1}ag$ for $g \in G$ are called elements conjugate to the element a. The set including all elements to conjugate to an element a of G, $\{g^{-1}ag, \forall g \in G\}$, is called a conjugacy class.

number of conjugacy classes" implies

$$\sum_n m_n = 4 \tag{2.173}$$

And from the above orthogonality relations we can write

$$\sum_\alpha [\chi_{D_\alpha(C_1)}]^2 = \sum_n m_n n^2 = N_G = 10 \tag{2.174}$$

Solving (3) and (4) we get $m_1 = 2$ and $m_2 = 2$. That is, there are two singlets and two doublets for D_5 group. They are denoted as $\mathbf{1}_+$, $\mathbf{1}_-$, $\mathbf{2}$, $\mathbf{2}'$.
The generators for the 2×2 representation in complex basis are

$$a = \begin{pmatrix} e^{\frac{i2\pi k}{5}} & 0 \\ 0 & e^{\frac{-i2\pi k}{5}} \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.175}$$

The character table [7] is given as

Conjugacy class	Order	χ_{1_1}	χ_{1_2}	χ_{2_1}	χ_{2_2}
C_1	1	1	1	2	2
$C_2^{(1)}$	5	1	1	$2\cos(\frac{2\pi}{5})$	$2\cos(\frac{4\pi}{5})$
$C_2^{(2)}$	5	1	1	$2\cos(\frac{4\pi}{5})$	$2\cos(\frac{2\pi}{5})$
C_5	2	1	-1	0	0

Table 2.6: Character table of D_5

2.3.9 Tensor Product

D_5 has four irreducible representations. Two two-dimensional representations denoted by $\mathbf{2}_1$ and $\mathbf{2}_2$, and two one-dimensional representations denoted by $\mathbf{1}_+$ and $\mathbf{1}_-$. Tensor product of irreps in real basis is given as following:

Product of doublets

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{\mathbf{2}_1} \otimes \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}_{\mathbf{2}_1} = [x_1y_1 + x_2y_2]_{\mathbf{1}_+} \oplus [x_1y_2 - x_2y_1]_{\mathbf{1}_-} \oplus \begin{bmatrix} x_1y_1 - x_2y_2 \\ x_1y_2 + x_2y_1 \end{bmatrix}_{\mathbf{2}_2}, \tag{2.176a}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{\mathbf{2}_2} \otimes \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}_{\mathbf{2}_2} = [x_1y_1 + x_2y_2]_{\mathbf{1}_+} \oplus [x_1y_2 - x_2y_1]_{\mathbf{1}_-} \oplus \begin{bmatrix} x_2y_2 - x_1y_1 \\ x_1y_2 + x_2y_1 \end{bmatrix}_{\mathbf{2}_1} , \tag{2.176b}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{\mathbf{2}_1} \otimes \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}_{\mathbf{2}_2} = \begin{bmatrix} x_1y_1 + x_2y_2 \\ x_1x_2' - x_2x_1' \end{bmatrix}_{\mathbf{2}_1} \oplus \begin{bmatrix} x_1x_1' - x_2x_2' \\ x_1y_2 + x_2y_1 \end{bmatrix}_{\mathbf{2}_2} . \tag{2.176c}$$

Product of doublet and singlet

$$\begin{bmatrix} w \end{bmatrix}_{\mathbf{1}_+} \otimes \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{\mathbf{2}_{1,2}} = \begin{bmatrix} wx_1 \\ wx_2 \end{bmatrix}_{\mathbf{2}_{1,2}} , \quad \begin{bmatrix} w \end{bmatrix}_{\mathbf{1}_-} \otimes \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{\mathbf{2}_{1,2}} = \begin{bmatrix} -wx_2 \\ wx_1 \end{bmatrix}_{\mathbf{2}_{1,2}} . \tag{2.177}$$

Product of singlets

$$\begin{bmatrix} w \end{bmatrix}_{\mathbf{1}_{s_1}} \otimes [w_2]_{\mathbf{1}_{s_2}} = [w_1w_2]_{\mathbf{1}_{s_1s_2}} . \tag{2.178}$$

where $\mathbf{1}_s$ is either $\mathbf{1}_+$ or $\mathbf{1}_-$.

I have not failed. I've just found 10,000 ways that won't work ...

Thomas A. Edison

3

Attempts Using D_5 symmetry

Now that we are equipped with the knowledge of group theory, we will use it to construct models using D_5 symmetry. This can be done by extending the SM scalar sector by adding SU(2) scalar doublets. We will start by writing the tensor products as found out in the previous chapter. Then we will build two Higgs doublet model(2HDM) and three Higgs doublet model(3HDM) in the upcoming sections.

D_5 has four irreducible representations. Two two-dimensional representations denoted by $\mathbf{2}_1$ and $\mathbf{2}_2$, and two one-dimensional representations denoted by $\mathbf{1}_+$ and $\mathbf{1}_-$. Tensor product of irreps in real basis is found out in Chapter-2. The results are summarized here again for reference.

Product of doublets

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{\mathbf{2}_1} \otimes \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}_{\mathbf{2}_1} = [x_1y_1 + x_2y_2]_{\mathbf{1}_+} \oplus [x_1y_2 - x_2y_1]_{\mathbf{1}_-} \oplus \begin{bmatrix} x_1y_1 - x_2y_2 \\ x_1y_2 + x_2y_1 \end{bmatrix}_{\mathbf{2}_2} , \quad (3.1a)$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{\mathbf{2}_2} \otimes \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}_{\mathbf{2}_2} = [x_1y_1 + x_2y_2]_{\mathbf{1}_+} \oplus [x_1y_2 - x_2y_1]_{\mathbf{1}_-} \oplus \begin{bmatrix} x_2y_2 - x_1y_1 \\ x_1y_2 + x_2y_1 \end{bmatrix}_{\mathbf{2}_1} , \tag{3.1b}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{\mathbf{2}_1} \otimes \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}_{\mathbf{2}_2} = \begin{bmatrix} x_1y_1 + x_2y_2 \\ x_1x_2' - x_2x_1' \end{bmatrix}_{\mathbf{2}_1} \oplus \begin{bmatrix} x_1x_1' - x_2x_2' \\ x_1y_2 + x_2y_1 \end{bmatrix}_{\mathbf{2}_2} . \tag{3.1c}$$

Product of doublet and singlet

$$\begin{bmatrix} w \end{bmatrix}_{\mathbf{1}_+} \otimes \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{\mathbf{2}_{1,2}} = \begin{bmatrix} wx_1 \\ wx_2 \end{bmatrix}_{\mathbf{2}_{1,2}} , \quad \begin{bmatrix} w \end{bmatrix}_{\mathbf{1}_-} \otimes \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{\mathbf{2}_{1,2}} = \begin{bmatrix} -wx_2 \\ wx_1 \end{bmatrix}_{\mathbf{2}_{1,2}} . \tag{3.2}$$

Product of singlets

$$\begin{bmatrix} w \end{bmatrix}_{\mathbf{1}_{s_1}} \otimes [w_2]_{\mathbf{1}_{s_2}} = [w_1w_2]_{\mathbf{1}_{s_1s_2}} . \tag{3.3}$$

where $\mathbf{1}_s$ is either $\mathbf{1}_+$ or $\mathbf{1}_-$.

3.1 Two Higgs Doublet Model

D_5 has two two-dimensional representations, so we have the freedom of assigning the three generations of fermions to different two-dimensional representations. One of the ways of assigning is putting the first generation in the one-dimensional representation, and the second and third generations in the two-dimensional representation as done in this paper [8]. Or we can multiply with the permutation matrices to get the required fields at the end.

We have tried other ways of transforming quark fields. Using one Higgs doublet is a trivial case. Shown below are different ways of transforming fields using two $SU(2)$ Higgs doublet. We can either put all the fields in the same doublet of D_5 or put different fields in different doublets of D_5.

Case 1: All the fields in the same doublet, say 2_1. The quark field transform as different representation of D_5 in the following way:

$$\mathbf{1}_+ : \quad Q_3 , \; n_{3R} , \qquad \mathbf{1}_- : \quad p_{3R} , \tag{3.4a}$$

$$\mathbf{2}_1 : \quad \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} , \; \begin{bmatrix} p_{1R} \\ P_{2R} \end{bmatrix} , \; \begin{bmatrix} n_{1R} \\ n_{2R} \end{bmatrix} , \tag{3.4b}$$

where Q_A's ($A = 1,2,3$) are the left handed $SU(2)$ quark doublets and p_{AR}'s and n_{AR}'s are the right handed up-type and down-type quark fields, respectively.
The two Higgs fields ϕ_1 and ϕ_2, which are $SU(2)$ doublets transform as

$$\mathbf{2}_1 : \quad \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} . \tag{3.4c}$$

The fermion mass matrix arises from the coupling $y_{ij}\,\overline{Q}_i\,\phi\,n_{jR}$ for down-type quarks and $y_{ij}\,\overline{Q}_i\,\tilde{\phi}\,p_{jR}$ for up-type quarks. ϕ is the Higgs field and $\tilde{\phi}$ is $\tilde{\phi} = i\sigma_2\phi^*$.
Using Eq. (2.176), Eq. (2.177) and Eq. (2.178), we can write the Yukawa Lagrangian, which will be invariant under D_5 symmetry(as done in section 4).

