METRICS ASSOCIATED WITH THE HURWITZ METRIC

Ph. D. Thesis

by

ARSTU

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submitted in partial fulfillment of the

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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **METRICS ASSOCIATED WITH THE HURWITZ METRIC** in the partial fulfillment of the requirements for the award of the degree of **DOCTOR OF PHILOSOPHY** and submitted in the **DISCIPLINE OF MATHEMATICS, Indian Institute of Technology Indore,** is an authentic record of my own work carried out during the time period from January 2016 to March 2021 under the supervision of Dr. Swadesh Kumar Sahoo, Associate Professor, Indian Institute of Technology Indore.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other institute.

19-03-2021

Signature of the student with date

(ARSTU)

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

19/3/2021

Signature of Thesis Supervisor with date

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ARSTU has successfully given his Ph.D. Oral Examination held on **October 20, 2021**.

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Signature of Thesis Supervisor with date

(Dr. SWADESH KUMAR SAHOO)

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ABSTRACT

KEYWORDS: The hyperbolic metric, the Hurwitz netric, the Kobayashi metric, the Carathéodory metric, the Gardiner-Lakic metric, Holomorphic covering map, Lipschitz domain, hyperbolic domain, hyperbolically covered domain, Hurwitz covering, generalized Hurwitz metric, Möbius invariant metric, conformal map, domain monotonicity, Generalized Schwarz-Pick lemma, local uniform convergence.

In this thesis, we study the Hurwitz metric and introduce metrics in connection with the Hurwitz metric by adopting the idea of the Kobayashi, Carathéodory, and Gardiner-Lakic metrics. We prove that the space with the distance induced by the Hurwitz metric is a complete metric space. Unit disk automorphism plays a crucial role to give a characterization of the Hurwitz metric through which we could define a generalized Hurwitz metric in the sense of Kobayashi in arbitrary subdomains of the complex plane. We study several important properties of this generalized metric, for instance, distance decreasing property, domain monotonicity etc. We establish that the Kobayashi density of the Hurwitz metric always exceeds the Hurwitz metric while the Carathéodory density of the Hurwitz density trails the Hurwitz density. We also study the situations where they coincide with each other. We define a subclass of the class of hyperbolic domains namely the class of hyperbolically covered domains. In the sequel, we study the local uniform convergence of the Hurwitz metric in a sequence of hyperbolically covered domains. Estimations of quotients of the Hurwitz metrics with of the hyperbolic metrics play important roles in this investigation. Furthermore, we study the continuity of the Hurwitz metric in arbitrary proper subdomains of the complex plane and introduce a new Möbius invariant metric which is bi-Lipschitz equivalent to the Hurwitz metric in hyperbolic domains. In addition, the lower semi-continuity of this Möbius invariant metric followed by bi-Lipschitz equivalence of this metric with the (quasi) hyperbolic metrics are investigated.

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CHAPTER 1

INTRODUCTION

This chapter is devoted to give motivation and set framework for the research work elaborated in the upcoming chapter of this thesis. More precisely, introductory discussion on the Hurwitz metric, the hyperbolic metric and the metrics associated with them are demonstrated.

The non-Euclidean geometry is a well known branch in geometric function theory. The idea of non-Euclidean geometry came to the picture when geometers started working on the proof of the Euclid's fifth postulate (also known as the parallel postulate), which states that *given a straight line and a point (not on the line), there exists a unique straight line passing through the given point and not intersecting the given line.* It was the Hungarian mathematician Bolyai (1802-1860) and the Russian mathematician Lobachevsky (1793-1856), who discovered the non-Euclidean geometry simultaneously and independently. It satisfies all of Euclid's axioms except the parallel postulate. However, around 1813, Gauss (1777-1855) considered the possibility of a geometry denying the Euclid fifth postulate, but he did not publish his work. Later in the nineteenth century Poincaré introduced a metric, namely, the hyperbolic metric, on the unit disk which is invariant under conformal self maps of the unit disk. Soon Poincaré's hyperbolic disk model vastly recognized by mathematicians across the globe.

Both of these models have significant amount of applications in establishing various well-known theorems in complex analysis. Metric geometry provides simple elegant and more natural proofs in function theory. It was natural for the mathematicians to find whether the hyperbolic metric can be defined elsewhere than in the unit disk and the upper half plane. The Riemann Mapping Theorem guarantees the existence of the hyperbolic metric on simply connected domains other than the whole complex plane. An outstanding generalization of the Riemann Mapping Theorem, namely the uniformization theorem is due to Poincaré and Köbe. For a proof of the Uniformization Theorem we refer to [1,2]. It generalizes the Riemann Mapping Theorem in the sense that, only simply connected Riemann surfaces are the unit disk, the complex plane and the Riemann sphere up to conformal homeomorphism. By a conformal homeomorphism, we mean a bijective holomorphic function. This theorem brings a revolution in the area of geometric function theory, in particular, in hyperbolic geometry. As a consequence of the uniformization theorem, the hyperbolic metric can be defined on a domain whose complement contains at least two points in the complex plane. However, except for a few cases, actual calculation of a given hyperbolic metric is notoriously difficult. To enhance the study of hyperbolic metric on non-trivial domains, a family of conformal metrics that are closely related to the hyperbolic metric was introduced by several mathematicians. Just to name a few, they are the Hurwitz metric [35], the Gardiner-Lakic metric [12], the Hahn metric [14], the quasihyperbolic metric [13], the K-P metric [31], the Ferrand metric [8], the Apollonian metric [3, 18], the Seittenranta metric [41, 48], the triangular ratio metric [22, 47], the visual angle metric [29, 53], the Cassinian metric [19, 23, 30] and many more can be found from the references therein.

In the literature, the problem of finding the extremal function (supremum of the modulus of derivative of a function evaluated at a fixed point) of a certain class of holomorphic functions is studied by various mathematicians. As pointed out, for instance in [24, p. 132] and [35], the hyperbolic density on a hyperbolic domain Ω can be understood through the extremal problem of maximizing $|f'(0)|$ over all holomorphic functions f that map the unit disk into the hyperbolic domain Ω. In 1981, Hahn [14] introduced a pseudo-differential metric for complex manifolds by means of an extremal problem. Two years later, Minda [37] reconsidered the Hahn metric in Riemann surfaces. Recently, Minda considered an extremal problem of Hurwitz [17] and introduced a new conformal metric, namely, the Hurwitz metric [35] in any proper subdomain of the complex plane $\mathbb{C}$. One of our main goals of this thesis is to further explore the Hurwitz metric in connection with the Kobayashi metric and the Carathéodory metric.

In 2007, Keen and Lakic [25] defined some new densities in arbitrary plane domains that generalize the hyperbolic density. They are namely the generalized Kobayashi density and the generalized Carathéodory density. They established various interesting properties. Two years later, Tavakoli [52] classified the plane domains where the Kobayashi density and the hyperbolic density agree.

One of the important problems in geometric function theory is to check bi-Lipschitz equivalence of the hyperbolic metric with the other conformal metrics and to characterize domains where they are bi-Lipschitz equivalent. Sharper bi-Lipschitz constant plays a vital role in estimating the hyperbolic metric on non-trivial domains. For instance see [5,21,43].

A noteworthy section of the hyperbolic geometry is to study the behaviour of metrics under certain classes of mappings, namely, the Möbius class, the conformal class, the holomorphic covering class, etc. As a consequence it is natural to check whether these classes or their sub-classes are the isometries of the distance function. If a metric is not invariant, we must study its quasi-invariance property.

Infinitesimal form of a metric space is the key object to study a metric space in hyperbolic geometry. Several crucial properties like domain monotonicity, completeness, isometries etc. can be deduced for a given distance function by working on its infinitesimal version. If a given metric is a positive function, it is sensible to talk the ratio of the metrics over two different domains. Note that the classical hyperbolic metric satisfies the domain monotonicity property. As a result, quotient of the hyperbolic density functions of a hyperbolic domain Ω_1 over another hyperbolic domain Ω_2 is always bounded above by 1 whenever $\Omega_1 \subset \Omega_2$. It is always challenging to get upper bound and sharper lower bound in such situations. In connection with this, [39] obtains upper as well as lower bounds for the quotient of the hyperbolic metrics, however, a more precise study on the related problem has been done by Minda in [36]. Based on these bounds, Minda studied local uniform convergence of the hyperbolic metric when corresponding sequences of domains converge in the sense of Carathéodory Kernel with some specific condition. However, a more general result is proved by Hejhal [15] in 1974.

A *covering space* $(\hat{\Omega}, h)$ of a given Riemann surface Ω is a Riemann surface together with a onto holomorphic map $h : \hat{\Omega} \to \Omega$ such that for every $w \in \Omega$ there exists a neighbourhood $U_w \subset \Omega$ of w whose inverse image can be written as a disjoint union of open sets V_α in $\hat{\Omega}$ and h restricted to each V_α is a homeomorphism to U_w. The function h here is known as *holomorphic covering map*. However, covering space can be defined in a more general setting (see [34,38]). A covering space $(\hat{\Omega}, h)$ of Ω is said to be a *regular covering space* [24] if, for all points $w \in \Omega$, every curve $\gamma(t)$ with $\gamma(0) = w$ has a lift to each $\hat{w}$ satisfying $h(\hat{w}) = w$ and the corresponding map h is said to be a *regular covering* *map*. Moreover, the regular covering space is defined as *universal covering space*, if it is simply connected and the corresponding map is called as a *universal covering map*. A plane domain whose universal covering space is the unit disk is said to be a *hyperbolic domain*. In other words, a plane domain with at least two points in its complement is a hyperbolic domain.

Throughout this thesis, we use relatively standard notations. Unless it is specified, we assume that Ω is an arbitrary domain and Y is a proper subdomain in $\mathbb{C}$. Symbolically, we write $\Omega \subset \mathbb{C}$ and $Y \subsetneq \mathbb{C}$. We denote $\mathcal{H}(\Omega, Y)$ by the set of all holomorphic functions from Ω into Y. For a fixed $w \in \Omega$, we define the following notation:

$$\mathcal{H}_s^w(\Omega, Y) = \{h \in \mathcal{H}(\Omega, Y),\ h(w) = s,\ h(z) \neq s \text{ for all } z \in \Omega \setminus \{w\}\}.$$

The open unit disk $\{z \in \mathbb{C} : |z| < 1\}$ is denoted by $\mathbb{D}$. The collection of all functions $h \in \mathcal{H}_s^0(\mathbb{D}, Y)$ such that $h'(0) > 0$ is known as the *Hurwitz family*. More about the Hurwitz family and several other classes of holomorphic functions analogous to the Hurwitz family are discussed in [35]. We set the notation $\mathbb{D}^* := \mathbb{D} \setminus \{0\}$, for the punctured unit disk. The upper half-plane of the complex plane is defined by $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. By $\partial\Omega$, we mean boundary of the set Ω and complement $\mathbb{C} \setminus \Omega$ of Ω in $\mathbb{C}$ is denoted by Ω^c.

1.1. The hyperbolic metric

Suppose f is a conformal homeomorphism of the unit disk onto itself. From the Schwarz-Pick lemma, it is easy to see that

$$\frac{|df(w)|}{1 - |f(w)|^2} = \frac{|dw|}{1 - |w|^2}.$$

Moreover, for a rectifiable path γ in $\mathbb{D}$ we have

$$\int_{f\circ\gamma} \frac{|df(w)|}{1 - |f(w)|^2} = \int_\gamma \frac{1}{1 - |w|^2}.$$

By adopting this idea, we now approach to define a length function which is invariant under conformal homeomorphism of $\mathbb{D}$. The *hyperbolic density* $\lambda_{\mathbb{D}}$ of the unit disk $\mathbb{D}$ is defined by the formula

$$\lambda_{\mathbb{D}}(w) = \frac{2}{1 - |w|^2}.$$

Let $w_1, w_2 \in \mathbb{D}$ be two distinct points and γ be a rectifiable path joining w_1 and w_2 in $\mathbb{D}$. Then the hyperbolic length of γ is defined as

$$\lambda_{\mathbb{D}}(\gamma) = \int_{\gamma} \lambda_{\mathbb{D}}(w)|dw|.$$

The hyperbolic distance $\lambda_{\mathbb{D}}(w_1, w_2)$ between two points w_1 and w_2 is defined as

$$\lambda_{\mathbb{D}}(w_1, w_2) = \inf_{\gamma} \lambda_{\mathbb{D}}(\gamma),$$

where infimum is taken over all rectifiable paths γ joining w_1 and w_2 in $\mathbb{D}$. The classical Riemann mapping theorem assures that any proper simply connected domain in $\mathbb{C}$ is conformally homeomorphic to the unit disk. As a consequence, one can push the hyperbolic metric from the unit disk to proper simply connected domains. If f is a conformal homeomorphism from $\mathbb{D}$ to a simply connected domain $\Omega \subsetneq \mathbb{C}$, then hyperbolic density on Ω is defined by

$$\lambda_{\Omega}(w)|f'(0)| = \lambda_{\mathbb{D}}(0),$$

where $f(0) = w$. However, as stated in the previous section, the hyperbolic metric has been defined on hyperbolic domains. It follows from the uniformization theorem that there exists a holomorphic covering map f from $\mathbb{D}$ onto a hyperbolic domain Ω. On a hyperbolic domain Ω, the *hyperbolic density* λ_{Ω} [24, p. 124] is obtained as

$$\lambda_{\Omega}(w) = \frac{\lambda_{\mathbb{D}}(t)}{|\pi'(t)|},$$

where $\pi : \mathbb{D} \to \Omega$ is a universal covering map with $\pi(t) = w$. The hyperbolic density function λ_{Ω} is a positive continuous function. Except for a few domains, the explicit formula of hyperbolic density is difficult to evaluate due to non-trivial universal covering map of the hyperbolic domains. Analogue to the case of the unit disk, the *hyperbolic distance* between two points w_1 and w_2 in Ω is defined by

$$\lambda_{\Omega}(w_1, w_2) = \inf \int_{\gamma} \lambda_{\Omega}(w)\,|dw|,$$

where the infumum is taken over all rectifiable paths γ joining w_1 and w_2 in Ω. The hyperbolic domain Ω with the distance function λ, that is $(\lambda_{\Omega}, \Omega)$ is a complete metric space and turns out to be locally equivalent to the Euclidean metric. The holomorphic covering map from a hyperbolic domain Ω_1 to another hyperbolic domain Ω_2 is an infinitesimal isometry of the hyperbolic metric. However, the generalized Schwarz-Pick lemma establishes the relation between the hyperbolic metrics over two different hyperbolic domains and is stated as follows. *If f is a holomorphic function from a hyperbolic domain Ω_1 into another hyperbolic domain Ω_2, then f is both an infinitesimal and global contraction with respect to the hyperbolic metrics on Ω_1 and Ω_2. That is*

$$\lambda_{\Omega_2}(f(w))|f'(w)| \leq \lambda_{\Omega_1}(w) \text{ for all } w \in \Omega_1$$

and

$$\lambda_{\Omega_2}(f(w_1), f(w_2)) \leq \lambda_{\Omega_1}(w_1, w_2) \text{ for all } w_1, w_2 \in \Omega_1,$$

respectively. The *domain monotonicity property*, that is larger domains have smaller hyperbolic metric is a direct consequence of the generalized Schwarz-Pick lemma.

1.2. The Hurwitz metric

In 2016, Minda introduced the notion of the Hurwitz metric $\eta_\Omega(s)|ds|$, defined by

$$\eta_\Omega(s) = \frac{2}{G'(0)} = \frac{2}{r_\Omega(s)},$$

where $G'(0) = \max\{|h'(0)| : \ h \in \mathcal{H}_s^0(\mathbb{D}, \Omega)\} =: r_\Omega(s)$. However, the core idea adopted by Minda is taken from the work of Hurwitz [17]. Hurwitz considered the extremal problem of maximizing the quantity $|f'(0)|$ over all holomorphic functions f in $\mathbb{D}$ with $f(0) = 0$, $f(w) \neq 0, 1$ for all $w \in \mathbb{D} \setminus \{0\}$. A sharp bound for $|f'(0)|$ was obtained by Carathéodory three years later and established that $|f'(0)| \leq \beta$ for $0 < \beta \leq 16$. The upper bound here is equivalent to the Hurwitz extremal problem on the unit disk. That is, if f is a holomorphic function in $\mathbb{D}$, $f(0) = 0$, $f(w) \neq 0$ for all $w \in \mathbb{D} \setminus \{0\}$ and $f'(0) \neq 0$ then $f(\mathbb{D})$ contains a Euclidean disk of radius $1/16$ centred at the origin.

Various basic properties of the Hurwitz metric are explored in [35] and [51]. We recall some of these here. For $w \in \Omega$, let $\gamma \subset \Omega \setminus \{w\}$ be a small positively oriented circle centred at w with radius r. This circle generates an infinite cyclic subgroup of the fundamental group $\pi_1(\Omega \setminus \{w\})$ to which there is an associated holomorphic covering map $g : \mathbb{D} \setminus \{0\} \to \Omega \setminus \{w\}$ and extends to a holomorphic function $g : \mathbb{D} \to \Omega$ with $g(0) = w$ and $g'(0) \neq 0$. This covering space $\mathbb{D} \setminus \{0\}$ of $\Omega \setminus \{w\}$ depends only on the free homotopy of the circle γ and is unique up to the pre-composition of rotation around the origin of $\mathbb{D}$. In other words, positivity of $g'(0)$ can be arranged as far as uniqueness is concerned. This unique function g is called the *Hurwitz covering map* associated with the point w. For an elaborative work on canonical doubly connected covering surface associated with each non-trivial free homotopy class in a region, we refer [40]. The *distance decreasing property of the Hurwitz metric* plays a vital role in many results of this thesis and is stated as follows: Suppose that Ω and $\triangle$ are proper subdomains of $\mathbb{C}$, $a \in \Omega$ and $b \in \triangle$. If h is a holomorphic function of Ω into $\triangle$ with $h(a) = b$ and $h(w) \neq b$ for $w \in \Omega \setminus \{a\}$, then

$$\eta_{\triangle}(b)|h'(a)| \leq \eta_{\Omega}(a).$$

Moreover, the equality holds if and only if h is a covering of $\Omega \setminus \{a\}$ onto $\triangle \setminus \{b\}$ that extends to a holomorphic map of Ω onto $\triangle$ with $h(a) = b$ and $h'(a) \neq 0$. Domain monotonicity and conformal invariance properties are the direct consequences of the distance decreasing property.

1.3. The Kobayashi metric

The Kobayashi metric was introduced by S. Kobayashi [26] in 1967 on a complex manifold. The natural motivation arises from the paper of Kobayashi is that, the intrinsic Kobayashi distance on complex manifolds is same as that of the distance generated by the infinitesimal version of the Kobayashi metric. Four years later, an affirmative answer to this question was given by Royden [45]. In 2007, Keen and Lakic defined some new densities in arbitrary plane domains that generalize the hyperbolic density. They are namely the generalized Kobayashi density [25] and the generalized Carathéodory density [24]. Following the paradigm set in previous sections, first we present the Kobayashi metric on infinitesimal level.

Let Ω be a domain in the complex plane. For every $w \in \Omega$, the *generalized Kobayashi density* κ_{Ω}^{Y} is given by

$$\kappa_{\Omega}^{Y}(w) = \inf \frac{\lambda_Y(t)}{|f'(t)|},$$

where λ_Y is the hyperbolic density on a hyperbolic domain $Y \subset \mathbb{C}$ and the infimum is taken over all $f \in \mathcal{H}(Y, \Omega)$ and all points $t \in Y$ such that $f(t) = w$. It has been already proved in the literature that the density κ_{Ω}^{Y} is a lower semi-continuous function, Kobayashi metric exceeds the hyperbolic metric on hyperbolic domains, and agrees with the hyperbolic metric whenever there exists a regular covering map from Y onto Ω. Generalized Schwarz-Pick lemma holds true as well for κ_{Ω}^{Y}. The Kobayashi metric induces a metric space and the distance between two points is given by:

$$\kappa_\Omega^Y(w_1, w_2) = \inf \int_\gamma \kappa_\Omega^Y(w)\,|dw|,$$

where the infimum is taken over all rectifiable paths γ in Ω joining w_1 and w_2.

1.4. The Carathéodory metric

C. Carathéodory [6] introduced the first intrinsic technique to define Carathéodory metric in two complex variables in 1927. However, adopting the idea of Carathéodory, the Carathéodory metric in $n-$complex variables is defined by Reiffen [44]. Alike the Kobayashi metric, the Carathéodory metric is less familiar in one complex variable. We refer [24, 27, 28] for a detailed discussion on the Carathéodory metric in $\mathbb{C}$.

Let Ω be an arbitrary domain and Y be a hyperbolic domain in $\mathbb{C}$. For $w \in \Omega$, the *generalized Carathéodory density* on Ω is defined as

$$c_\Omega^Y(w) = \sup \lambda_Y(f(w))|f'(w)|,$$

where the supremum is taken over all $f \in \mathcal{H}(\Omega, Y)$. The Carathéodory density c_Ω^Y trails the hyperbolic density λ_Ω on hyperbolic domains. As a consequence, it also trails the Kobayashi density. However they coincides, whenever there exists a regular covering map from Ω onto Y. Note that, c_Ω^Y is an upper semi-continuous function. This density function induces a pseudo-metric space and the pseudo distance is given by:

$$c_\Omega^Y(w_1, w_2) = \inf \int_\gamma c_\Omega(w)\,|dw|,$$

where the infimum is over all rectifiable paths γ joining w_1 and w_2 in Ω.

1.5. The Gardiner-Lakic and quasihyperbolic metrics

In association with the hyperbolic metric, Gardiner-Lakic [12] introduced a metric $\overline{\lambda}_\Omega(w)\,|dw|$ which is defined by

$$\overline{\lambda}_\Omega(w) = \sup_{a,b \in \Omega^c} \lambda_{\mathbb{C}\setminus\{a,b\}}(w)$$

for $w \in \Omega$ and distinct points $a, b \in \Omega^c$. Further, Gardiner and Lakic proved in the same paper (see [12, Theorem 3]) that the $\overline{\lambda}_\Omega$-metric is bi-Lipschitz equivalent to the hyperbolic metric λ_Ω. However, discussions on various improvements in the upper bi-Lipschitz constant are taken places later in [21, 50]. This metric is Möbius invariant and a continuous function.

Let $\Omega \subsetneq \mathbb{C}$ be a domain. For $w \in \Omega$, the *quasihyperbolic density* is defined by

$$\frac{1}{\delta_\Omega(w)} = \frac{1}{|w-p|},$$

where the infimum is taken over all $p \in \Omega^c$. The quasihyperbolic metric was defined by Geharing and Palka [13] in their paper on the proper subdomains of $\mathbb{R}^n$. Due to its comparable nature to the hyperbolic metric and other metrics, several mathematicians (see [10], [20], [46]) got attracted towards the quasihyperbolic metric. Note that the Hurwitz metric is bi-Lipschitz equivalent to the quasihyperbolic metric on proper domains of the complex plane. It is quite interesting to study the quasihyperbolic metric in the Gardiner-Lakic sense, which we define as follows:

$$\frac{1}{\overline{\delta}_\Omega(w)} = \sup \frac{1}{\delta_{\mathbb{C}\setminus\{w_1,w_2\}}(w)},$$

where the supremum is taken over all *distinct* pair of points $w_1, w_2 \in \Omega^c$. However, surprisingly, we demonstrate here that $1/\overline{\delta_\Omega}$ and $1/\delta_\Omega$ agree on the hyperbolic domains. Indeed, we have

$$\frac{1}{\overline{\delta}_\Omega(w)} = \sup_{w_1,w_2\in\Omega^c} \frac{1}{\delta_{\mathbb{C}\setminus\{w_1,w_2\}}(w)} = \sup_{w_1,w_2\in\Omega^c} \left\{ \frac{1}{|w-w_1|}, \frac{1}{|w-w_2|} \right\} = \sup_{w_1\in\Omega^c} \frac{1}{|w-w_1|} = \frac{1}{\delta_\Omega}.$$

This justifies the introduction of the Gardiner-Lakic version of the Hurwitz metric in Chapter 5 instead of the quasihyperbolic metric.

1.6. Structure of the thesis

This thesis contains six chapters including this chapter. Present chapter gives some historical background of the non-Euclidean geometry, precisely the hyperbolic geometry. The hyperbolic metric was first defined over the unit disk, later it was extended to simply connected domains and further to hyperbolic domains. In doing so the Riemann mapping theorem and the uniformization theorem played crucial roles. The Kobayashi metric, the Carathéodory metric, the Hurwitz metric and the Gardiner-Lakic metric are discussed in details. Many important properties like infinitesimal isometries, domain monotonicity, etc. are demonstrated.

Chapter 2 is devoted to establishing the distance function originated from the Hurwitz metric and in fact we prove that it is a complete metric space. We do give an alternative definition of the Hurwitz metric, resulting a natural generalization of the Hurwitz metric by adopting the idea of the Kobayashi metric. Various basic properties, for example the distance decreasing property, domain monotonicity, comparison with the Hurwitz and the hyperbolic metric are demonstrated on the hyperbolic domains. At the end of this chapter, we prove that the generalized Hurwitz density and the Hurwtiz density coincide on the Hurwitz non-Lipschitz domains.

In Chapter 3, we define the Carathéodory density of the Hurwitz metric in arbitrary domains of the complex plane. This density function agrees with the Hurwitz density as well the Kobayashi density of the Hurwitz metric on proper simply connected domains while lags behind the Hurwitz metric on non-simply connected domains. We do establish the distance decreasing property of the Carathéodory density of the Hurwitz metric for a special class of holomorphic functions. Conformal invariance and the domain monotonicity properties are thus direct consequences of the distance decreasing properties of the Carathéodory density of the Hurwitz metric. The distance function induced by the Carathéodory density of the Hurwitz metric with some restriction on the domain is demonstrated at the end of the chapter.

We give the notion of hyperbolically covered domains which in fact hyperbolic domains with some conditions in Chapter 4. Upper and lower bounds of the quotients of the Hurwitz metrics are obtained in connection with the hyperbolic disks and for the proper subdomains of the complex plane when one is contained in another. As a result, we discuss the convergence of the Hurwitz densities in the Carathéodory sense for hyperbolically covered domains.

Chapter 5 deals with the continuity of the Hurwitz metric on arbitrary proper subdomains of the complex plane and a new Möbius invariant metric, which is defined in a similar fashion as that of Gardiner-Lakic metric. We establish the lower semi-continuity of this metric and prove some other basic properties of this new metric. The bi-Lipschitz equivalence of this metric with the Hurwitz metric and the quasihyperbolic metric are discussed. we also establish the sharper bi-Lipschitz constant for the case of the Hurwitz metric whenever the domain is hyperbolic with connected boundary.