$$\begin{aligned} -\mathscr{L} = {} & A_d(\overline{Q}_1\phi_1 n_{3R} + \overline{Q}_2\phi_2 n_{3R}) + B_d(\overline{Q}_3\phi_1 n_{1R} + \overline{Q}_3\phi_2 n_{2R}) \\ & + A_u(\overline{Q}_1\tilde{\phi}_2 p_{3R} - \overline{Q}_2\tilde{\phi}_1 p_{1R}) + B_u(\overline{Q}_3\tilde{\phi}_1 p_{1R} + \overline{Q}_3\tilde{\phi}_2 p_{2R}) + h.c.\,. \end{aligned} \tag{3.5}$$

where A_u,, A_d, B_u and B_d are the Yukawa couplings. Let $\langle\phi_k\rangle = \frac{v_k}{\sqrt{2}}$ represents the vacuum expectation values of ϕ_k, $k = 1,2$. The mass matrices arising from the above Lagrangian are written as

$$M_d = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & A_d v_1 \\ 0 & 0 & A_d v_2 \\ B_d v_1 & B_d v_2 & 0 \end{pmatrix}, \tag{3.6a}$$

$$M_u = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & A_u v_2 \\ 0 & 0 & -A_u v_1 \\ B_u v_1 & B_u v_2 & 0 \end{pmatrix}. \tag{3.6b}$$

The biunitary transformation to obtain the diagonal matrices are:

$$D_u = V_L\, M_u\, V_R^{\dagger}, \tag{3.7a}$$

$$D_d = U_L\, M_d\, U_R^{\dagger}. \tag{3.7b}$$

Here, the matrices relate the gauge basis(n, p) to mass basis(u,d) in the following way:

$$u_L = V_L\, p_L, \qquad u_R = V_R\, p_R\,, \tag{3.8a}$$

$$d_L = U_L\, n_L, \qquad d_R = U_R\, n_R\,. \tag{3.8b}$$

where u and d represent the physical up and down quark respectively. The Cabibbo-Kobayashi-Maskawa(CKM) matrix is obtained from the gauge interaction term and given by

$$V_{CKM} = V_L \cdot U_L^\dagger \,. \tag{3.9}$$

From Eq. (3.7b)

$$D_d^\dagger = (U_L \cdot M_d \cdot U_R^\dagger)^\dagger = U_R \cdot M_d^\dagger \cdot U_L^\dagger \,, \tag{3.10}$$

$$D_d^2 = U_L \cdot M_d^2 \cdot U_L^\dagger = \mathrm{diag}(\mathrm{m_d^2}, \mathrm{m_s^2}, \mathrm{m_b^2}), . \tag{3.11}$$

Similarly,

$$D_u^2 = V_L \cdot M_u^2 \cdot V_L^\dagger = \mathrm{diag}(\mathrm{m_u^2}, \mathrm{m_c^2}, \mathrm{m_t^2}), . \tag{3.12}$$

From Eq. (3.6),

$$M_d M_d^\dagger = \frac{1}{2}\begin{pmatrix} A_d^2 v_1^2 & A_d^2 v_1 v_2 & 0 \\ A_d^2 v_1 v_2 & A_d^2 v_2^2 & 0 \\ 0 & 0 & B_d^2 v^2 \end{pmatrix} , \tag{3.13a}$$

$$M_u M_u^\dagger = \frac{1}{2}\begin{pmatrix} A_u^2 v_2^2 & -A_u^2 v_1 v_2 & 0 \\ -A_u^2 v_1 v_2 & A_u^2 v_1^2 & 0 \\ 0 & 0 & B_u^2 v^2 \end{pmatrix} . \tag{3.13b}$$

Diagonalizing the above matrices will give us the value of V_L and U_L which can be used to find the CKM matrix. We define,

$$U_\beta = \begin{pmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{3.14}$$

where β is defined as following:

$$\tan\beta \equiv v_2/v_1 \implies \cos\beta = v_1/v \qquad \sin\beta = v_2/v \,. \tag{3.15}$$

And $v = \sqrt{v_1^2 + v_2^2}$ is the total VEV. U_β diagonalizes the mass matrices as follows:

$$U_\beta \cdot M_u^2 \cdot U_\beta^\dagger = \mathrm{diag}(0, \mathrm{A_u^2 v^2/2, B_u^2 v_2^2/2}) \equiv \mathrm{diag}(\mathrm{m_{p_1}^2, m_{p_2}^2, m_{p_3}^2})\,, \tag{3.16a}$$

$$U_\beta \cdot M_d^2 \cdot U_\beta^\dagger = \mathrm{diag}(\mathrm{A_d^2 v^2/2}, 0, \mathrm{B_d^2 v_2^2/2}) \equiv \mathrm{diag}(\mathrm{m_{d_1}^2, m_{d_2}^2, m_{d_3}^2})\,. \tag{3.16b}$$

We have the freedom to rearrange these eigenvalues to assign them to m_d, m_s and m_b according to our convenience. We concisely take into account all such possibilities by writing

$$\begin{pmatrix} d \\ s \\ b \end{pmatrix} = u \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}, \tag{3.17}$$

where, u is a permutation matrix with 0s and 1s in the appropriate places. For example, if we assign $d \equiv d_1$, $s \equiv d_2$ and $s \equiv d_3$, then $u = I_3$, the 3×3 identity matrix. Thus, using Eq. (3.16b), the diagonal mass matrix in the (d, s, b) basis may be written as:

$$D_d^2 = \mathrm{diag}(m_d^2, m_s^2, m_b^2) = u \cdot U_\beta \cdot M_d^2 \cdot U_\beta^\dagger \cdot u^\dagger\,. \tag{3.18}$$

Take matrix u as

$$u = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.19}$$

such that lowest mass eigenvalue is assigned to the down quark.

And for the up-sector

$$D_u^2 = \mathrm{diag}(\mathrm{m_u^2, m_c^2, m_t^2}) = U_\beta \cdot M_u^2 \cdot U_\beta^\dagger \tag{3.20}$$

Now, comparing Eqs. (3.18) and (3.11), and Eqs. (3.18) and (3.12)

$$V_L = U_\beta\,, \quad U_L = u \cdot U_\beta\,. \tag{3.21}$$

The CKM matrix is then calculated as follows:

$$V_{CKM} = V_L \cdot U_L^\dagger = U_\beta \cdot U_\beta^\dagger \cdot u^\dagger = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.22}$$

And the mass of up quark and down quark is coming zero.

We will get the similar result if we put all the fields in the 2_2 doublet instead of 2_1. Also, there will not be much change if we assign the down-type right-handed field to 1_- instead of 1_+.

Case 2: Different fields in different doublets. The quark fields are assigned to different representation of D_5 in the following way:

$$\mathbf{1}_+ : \quad Q_3 \qquad \mathbf{1}_- : \quad p_{3R}, \ n_{3R} \tag{3.23a}$$

$$\mathbf{2}_1 : \quad \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}, \ \begin{bmatrix} p_{1R} \\ P_{2R} \end{bmatrix} \qquad \mathbf{2}_2 : \quad \begin{bmatrix} n_{1R} \\ n_{2R} \end{bmatrix} \tag{3.23b}$$

where Q_A's $(A = 1, 2, 3)$ are the left handed $SU(2)$ quark doublets and p_{AR}'s and n_{AR}'s are the right handed up-type and down-type quark fields, respectively.

The two Higgs fields ϕ_1 and ϕ_2, which are $SU(2)$ doublets, transform as

$$\mathbf{2}_1 : \quad \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \tag{3.23c}$$

The Yukawa Lagrangian which is invariant under D_5 symmetry is then written as

$$\begin{aligned} \mathscr{L} = {} & A_d(\overline{Q}_1\phi_2 n_{3R} - \overline{Q}_2\phi_1 n_{3R}) + B_d(\overline{Q}_1\phi_1 n_{1R} - \overline{Q}_2\phi_2 n_{1R} + \overline{Q}_1\phi_2 n_{2R} + \overline{Q}_2\phi_1 n_{2R}) \\ & + A_u(\overline{Q}_1\tilde{\phi}_2 p_{3R} - \overline{Q}_2\tilde{\phi}_1 p_{1R}) + B_u(\overline{Q}_3\tilde{\phi}_1 p_{1R} + \overline{Q}_3\tilde{\phi}_2 p_{2R}) + h.c. \end{aligned} \tag{3.24}$$

where A_u,, A_d, B_u and B_d are the Yukawa couplings. Let $\langle\phi_k\rangle = \frac{v_k}{\sqrt{2}}$ represents the vacuum expectation values of ϕ_k, $k = 1, 2$. The mass matrices arising from the above Lagrangian are written as

$$M_u = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & A_u v_2 \\ 0 & 0 & -A_u v_1 \\ B_u v_1 & B_u v_2 & 0 \end{pmatrix} \tag{3.25a}$$

$$M_d = \frac{1}{\sqrt{2}} \begin{pmatrix} B_d\, v_1 & B_d\, v_1 & A_d v_2 \\ B_d\, v_1 & B_d\, v_1 & -A_d v_1 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.25b}$$

Again after diagonalizing the mass matrices, we are getting the mass of up quark and down quark to be zero. From the above cases, we can see that two $SU(2)$ Higgs doublets are not sufficient to get the required results, so we will introduce another Higgs doublet ϕ_3 to our model and see what happens.