CHAPTER 2

A GENERALIZED HURWITZ METRIC

In this chapter we study the Hurwitz metric space and establish that it is a complete metric space. The conformal map of the unit disk onto it serves as a tool to give an alternate definition of the Hurwitz metric which further lead us to define a generalized Hurwitz metric in the sense of Kobayashi. We develop many basic properties of this generalized Hurwitz metric and characterize the domains where it agrees with the Hurwitz metric. The distance decreasing property, which is stated in Section 1.2, of the Hurwitz density for the holomorphic function plays a crucial role to prove our results in this chapter.

2.1. The Hurwitz Metric

Firstly, we present here a characterization of the Hurwitz density which gives us ideas to introduce the notion of generalized Hurwitz density in the next section. Let F be the extremal function for the extremal problem $\max\{|h'(t)| : h \in \mathcal{H}_t^s(\mathbb{D}, \Omega)\}$. Then, we have

$$\begin{aligned} F'(t) &= \max\{|h'(t)| : h \in \mathcal{H}_t^s(\mathbb{D}, \Omega)\} \\ &= \max\{|h'(t)| = |(f \circ T)'(t)| : f \in \mathcal{H}_0^s(\mathbb{D}, \Omega) \text{ and } T\text{, the Mobius transformation} \\ &\qquad\qquad \text{of } \mathbb{D} \text{ onto itself with } T(t) = 0 \text{ and } T'(t) > 0\}. \end{aligned}$$

Since $T(z) = (z - t)/(1 - \bar{t}z)$, it follows that

$$F'(t) = \max\left\{\frac{|f'(0)|}{1 - |t|^2} : \ f \in \mathcal{H}_0^s(\mathbb{D}, \Omega)\right\}.$$

Since the hyperbolic density on $\mathbb{D}$ is given by $\lambda_{\mathbb{D}}(t) = 2/(1-|t|^2)$, by the notations defined in the previous section, we have

$$F'(t) = \frac{\lambda_{\mathbb{D}}(t)}{\eta_\Omega(s)}.$$

By using this argument, we provide here an alternate definition of the Hurwitz density as follows:

Definition 2.1. The *Hurwitz density* on a proper subdomain Ω of $\mathbb{C}$ is defined as

$$\eta_\Omega(w) = \frac{\eta_{\mathbb{D}}(s)}{g'(s)}, \tag{2.1}$$

where $g = h \circ T$ such that T is the Möbius transformation from $\mathbb{D}$ onto $\mathbb{D}$ with $T(s) = 0,\ T'(s) > 0$ and h is the Hurwitz covering map from $\mathbb{D}$ onto Ω with $h(0) = w$.

To define the Hurwitz distance between any two points in Ω we integrate the density η_Ω and obtain the following definition:

Definition 2.2. [Hurwitz distance] For w_1, w_2 in Ω, we define

$$\eta_\Omega(w_1, w_2) = \inf \int_\gamma \eta_\Omega(w)|dw|,$$

where the infimum is taken over all rectifiable paths γ in Ω joining w_1 and w_2.

Note that we are using the same notation for the Hurwitz density as well as the Hurwitz distance between any two points where the distinction can be observed by seeing the number of parameters. However, now onward, for simplicity, we sometimes use the notation η_Ω for $\eta_\Omega(w_1, w_2)$. To justify our above definition we indeed prove that η_Ω defines a metric when the domain Ω is assumed to be hyperbolic.

Theorem 2.3. *If Ω is a hyperbolic domain, then (Ω, η_Ω) is a complete metric space.*

Proof. By the definition of η_Ω, symmetry and triangle inequality follow directly. Therefore to prove that (Ω, η_Ω) is a metric space, we need to prove strictly positivity of the Hurwitz distance between any two distinct points. Let w_1, w_2 be any two distinct points in Ω. Since $\eta_\Omega(w_1, w_2)$ is the infimum of the Hurwitz length of all rectifiable curves joining w_1 and w_2 in Ω, for any $\epsilon > 0$ there exists a rectifiable path γ such that

$$\eta_\Omega(w_1, w_2) \geq \int_\gamma \eta_\Omega(w)|dw| - \epsilon.$$

Note that in a hyperbolic domain Ω, the inequality $\eta_\Omega \geq \lambda_\Omega$ is well-known; see [35]. Then we have

$$\eta_\Omega(w_1, w_2) \geq \int_\gamma \lambda_\Omega(w)|dw| - \epsilon \geq \lambda_\Omega(w_1, w_2) - \epsilon.$$

Letting $\epsilon \to 0$, we obtain

$$\eta_\Omega(w_1, w_2) \geq \lambda_\Omega(w_1, w_2) > 0. \tag{2.2}$$

To prove the completeness, we use the following fact (see [4, Theorem 2.5.28, p. 52]): *a locally compact length (metric) space X is complete if and only if every closed disc in X is compact (see also [4, p. 28]).* Because λ_Ω is complete, each closed hyperbolic disk $\overline{D}_{\lambda_\Omega}(a, r) = \{w \in \Omega : \lambda_\Omega(a, w) \leq r\}$ is compact. Because $\lambda_\Omega \leq \eta_\Omega$, $\overline{D}_{\eta_\Omega}(a, r) \subset \overline{D}_{\lambda_\Omega}(a, r)$. A closed subset of a compact set is compact, so $\overline{D}_{\eta_\Omega}(a, r)$ is compact. □

The following remark assures that there exists a non-hyperbolic domain for which Theorem 5.3 still satisfies.

Remark 2.4. Let $\Omega = \mathbb{C} \setminus \{0\}$. Then, the Hurwitz density has the elementary formula $\eta_\Omega(w) = 1/8|w|$. This is nothing but a scalar multiplication of the classical quasihyperbolic metric of Ω. The completeness property now follows from the fact that the quasihyperbolic metric space is complete.

We know that the holomorphic functions are global as well as infinitesimal contraction functions with respect to the hyperbolic metric. In analogy to this we now prove that the one-to-one holomorphic functions are global contraction functions for the Hurwitz metric as well.

Proposition 2.5. *Let Ω and Y be proper subdomains of $\mathbb{C}$ and h be an injective holomorphic function from Ω to Y. Then we have the inequality*

$$\eta_Y(h(w_1), (w_2)) \leq \eta_\Omega(w_1, w_2)$$

for all w_1, w_2 in Ω. Equality holds in the above inequality if h is an conformal homeomorphism.

Proof. By the definition of $\eta_\Omega(w_1, w_2)$, for any $\epsilon > 0$ there exists a path γ joining w_1 and w_2 in Ω such that

$$\int_\gamma \eta_\Omega(w)|dw| \leq \eta_\Omega(w_1, w_2) + \epsilon.$$

By Definition 2.2, it follows clearly that

$$\eta_Y(h(w_1), (w_2)) \leq \int_{h(\gamma)} \eta_Y(z)|dz| = \int_\gamma \eta_Y(h(w))|h'(w)||dw|. \tag{2.3}$$

Since h is one-to-one holomorphic function, by Theorem A, we have

$$\eta_Y(h(w))|h'(w)| \leq \eta_\Omega(w) \tag{2.4}$$

for every w in Ω. Combining (2.3) and (2.4), we obtain

$$\eta_Y(h(w_1),(w_2)) \leq \int_\gamma \eta_\Omega(w)|dw| \leq \eta_\Omega(w_1,w_2)+\epsilon.$$

Letting $\epsilon \to 0$, we conclude what we wanted to prove. □

2.2. The Generalized Hurwitz Metric

In Section 2.1 we discussed the alternate definition of the Hurwitz density. By adopting the idea of generalized Kobayashi density we are going to define and study the generalized Hurwitz density in this section. The distance decreasing property of the Hurwitz density implies that for any holomorphic function h from $\mathbb{D}$ to Ω with $h(s)=w,\ h(t)\neq w$ for all t in $\mathbb{D}\setminus\{s\}$ and $h'(s)\neq 0$, we have the inequalities

$$\eta_\Omega(h(s))|h'(s)| \leq \eta_{\mathbb{D}}(s),$$

and

$$\eta_\Omega(h(s)) \leq \frac{\eta_{\mathbb{D}}(s)}{|h'(s)|}.$$

Since the formula (2.1) provides an existence of a holomorphic function h for which the equality holds, we have

$$\eta_\Omega(w) = \inf \frac{\eta_{\mathbb{D}}(s)}{|h'(s)|},$$

where the infimum is taken over all holomorphic functions h from $\mathbb{D}$ to Ω with $h(s)=w,\ h(t)\neq w$ for all $t\in\mathbb{D}\setminus\{s\},\ h'(s)\neq 0$. This leads to the notion of introducing generalized Hurwitz density for an arbitrary domain Ω.

Definition 2.6. For any domain $\Omega\subset\mathbb{C}$, the *generalized Hurwitz density* is defined as

$$\eta_\Omega^{\mathbb{D}}(w) = \inf \frac{\eta_{\mathbb{D}}(s)}{|h'(s)|},$$

where the infimum is taken over all $h\in\mathcal{H}(\mathbb{D},\Omega)$ with $h(s)=w,\ h(t)\neq w$ for all $t\in\mathbb{D}\setminus\{s\},\ h'(s)\neq 0$, and all s in $\mathbb{D}$.

Remark 2.7. If $\Omega \subsetneq \mathbb{C}$ and $a \in \Omega$, then there exists a Hurwitz covering map g from $\mathbb{D}$ to Ω which realizes the infimum; thus $\eta_\Omega(a) = \eta_\Omega^{\mathbb{D}}(a)$ for all $a \in \Omega$.

Note that, in Definition 2.6 it is not required to choose the domain Ω to be a proper subdomain of the complex plane $\mathbb{C}$. In the following theorem, we calculate $\eta_\Omega^{\mathbb{D}}$, when $\Omega = \mathbb{C}$.

Theorem 2.8. *Suppose that Ω is the whole complex plane, then the generalized Hurwitz density $\eta_\Omega^{\mathbb{D}}(w)$ is identically equal to zero for all elements w in Ω.*

Proof. Let $w \in \Omega$ be an arbitrary element and n be any positive integer. Setting $h_n(s) = (s - t)n + w$. Clearly, h_n is a sequence of holomorphic functions from $\mathbb{D}$ into $\mathbb{C}$ with $h_n(t) = w$ and $h_n(s) \neq w$ for all $s \in \mathbb{D} \setminus \{t\}$. By Definition 2.6, we have

$$\eta_\Omega^{\mathbb{D}}(w) \leq \frac{\eta_{\mathbb{D}}(t)}{|h_n'(t)|}.$$

Since the Hurwitz and the hyperbolic densities coincide on simply connected domains, we have

$$\eta_\Omega^{\mathbb{D}}(w) \leq \frac{\lambda_{\mathbb{D}}(t)}{n}.$$

Letting n goes to infinity, we obtain that $\eta_\Omega^{\mathbb{D}}(w) = 0$. □

The definition of the generalized Hurwitz density can further be generalized by changing the fixed domain $\mathbb{D}$ to an arbitrary proper subdomain Y of $\mathbb{C}$, that is, by pushing forward the Hurwitz density on Y to Ω by a holomorphic function having some special property. Here, we call Y as the *basepoint domain.* This idea leads to the following definition.

Definition 2.9. Let $\Omega \subset \mathbb{C}$ be arbitrary. For all $s \in Y$, the *generalized Hurwitz density* η_Ω^Y for the basepoint domain Y is defined as

$$\eta_\Omega^Y(w) = \inf \frac{\eta_Y(s)}{|h'(s)|},$$

where η_Y is the Hurwitz density on Y and the infimum is taken over all holomorphic functions h from Y to Ω with $h(s) = w,\ h(t) \neq w$ for all $t \in Y \setminus \{s\},\ h'(s) \neq 0$.

In view of the nature of Definition 2.9, it is here appropriate to remark that η_Ω^Y can be $+\infty$ at some points, or even at every point.

We will now prove some expected elementary properties of η_Ω^Y. We start by comparing the Hurwitz and the generalized Hurwitz densities on proper subdomains of the complex plane.

Proposition 2.10. *Let $Y \subset \mathbb{C}$ be a domain and Ω be a proper subdomain of $\mathbb{C}$. Then for every point w in Ω, we have*

$$\eta_\Omega^Y(w) \geq \eta_\Omega(w).$$

Proof. Let $a \in Y$ and h be any holomorphic function from Y to Ω with $h(a) = b,\ h(s) \neq b$ for all $s \in Y \setminus \{a\}$ and $h'(a) \neq 0$. Then by distance decreasing property of the Hurwitz density, we have

$$\eta_\Omega(h(a))|h'(a)| \leq \eta_Y(a),$$

and

$$\eta_\Omega(b) \leq \frac{\eta_Y(a)}{|h'(a)|}.$$

Taking the infimum on both sides over $h \in \mathcal{H}(Y, \Omega)$ with $h(a) = b,\ h(s) \neq b$ for all $s \in Y \setminus \{a\},\ h'(a) \neq 0$, we have

$$\eta_\Omega^Y(b) \geq \eta_\Omega(b).$$

Since $b \in \Omega$ is arbitrary, the above inequality holds true for every $b \in \Omega$. $\square$

One naturally asks the comparison between the classical generalized Kobayashi density and the generalized Hurwitz density.

We immediately have

Corollary 2.11. *If Y and Ω are hyperbolic domains, then $\eta_\Omega^Y \geq \kappa_\Omega^Y$.*

Corollary 2.12. *For proper subdomains Ω and Y of $\mathbb{C}$, we have $\eta_\Omega^Y(w) \geq 0$ for all w in Ω.*

There are certain situations where the classical generalized Kobayashi density agrees with the hyperbolic density, see for instance [25]. This motivates us to investigate the situations under which the generalized Hurwitz density coincides with the Hurwitz density. The following proposition justifies one such case and a few more situations will be covered in the next section.