3.2 Three Higgs Doublet Model

We have three $SU(2)$ Higgs doublet ϕ_i, three left-handed $SU(2)$ quark doublet Q_{AL}, ad three right-handed $SU(2)$ quark singlet for the up and down sector, p_{AR} and n_{AR}, respectively. We can transform the fields as D_5 doublets and singlets. There are sixteen ways in which the fields transform as doublet, $2_{\mathbf{1}}$ or $2_{\mathbf{2}}$. They are shown below:

$$1111: \quad \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}_{2_{\mathbf{1}}}, \quad \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}_{2_{\mathbf{1}}}, \quad \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}_{2_{\mathbf{1}}}, \quad \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}_{2_{\mathbf{1}}}, \tag{3.26a}$$

$$1112: \quad \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}_{2_{\mathbf{1}}}, \quad \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}_{2_{\mathbf{1}}}, \quad \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}_{2_{\mathbf{1}}}, \quad \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}_{2_{\mathbf{2}}}, \tag{3.26b}$$

$$1121: \quad \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}_{2_{\mathbf{1}}}, \quad \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}_{2_{\mathbf{1}}}, \quad \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}_{2_{\mathbf{2}}}, \quad \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}_{2_{\mathbf{1}}}, \tag{3.26c}$$

$$1122: \quad \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}_{2_{\mathbf{1}}}, \quad \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}_{2_{\mathbf{1}}}, \quad \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}_{2_{\mathbf{2}}}, \quad \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}_{2_{\mathbf{2}}}, \tag{3.26d}$$

$$1211: \quad \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}_{2_{\mathbf{1}}}, \quad \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}_{2_{\mathbf{2}}}, \quad \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}_{2_{\mathbf{1}}}, \quad \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}_{2_{\mathbf{1}}}, \tag{3.26e}$$

$$1212: \quad \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}_{2_{\mathbf{1}}}, \quad \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}_{2_{\mathbf{2}}}, \quad \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}_{2_{\mathbf{1}}}, \quad \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}_{2_{\mathbf{2}}}, \tag{3.26f}$$

$$1221: \quad \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}_{2_{\mathbf{1}}}, \quad \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}_{2_{\mathbf{2}}}, \quad \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}_{2_{\mathbf{2}}}, \quad \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}_{2_{\mathbf{1}}}, \tag{3.26g}$$

$$1222: \quad \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}_{2_{\mathbf{1}}}, \quad \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}_{2_{\mathbf{2}}}, \quad \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}_{2_{\mathbf{2}}}, \quad \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}_{2_{\mathbf{2}}}, \tag{3.26h}$$

$$2111: \quad \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}_{\mathbf{2_2}}, \quad \begin{pmatrix}Q_1\\ Q_2\end{pmatrix}_{\mathbf{2_1}}, \quad \begin{pmatrix}p_1\\ p_2\end{pmatrix}_{\mathbf{2_1}}, \quad \begin{pmatrix}n_1\\ n_2\end{pmatrix}_{\mathbf{2_1}}, \tag{3.26i}$$

$$2112: \quad \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}_{\mathbf{2_2}}, \quad \begin{pmatrix}Q_1\\ Q_2\end{pmatrix}_{\mathbf{2_1}}, \quad \begin{pmatrix}p_1\\ p_2\end{pmatrix}_{\mathbf{2_1}}, \quad \begin{pmatrix}n_1\\ n_2\end{pmatrix}_{\mathbf{2_2}}, \tag{3.26j}$$

$$2121: \quad \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}_{\mathbf{2_2}}, \quad \begin{pmatrix}Q_1\\ Q_2\end{pmatrix}_{\mathbf{2_1}}, \quad \begin{pmatrix}p_1\\ p_2\end{pmatrix}_{\mathbf{2_2}}, \quad \begin{pmatrix}n_1\\ n_2\end{pmatrix}_{\mathbf{2_1}}, \tag{3.26k}$$

$$2122: \quad \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}_{\mathbf{2_2}}, \quad \begin{pmatrix}Q_1\\ Q_2\end{pmatrix}_{\mathbf{2_1}}, \quad \begin{pmatrix}p_1\\ p_2\end{pmatrix}_{\mathbf{2_2}}, \quad \begin{pmatrix}n_1\\ n_2\end{pmatrix}_{\mathbf{2_2}}, \tag{3.26l}$$

$$2211: \quad \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}_{\mathbf{2_2}}, \quad \begin{pmatrix}Q_1\\ Q_2\end{pmatrix}_{\mathbf{2_2}}, \quad \begin{pmatrix}p_1\\ p_2\end{pmatrix}_{\mathbf{2_1}}, \quad \begin{pmatrix}n_1\\ n_2\end{pmatrix}_{\mathbf{2_1}}, \tag{3.26m}$$

$$2212: \quad \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}_{\mathbf{2_2}}, \quad \begin{pmatrix}Q_1\\ Q_2\end{pmatrix}_{\mathbf{2_2}}, \quad \begin{pmatrix}p_1\\ p_2\end{pmatrix}_{\mathbf{2_1}}, \quad \begin{pmatrix}n_1\\ n_2\end{pmatrix}_{\mathbf{2_2}}, \tag{3.26n}$$

$$2221: \quad \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}_{\mathbf{2_2}}, \quad \begin{pmatrix}Q_1\\ Q_2\end{pmatrix}_{\mathbf{2_2}}, \quad \begin{pmatrix}p_1\\ p_2\end{pmatrix}_{\mathbf{2_2}}, \quad \begin{pmatrix}n_1\\ n_2\end{pmatrix}_{\mathbf{2_1}}, \tag{3.26o}$$

$$2222: \quad \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}_{\mathbf{2_2}}, \quad \begin{pmatrix}Q_1\\ Q_2\end{pmatrix}_{\mathbf{2_2}}, \quad \begin{pmatrix}p_1\\ p_2\end{pmatrix}_{\mathbf{2_2}}, \quad \begin{pmatrix}n_1\\ n_2\end{pmatrix}_{\mathbf{2_2}}. \tag{3.26p}$$

And the ways in which the fields transform as singlets are

$$\begin{pmatrix}\phi_3\end{pmatrix}_{1_+}, \quad \begin{pmatrix}Q_3\end{pmatrix}_{1_+}, \quad \begin{pmatrix}p_3\end{pmatrix}_{1_+}, \quad \begin{pmatrix}n_3\end{pmatrix}_{1_+}, \tag{3.27}$$

$$\begin{pmatrix}\phi_3\end{pmatrix}_{1_+}, \quad \begin{pmatrix}Q_3\end{pmatrix}_{1_+}, \quad \begin{pmatrix}p_3\end{pmatrix}_{1_+}, \quad \begin{pmatrix}n_3\end{pmatrix}_{1_-}, \tag{3.28}$$

$$\begin{pmatrix}\phi_3\end{pmatrix}_{1_+}, \quad \begin{pmatrix}Q_3\end{pmatrix}_{1_+}, \quad \begin{pmatrix}p_3\end{pmatrix}_{1_-}, \quad \begin{pmatrix}n_3\end{pmatrix}_{1_+}, \tag{3.29}$$

$$\begin{pmatrix}\phi_3\end{pmatrix}_{1_+}, \quad \begin{pmatrix}Q_3\end{pmatrix}_{1_+}, \quad \begin{pmatrix}p_3\end{pmatrix}_{1_-}, \quad \begin{pmatrix}n_3\end{pmatrix}_{1_-}, \tag{3.30}$$

$$\begin{pmatrix}\phi_3\end{pmatrix}_{1_+}, \quad \begin{pmatrix}Q_3\end{pmatrix}_{1_-}, \quad \begin{pmatrix}p_3\end{pmatrix}_{1_+}, \quad \begin{pmatrix}n_3\end{pmatrix}_{1_+}, \tag{3.31}$$

$$\begin{pmatrix}\phi_3\end{pmatrix}_{1_+}, \quad \begin{pmatrix}Q_3\end{pmatrix}_{1_-}, \quad \begin{pmatrix}p_3\end{pmatrix}_{1_+}, \quad \begin{pmatrix}n_3\end{pmatrix}_{1_-}, \tag{3.32}$$

$$\begin{pmatrix}\phi_3\end{pmatrix}_{1_+}, \quad \begin{pmatrix}Q_3\end{pmatrix}_{1_-}, \quad \begin{pmatrix}p_3\end{pmatrix}_{1_-}, \quad \begin{pmatrix}n_3\end{pmatrix}_{1_+}, \tag{3.33}$$

$$\begin{pmatrix}\phi_3\end{pmatrix}_{1_+}, \quad \begin{pmatrix}Q_3\end{pmatrix}_{1_-}, \quad \begin{pmatrix}p_3\end{pmatrix}_{1_-}, \quad \begin{pmatrix}n_3\end{pmatrix}_{1_-}, \tag{3.34}$$

$$\begin{pmatrix}\phi_3\end{pmatrix}_{1_-}, \quad \begin{pmatrix}Q_3\end{pmatrix}_{1_+}, \quad \begin{pmatrix}p_3\end{pmatrix}_{1_+}, \quad \begin{pmatrix}n_3\end{pmatrix}_{1_+}, \tag{3.35}$$

$$\begin{pmatrix}\phi_3\end{pmatrix}_{1_-}, \quad \begin{pmatrix}Q_3\end{pmatrix}_{1_+}, \quad \begin{pmatrix}p_3\end{pmatrix}_{1_+}, \quad \begin{pmatrix}n_3\end{pmatrix}_{1_-}, \tag{3.36}$$

$$\begin{pmatrix}\phi_3\end{pmatrix}_{1_-}, \quad \begin{pmatrix}Q_3\end{pmatrix}_{1_+}, \quad \begin{pmatrix}p_3\end{pmatrix}_{1_-}, \quad \begin{pmatrix}n_3\end{pmatrix}_{1_+}, \tag{3.37}$$

$$\begin{pmatrix}\phi_3\end{pmatrix}_{1_-}, \quad \begin{pmatrix}Q_3\end{pmatrix}_{1_+}, \quad \begin{pmatrix}p_3\end{pmatrix}_{1_-}, \quad \begin{pmatrix}n_3\end{pmatrix}_{1_-}, \tag{3.38}$$

$$\begin{pmatrix}\phi_3\end{pmatrix}_{1_-}, \quad \begin{pmatrix}Q_3\end{pmatrix}_{1_-}, \quad \begin{pmatrix}p_3\end{pmatrix}_{1_+}, \quad \begin{pmatrix}n_3\end{pmatrix}_{1_+}, \tag{3.39}$$

$$\begin{pmatrix}\phi_3\end{pmatrix}_{1_-}, \quad \begin{pmatrix}Q_3\end{pmatrix}_{1_-}, \quad \begin{pmatrix}p_3\end{pmatrix}_{1_+}, \quad \begin{pmatrix}n_3\end{pmatrix}_{1_-}, \tag{3.40}$$

$$\begin{pmatrix}\phi_3\end{pmatrix}_{1_-}, \quad \begin{pmatrix}Q_3\end{pmatrix}_{1_-}, \quad \begin{pmatrix}p_3\end{pmatrix}_{1_-}, \quad \begin{pmatrix}n_3\end{pmatrix}_{1_+}, \tag{3.41}$$

$$\begin{pmatrix}\phi_3\end{pmatrix}_{1_-}, \quad \begin{pmatrix}Q_3\end{pmatrix}_{1_-}, \quad \begin{pmatrix}p_3\end{pmatrix}_{1_-}, \quad \begin{pmatrix}n_3\end{pmatrix}_{1_-}. \tag{3.42}$$

$$\tag{3.43}$$

Remember here 1, 2 and 3 do not represent the generations. We can always multiply with the permutation matrix and put the generations in the desired place. For one set of doublet fields, say 1111, there are sixteen ways of arranging the fields transforming as singlets. So we have sixteen sets of doublet fields; therefore, we have 256 possible ways of transforming the fields. Some examples are shown in the following subsections.

3.2.1 Case: 1221 ++++

The fields transforms as following:

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}_{2_1}, \quad \begin{pmatrix}Q_1\\ Q_2\end{pmatrix}_{2_2}, \quad \begin{pmatrix}p_1\\ p_2\end{pmatrix}_{2_2}, \quad \begin{pmatrix}n_1\\ n_2\end{pmatrix}_{2_1}, \tag{3.44a}$$

$$\begin{pmatrix}\phi_3\end{pmatrix}_{1_+}, \quad \begin{pmatrix}Q_3\end{pmatrix}_{1_+}, \quad \begin{pmatrix}p_3\end{pmatrix}_{1_+}, \quad \begin{pmatrix}n_3\end{pmatrix}_{1_+}. \tag{3.44b}$$

Down type Yukawa Lagrangian:

$$\begin{aligned} -\mathscr{L}_d = \quad & A_d\{(\bar{Q}_{1L}\phi_1 + \bar{Q}_{2L}\phi_2)n_{1R} + (\bar{Q}_{2L}\phi_1 - \bar{Q}_{1L}\phi_2)n_{2R}\} + \\ & B_d\bar{Q}_{3L}(\phi_1 n_{1R} + \phi_2 n_{2R}) + C_d\bar{Q}_{3L}\phi_3 n_{3R}\,. \end{aligned} \tag{3.45}$$

Down type Yukawa Lagrangian:

$$\begin{aligned} -\mathscr{L}_u = \quad & A_u\{(\bar{Q}_{2L}\tilde{\phi}_2 - \bar{Q}_{1L}\tilde{\phi}_1)p_{1R} + (\bar{Q}_{1L}\tilde{\phi}_2 + \bar{Q}_{2L}\tilde{\phi}_1)p_{2R}\} + \\ & B_u\{\bar{Q}_{1L}\tilde{\phi}_3 p_{1R} + \bar{Q}_{2L}\tilde{\phi}_3 p_{2R}\} + C_u\bar{Q}_{3L}\tilde{\phi}_3 p_{3R}\,. \end{aligned} \tag{3.46}$$

The mass matrices are then written as

$$M_d = \begin{pmatrix} A_d v_1 & -A_d v_2 & 0 \\ A_d v_2 & A_d v_1 & 0 \\ B_d v_1 & B_d v_2 & C_d v_3 \end{pmatrix} \tag{3.47a}$$

$$M_u = \begin{pmatrix} -A_u v_1 + B_u v_3 & A_u v_2 & 0 \\ A_u v_2 & A_u v_1 + B_u v_3 & 0 \\ 0 & 0 & C_u v_3 \end{pmatrix} \tag{3.47b}$$

Our mass squared matrices are:

$$M_d^2 = \begin{pmatrix} A_d^2(v_1^2+v_2^2) & 0 & A_d B_d(v_1^2-v_2^2) \\ 0 & A_d^2(v_1^2+v_2^2) & 2A_d B_d v_1 v_2 \\ A_d B_d(v_1^2-v_2^2) & 2A_d B_d v_1 v_2 & B_d^2(v_1^2+v_2^2)+C_d^2 v_3^2 \end{pmatrix}, \tag{3.48a}$$

$$M_u^2 = \begin{pmatrix} A_u^2 v_2^2 + (A_u v_1 - B_u v_3)^2 & 2A_u B_u v_2 v_3 & 0 \\ 2A_u B_u v_2 v_3 & A_u^2 v_2^2 + (A_u v_1 + B_u v_3)^2 & 0 \\ 0 & 0 & C_u^2 v_3^2 \end{pmatrix}. \tag{3.48b}$$

Redefining the VEVs in terms of the total VEV and the polar angles:

$$v_1 = v\cos\beta_1 \sin\beta_2\,, \quad v_2 = v\sin\beta_1\sin\beta_2, v_3 = v\cos\beta_2\,. \tag{3.49}$$

where total VEV, $v = \sqrt{v_1^2+v_2^2+v_3^2} = 174$ GeV in our case.

Diagonalizing Mass Matix In Down Sector After the redefinition, Eq. (3.48a) is written as

$$M_d^2 = \begin{pmatrix} A_d^2 \sin\beta_2^2 & 0 & A_d B_d \cos 2\beta_1 \sin^2\beta_2 \\ 0 & A_d^2\sin\beta_2^2 & A_d B_d \sin 2\beta_1 \sin^2\beta_2 \\ A_d B_d \cos 2\beta_1 \sin^2\beta_2 & A_d B_d \sin 2\beta_1 \sin^2\beta_2 & B_d^2 \sin^2\beta_2 + C_d^2\cos^2\beta_2 \end{pmatrix}. \tag{3.50}$$

We define,

$$O_d = \begin{pmatrix} \sin 2\beta_1 & -\cos 2\beta_1 & 0 \\ \cos 2\beta_1 & \sin 2\beta_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.51}$$

Then, one can check

$$(M_d^2)_{\text{Block}} = O_d M_d^2 O_d^T = \begin{pmatrix} A_d \sin^2\beta_2 & 0 & 0 \\ 0 & A_d \sin^2\beta_2 & A_d B_d \sin^2\beta_2 \\ 0 & A_d B_d \sin^2\beta_2 & B_d \sin^2\beta_2 C_d \cos^2\beta_2 \end{pmatrix}. \tag{3.52}$$

For a symmetric 2×2 matrix

$$K = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \tag{3.53}$$

the orthogonal matrix which diagonalizes it, is given as

$$O = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{3.54}$$

where θ is

$$\theta = \frac{1}{2}\tan^{-1}\left|\frac{2b}{a-c}\right|. \tag{3.55}$$

It will diagonalize the symmetric matrix in the following way:

$$D = OKO^T \tag{3.56}$$

So, we have

$$D_d^2 = O_\theta (M_d^2)_{\text{Block}} O_\theta^T \implies (M_d^2)_{\text{Block}} = O_\theta^T D_d^2 O_\theta \tag{3.57}$$

where

$$O_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}, \quad D_d^2 = \begin{pmatrix} m_x^2 & 0 & 0 \\ 0 & m_y^2 & 0 \\ 0 & 0 & m_z^2 \end{pmatrix}. \tag{3.58}$$

Now,

$$O_\theta^T D_d^2 O_\theta = \begin{pmatrix} m_x^2 & 0 & 0 \\ & m_y^2\cos^2\theta + m_z^2\sin^2\theta & (m_y^2 - m_z^2)\cos\theta\sin\theta \\ 0 & (m_y^2 - m_z^2)\cos\theta\sin\theta & m_z^2\cos^2\theta + m_y^2\sin^2\theta \end{pmatrix} . \tag{3.59}$$

Comparing Eqs. (3.52) and (3.59), we get the following equations:

$$m_x^2 = A_d\sin^2\beta_2\,, \tag{3.60a}$$
$$m_y^2\cos^2\theta + m_z^2\sin^2\theta = A_d\sin^2\beta_2\,, \tag{3.60b}$$
$$m_z^2\cos^2\theta + m_y^2\sin^2\theta = B_d\sin^2\beta_2 C_d\cos^2\beta_2\,, \tag{3.60c}$$
$$(m_y^2 - m_z^2)\cos\theta\sin\theta = A_d B_d\sin^2\beta_2\,. \tag{3.60d}$$

From Eq. (3.60b), we can write

$$m_y^2\cos^2\theta + m_z^2\sin^2\theta = m_x^2(\cos^2\theta + \sin^2\theta) \tag{3.61a}$$
$$(m_y^2 - m_x^2)\cos^2\theta + (m_z^2 - m_x^2)\sin^2\theta = 0\,. \tag{3.61b}$$