Proposition 2.13. *Let Ω and Y be proper subdomains of $\mathbb{C}$. If for every $b \in \Omega$, there exists a holomorphic covering map h_b from $Y \setminus \{a\}$ onto $\Omega \setminus \{b\}$ that extends to a holomorphic map of Y onto Ω with $h_b(a) = b$ and $h_b'(a) \neq 0$, then*

$$\eta_\Omega^Y(w) = \eta_\Omega(w)$$

for all w in Ω. In particular, we also have

$$\eta_\Omega^\Omega(w) = \eta_\Omega(w)$$

for every w in Ω.

Proof. Since h_b is a holomorphic map from Y to Ω with $h_b(a) = b$, $h_b(w) \neq b$ for all w in $Y \setminus \{a\}$ and $h_b'(a) \neq 0$, by the definition of generalized Hurwitz density, we have

$$\eta_\Omega^Y(b) \leq \frac{\eta_Y(a)}{|h_b'(a)|.} \tag{2.1}$$

In addition, by Theorem A, we obtain

$$\eta_\Omega(b)|h_b'(a)| = \eta_Y(a). \tag{2.2}$$

Combining (2.1) and (2.2), we obtain

$$\eta_\Omega^Y(b) \leq \eta_\Omega(b).$$

Since b is an arbitrary point, it follows that $\eta_\Omega^Y(w) \leq \eta_\Omega(w)$ for all $w \in \Omega$. On the other hand, by Proposition 2.10 it follows that $\eta_\Omega^Y(w) \geq \eta_\Omega(w)$. Hence the proof is complete. $\square$

Proposition 2.13 is stronger, because for non-simply connected domains Y and Ω the proposition certainly holds (see for instance Example 2.18). To demonstrate this, we use the distance decreasing property of the generalized Hurwitz density, which is proved below (see Theorem 2.15). However, for simply connected domains we have the following special situation.

Corollary 2.14. *If $Y \subsetneq \mathbb{C}$ is a simply connected domain and $\Omega \subsetneq \mathbb{C}$ is any domain, then*

$$\eta_\Omega^{\mathbb{D}} \equiv \eta_\Omega^Y \equiv \eta_\Omega.$$

Proof. Since $Y \subsetneq \mathbb{C}$ is a simply connected domain, by Riemann Mapping Theorem there exists a conformal homeomorphism T from Y onto $\mathbb{D}$. Furthermore, $\Omega \subsetneq \mathbb{C}$ implies that for every point $w \in \Omega$ there is a Hurwitz covering map g_w from $\mathbb{D}$ onto Ω with $g_w(0) = w$.

Hence, by using the composed map $g \circ T$ from Y onto Ω in Proposition 2.13, we conclude our result. $\square$

It is well-known that the Hurwitz, the hyperbolic as well as the generalized Kobayashi metrics have distance decreasing properties. The following result provides a similar property for the generalized Hurwitz metric.

Theorem 2.15. (Distance decreasing property of the generalized Hurwitz density) *Let Ω and $\triangle$ be any subdomains of $\mathbb{C}$ and $Y \subsetneq \mathbb{C}$ be a domain. If h is a holomorphic function from Ω to $\triangle$ with $h(a) = b$, $h'(a) \neq 0$ and $h(w) \neq b$ for all $w \in \Omega \setminus \{a\}$, then*

$$\eta_{\triangle}^{Y}(h(a))|h'(a)| \leq \eta_{\Omega}^{Y}(a).$$

Proof. By the definition of generalized Hurwitz density, for every $\epsilon > 0$ there exists a point $c \in Y$ and a holomorphic map g from Y to Ω with $g(c) = a$, $g(s) \neq a$ for all $s \in Y \setminus \{c\}$, $g'(c) \neq 0$ and

$$\eta_{\Omega}^{Y}(a) \geq \frac{\eta_Y(c)}{|g'(c)|} - \epsilon. \tag{2.3}$$

Note that, $h \circ g$ maps Y to $\triangle$ such that $(h \circ g)(c) = b$, $(h \circ g)(s) \neq b$ for all $s \in Y \setminus \{c\}$ and $(h \circ g)'(c) = h'(g(c))g'(c) = h'(a)g'(c) \neq 0$. Therefore, using $h \circ g$ in the definition of $\eta_{\triangle}^{Y}(h(a))$, we have

$$\eta_{\triangle}^{Y}(h \circ g)(c) \leq \frac{\eta_Y(c)}{|(h \circ g)'(c)|}.$$

By (2.3) and using the chain rule, it follows that

$$\eta_{\triangle}^{Y}(h \circ g)(c)|h'(a)| \leq \frac{\eta_Y(c)}{|g'(c)|} \leq \eta_{\Omega}^{Y}(a) + \epsilon.$$

Letting ϵ goes to zero, we obtain

$$\eta_{\triangle}^{Y}(h(a))|h'(a)| \leq \eta_{\Omega}^{Y}(a).$$

This completes the proof. $\square$

Corollary 2.16. (*Conformal invariance property*) *Let $Y \subsetneq \mathbb{C}$ be a domain. If f is a conformal mapping from a domain $\Omega \subset \mathbb{C}$ onto another domain $\triangle \subset \mathbb{C}$, then we have*

$$\eta_{\triangle}^{Y}(h(a))|h'(a)| = \eta_{\Omega}^{Y}(a),$$

for all $a \in \Omega$.

Corollary 2.17. (*Domain monotonicity property*) *If* $\Omega \subset \triangle$ *and* Y *are domains as in Theorem* 2.15*, then* $\eta_{\triangle}^{Y}(a) \leq \eta_{\Omega}^{Y}(a)$ *for all* $a \in \Omega$.

We now provide an example which demonstrate Proposition 2.13 in non-simply connected domains.

Example 2.18. Let $Y = \mathbb{D}^* := \mathbb{D} \setminus \{0\}$, the punctured unit disk and $\Omega = \mathbb{C}^* := \mathbb{C} \setminus \{0\}$, the punctured plane. We shall prove that for all $w \in \mathbb{C}^*$

$$\eta_{\mathbb{C}^*}^{\mathbb{D}^*}(w) = \eta_{\mathbb{C}^*}(w).$$

As stated in [42, p. 322] (see also [35, (4.1)]), the Hurwitz covering map from $\mathbb{D}$ onto $\mathbb{C} \setminus \{1\}$ is obtained by the infinite product representation

$$g(w) = 16w \prod_{n=1}^{\infty} \Big(\frac{1+w^{2n}}{1+w^{2n-1}}\Big)^8, \quad |w| < 1. \tag{2.4}$$

For $s \in \mathbb{D}$, it is well known that the map $T(z) = (z-s)/(1-\overline{s}z)$ defines a Möbius transformation of $\mathbb{D}$ onto itself. Clearly, $T(s) = 0, T'(s) = (1+|s|^2)/(1-|s|^2) > 0$. Since g is a holomorphic covering map and T is a one-one holomorphic map on $\mathbb{D}$, the composition $g \circ T$ is also a holomorphic covering map satisfying $(g \circ T)(s) = 0$ and $(g \circ T)(t) \neq 0$ for all $t \in \mathbb{D} \setminus \{s\}$. Furthermore, $(g \circ T)'(s) > 0$ which follows from the chain rule and the fact that $g'(0) = 16 > 0$ and $T'(s) > 0$. By the distance decreasing property we have

$$\eta_{\mathbb{C}\setminus\{1\}}((g \circ T)(s))(g \circ T)'(s) = \eta_{\mathbb{D}}(s). \tag{2.5}$$

Restricting the function $g \circ T$ onto $\mathbb{D}^*$ and plugging it in the definition of $\eta_{\mathbb{C}\setminus\{1\}}^{\mathbb{D}^*}(0)$ we obtain

$$\eta_{\mathbb{C}\setminus\{1\}}^{\mathbb{D}^*}(0) \leq \frac{\eta_{\mathbb{D}^*}(s)}{(g \circ T)'(s)}.$$

Now, we choose a sequence $s_n \in \mathbb{D}^*$ such that $|s_n| \to 1$. By using the same argument as above, we can find Möbius transformations T_n from $\mathbb{D}$ onto itself with $T_n(s_n) = 0$ and $T_n'(s_n) > 0$. Therefore, it follows from (2.5) that

$$\eta_{\mathbb{C}\setminus\{1\}}^{\mathbb{D}^*}(0) \leq \frac{\eta_{\mathbb{D}^*}(s_n)}{(g \circ T_n)'(s_n)} = \frac{\eta_{\mathbb{D}^*}(s_n)}{\eta_{\mathbb{D}}(s_n)} \eta_{\mathbb{C}\setminus\{1\}}(0),$$

since $(g \circ T_n)(s_n) = 0$. We notice from [35, Section 2] that the Hahn density of the punctured unit disk obtained by (see [37, (2)])

$$S_{\mathbb{D}^*}(s_n) = \frac{1 + |s_n|}{4|s_n|(1 - |s_n|)}$$

exceeds the Hurwitz density. Thus, we obtain

$$\eta_{\mathbb{C}\setminus\{1\}}^{\mathbb{D}^*}(0) \leq \frac{\eta_{\mathbb{D}^*}(s_n)}{\eta_{\mathbb{D}}(s_n)}\eta_{\mathbb{C}\setminus\{1\}}(0) \leq \frac{S_{\mathbb{D}^*}(s_n)}{\eta_{\mathbb{D}}(s_n)}\eta_{\mathbb{C}\setminus\{1\}}(0) = \frac{1 + |s_n|}{4|s_n|(1 - |s_n|)}(1 - |s_n|^2)\eta_{\mathbb{C}\setminus\{1\}}(0).$$

Now, letting $|s_n| \to 1$ we have $\eta_{\mathbb{C}\setminus\{1\}}^{\mathbb{D}^*}(0) \leq \eta_{\mathbb{C}\setminus\{1\}}(0)$. The reverse inequality is followed by Proposition 2.10. Now, by using the holomorphic functions $f(w) = 1 - w$ and $h(w) = bw$ (for some complex constant b) in the distance decreasing property for the generalized Hurwitz density, it follows that, both the metrics coincide on $\mathbb{C}^*$. That is, $\eta_{\mathbb{C}^*}^{\mathbb{D}^*}(w) = \eta_{\mathbb{C}^*}(w)$ for all $w \in \mathbb{C}^*$. $\square$

Next we define the generalized Hurwitz distance between two points in a domain.

Definition 2.19. Let $\Omega \subset \mathbb{C}$ and $Y \subsetneq \mathbb{C}$ be domains. For $w_1, w_2 \in \Omega$, define

$$\eta_{\Omega}^{Y}(z_1, z_2) = \inf \int_{\gamma} \eta_{\Omega}^{Y}(w)|dw|,$$

where the infimum is taken over all rectifiable paths γ in Ω joining z_1 to z_2.

Proof of the following theorem is similar to that of Theorem 5.3.

Theorem 2.20. *Let $Y \subsetneq \mathbb{C}$ be a domain. If Ω is a hyperbolic domain, then $(\Omega, \eta_{\Omega}^{Y})$ is a complete metric space.*

We do have also the distance decreasing property in the global sense whose proof follows the steps of the proof of Proposition 2.5.

Theorem 2.21. *Let Y be a proper subdomain of $\mathbb{C}$ and Ω, $\triangle$ be any subdomain of $\mathbb{C}$. If h is a one-to-one holomorphic map from Ω to $\triangle$, then*

$$\eta_{\triangle}^{Y}(h(w_1), (w_2)) \leq \eta_{\Omega}^{Y}(w_1, w_2),$$

for all w_1, w_2 in Ω.

Note that, till now we have derived all the results of the generalized Hurwitz density η_{Ω}^{Y} for a base domain Y. In the next theorem we will see the comparison between generalized Hurwitz densities when the range domain is fixed while the source domain is varying.

Theorem 2.22. *Let $Y_1,\ Y_2$ be proper subdomains of $\mathbb{C}$ and Ω be any subdomain of $\mathbb{C}$. If for every point $b \in Y_2$, there exists a point $a \in Y_1$ and a holomorphic covering map h_b from $Y_1 \setminus \{a\}$ onto $Y_2 \setminus \{b\}$ which extends to a holomorphic map from Y_1 onto Y_2 with $h_b(a) = b,\ h_b(w) \neq b$ for any $w \in Y_1 \setminus \{a\}$ and $h_b'(a) \neq 0$, then*

$$\eta_\Omega^{Y_1}(\zeta) \leq \eta_\Omega^{Y_2}(\zeta),$$

for all ζ in Ω.