Notice that $\sin\theta$ and $\cos\theta$ can't be simultaneously zero. Therefore, $(m_y^2 - m_x^2)$ and $(m_z^2 - m_x^2)$ cannot have the same sign. This means m_x must lie between m_y and m_z. Thus we must have $m_x = m_s$ =mass of strange quark. This implies two possible hierarchies:

$$m_y < m_x < m_z \implies m_y = m_d\,, \quad m_z = m_b\,. \tag{3.62}$$
$$m_z < m_x < m_y \implies m_y = m_b\,, \quad m_z = m_d\,. \tag{3.63}$$

The full diagonalization looks like

$$D_d^2 = O_\theta O_d M_d^2 O_d^T O_\theta^T\,. \tag{3.64}$$

Reshuffling of eigenvalues is done by the permutation matrix u. Thus, the left-handed diagonalizing matrix in the down sector is:

$$U_L = u O_\theta O_d\,. \tag{3.65}$$

Diagonalizing the mass matrix in the up-sector: Matrix M_u^2 is a block diagonal matrix which can be diagonalized using an orthogonal matrix O_θ^u, where

$$O_\theta^u = \begin{pmatrix} \cos\theta_u & \sin\theta_u & 0 \\ -\sin\theta_u & \cos\theta_u & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.66}$$

The orthogonal rotation angle is found out using Eq. (3.55) as following:

$$\tan 2\theta_u = \frac{2A_uB_uv_2v_3}{\{A_uv_2^2 + (A_uv_1 + B_uv_3)^2\} - \{A_uv_2^2 + (A_uv_1 - B_uv_3)^2\}} = \frac{v_2}{v_1} \tag{3.67}$$

From Eq. (3.49),

$$\frac{v_2}{v_1} = \tan\beta_1 \implies \theta_u = \frac{\beta_1}{2} \tag{3.68}$$

O_θ^u then becomes

$$O_\theta^u = \begin{pmatrix} \cos\beta_1/2 & \sin\beta_1/2 & 0 \\ -\sin\beta_1/2 & \cos\beta_1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.69}$$

One can easily check that

$$D_u^2 = O_\theta^{uT} M_u^2 O_\theta^u = \begin{pmatrix} (B_u^2\cos\beta_2 - A_u\sin\beta_2)^2 & 0 & 0 \\ 0 & (B_u^2\cos\beta_2 + A_u\sin\beta_2)^2 & 0 \\ 0 & 0 & C_u\cos^2\beta_2 \end{pmatrix}. \tag{3.70}$$

Again we will the permutation matrix u' to assign these eigenvalues to m_u^2, m_c^2 and m_t^2. The left-handed diagonalizing matrix for the up-sector is then given as:

$$V_L = u' O_u^T\,. \tag{3.71}$$

CKM Matrix The CKM matrix is given as

$$V_{\text{CKM}} = V_L U_L^\dagger = u' O_\theta^{uT} (u O_\theta O_d)^\dagger\,. \tag{3.72}$$

Let's try for the first hierarchy, $m_x = m_s$, $m_y = m_d$ and $m_z = m_b$. For this case the permutation matrix in the down sector is found out as

$$\begin{pmatrix} d \\ s \\ b \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} m_x \\ m_y \\ m_z \end{pmatrix} = \begin{pmatrix} m_y \\ m_x \\ m_z \end{pmatrix} \implies u = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{3.73}$$

We have six permutation matrices, u', in the up sector for this hierarchy. For the first case let u' be the identity matrix. From Eq. (3.51),Eqs. (3.58) and (3.69), we get

$$V_{\text{CKM}} = \begin{pmatrix} \cos(3\beta_1/2)\cos\theta & \sin(3\beta_1/2) & -\cos(3\beta_1/2)\sin\theta \\ \cos\theta\sin(3\beta_1/2) & -\cos(3\beta_1/2) & -\sin(3\beta_1/2)\sin\theta \\ \sin\theta & 0 & \cos\theta \end{pmatrix} . \tag{3.74}$$

We are getting the 32 element of the matrix to be zero. We seem to get one of the CKM matrix elements to be zero for all the possible permutations $(6+6=12)$. Some of the CKM matrix elements are close to zero but not exactly zero, so we moved onto trying other models.

3.2.2 Case: 1121 $++++$

The fields transform as following:

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}_{\mathbf{2_1}} , \quad \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}_{\mathbf{2_1}} , \quad \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}_{\mathbf{2_2}} , \quad \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}_{\mathbf{2_1}} , \tag{3.75a}$$

$$\begin{pmatrix} \phi_3 \end{pmatrix}_{1_+} , \quad \begin{pmatrix} Q_3 \end{pmatrix}_{1_+} , \quad \begin{pmatrix} p_3 \end{pmatrix}_{1_+} , \quad \begin{pmatrix} n_3 \end{pmatrix}_{1_+} . \tag{3.75b}$$

The Yukawa Lagrangian is then written as

$$\begin{aligned} -\mathscr{L}_Y = \quad & A_d(\bar{Q}_1\phi_1 n_3 + \bar{Q}_2\phi_2 n_3) + B_d(\bar{Q}_1\phi_3 n_1 + \bar{Q}_2\phi_3 n_2) + C_d(\bar{Q}_3\phi_1 n_1 + \bar{Q}_3\phi_2 n_2) \\ & + D_d\bar{Q}_3\phi_3 n_3 \\ & + A_u(\bar{Q}_1\tilde{\phi}_1 p_3 + \bar{Q}_2\tilde{\phi}_2 p_3) + B_u(\bar{Q}_1\tilde{\phi}_1 p_1 - \bar{Q}_2\tilde{\phi}_2 p_1 + \bar{Q}_1\tilde{\phi}_2 p_2 + \bar{Q}_2\tilde{\phi}_1 p_2) \\ & + C_u\bar{Q}_3\tilde{\phi}_3 p_3 \, . \end{aligned} \tag{3.76}$$

The mass matrices that follows from the above equation are

$$M_d = \begin{pmatrix} B_d v_3 & 0 & A_d v_1 \\ 0 & B_d v_3 & A_d v_2 \\ C_d v_1 & C_d v_2 & D_d v_3 \end{pmatrix}, \quad M_u = \begin{pmatrix} B_u v_1 & B_u v_2 & A_u v_1 \\ -B_u v_2 & B_u v_1 & A_u v_2 \\ 0 & 0 & C_u v_3 \end{pmatrix}. \tag{3.77}$$

Redefining the VEVs in terms of the total VEV and the polar angles:

$$v_1 = v\cos\beta_1 \sin\beta_2\,, \quad v_2 = v\sin\beta_1\sin\beta_2, v_3 = v\cos\beta_2\,. \tag{3.78}$$

where total VEV, $v = \sqrt{v_1^2 + v_2^2 + v_3^2} = 174$ Gev in our case.

The mass squared matrices are then

$$M_d^2 = \begin{pmatrix} B_d^2\cos^2\beta_2 + A_d^2\cos^2\beta_1\sin^2\beta_2 & A_d^2\cos\beta_1\sin\beta_1\sin^2\beta_2 & (B_dC_d + A_dD_d)\cos\beta_1\cos\beta_2\sin\beta_2 \\ A_d^2\cos\beta_1\sin\beta_1\sin^2\beta_2 & B_d^2\cos^2\beta_2 + A_d^2\sin^2\beta_1\sin^2\beta_2 & (B_dc_d + A_dD_d)\cos\beta_2\sin\beta_1\sin\beta_2 \\ (B_dc_d + A_dD_d)\cos\beta_1\cos\beta_2\sin\beta_2 & (B_dC_d + A_dD_d)\cos\beta_2\sin\beta_1\sin\beta_2 & \frac{1}{2}(C_d^2 + D_d^2 + (-C_d^2 + D_d^2)\cos 2\beta_2) \end{pmatrix} \tag{3.79a}$$

$$M_u^2 = \begin{pmatrix} \frac{1}{2}(A_u^2 + 2B_u^2 + A_u^2\cos 2\beta_1)\sin^2\beta_2 & A_u^2\cos\beta_1\sin\beta_1\sin^2\beta_2 & A_uC_u\cos\beta_1\cos\beta_2\sin\beta_2 \\ A_u^2\cos\beta_1\sin\beta_1\sin^2\beta_2 & 1/2(A_u^2 + 2B_u^2 - A_u^2\cos 2\beta_1)\sin^2\beta_2 & A_uC_u\cos\beta_2\sin\beta_1\sin\beta_2 \\ A_uC_u\cos\beta_1\cos\beta_2\sin\beta_2 & A_uC_u\cos\beta_2\sin\beta_1\sin\beta_2] & C_u^2\cos^2\beta_2 \end{pmatrix}. \tag{3.79b}$$

Diagonalizing The Mass Matrices Both M_d^2 and M_u^2 can be block diagonalized using O_{β_1} as following:

$$(M_d^2)_{\text{Block}} = O_{\beta_1} M_d^2 O_{\beta_1}^T\,, \quad (M_u^2)_{\text{Block}} = O_{\beta_1} M_d^2 O_{\beta_1}^T \tag{3.80}$$

where O_{β_1} is given as

$$O_{\beta_1} = \begin{pmatrix} \sin\beta_1 & -\cos\beta_1 & 0 \\ \cos\beta_1 & \sin\beta_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.81}$$

And the block diagonal matrices are given as

$$(M_d^2)_B = \begin{pmatrix} B_d^2\cos\beta_2^2 & 0 & 0 \\ 0 & A_d^2\sin^2\beta_2 + B_d^2\cos^2\beta_2 & (B_dC_d + A_dD_d)\cos\beta_2\sin\beta_2 \\ 0 & (B_dC_d + A_dD_d)\cos\beta_2\sin\beta_2 & C_d^2\sin^2\beta_2 + D_d^2\cos^2\beta_2 \end{pmatrix} \tag{3,82a}$$

$$(M_u^2)_B = \begin{pmatrix} B_u^2 \sin\beta_2^2 & 0 & 0 \\ 0 & (A_u^2 + B_u^2)\sin^2\beta_2 & A_u C_u \cos\beta_2 \sin\beta_2 \\ 0 & A_u C_u \cos\beta_2 \sin\beta_2 & C_u^2 \cos^2\beta_2 \end{pmatrix} . \tag{3.82b}$$

These block diagonal matrices can be further diagonalized using orthogonal matrices O_ϕ^d and O_ϕ^u, respectively, as following:

$$D_d^2 = O_\phi^d (M_d^2)_{\text{Block}} O_\phi^{dT} , \quad D_u^2 = O_\phi^u (M_u^2)_{\text{Block}} O_\phi^{uT} . \tag{3.83}$$

where,

$$O_\phi^d = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_d & \sin\phi_d \\ 0 & -\sin\phi_d & \cos\phi_d \end{pmatrix} , \quad O_\phi^u = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_u & \sin\phi_u \\ 0 & -\sin\phi_u & \cos\phi_u \end{pmatrix} . \tag{3.84}$$

We can then write the left-handed diagonalizing matrix in the up and down sector as

$$U_L = O_\phi^d O_{\beta_1} , \quad V_L = O_\phi^u O_{\beta_1} \tag{3.85}$$

CKM Matrix Using Eq. (3.85), we can write the CKM matrix as

$$V_{\text{CKM}} = V_L U_L^\dagger = O_\phi^u O_{\beta_1} (O_\phi^d O_{\beta_1})^\dagger = O_\phi^u O_\phi^{d\dagger} \tag{3.86}$$

From Eq. (3.84), we get

$$V_{\text{CKM}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi_u - \phi_d) & \sin(\phi_u - \phi_d) \\ 0 & -\sin(\phi_u - \phi_d) & \cos(\phi_u - \phi_d) \end{pmatrix} \tag{3.87}$$

3.3 Summary

- First, we constructed two Higgs doublet models using D_5 symmetry. However, we were not getting desired results for the 2HDM, so we moved onto 3HDM.
- For 3HDM, we found out that the CKM matrices that we were getting were either like Eq. (3.74) or Eq. (3.87) for the cases in which we consider all the singlets to transform like $\mathbf{1}_+$.

Also, the results did not change much if the singlets transform as $\mathbf{1}_-$. After considering all the possibilities, we found no suitable transformation of fields for which we may get a sensible result.

- We can see why we are getting these results from the tensor product of the D_5 group. Lets look at Eq. (3.1a) and Eq. (3.1b), the product of same doublets gives us a different doublet. This was a problem because, from Eq. (3.1c), we can see that the product of two different doublets did not result in a singlet. We could have gotten a different Yukawa Lagrangian if this was not the case, and then things would have turned out differently.

Success is stumbling from failure to failure
with no loss of enthusiasm ...

Winston S. Churchill

4

$2+1+1$ Model

In the first chapter, we discussed SM and the need to go BSM. We discussed the flavor puzzle, hierarchy in fermion masses, and mixings. This chapter will give an extension of SM with D_4 symmetry, which can explain the mystery of masses and mixings. First, we will start with the basics of the D_4 group. The discrete group D_4 has five irreducible representations: $\mathbf{1}_{++}$, $\mathbf{1}_{--}$, $\mathbf{1}_{-+}$, $\mathbf{1}_{+-}$, and $\mathbf{2}$. We opt to work in the real basis of D_4, where the tensor products are given by [9]

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{\mathbf{2}} \otimes \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}_{\mathbf{2}} = \Big[x_1y_1 + x_2y_2\Big]_{\mathbf{1}_{++}} \oplus \Big[x_1y_2 - x_2y_1\Big]_{\mathbf{1}_{--}} \oplus \Big[x_1y_2 + x_2y_1\Big]_{\mathbf{1}_{-+}} \oplus \Big[x_1y_1 - x_2y_2\Big]_{\mathbf{1}_{+-}} \tag{4.1a}$$

$$\mathbf{1}_{r,s} \otimes \mathbf{1}_{r',s'} = \mathbf{1}_{r\cdot r',s\cdot s'} \tag{4.1b}$$

These tensor products were obtained by choosing the following basis for the D_4 symmetry:

$$a = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{4.2}$$

where a is of order 4 and b is of order 2.

4.1 The Model

Here we will describe the D_4 transformations of different fields in our model. The i-th generation of left-handed quark doublet is denoted by Q_i. The right handed charged quark singlet is denoted by p_i in the up sector and by n_i in the down sector. These fields transform under the D_4 symmetry as follows:

$$\mathbf{2} : \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}, \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}, \tag{4.3a}$$

$$\mathbf{1}_{++} : n_1\,, \quad \mathbf{1}_{--} : n_2\,, n_3\,, \phi_u\,, \quad \mathbf{1}_{-+} : p_2\,, p_3\,, \phi_d\,, \quad \mathbf{1}_{+-} : Q_3\,, p_1\,, \tag{4.3b}$$

where ϕ_1, ϕ_2, ϕ_u and ϕ_d are the four scalar doublets.

The D_4 invariant Yukawa Lagrangian in the up quark and down quark sector is given by

$$\begin{aligned} -\mathscr{L}_u = \quad & A_u(\bar{Q}_1\tilde{\phi}_1 - \bar{Q}_2\tilde{\phi}_2)p_1 + B_u(\bar{Q}_1\tilde{\phi}_2 + \bar{Q}_2\tilde{\phi}_1)p_2 + C_u(\bar{Q}_1\tilde{\phi}_2 + \bar{Q}_2\tilde{\phi}_1)p_3 \\ & + X_u\bar{Q}_3\phi_u p_2 + Y_u\bar{Q}_3\phi_u p_3\,. \end{aligned} \tag{4.4a}$$

$$\begin{aligned} -\mathscr{L}_d = \quad & A_d(\bar{Q}_1\phi_1 + \bar{Q}_2\phi_2)n_1 + B_d(\bar{Q}_1\phi_2 - \bar{Q}_2\phi_1)n_2 + C_d(\bar{Q}_1\phi_2 - \bar{Q}_2\phi_1)n_3 \\ & + X_d\bar{Q}_3\phi_d n_2 + Y_d\bar{Q}_3\phi_d n_3\,. \end{aligned} \tag{4.4b}$$

where $A_{u,d}$, $B_{u,d}$, $C_{u,d}$, $X_{u,d}$ and $Y_{u,d}$ are the Yukawa couplings. Let $\langle\phi_k\rangle = v_k$ represents the vacuum expectation values of ϕ_k, where $k = 1,\, 2\,, u$ and d. The mass matrices arising from the above Lagrangian in the up and down sector are written as

$$M_u = \begin{pmatrix} A_u v_1 & B_u v_2 & C_u v_2 \\ -A_u v_2 & B_u v_1 & C_u v_1 \\ 0 & X_u v_u & Y_u v_u \end{pmatrix}, \quad M_d = \begin{pmatrix} A_d v_1 & B_d v_2 & C_d v_2 \\ A_d v_2 & -B_d v_1 & -C_d v_1 \\ 0 & X_d v_d & Y_d v_d \end{pmatrix}. \tag{4.5}$$

Total VEV v is given as

$$v=\sqrt{v_{12}^2+v_u^2+v_d^2}=174\ \text{GeV}\,. \tag{4.6}$$

Here, $v_{12}^2=v_1^2+v_2^2$ is the total VEV responsible for light quark masses, as we will see in the later section.

The diagonal mass matrices can be obtained via the following biunitary transformations:

$$D_u = V_L M_u V_R^\dagger = \text{diag}(m_u,\ m_c,\ m_t), \tag{4.7a}$$

$$D_d = U_L M_d U_R^\dagger = \text{diag}(m_d,\ m_s,\ m_b), \tag{4.7b}$$

The CKM matrix is then given by

$$V_{\text{CKM}}=V_L U_L^\dagger\,. \tag{4.8}$$

The matrices, V_L and U_L can be obtained by diagonalizing $M_u M_u^\dagger$ and $M_d M_d^\dagger$ respectively, which can be calculated from Eq. (4.5) as follows:

$$M_u M_u^\dagger=\begin{pmatrix} A_u^2v_1^2+(B_u^2+C_u^2)v_2^2 & (-A_u^2+B_u^2+C_u^2)v_1v_2 & (C_uY_u+B_uX_u)v_2v_u \\ (-A_u^2+B_u^2+C_u^2)v_1v_2 & (B_u^2+C_u^2)v_1^2+A_u^2v_2^2 & (C_uY_u+B_uX_u)v_1v_u \\ (C_uY_u+B_uX_u)v_2v_u & (C_uY_u+B_uX_u)v_1v_u & (Y_u^2+{X_u}^2)v_u^2 \end{pmatrix}, \tag{4.9a}$$

$$M_d M_d^\dagger=\begin{pmatrix} A_d^2v_1^2+(B_d^2+C_d^2)v_2^2 & (A_d^2-B_d^2-C_d^2)v_1v_2 & (C_dY_d+B_dX_d)v_2v_d \\ (A_d^2-B_d^2-C_d^2)v_1v_2 & (B_d^2+C_d^2)v_1^2+A_d^2v_2^2 & -(C_dY_d+B_dX_d)v_1v_d \\ (C_dY_d+B_dY_d\prime)v_2v_d & -(C_dY_d+B_dX_d)v_1v_d & (Y_d^2+{X_d}^2)v_d^2 \end{pmatrix}. \tag{4.9b}$$

4.2 Diagonalization Procedure

Our next step is to diagonalize the matrices. If we define

$$v_1=v_{12}\cos\beta\,,\quad v_2=v_{12}\sin\beta\,. \tag{4.10}$$

Then it is found that the following matrix O_β block diagonalizes the matrices in Eqs. (4.9a) and (4.9b).

$$O_\beta = \begin{pmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{4.11}$$

This is possible because one of the eigenvectors of $M_u M_u^\dagger$ is $(-\cot\beta,\ 1,\ 0)$ and $M_d M_d^\dagger$ is $(\cot\beta,\ 1,\ 0)$. The above found orthogonal matrix can then be easily constructed from the given eigenvectors.