Proof. Let ζ be any arbitrary point in Ω and ϵ be a positive real number. By definition of $\eta_\Omega^{Y_2}$, there exists a point b in Y_2 and a holomorphic function g from Y_2 to Ω with $g(b) = \zeta,\ g(s) \neq \zeta$ for any $s \in Y_2 \setminus \{b\},\ g'(b) \neq 0$ such that

$$\eta_\Omega^{Y_2}(\zeta) \geq \frac{\eta_{Y_2}(b)}{|g'(b)|} - \epsilon. \tag{2.6}$$

Since for every point $b \in Y_2$, there exists a point a in Y_1 and a holomorphic covering h_b from Y_1 onto Y_2 with $h_b(a) = b,\ h_b(w) \neq b$ for any $w \in Y_1 \setminus \{a\},\ g'(a) \neq 0$, by Theorem A we have

$$\eta_{Y_2}(b)|h_b'(a)| = \eta_{Y_1}(a). \tag{2.7}$$

Note that, the composition $g \circ h_b$ is a holomorphic function from Y_1 to Ω with $(g \circ h_b)(a) = \zeta,\ (g \circ h_b)(w) \neq \zeta$ for any w in Y_1 and $(g \circ h_b)'(a) = g'(b)h_b'(a) \neq 0$. Therefore, by the definition of $\eta_\Omega^{Y_1}$, we obtain

$$\eta_\Omega^{Y_1}(a) \leq \frac{\eta_{Y_1}(a)}{|(g \circ h_b)'(a)|} = \frac{\eta_{Y_1}(a)}{|g'(b)h_b'(a)|}. \tag{2.8}$$

By (2.6), (2.7), (2.8), it follows that

$$\eta_\Omega^{Y_2}(\zeta) \geq \frac{\eta_{Y_2}(b)}{|g'(b)|} - \epsilon = \frac{\eta_{Y_1}(a)}{|g'(b)||h_b'(a)|} - \epsilon \geq \eta_\Omega^{Y_1}(\zeta) - \epsilon.$$

Letting ϵ goes to zero, we have $\eta_\Omega^{Y_2}(\zeta) \geq \eta_\Omega^{Y_1}(\zeta)$, which completes the proof of our result. □

We look forward for the existence of non-simply connected domains Y_1 and Y_2 validating the statement of Theorem 2.22, however, they remain open due to their non-trivial nature.

Corollary 2.23. *If $Y_1 \subsetneq \mathbb{C}$ is a simply connected domain, then for all proper subdomains Y_2 and Ω of $\mathbb{C}$, we have*

$$\eta_\Omega(w) = \eta_\Omega^{\mathbb{D}}(w) = \eta_\Omega^{Y_1}(w) \leq \eta_\Omega^{Y_2}(w),$$

for all w in Ω.

Proof. Since $Y_1 \subsetneq \mathbb{C}$ is a simply connected domain, by Riemann Mapping Theorem, there exists a conformal homeomorphism f from Y_1 onto $\mathbb{D}$. Furthermore, there exists a Hurwitz covering map T_b from $\mathbb{D}$ onto Y_2 for every b in Y_2. Thus, the composed map $T_b \circ f$ is a holomorphic covering from $Y_1 \setminus \{f^{-1}(0)\}$ onto $Y_2 \setminus \{b\}$, which extends to a holomorphic function from Y_1 to Y_2 with $(T_b \circ f)(f^{-1}(0)) = b,\ (T_b \circ f)'(f^{-1}(0)) \neq 0$ and $(T_b \circ f)(s) \neq b$ for all s in $Y_1 \setminus \{f^{-1}(0)\}$. Taking $h_b = T_b \circ f$ in Theorem 2.22, we obtain the desired result. □

Two subdomains Y_1 and Y_2 of $\mathbb{C}$ are conformally equivalent if there exists a holomorphic bijection f from Y_1 to Y_2.

Corollary 2.24. *Let $\Omega \subset \mathbb{C}$ be any arbitrary domain. If $Y_1 \subsetneq \mathbb{C}$ and $Y_2 \subsetneq \mathbb{C}$ are conformally equivalent domains, then*

$$\eta_\Omega^{Y_1}(w) = \eta_\Omega^{Y_2}(w)$$

for all w in Ω.

2.3. Lipschitz Domain

In this section, one of our main objectives is to study the situations, in terms of the Hurwitz non-Lipschitz domains, when the Hurwitz density coincides with the generalized Hurwitz density. The following notations are useful in the definition of Hurwitz Lipschitz domains. Let Y be a hyperbolic domain and Ω be a subdomain of Y. If i is the inclusion map from Ω to Y, the global contraction constant $gl_\eta(\Omega, Y)$ is defined by

$$gl_\eta(\Omega, Y) := \sup_{w_1, w_2 \in \Omega,\ w_1 \neq w_2} \frac{\eta_Y(w_1, w_2)}{\eta_\Omega(w_1, w_2)}.$$

If Y is any proper subdomain of $\mathbb{C}$, then the infinitesimal contraction constant is defined as

$$l_\eta(\Omega, Y) := \sup_{w\in\Omega} \frac{\eta_Y(w)}{\eta_\Omega(w)}.$$

Since the inclusion map is an *injective* holomorphic function from Ω to Y, by the distance decreasing property of Hurwitz density, we have $\eta_Y(w) \le \eta_\Omega(w)$ for every w in Ω. Furthermore, by Proposition 2.5 it follows that $\eta_Y(w_1, w_2) \le \eta_\Omega(w_1, w_2)$ for all w_1 and w_2 in Ω. Thus, both infinitesimal and global contraction constants are less than or equal to 1.

Theorem 2.25. *Let $Y \subsetneq \mathbb{C}$ be a domain. If Ω is a subdomain of Y, then $gl_\eta(\Omega, Y) \le l_\eta(\Omega, Y) \le 1$. Furthermore, the inclusion map i from Ω to Y is a strict infinitesimal contraction map, whenever Ω is a proper subdomain of Y.*

Proof. Let w_1, w_2 be any two points in Ω and $\gamma \subset \Omega$ be any path joining w_1 and w_2 such that

$$\eta_\Omega(w_1, w_2) = \int_\gamma \eta_\Omega(w)|dw|.$$

By the definition of $\eta_Y(w_1, w_2)$, it follows that

$$\eta_Y(w_1, w_2) \le \int_\gamma \frac{\eta_Y(w)}{\eta_\Omega(w)} \eta_\Omega(w)|dw| \le l_\eta(\Omega, Y)\, \eta_\Omega(w_1, w_2).$$

Thus, $gl_\eta(\Omega, Y) \le l_\eta(\Omega, Y)$.

The proof of the second part of our theorem follows from [35, Theorem 6.1]. □

Definition 2.26. Let Ω be a subdomain of a domain Y in $\mathbb{C}$. Then Ω is called a *Hurwitz Lipschitz subdomain* of Y, if the inclusion map from Ω to Y is a strict infinitesimal contraction. That is, the infinitesimal contraction constant l_Ω is strictly less than 1.

By Proposition 2.5, for any proper subdomain Ω of $\mathbb{C}$, we have $\eta_\Omega^Y \ge \eta_\Omega$. However, in the following theorem, we find a condition on Y so that for every proper subdomain Ω of $\mathbb{C}$, the Hurwitz and the generalized Hurwitz densities coincide. We adopt the proof technique from [52, Theorem 2.1] where the author compares the hyperbolic density with the Kobayashi density.

Theorem 2.27. *If Ω is any proper subdomain of $\mathbb{C}$ and Y is a Hurwitz non-Lipschitz subdomain of $\mathbb{D}$, then we have*

$$\eta_\Omega^Y(w) = \eta_\Omega(w),$$

for every w in Ω.

Proof. To show that $\eta_\Omega^Y = \eta_\Omega$, we only need to show that $\eta_\Omega^Y \leq \eta_\Omega$, as the relation $\eta_\Omega^Y \geq \eta_\Omega$ always holds. Since $\Omega \subsetneq \mathbb{C}$, for every point $w \in \Omega$ there exists a Hurwitz covering map g from $\mathbb{D}$ onto Ω with $g(0) = w$, $g(s) \neq w$ for all $s \in \mathbb{D} \setminus \{0\}$ and $g'(0) \neq 0$. We now pre-compose g with a Möbius transformation T of $\mathbb{D}$ that maps s in Y to the origin. Therefore $g \circ T$ is a holomorphic covering of Ω from $\mathbb{D} \setminus \{s\}$ onto $\Omega \setminus \{w\}$, which extends to a holomorphic function from $\mathbb{D}$ to Ω with $(g \circ T)(s) = w$, $(g \circ T)(t) \neq w$ for all t in $\mathbb{D} \setminus \{s\}$ and $(g \circ T)'(s) = g'(0)T'(s) \neq 0$. Thus, by Theorem A, we have

$$\eta_\Omega(w)|(g \circ T)'(s)| = \eta_\mathbb{D}(s). \tag{2.1}$$

Now, let h be the restriction of $g \circ T$ on Y. By the definition of η_Ω^Y, we obtain

$$\eta_\Omega^Y(w) \leq \frac{\eta_Y(s)}{|h'(s)|} = \frac{\eta_Y(s)}{|(g \circ T)'(s)|}.$$

On the other hand, by the help of (2.1), we have

$$\eta_\Omega^Y(w) \leq \frac{\eta_Y(s)}{\eta_\mathbb{D}(s)}\eta_\Omega(w).$$

Since Y is a non-Lipschitz Hurwitz subdomain of $\mathbb{D}$, by choosing s in Y appropriately, $\eta_Y(s)/\eta_\mathbb{D}(s)$ can be made as close to 1 as we wish. Thus, we can say that

$$\eta_\Omega^Y(w) \leq \eta_\Omega(w).$$

Since $w \in \Omega$ is an arbitrary element, therefore we have $\eta_\Omega^Y(w) = \eta_\Omega(w)$ for all w in Ω. $\square$

Through the following example we also demonstrate Theorem 2.27 by finding a suitable Hurwitz non-Lipschitz domain Y and a proper subdomain Ω of $\mathbb{C}$.

Example 2.28. We first demonstrate that $Y = \mathbb{D}^*$ is a Hurwitz non-Lipschitz subdomain of $\mathbb{D}$. Consider

$$\frac{\eta_\mathbb{D}(w)}{\eta_{\mathbb{D}^*}(w)} \geq \frac{\lambda_\mathbb{D}(w)}{S_{\mathbb{D}^*}(w)} = \frac{1}{1-|w|^2}\frac{4|w|(1-|w|)}{1+|w|}, \tag{2.2}$$

where second inequality follows from the fact that the Hurwitz and the Hyperbolic densities agree on the simply connected domains and the Hahn density S_Ω is always greater than the Hurwitz density on proper domains of $\mathbb{C}$. Letting w goes to 1 in (2.2), we obtain

$$\lim_{|w|\to 1} \frac{\eta_{\mathbb{D}}(w)}{\eta_{\mathbb{D}^*}(w)} = 1,$$

where the equality follows from the domain monotonicity property of the Hurwitz density. This shows that the punctured unit disk is a Hurwitz non-Lipschitz domain.

By choosing $\Omega = \mathbb{C}^*$, from Example 2.18 we have $\eta_{\mathbb{C}^*}^{\mathbb{D}^*}(w) = \eta_{\mathbb{C}^*}(w)$ for all $w \in \mathbb{C}^*$.

In order to generalize Theorem 2.27 for a broader class of domains in $\mathbb{C}$, we now discuss the notion of quasi-bounded domains as follows.

Definition 2.29. A domain Y in $\mathbb{C}$ is said to be *quasi-bounded* if the smallest simply connected plane domain containing Y is a proper subset of $\mathbb{C}$. We denote the smallest simply connected domain by $\hat{Y}$.

Example 2.30. All bounded domains in $\mathbb{C}$ are quasi-bounded.

The following theorem is an analogue of [52, Theorem 2.2] from the notion of hyperbolic density to the notion of Hurwitz density.

Theorem 2.31. *If Ω is any proper subdomain of the complex plane and Y is quasi-bounded, non-Lipschitz Hurwitz subdomain of $\hat{Y}$, then*

$$\eta_\Omega^Y(w) = \eta_\Omega(w),$$

for all w in Ω.

Proof. Since, $\hat{Y}$ is simply connected, by Riemann Mapping Theorem there exists a conformal homeomorphism h from $\hat{Y}$ onto $\mathbb{D}$. Note that the conformal mappings are isometries for the Hurwitz metric (see [35, Corollary 6.2]), and thus we have

$$\eta_{\mathbb{D}}(h(s))|h'(s)| = \eta_{\hat{Y}}(s) \tag{2.3}$$

for all s in $\hat{Y}$. Furthermore, the restriction of h to Y, resulting a conformal homeomorphism from Y onto $h(Y)$. Therefore, we have

$$\eta_{h(Y)}(s)|h'(s)| = \eta_Y(s). \tag{2.4}$$

By using (2.3), (2.4) and the definition of $l_\eta(Y, \hat{Y})$, we obtain

$$l_\eta(Y, \hat{Y}) = \sup_{z\in Y} \frac{\eta_Y(z)}{\eta_{\hat{Y}}(z)} = \sup_{z\in h(Y)} \frac{\eta_{h(Y)}(z)}{\eta_{\mathbb{D}}(z)} = l_\eta(h(Y), \mathbb{D}).$$

Thus, Y is a non-Lipschitz Hurwitz subdomain of $\hat{Y}$ if and only if $h(Y)$ is non-Lipschitz Hurwitz subdomain of $\mathbb{D}$. By Theorem 2.27, it follows that

$$\eta_\Omega^{h(Y)}(w) = \eta_\Omega(w) \tag{2.5}$$

for all w in Ω. Since Y and $h(Y)$ are conformally homeomorphic by the map h, by Corollary 2.24, we have

$$\eta_\Omega^{Y}(w) = \eta_\Omega^{h(Y)}(w) \tag{2.6}$$

for all w in Ω. Combining (2.5) and (2.6), we obtain

$$\eta_\Omega^{Y}(w) = \eta_\Omega(w),$$

as desired. $\square$

CHAPTER 3

CARATHÉODORY DENSITY OF THE HURWITZ METRIC

This chapter deals with Carathéodory density of the Hurwitz metric for arbitrary plane domains. Several basic properties in connection with the Hurwitz metric and the Kobayashi density of the Hurwitz metric are established.