The block diagonal matrices in the up and down sector are given as

$$(M_u M_u^\dagger)_{\text{Block}} = O_\beta^\dagger M_u M_u^\dagger O_\beta = \begin{pmatrix} A_u^2 v_{12}^2 & 0 & 0 \\ 0 & (B_u^2 + C_u^2) v_{12}^2 & (C_u Y_u + B_u X_u) v_{12} v_u \\ 0 & (C_u Y_u + B_u X_u) v_{12} v_u & (Y_u^2 + {X_u}^2) v_u^2 \end{pmatrix} , \tag{4.12a}$$

$$(M_d M_d^\dagger)_{\text{Block}} = O_\beta M_d M_d^\dagger {O_\beta}^\dagger = \begin{pmatrix} A_d^2 v_{12}^2 & 0 & 0 \\ 0 & (B_d^2 + C_d^2) v_{12}^2 & -(C_d Y_d + B_d X_d) v_{12} v_d \\ 0 & -(C_d Y_d + B_d X_d) v_{12} v_d & (Y_d^2 + {X_d}^2) v_d^2 \end{pmatrix} . \tag{4.12b}$$

Diagonalization of 2×2 orthogonal matrix A symmetric 2×2 matrix is given as

$$K = \begin{pmatrix} a & b \\ b & c \end{pmatrix} . \tag{4.13}$$

Let the orthogonal matrix which diagonalizes it is

$$O = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} . \tag{4.14}$$

Then, using similarity transformation, we can write

$$D = OKO^T \implies K = O^T D O , \tag{4.15}$$

where D is the diagonal matrix.

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} . \tag{4.16}$$

From the above equation, we get the following relations:

$$\lambda_1 \cos^2\theta + \lambda_2 \sin^2\theta = a\,, \tag{4.17a}$$

$$-\lambda_1 \sin\theta\cos\theta + \lambda_2 \sin\theta\cos\theta = b\,, \tag{4.17b}$$

$$\lambda_1 \sin^2\theta + \lambda_2 \cos^2\theta = c\,. \tag{4.17c}$$

Solving Eq. (4.17) for θ, we get

$$\theta = \frac{1}{2}\tan^{-1}\left|\frac{2b}{c-a}\right| . \tag{4.18}$$

For diagonalizing the block diagonal matrices in Eqs. (4.12a) and (4.12b), we define

$$O_\theta^u = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_u & -\sin\theta_u \\ 0 & \sin\theta_u & \cos\theta_u \end{pmatrix} , \quad O_\theta^d = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_d & -\sin\theta_d \\ 0 & \sin\theta_d & \cos\theta_d \end{pmatrix} . \tag{4.19}$$

O_θ^u and O_θ^d are the orthogonal matrices which diagonalizes the block diagonal matrices as follows:

$$D_u^2 = O_\theta^u (M_u M_u^\dagger)_{\text{Block}} O_\theta^{u\dagger} , \quad D_d^2 = O_\theta^d (M_d M_d^\dagger)_{\text{Block}} O_\theta^{d\dagger} . \tag{4.20}$$

θ_u and θ_d are found out using Eq. (4.18) as

$$|\tan 2\theta_u| = \left| \frac{2(C_u Y_u + B_u X_u) v_{12} v_u}{(Y_u^2 + {X_u}^2) v_u^2 - (B_u^2 + C_u^2) v_{12}^2} \right| , \tag{4.21a}$$

$$|\tan 2\theta_d| = \left| \frac{-2(C_d Y_d + B_d X_d) v_{12} v_d}{(Y_d^2 + {X_d}^2) v_d^2 - (B_d^2 + C_d^2) v_{12}^2} \right| . \tag{4.21b}$$

The full diagonalization of $M_u M_u^\dagger$ and $M_d M_d^\dagger$ can be done as follows:

$$D_u^2 = O_\theta^u O_\beta^\dagger M_u M_u^\dagger O_\beta O_\theta^{u\dagger} = \text{diag}(m_u^2, m_c^2, m_t^2)\,, \tag{4.22a}$$

$$D_d^2 = O_\theta^d O_\beta M_d M_d^\dagger {O_\beta}^\dagger O_\theta^{d\dagger} = \text{diag}(m_d^2, m_s^2, m_b^2)\,. \tag{4.22b}$$

According to our convention in Eq. (4.7), the matrix V_L should diagonalize $M_u M_u^\dagger$ as follows:

$$\mathrm{diag}(m_u^2, m_c^2, m_t^2) = V_L M_u M_u^\dagger V_L^\dagger\,. \tag{4.23}$$

Comparing Eqs. (4.22a) and (4.23), we can extract V_L as follows:

$$V_L = O_\theta^u O_\beta^\dagger\,. \tag{4.24}$$

Similarly, the matrix U_L should diagonalize $M_d M_d^\dagger$ as follows:

$$\mathrm{diag}(m_d^2, m_s^2, m_b^2) = U_L M_d M_d^\dagger U_L^\dagger\,. \tag{4.25}$$

Comparing Eqs. (4.22b) and (4.25), U_L is written as follows:

$$U_L = O_\theta^d O_\beta\,. \tag{4.26}$$

4.3 CKM Matrix

From Eq. (4.8), the CKM matrix is given as

$$V_{\mathrm{CKM}} = V_L U_L^\dagger = O_\theta^u O_\beta^\dagger (O_\theta^d O_\beta)^\dagger. \tag{4.27}$$

Substituting the matrices from Eq. (4.11) and Eq. (4.19) in Eq. (4.27), we get

$$V_{\mathrm{CKM}} = \begin{pmatrix} \cos 2\beta & -\cos\theta_d \sin 2\beta & -\sin 2\beta \sin\theta_d \\ \cos\theta_u \sin 2\beta & \cos 2\beta \cos\theta_d \cos\theta_u + \sin\theta_d \sin\theta_u & \cos 2\beta \cos\theta_u \sin\theta_d - \cos\theta_d \sin\theta_u \\ \sin 2\beta \sin\theta_u & -\cos\theta_u \sin\theta_d + \cos 2\beta \cos\theta_d \sin\theta_u & \cos\theta_d \cos\theta_u + \cos 2\beta \sin\theta_d \sin\theta_u \end{pmatrix}. \tag{4.28}$$

CKM matrix in Wolfenstein parameterization(Eq. (1.49)) is given as

$$V_{\mathrm{CKM}} \approx \begin{pmatrix} 1-\lambda^2/2 & -\lambda & \mathcal{O}\left(\lambda^3\right) \\ \lambda & 1-\lambda^2/2 & \mathcal{O}\left(\lambda^2\right) \\ \mathcal{O}\left(\lambda^3\right) & \mathcal{O}\left(\lambda^2\right) & 1 \end{pmatrix} \tag{4.29}$$

where $\lambda \approx 0.22$ is the Cabibbo mixing parameter.

To make the connection between Eqs. (4.28) and (4.29) apparent, we assume that v_{12} is responsible for the masses of the first two generations and v_u and v_d are accountable for the mass of the third generation, then

$$v_{12} \ll v_{u,d}\,, \quad v_{12} \sim \mathcal{O}\left(1\text{GeV}\right)\,, \quad v_{u,d} \sim \mathcal{O}\left(100\text{GeV}\right)\,. \tag{4.30}$$

Therefore, from Eq. (4.21), we can write

$$\tan 2\theta_u \approx 2\theta_u \approx \frac{2(C_uY_u + B_uX_u)v_{12}v_u}{(Y_u^2 + {X_u}^2)v_u^2}\,, \tag{4.31a}$$

$$\tan 2\theta_d \approx 2\theta_d \approx \frac{-2(C_dY_d + B_dX_d)v_{12}v_d}{(Y_d^2 + {X_d}^2)v_d^2}\,. \tag{4.31b}$$

Also, if we take the order of the Yukawa couplings to be same then

$$\theta_u \approx \sin\theta_u \approx \frac{(C_uY_u + B_uX_u)v_{12}}{(Y_u^2 + {X_u}^2)v_u} \approx \mathcal{O}\left(\frac{v_{12}}{v_u}\right) \approx \mathcal{O}\left(\lambda^2\right)\,, \tag{4.32a}$$

$$\theta_d \approx \sin\theta_d \approx \frac{-(C_dY_d + B_dX_d)v_{12}}{(Y_d^2 + {X_d}^2)v_d} \approx \mathcal{O}\left(\frac{v_{12}}{v_d}\right) \approx \mathcal{O}\left(\lambda^2\right)\,. \tag{4.32b}$$

Moreover, if we identify $\sin 2\beta$ as the Cabibbo mixing, namely,

$$\sin 2\beta = \lambda\,, \tag{4.33}$$

then Eq. (4.28) resembles exactly to Eq. (4.29). These intuitive results are validated by finding out the best fit values for β, θ_u and θ_d.