3.1. Carathéodory density of the Hurwitz metric

In the second chapter, by adopting the idea of the Kobayashi metric, we generalized the Hurwitz density for arbitrary domains $\Omega \subset \mathbb{C}$ and $Y \subsetneq \mathbb{C}$, namely the generalized Hurwitz density 2.6. This section is devoted to the introduction of a new density that generalizes the Hurwitz density in the sense of Carathéodory. This is defined as follows:

Definition 3.1. Let $w \in \Omega \subsetneq \mathbb{C}$. For an element $s \in \mathbb{D}$, we define a new quantity

$$\mathscr{C}_{\Omega}^{\mathbb{D},s}(w) = \sup \eta_{\mathbb{D}}(h(w))|h'(w)|, \tag{3.1}$$

where the supremum is taken over all $h \in \mathcal{H}(\Omega, \mathbb{D})$ such that $h(w) = s$, $h(z) \neq s$ for all $z \in \Omega \setminus \{w\}$, i.e. for all $h \in \mathcal{H}_w^s(\Omega, \mathbb{D})$. We call this quantity by *the Carathéodory density of the Hurwitz metric of Ω relative to $\mathbb{D}$. Setting $\mathscr{C}_{\Omega}^{\mathbb{D}} := \mathscr{C}_{\Omega}^{\mathbb{D},0}$.*

Remark 3.2.

1. Note that on simply connected domains the Hurwitz density agrees with the hyperbolic density, so one can replace $\eta_{\mathbb{D}}$ by the hyperbolic density $\lambda_{\mathbb{D}}$ in Definition 3.1.
2. If $\Omega = \mathbb{C}$, then by Liouville's Theorem, the only holomorphic function from Ω into $\mathbb{D}$ is a constant function, which does not belong to the class $\mathcal{H}_w^s(\Omega, \mathbb{D})$. Hence, it can be defined that $\mathscr{C}_{\mathbb{C}}^{\mathbb{D},s}(w) = 0$ when the set $\mathcal{H}_w^s(\Omega, \mathbb{D})$ becomes empty. It suggests us to assume that $\mathcal{H}_w^s(\Omega, Y) \neq \emptyset$ throughout the paper for an arbitrary base domain $Y \subsetneq \mathbb{C}$.

The first basic property of the Carathéodory density of the Hurwitz metric $\mathscr{C}_{\Omega}^{\mathbb{D},s}$ is that the supremum is attained by some holomorphic function $h \in \mathcal{H}_w^s(\Omega, \mathbb{D})$ in (3.1).

Proposition 3.3. *Let $\Omega \subsetneq \mathbb{C}$ be a domain and $\mathcal{H}_w^0(\Omega, \mathbb{D}) \neq \emptyset$. Then, the Carathéodory density of the Hurwitz metric $\mathscr{C}_\Omega^{\mathbb{D}}$ can be computed by the formula:*

$$\mathscr{C}_\Omega^{\mathbb{D}}(w) = 2 \max\{|h'(w)| : h \in \mathcal{H}_w^0(\Omega, \mathbb{D})\}.$$

Proof. Since the members of the family $\mathcal{H}_w^0(\Omega, \mathbb{D})$ are uniformly bounded by 1, by Montel's Theorem, $\mathcal{H}_w^0(\Omega, \mathbb{D})$ is a normal family. By Definition 3.1 there exists a sequence of holomorphic functions $h_n \in \mathcal{H}_w^0(\Omega, \mathbb{D})$ such that $2|h_n'(w)| \to \mathscr{C}_\Omega^{\mathbb{D}}(w)$, since $h_n(w) = 0$ and $\eta_{\mathbb{D}}(0) = \lambda_{\mathbb{D}}(0) = 2$. Furthermore, by the open mapping theorem, there exists a subsequence h_{n_k} of h_n which converges to either an open map h or a constant map. Since $h_n \in \mathcal{H}(\Omega, \mathbb{D})$, it follows that $|h(z)| \leq 1$ for all $z \in \Omega$. Note that, if $h(z)$ attains 1 for some $z \in \Omega$, then by the maximum modulus principle, $|h| = 1$, contradicting to the fact that $h(w) = 0$. Moreover, by Hurwitz Theorem, there exists an $N \in \mathbb{N}$ such that h_{n_k} and h have the same number of zeros for all $n_k \geq N$ in some neighbourhood of w. Since $h(z) \neq 0$ for all $z \in \Omega \setminus \{w\}$, we conclude by the uniqueness of limit that $2|h'(w)| = \mathscr{C}_\Omega^{\mathbb{D}}(w)$, which completes the proof. □

Remark 3.4. By a suitable composition of the disk automorphism with the function obtained in Proposition 3.3, we can prove the existence of the holomorphic function h in Definition 3.1 when $s \neq 0$.

Alike to the case of coinciding of the hyperbolic and the Carathéodory densities on simply connected domains, we now prove that the Hurwitz density η_Ω and the Carathéodory density of the Hurwitz metric $\mathscr{C}_\Omega^{\mathbb{D},s}$ too agree on a simply connected domain Ω.

Proposition 3.5. *If $\Omega \subsetneq \mathbb{C}$ is a simply connected domain, then the Carathéodory density of the Hurwitz metric $\mathscr{C}_\Omega^{\mathbb{D},s}$ coincides with the Hurwitz density η_Ω as well as with the Kobayashi density of the Hurwitz metric $\eta_\Omega^{\mathbb{D}}$. That is, we have $\mathscr{C}_\Omega^{\mathbb{D},s} \equiv \eta_\Omega \equiv \eta_\Omega^{\mathbb{D}}$.*

Proof. By the distance decreasing property of the Hurwitz density (see [35, Theorem 6.1]), for a point $w \in \Omega$ and for any $h \in \mathcal{H}_w^s(\Omega, \mathbb{D})$ we have $\eta_{\mathbb{D}}(h(w))|h'(w)| \leq \eta_\Omega(w)$. By taking supremum over all $h \in \mathcal{H}_w^s(\Omega, \mathbb{D})$, on the one hand, we obtain $\mathscr{C}_\Omega^{\mathbb{D},s}(w) \leq \eta_\Omega(w)$. On the other hand, to prove the reverse inequality, we consider the conformal homeomorphism $f : \Omega \to \mathbb{D}$ which is guaranteed by Riemann Mapping Theorem. By [35, Corollary 6.2], it follows that

$$\eta_\Omega(w) = \eta_{\mathbb{D}}(h(w))|h'(w)| \leq \mathscr{C}_\Omega^{\mathbb{D},s}(w),$$

where the inequality holds by Definition 3.1. Thus, we have the identity $\mathscr{C}_\Omega^{\mathbb{D},s} \equiv \eta_\Omega$.

The second required identity follows from Corollary 2.14, completing the proof. $\square$

Due to Corollary 2.14, the Kobayashi density of the Hurwitz metric $\eta_\Omega^{\mathbb{D}}$ and the Hurwitz density η_Ω both agree on any domain Ω, whereas, in the following result we show that on non-simply connected domains the Carathéodory density of the Hurwitz metric $\mathscr{C}_\Omega^{\mathbb{D},s}$ is strictly less than the Hurwitz density η_Ω.

Proposition 3.6. *Let $\Omega \subsetneq \mathbb{C}$ be a non-simply connected domain and $\mathscr{C}_\Omega^{\mathbb{D}} > 0$. Then for an element $w \in \Omega$ we have the strict inequality: $\mathscr{C}_\Omega^{\mathbb{D}}(w) < \eta_\Omega(w)$.*

Proof. Let $w \in \Omega$. Since $\Omega \subsetneq \mathbb{C}$, there exists a Hurwitz covering map $g : \mathbb{D} \to \Omega$ with $g(0) = w$. By Proposition 3.3, there exists a function $h \in \mathcal{H}_w^0(\Omega, \mathbb{D})$ such that

$$\mathscr{C}_\Omega^{\mathbb{D}}(w) = 2|h'(w)| = \eta_{\mathbb{D}}(h(w))|h'(w)| \tag{3.2}$$

holds, since $h(w) = 0$ and $\eta_{\mathbb{D}}(0) = \lambda_{\mathbb{D}}(0) = 2$. Thus, we observe that the composition $h \circ g$ is a holomorphic function from $\mathbb{D}$ to $\mathbb{D}$ that fixes the origin. Since Ω is non-simply connected, the covering map g can not be one-one and hence the composition $h \circ g$ can never be conformal. Thus, by the classical Schwarz lemma we conclude the strict inequality

$$\lambda_{\mathbb{D}}((h \circ g)(0))|(h \circ g)'(0)| < \lambda_{\mathbb{D}}(0).$$

Note that the hyperbolic density coincides with the Hurwitz density on simply connected hyperbolic domains (see [35, p. 15]). Therefore, it follows that

$$\eta_{\mathbb{D}}((h \circ g)(0))|(h \circ g)'(0)| < \eta_{\mathbb{D}}(0). \tag{3.3}$$

Since g is a Hurwitz covering map, by [35, Theorem 6.1], we have the equality

$$\eta_\Omega(g(0))|g'(0)| = \eta_{\mathbb{D}}(0). \tag{3.4}$$

Combining (3.3) and (3.4), we obtain from (3.2) that

$$\mathscr{C}_\Omega^{\mathbb{D}}(w) = \eta_{\mathbb{D}}(h(w))|h'(w)| = \eta_{\mathbb{D}}((h \circ g)(0))\frac{|(h \circ g)'(0)|}{|g'(0)|} < \frac{\eta_{\mathbb{D}}(0)}{|g'(0)|} = \eta_\Omega(w),$$

where the second equality follows by the chain rule. $\square$

Since the Hurwitz density can be defined on a proper subdomain of the complex plane, a natural way of further generalizing the Carathéodory density of the Hurwitz metric $\mathscr{C}_\Omega^{\mathbb{D},s}$ by changing the base domain from the unit disk to a proper subdomain Y of $\mathbb{C}$. The definition is as follows:

Definition 3.7. Let $Y \subsetneq \mathbb{C}$ and $\Omega \subset \mathbb{C}$ be domains. For $w \in \Omega$ and $s \in Y$, the *Carathéodory density of the Hurwitz metric of* Ω *relative to the base domain* Y is defined as

$$\mathscr{C}_\Omega^{Y,s}(w) = \sup \eta_Y(h(w))|h'(w)|,$$

where the supremum is taken over all $h \in \mathcal{H}(\Omega, Y)$ such that $h(w) = s$, $h(z) \neq s$ for all $z \in \Omega \setminus \{w\}$, i.e. for all $h \in \mathcal{H}_w^s(\Omega, Y)$.

In Chapter 1 we have noticed that the Kobayashi density of the Hurwitz metric η_Ω^Y exceeds over the Hurwitz density η_Ω whereas in case of the Carathéodory density of the Hurwitz metric $\mathscr{C}_\Omega^{Y,s}$, we prove that it lacks the Hurwitz density on proper subdomains of $\mathbb{C}$.

Proposition 3.8. *Let* Ω *and* Y *be proper subdomains of the complex plane* $\mathbb{C}$*. If for an element* $s \in Y$*, we assume* $\mathscr{C}_\Omega^{Y,s} > 0$ *then*

$$\eta_\Omega(w) \geq \mathscr{C}_\Omega^{Y,s}(w)$$

holds for every $w \in \Omega$*.*

Proof. By the distance decreasing property of the Hurwitz density, for $w \in \Omega, s \in Y$ and for any $h \in \mathcal{H}_w^s(\Omega, Y)$ we have

$$\eta_Y(h(w))|h'(w)| \leq \eta_\Omega(w).$$

Taking the supremum over all $h \in \mathcal{H}_w^s(\Omega, Y)$ on both sides, we obtain

$$\mathscr{C}_\Omega^{Y,s}(w) \leq \eta_\Omega(w).$$

Since $w \in \Omega$ was arbitrary, we conclude the proof as desired. □

Recall that the Hurwitz density and the hyperbolic density agree on simply connected domains. Analogous to this, we now prove that upon some specific conditions the Carathéodory density of the Hurwitz metric $\mathscr{C}_\Omega^{Y,s}$ and the Hurwitz density η_Ω coincide and in a more special situation, they also coincide with the Kobayashi density of the Hurwitz metric η_{Ω}^{Y}.

Proposition 3.9. *Let $\Omega, Y \subsetneq \mathbb{C}$ be domains. Suppose that for every $s \in Y$ there exists a point $w \in \Omega$ and a holomorphic covering map $g_s : \Omega \setminus \{w\} \to Y \setminus \{s\}$ which extends to a holomorphic function $g : \Omega \to Y$ with $g(w) = s$ and $g'(w) \neq 0$. If $\mathscr{C}_{\Omega}^{Y,s} > 0$, then*

$$\mathscr{C}_{\Omega}^{Y,s} \equiv \eta_{\Omega}.$$

In particular, when $Y = \Omega$, we have

$$\mathscr{C}_{\Omega}^{\Omega,w} \equiv \eta_{\Omega} \equiv \zeta_{\Omega}^{\Omega}.$$

Proof. By the distance decreasing property of the Hurwitz density (see the second part of [35, Theorem 6.1]), we have

$$\eta_Y(g(w))|g'(w)| = \eta_{\Omega}(w).$$

Now, plugging the holomorphic covering map g_s into Definition 3.7, on the one hand we obtain

$$\mathscr{C}_{\Omega}^{Y,s}(w) \geq \eta_Y(g(w))|g'(w)| = \eta_{\Omega}(w).$$

On the other hand, the reverse inequality follows from Proposition 3.8. Since w is arbitrary, the Carathéodory density of the Hurwitz metric $\mathscr{C}_{\Omega}^{Y,s}$ and the Hurwitz density η_{Ω} both agree over Ω.

The proof of the second part is a combination of the above identity that we just proved and the identity proved in [**?**, Proposition 3.9]. $\square$

An instant corollary to Proposition 3.9 is that on simply connected domains both the Hurwitz density η_{Ω} and the Carath'eodory density of the Hurwitz metric $\mathscr{C}_{\Omega}^{Y,s}$ agree.

Corollary 3.10. *If $\Omega \subsetneq \mathbb{C}$ is a simply connected domain and $Y \subsetneq \mathbb{C}$ is an arbitrary domain, then*

$$\mathscr{C}_{\Omega}^{Y,s} \equiv \eta_{\Omega},$$

where $s \in Y$.

Proof. Since $Y \subsetneq \mathbb{C}$, there exists a Hurwitz covering map $g : \mathbb{D} \to Y$. Now, $\Omega \subsetneq \mathbb{C}$ being a simply connected domain, by Riemann Mapping Theorem, we would get a conformal mapping $h : \Omega \to \mathbb{D}$ with $h(w) = 0$ and $h'(w) > 0$ for some $w \in \Omega$. Then the composition

$g \circ h$ is a holomorphic covering map from $\Omega \setminus \{w\}$ onto $Y \setminus \{s\}$ for some $s \in Y$ that can be extended from Ω onto Y by taking w to s. The proof now follows by Proposition 3.9. $\square$

Recall that the hyperbolic density λ_Ω, the Hurwitz density η_Ω and the Kobayashi density of the Hurwitz metric η_Ω^Y satisfy the distance decreasing property. Note that, in case of the hyperbolic metric the distance decreasing property is also known as the generalized Schwarz-Pick lemma. Alike to these properties we here show that the Carathéodory density of the Hurwitz metric $\mathscr{C}_\Omega^{Y,s}$ too satisfies the distance decreasing property.

Theorem 3.11. (*Distance decreasing property*) *Let $\Omega_1, \Omega_2 \subset \mathbb{C}$ and $Y \subsetneq \mathbb{C}$ be domains. If there exists a holomorphic function f from Ω_1 into Ω_2 with $f(a) = b$, $f(s) \neq b$ for all $s \in \Omega_1 \setminus \{a\}$, then*

$$\mathscr{C}_{\Omega_2}^{Y,c}(f(a))|f'(a)| \leq \mathscr{C}_{\Omega_1}^{Y,c}(a),$$

where $c \in Y$.

Proof. If $\mathcal{H}_b^c(\Omega_2, Y) = \emptyset$, then $\mathscr{C}_{\Omega_2}^{Y,c} = 0$ and hence there is nothing to prove. Therefore, without loss of generality we assume that $\mathcal{H}_b^c(\Omega_2, Y) \neq \emptyset$.

By the definition of $\mathscr{C}_{\Omega_2}^{Y,c}(b)$, for every $\epsilon > 0$ there exists a holomorphic function h from Ω_2 into Y with $h(b) = c, h(s) \neq c$ for all $s \in \Omega_2 \setminus \{b\}$ for some $c \in Y$, such that

$$\mathscr{C}_{\Omega_2}^{Y,c}(b) - \epsilon \leq \eta_Y(h(b))|h'(b)|. \tag{3.5}$$

Suppose that f is a holomorphic function from Ω_1 into Ω_2 with $f(a) = b$, $f(s) \neq b$ for all $s \in \Omega_1 \setminus \{a\}$. Now the composition function $h \circ f \in \mathcal{H}(\Omega_1, Y)$ satisfies $(h \circ f)(a) = c$. Furthermore, $(h \circ f)(t) \neq c$ for all $t \in \Omega_1 \setminus \{a\}$ as $b \notin f(\Omega_1) \setminus \{a\}$ and $c \notin h(\Omega_2) \setminus \{b\}$. Now, by plugging the map $h \circ f$ into the definition of $\mathscr{C}_{\Omega_1}^{Y,c}(a)$, it follows that

$$\mathscr{C}_{\Omega_1}^{Y,c}(a) \geq \eta_Y((h \circ f)(a))|(h \circ f)'(a)| = \eta_Y(h(b))|h'(b)||f'(a)|. \tag{3.6}$$

Combining (3.5) and (3.6), we obtain

$$\mathscr{C}_{\Omega_1}^{Y,c}(a) \geq (\mathscr{C}_{\Omega_2}^{Y,c}(b) - \epsilon)|f'(a)|$$

which holds for every $\epsilon > 0$. Letting $\epsilon \to 0$, we have the desired inequality. $\square$

As a direct consequence of Theorem 3.11, we obtain the conformal invariance property and monotonicity property of the Carathéodory density of the Hurwitz metric $\mathscr{C}_\Omega^{Y,s}$ as follows:

Corollary 3.12. (*Conformal invariance property*) *If f is a conformal mapping from a domain $\Omega_1 \subset \mathbb{C}$ onto another domain $\Omega_2 \subset \mathbb{C}$, then for a base domain $Y \subsetneq \mathbb{C}$ we have*

$$\mathscr{C}_{\Omega_2}^{Y,s}(f(w))|f'(w)| = \mathscr{C}_{\Omega_1}^{Y,s}(w),$$

for all $w \in \Omega_1$ and $s \in Y$.

Corollary 3.13. (*Domain monotonicity property*) *If $\Omega_1 \subsetneq \Omega_2$ and Y are domains as in Theorem 3.11, then $\mathscr{C}_{\Omega_2}^{Y,s}(w) \leq \mathscr{C}_{\Omega_1}^{Y,s}(w)$ for all $w \in \Omega_1$ and $s \in Y$.*

Until now, we studied the properties of the Carathéodory density of the Hurwitz metric $\mathscr{C}_{\Omega}^{Y,s}$ by fixing the base domain Y. For two different base domains, the comparison result is given below.

Theorem 3.14. *Let $Y_1, Y_2 \subsetneq \mathbb{C}$ and $\Omega \subset \mathbb{C}$ be subdomains. If for every point $b \in Y_2$ there exists a point $a \in Y_1$ and a holomorphic covering map $g_b : Y_1 \setminus \{a\} \to Y_2 \setminus \{b\}$ which extends to the holomorphic function with $g_b(a) = b$ and $g_b'(a) \neq 0$, then*

$$\mathscr{C}_{\Omega}^{Y_1,a}(w) \leq \mathscr{C}_{\Omega}^{Y_2,b}(w)$$

for all $w \in \Omega$.

Proof. By the distance decreasing property for Hurwitz density, it follows that

$$\eta_{Y_2}(g_b(a))|g_b'(a)| = \eta_{Y_1}(a) \tag{3.7}$$

since g_b is the extended holomorphic covering map from Y_1 onto Y_2. Let $\epsilon > 0$ be arbitrary.

If $\mathcal{H}_w^a(\Omega, Y_1) = \emptyset$, then $\mathscr{C}_{\Omega}^{Y_1,a} = 0$ and hence there is nothing to prove. Therefore, without loss of generality we assume that $\mathcal{H}_w^a(\Omega, Y_1) \neq \emptyset$.

By the definition of $\mathscr{C}_{\Omega}^{Y_1,a}$, for $a \in Y_1$ and $w \in \Omega$, there exists a function $h \in \mathcal{H}_w^a(\Omega, Y_1)$ such that

$$\mathscr{C}_{\Omega}^{Y_1,a}(w) \leq \eta_{Y_1}(h(w))|h'(w)| + \epsilon. \tag{3.8}$$

Now we notice that the composed function $g_b \circ h \in \mathcal{H}(\Omega, Y_2)$ satisfies $(g_b \circ h)(w) = b$, $(g_b \circ h)(z) \neq b$ for all $z \in \Omega \setminus \{w\}$. Hence, $g_b \circ h \in \mathcal{H}_w^b(\Omega, Y_2)$. Applying $g_b \circ h$ in the definition of $\mathscr{C}_{\Omega}^{Y_2,b}(w)$, we conclude that

$$\mathscr{C}_{\Omega}^{Y_2,b}(w) \geq \eta_{Y_2}((g_b \circ h)(w))|(g_b \circ h)'(w)| = \eta_{Y_2}(g_b(a))|g_b'(a)||h'(w)| \tag{3.9}$$

for all $w \in \Omega$. Combining (3.7), (3.8), (3.9) and applying the chain rule, we obtain

$$\mathscr{C}_\Omega^{Y_1,a}(w) \le \eta_{Y_1}(a)|h'(w)| + \epsilon = \eta_{Y_2}(g_b(a))|g_b'(a)||h'(w)| + \epsilon \le \mathscr{C}_\Omega^{Y_2,b}(w) + \epsilon$$

for all $w \in \Omega$. Since ϵ is arbitrary, we can let it approach to zero to obtain the desired inequality. $\square$

Corollary 3.15. *If Y_1 and Y_2 are conformally equivalent proper subdomains of $\mathbb{C}$ and Ω is an arbitrary subdomain of $\mathbb{C}$, then*

$$\mathscr{C}_\Omega^{Y_1,a}(w) = \mathscr{C}_\Omega^{Y_2,b}(w)$$

holds for every $w \in \Omega$ and for some $a \in Y_1, b \in Y_2$.

Proof. We consider the inverse image of the conformal mapping in Theorem 3.14 to obtain the reverse inequality $\mathscr{C}_\Omega^{Y_1,a}(w) \ge \mathscr{C}_\Omega^{Y_2,b}(w)$. $\square$

3.2. A distance function

In this section, we consider the usual distance function associated with the Carathéodory density of the Hurwitz metric $\mathscr{C}_\Omega^{Y,s}$ for the domains $Y \subsetneq \mathbb{C}$ and $\Omega \subset \mathbb{C}$.

Definition 3.16. Let $Y \subsetneq \mathbb{C}$ and $\Omega \subset \mathbb{C}$ be domains. For $w_1, w_2 \in \Omega$ and $s \in Y$ define

$$\mathscr{C}_\Omega^{Y}(w_1, w_2) = \inf \int_\gamma \mathscr{C}_\Omega^{Y,s}(w)|dw|,$$

where the infimum is taken over all rectifiable paths $\gamma \subset \Omega$ joining w_1 to w_2. If $\mathscr{C}_\Omega^{Y}$ defines a metric, then we say $(\Omega, \mathscr{C}_\Omega^{Y})$ a metric space.

It is easy to see from Definition 3.16 that $\mathscr{C}_\Omega^{Y}(w_1, w_1) = 0$ and $\mathscr{C}_\Omega^{Y}(w_1, w_2) = \mathscr{C}_\Omega^{Y}(w_2, w_1)$ for any $w_1, w_2 \in \Omega$. Further, it can also be verified that $\mathscr{C}_\Omega^{Y}$ satisfies the triangle inequality. Hence, at least we can say that $\mathscr{C}_\Omega^{Y}$ is a pseudo-metric. At present we do not know whether $\mathscr{C}_\Omega^{Y}$ defines a metric or not. However, we have a partial solution to this whenever $\Omega \subset Y$.

Theorem 3.17. *If $\Omega \subset Y \subsetneq \mathbb{C}$ are domains, then $(\mathscr{C}_\Omega^{Y}, \Omega)$ becomes a metric space.*

Proof. Since $\mathscr{C}_\Omega^Y$ is a pseudo-metric on Ω, it is enough to show that $\mathscr{C}_\Omega^Y(w_1, w_2) > 0$ for two distinct points $w_1, w_2 \in \Omega$. Let γ be an arbitrary rectifiable curve joining w_1 to w_2 in Ω. Since $\Omega \subset Y$, plugging the inclusion mapping $i \in \mathcal{H}_w^w(\Omega, Y)$ into the definition of $\mathscr{C}_\Omega^{Y,w}(w)$, we conclude that

$$\int_\gamma \mathscr{C}_\Omega^{Y,w}(w)|dw| \geq \int_\gamma \eta_Y(i(w))|i'(w)||dw| = \int_\gamma \eta_Y(w)|dw|.$$

By the Definition 2.2 of Hurwitz distance between two points, it follows that

$$\int_\gamma \mathscr{C}_\Omega^{Y,w}(w)|dw| > \eta_Y(w_1, w_2).$$

Now, taking the infimum over γ, we obtain

$$\mathscr{C}_\Omega^Y(w_1, w_2) \geq \eta_Y(w_1, w_2) > 0,$$

where the last inequality follows from Theorem 5.3. Hence $(\Omega, \mathscr{C}_\Omega^Y)$ defines a metric space, completing the proof. $\square$

CHAPTER 4

THE HURWITZ METRIC ON HYPERBOLICALLY COVERED DOMAINS

This chapter attempts to obtain upper and lower bounds of quotients of the Hurwitz metrics in some special kind of domains, namely hyperbolically covered domains. As a result, we show that whenever there is a sequence of hyperbolically covered domains converging to some hyperbolically covered domain and contained in it, in the sense of Carathéodory kernel convergence, the corresponding sequence of the Hurwitz densities converges locally uniformly.

4.1. Hyperbolically covered domain

Hyperbolically covered domains are defined in this section, in terms of evenly covered sets.

Let Ω_1, Ω_2 be topological spaces and $h : \Omega_1 \to \Omega_2$ be a continuous surjective function. The open set $U \subset \Omega_2$ is said to be *evenly covered* [38, p. 336] by h if the inverse image $h^{-1}(U)$ can be written as the union of disjoint open sets $V_\alpha \subset \Omega_1$ such that for each α, the restriction of h to V_α is a homeomorphism of V_α onto U.