4.4 Best Fit Value

The CKM mixing angles θ_{ij} [10] extracted from the CKM matrix in standard parametrization(refer to Eq. (1.47)) are given as:

$$\theta_{13} = \arcsin(|V_{13}|)\,, \tag{4.34a}$$

$$\theta_{12} = \begin{cases} \arctan\left(\frac{V_{12}}{V_{11}}\right) & \text{if } V_{11} \neq 0 \\ \frac{\pi}{2} & \text{else} \end{cases} \tag{4.34b}$$

$$\theta_{23} = \begin{cases} \arctan\left(\frac{V_{22}}{V_{33}}\right) & \text{if } V_{33} \neq 0 \\ \frac{\pi}{2} & \text{else} \end{cases} \tag{4.34c}$$

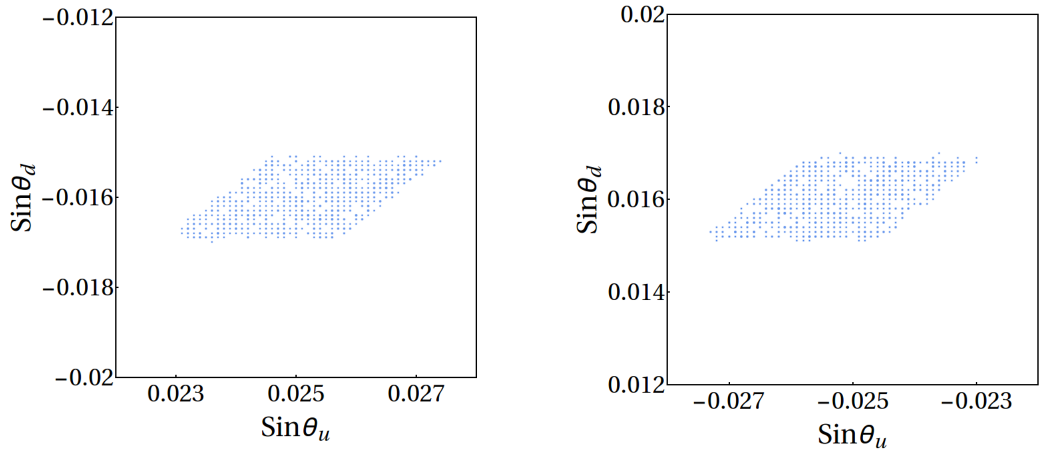


FIGURE 4.1: Plot of $\sin\theta_u$ versus $\sin\theta_d$

The values of mixing angles are taken from the particle data group [5], listed as follows:

$$\sin\theta_{12} = 0.22650 \pm (0.00096)\,, \quad \sin\theta_{13} = 0.00361^{+0.00022}_{-0.00018}\,, \quad \sin\theta_{23} = 0.04053^{+0.00166}_{-0.00122}\,. \quad (4.35)$$

We used Eq. (4.34) for the CKM matrix in Eq. (4.28) to get the relation between the mixing angles $\theta_{12,13,23}$ and β, $\theta_{u,d}$. We then generated a set of random numbers for β and $\theta_{u,d}$ and extracted out the values which satisfy the relation and for which the mixing angles lie within the range given in Eq. (4.35).

$$\sin 2\beta \approx 0.226\,, \qquad \sin\theta_u \approx \pm 0.025\,, \qquad \sin\theta_d \approx \mp 0.016\,. \qquad (4.36)$$

4.5 Mass Eigenvalues

The first generation quark masses, given from Eq. (4.12), are

$$m_u^2 = A_u^2 v_{12}^2\,, \qquad m_d^2 = A_d^2 v_{12}^2\,. \qquad (4.37)$$

Masses of second and third generation of quarks are found out by diagonalizing the remaining 2×2 block. The traces in the up and down sector are written as:

$$m_c^2 + m_t^2 \;=\; (B_u^2 + C_u^2)v_{12}^2 + (Y_u^2 + {X_u}^2)v_u^2\,, \qquad (4.38a)$$

$$m_s^2 + m_b^2 \;=\; (B_d^2 + C_d^2)v_{12}^2 + (Y_d^2 + {X_d}^2)v_d^2\,. \tag{4.38b}$$

Keeping in mind the hierarchies, $v_{u,d} \gg v_{12}$, $m_t \gg m_c$ and $m_b \gg m_s$, the above relations can be approximated to express the top quark mass and bottom quark mass as

$$m_t^2 \;\approx\; (Y_u^2 + {X_u}^2)v_u^2\,, \tag{4.39}$$

$$m_b^2 \;\approx\; (Y_d^2 + {X_d}^2)v_d^2\,. \tag{4.40}$$

Again, from the determinant in Eq. (4.12), we can write

$$m_c^2 m_t^2 \;=\; (B_u Y_u - C_u X_u)^2 v_{12}^2 v_u^2\,, \tag{4.41a}$$

$$m_s^2 m_b^2 \;=\; (B_d Y_d - C_d X_d)^2 v_{12}^2 v_d^2\,. \tag{4.41b}$$

Using Eqs. (4.39) and (4.40) in Eq. (4.41a), we can find out the approximate mass eigenvalues of charm quark and strange quark as following:

$$m_c^2 \;\approx\; \frac{(B_u Y_u - C_u X_u)^2}{(Y_u^2 + {X_u}^2)} v_{12}^2\,, \tag{4.42}$$

$$m_s^2 \;\approx\; \frac{(B_d Y_d - C_d X_d)^2}{(Y_d^2 + {X_d}^2)} v_{12}^2\,. \tag{4.43}$$

At this point, we wish to emphasize that a natural outcome of our model is

$$\frac{m_c}{m_t} \approx \frac{m_s}{m_b} \approx \frac{v_{12}}{v_{u,d}} \sim \mathcal{O}\left(\lambda^2\right)\,, \tag{4.44}$$

which agrees with the observations.

4.6 Conclusion

- The hierachy of the Yukawa couplings is diluted by two orders of magnitude, at least. Note that, in the SM, $m_t = 174$ GeV and $m_{u,d} \sim \mathcal{O}\left(10^{-3}\ \text{GeV}\right)$. This means, the quark Yukawa couplings span five orders of magnitudes. We dampen this problem by assuming that the first two generations of quarks receive their masses from v_{12} which is of $\mathcal{O}\left(1\ \text{GeV}\right)$. This implies, the first generation Yukawas are, at worst, of $\mathcal{O}\left(10^{-3}\right)$ whereas the third generation Yukawas can be of $\mathcal{O}\left(1\right)$.

- We have introduced $\phi_{u,d}$ dedicated for masses of the third generation of quarks. Quite naturally, we expect, $v_{u,d} \sim \mathcal{O}\left(100 \text{ GeV}\right)$ so that the top-Yukawa is of $\mathcal{O}\left(1\right)$. Thus, we should have the ratio $v_{12}/v_{u,d} \sim \mathcal{O}\left(\lambda^2\right)$. It is very interesting to note that, this automatically conforms to $m_2/m_3 \approx v_{12}/v_{u,d} \sim \mathcal{O}\left(\lambda^2\right)$ where m_k is the mass for the k-th generation of quark. Quite clearly, this is a natural upshot of our model.

- We have connected the quark mixings with the dynamics of the scalar sector. We have shown that the Cabibbo part of the quark-mixing stems purely from the ratio v_2/v_1. The smallness of the off-Cabibbo elements of the CKM matrix is further connected to the VEV hierarchy $v_{12} \ll v_{u,d}$. In other way, we are suggesting that the fact that the third generation of quarks are much heavier than the first two generations, is intimately connected to the smallness of the off-Cabibbo elements.

Bibliography

[1] M. Kobayashi et al., *M. kobayashi and t. maskawa*, *Prog. Theor. Phys* **49** (1973) 652.

[2] N. Cabibbo, *Unitary symmetry and leptonic decays*, *Physical Review Letters* **10** (1963), no. 12 531.

[3] L.-L. Chau and W.-Y. Keung, *Comments on the parametrization of the kobayashi-maskawa matrix*, *Physical Review Letters* **53** (1984), no. 19 1802.

[4] H. Harari and M. Leurer, *Recommending a standard choice of cabibbo angles and km phases for any number of generations*, *Physics Letters B* **181** (1986), no. 1-2 123–128.

[5] **Particle Data Group** Collaboration, P. Zyla, R. Barnett, J. Beringer, O. Dahl, D. Dwyer, D. Groom, C.-J. Lin, K. Lugovsky, E. Pianori, et al., *Review of particle physics*, *Progress of Theoretical and Experimental Physics* **2020** (2020), no. 8 083C01.

[6] L. Wolfenstein, *Parametrization of the kobayashi-maskawa matrix*, *Physical Review Letters* **51** (1983), no. 21 1945.

[7] H. Ishimori, T. Kobayashi, H. Ohki, Y. Shimizu, H. Okada, and M. Tanimoto, *Non-Abelian Discrete Symmetries in Particle Physics*, *Prog. Theor. Phys. Suppl.* **183** (2010) 1–163, [arXiv:1003.3552].

[8] C. Hagedorn, M. Lindner, and F. Plentinger, *Discrete flavor symmetry d 5*, *Physical Review D* **74** (2006), no. 2 025007.

[9] D. Das, *Relating the Cabibbo angle to* $\tan\beta$ *in a two Higgs-doublet model*, *Phys. Rev. D* **100** (2019), no. 7 075004, [arXiv:1908.0396].

[10] S. Antusch, J. Kersten, M. Lindner, and M. Ratz, *Running neutrino masses, mixings and cp phases: Analytical results and phenomenological consequences*, *Nuclear Physics B* **674** (2003), no. 1-2 401–433.