The following definition is our main concern in this paper:

Definition 4.1. Let Ω be a hyperbolic domain and g be the holomorphic covering map from $\mathbb{D}$ onto Ω. For each $w \in \Omega$, if the hyperbolic disks centred at w in Ω are the evenly covered neighbourhoods of w, then we call such Ω to be *hyperbolically covered*.

It is easy to see that the proper simply connected domains of $\mathbb{C}$ are hyperbolically covered. However, we illustrate that there exits a non-simply connected domain which is hyperbolically covered.

Example 4.2. Consider the punctured disk $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$. It can be easily seen that the function $w = g(z) = e^{iz}$ is the covering map from the upper half plane $\mathbb{H}$ onto $\mathbb{D}^*$ and each hyperbolic disk centred at w is an evenly covered neighbourhood of w. Indeed, on every strip of length 2π, the function g is a bijection. As a consequence, by Definition 4.1, $\mathbb{D}^*$ is a hyperbolically covered domain.

Remark 4.3. Let Ω be a hyperbolic domain. If there exists a point $w \in \Omega$ and a hyperbolic disk centred at w having large enough radius which properly contains all possible evenly covered neighbourhoods of w, then Ω is not hyperbolically covered. By adopting this idea, we do believe that there exists non hyperbolically covered domains. However, at present, we do not have a specific example.

4.2. Quotient of the Hurwitz densities

For a point $w \in \Omega$, the open hyperbolic disk of radius ρ centred at w is denoted by $D_\Omega(w, \rho)$ and is defined by

$$D_\Omega(w, \rho) = \{z \in \Omega : \lambda_\Omega(w, z) < \rho\}.$$

We frequently use the notation $D_\Omega(w, \rho)$ throughout this chapter.

Suppose that $\Omega_1 \subsetneq \Omega_2$ are proper subdomains of $\mathbb{C}$. By distance decreasing property [35, Theorem 6.1] of the Hurwitz metric it follows that $\eta_{\Omega_1}/\eta_{\Omega_2} > 1$. We now formulate a tool in terms of lemma to estimate upper and sharper lower bounds for the quantity $\eta_{\Omega_1}/\eta_{\Omega_2}$.

Lemma 4.4. *Suppose that $\Omega \subsetneq \mathbb{C}$ be a hyperbolically covered domain.*

(1) *Let $w \in \Omega$ and $\rho > 0$ be such that the hyperbolic disk $D_\Omega(w, \rho) \subset \Omega$. Then we have*

$$\frac{\eta_{D_\Omega(w,\rho)}(w)}{\eta_\Omega(w)} \leq 1 + \frac{2}{e^\rho - 1}.$$

Equality holds if, and only if, Ω is simply connected.

(2) *Let Ω be a hyperbolic domain. If $w \in \Omega \setminus \{a\}$, where $a \in \Omega$ and $\rho(w) = \lambda_\Omega(w, a)$, then*

$$\frac{2e^{\rho(w)}}{\log \frac{e^{\rho(w)}+1}{e^{\rho(w)}-1}(e^{2\rho(w)} - 1)} \leq \frac{\eta_{\Omega\setminus\{a\}}(w)}{\eta_\Omega(w)}.$$

Proof. (1) Since Ω is a hyperbolic domain in $\mathbb{C}$, for a point $w \in \Omega$, there exists a holomorphic covering map g from $\mathbb{D}$ onto Ω with $g(0) = w$. Suppose that E is the inverse image $g^{-1}(D_\Omega(w,\rho))$ which contains 0 such that $g^{-1}(w) = 0$. Therefore, the function g restricted to $E \setminus \{0\}$, that is $g|_{E\setminus\{0\}}$ is a holomorphic covering map from $E \setminus \{0\}$ to $D_\Omega(w,\rho) \setminus \{w\}$ which naturally extends to the holomorphic function with $g|_E(0) = w$. Hence, $g|_E \in \mathcal{H}(E, D_\Omega(w,\rho))$ with $g|_E(0) = w$, $g|_E(z) \neq w$ for all $z \in E \setminus \{0\}$ and $g'(0) > 0$. Thus, by the distance decreasing property of the Hurwitz density, we have

$$\eta_{D_\Omega(w,\rho)}(g(0))g'(0) = \eta_E(0). \tag{4.1}$$

Since holomorphic covering maps are isometries of the hyperbolic densities, we have

$$\lambda_\Omega(g(0))g'(0) = \lambda_{\mathbb{D}}(0). \tag{4.2}$$

It is well-known that $\eta_\Omega \geq \lambda_\Omega$, whenever Ω is a hyperbolic domain. From (5.2) and (5.9), we conclude that

$$\frac{\eta_{D_\Omega(w,\rho)}(w)}{\eta_\Omega(w)} \leq \frac{\eta_E(0)}{\lambda_\Omega(w)g'(0)} = \frac{\eta_E(0)}{\lambda_{\mathbb{D}}(0)} = \frac{1}{2}\eta_E(0).$$

Since hyperbolic disk are preserved under holomorphic covering maps, it can be noticed that $g(D_{\mathbb{D}}(0,\rho)) = D_\Omega(w,\rho)$. This, together with the relation $g(E) = D_\Omega(w,\rho)$, yields $D_{\mathbb{D}}(0,\rho) = E$. Hence, we have

$$\frac{\eta_{D_\Omega(w,\rho)}(w)}{\eta_\Omega(w)} \leq \frac{1}{2}\eta_E(0) = \frac{1}{2}\eta_{D_{\mathbb{D}}(0,\rho)}(0) = \frac{1}{2}\lambda_{D_{\mathbb{D}}(0,\rho)} = 1 + \frac{2}{e^\rho - 1},$$

since the hyperbolic and the Hurwitz densities agree in simply connected domains.

(2) If $\Omega \subsetneq \mathbb{C}$ is a simply connected domain, then the result follows from [36, Lemma 1(b)] and the fact that the hyperbolic density and the Hurwitz density coincides on simply connected domains.

We now prove the result for a non-simply connected domain Ω. Since $\Omega \subsetneq \mathbb{C}$ and $w \in \Omega \setminus \{a\}$, there exists a holomorphic covering map g from $\mathbb{D} \setminus \{s_0\}$ onto $\Omega \setminus \{w\}$ which extends to the holomorphic function such that $g(s_0) = w$, $g(s) \neq w$ for all $s \in \mathbb{D} \setminus \{s_0\}$ and $g'(s_0) \neq 0$ for some $s_0 \in \mathbb{D}$. By [35, Theorem 6.1], it follows that

$$\eta_\Omega(g(s_0))|g'(s_0)| = \eta_{\mathbb{D}}(s_0). \tag{4.3}$$

Let Ω^* be a component of $g^{-1}(\Omega \setminus \{a\})$ such that $0 \notin \Omega^*$. The restricted function $g|_{\Omega^*}$ on Ω^* is holomorphic and satisfies $g(s_0) = w, g(s) \neq w$ for all $s \in \Omega^* \setminus \{s_0\}$ and $g'(s_0) > 0$.

Therefore, $g|_{\Omega^*} \in \mathcal{H}(\Omega^*, \Omega \setminus \{a\})$ with $g|_{\Omega^*}(s_0) = w,\ g|_{\Omega^*}(z) \neq w$ for all $z \in \Omega^* \setminus \{s_0\}$ and $g|_{\Omega^*}(s_0) > 0$. Now, by the distance decreasing property of the Hurwitz density, we obtain

$$\eta_{\Omega\setminus\{a\}}(g|_{\Omega^*}(s_0))g'|_{\Omega^*}(s_0) = \eta_{\Omega^*}(s_0). \tag{4.4}$$

From equations (5.1) and (4.4), we obtain

$$\frac{\eta_{\Omega\setminus\{a\}}(w)}{\eta_\Omega(w)} = \frac{\eta_{\Omega^*}(s_0)}{\eta_{\mathbb{D}}(s_0)} \geq \frac{\eta_{\mathbb{D}^*}(s_0)}{\eta_{\mathbb{D}}(s_0)} \geq \frac{\lambda_{\mathbb{D}^*}(s_0)}{\lambda_D(s_0)},$$

where the first inequality follows from the domain monotonicity property of the Hurwitz density and the last inequality is due to the fact that the hyperbolic density lags behind the Hurwitz density and agrees over simply connected domains. Now choosing $s_0 = \tilde{z}$ and adopting the steps from the proof of [36, Lemma 1(b)], the desired inequality is obtained. $\square$

Theorem 4.5. *Let Ω_2 be a hyperbolically covered domain and $\Omega_1 \subseteq \Omega_2 \subsetneq \mathbb{C}$. Then for all $w \in \Omega_1$*

$$1 < \frac{2e^{\rho(w)}}{\log \frac{e^{\rho(w)}+1}{e^{\rho(w)}-1}(e^{2\rho(w)} - 1)} \leq \frac{\eta_{\Omega_1}(w)}{\eta_{\Omega_2}(w)} \leq 1 + \frac{2}{e^{\rho(w)} - 1},$$

where $\rho(w) = \lambda_{\Omega_2}(w, \Omega_2 \setminus \Omega_1) = \inf\{\lambda_{\Omega_2}(w, z) : z \in \Omega_1^c\}$. The last inequality becomes equality when Ω_2 is simply connected and $\Omega_1 = D_{\Omega_2}(w, \rho(w))$.

Proof. Setting $\rho = \rho(w)$. The definition of ρ leads to $D_{\Omega_2}(w, \rho) \subset \Omega_1$. Now, by the domain monotonicity property of the Hurwitz metric it follows that $\eta_{D_{\Omega_2}(w,\rho)} \geq \eta_{\Omega_1}$. As a consequence of Lemma 5.6(1), we obtain

$$\frac{\eta_{\Omega_1}(w)}{\eta_{\Omega_2}(w)} \leq \frac{\eta_{D_{\Omega_2}(w,\rho)}(w)}{\eta_{\Omega_2}(w)} \leq 1 + \frac{2}{e^\rho - 1},$$

completing the proof of the right side inequality. Case of equality in the upper bound directly follows by Lemma 5.6.

To obtain the lower bound, consider an element $a \in \Omega_2 \setminus \Omega_1$ such that $\lambda_{\Omega_2}(w, a) = \rho(w)$. Since $\Omega_1 \subset \Omega_2 \setminus \{a\}$, the domain monotonicity property of the Hurwitz density and Lemma 5.6(2) together produce

$$\frac{\eta_{\Omega_1}(w)}{\eta_{\Omega_2}(w)} \geq \frac{\eta_{\Omega_2\setminus\{a\}}(w)}{\eta_{\Omega_2}(w)} \geq \frac{2e^{\rho(w)}}{\log \frac{e^{\rho(w)}+1}{e^{\rho(w)}-1}(e^{2\rho(w)} - 1)},$$

completing the proof. $\square$

Corollary 4.6. *Let $\rho_1, \rho_2 > 0, \rho = \rho_1 + \rho_2$ and Ω be a hyperbolically covered domain, for all $w \in D_\Omega(a, \rho_1)$ we have*

$$\frac{\eta_{D_\Omega(a,\rho)}(w)}{\eta_\Omega(w)} < 1 + \frac{2}{e^{\rho_2} - 1},$$

where $a \in \Omega$. Moreover, if Ω_1 is a domain such that $D_\Omega(a, \rho) \subset \Omega_1 \subset \Omega$, then

$$1 < \frac{\eta_{\Omega_1}(w)}{\eta_\Omega(w)} < 1 + \frac{2}{e^{\rho_2} - 1},$$

for all $w \in D_\Omega(a, \rho_1)$.

Proof. For an arbitrary $w \in D_\Omega(a, \rho_1)$, setting $\rho(w) = \lambda_\Omega(w, \Omega \setminus \{D_\Omega(a, \rho)\})$. Now the proof follows from the Theorem 4.5 and [36, Theorem 2]. □

It is well known in literature (see [15]) that universal covering map π_n of Ω_n converges to Ω locally uniformly in Ω whenever Ω_n converges to Ω in the sense of Carathéodory kernel. Hence $\lambda_{\Omega_n} \to \lambda_\Omega$ locally uniformly in Ω. We prove an analogue result for the Hurwitz metric when $\Omega_n \subset \Omega$ for $n \in \mathbb{N}$. For the situation $\Omega_n \subset \Omega$, we have $\Omega_n \to \Omega$ in the sense of *Carathéodory kernel convergence*, if, and only if, for every compact set $K \subset \Omega$ there exists a $N(K)$ such that $K \subset \Omega_n$ for all $n > N(K)$.

Theorem 4.7. *Let $\Omega_n \subset \Omega$, for $n \in \mathbb{N}$, be a sequence of hyperbolically covered domains converges to the hyperbolically covered domain Ω in the sense of Carathéodory. Then $\eta_{\Omega_n}/\eta_\Omega$ converges to 1 locally uniformly in Ω.*

Proof. Let $U \subset \Omega$ be a compact set. For a given $\epsilon > 0$ and $a \in U$, we can choose $U = \overline{D}_\Omega(a, \rho_1)$ for some $\rho_1 > 0$. Furthermore, the quantity $\rho_2 > 0$ is chosen so that $2/(e^{\rho_2} - 1) < \epsilon$ and $\rho = \rho_1 + \rho_2$. Since $\overline{D}_\Omega(a, \rho) \subset \Omega$ is a compact set, there exists a number $M \in \mathbb{N}$ such that $\overline{D}_\Omega(a, \rho) \subset \Omega_n$ for all $n \geq M$. Now, the proof follows from Corollary 5.10. □

CHAPTER 5

Continuity and bi-Lipschitz property

In this chapter, we adopt the definition of the Gardiner-Lakic metric of the hyperbolic metric to define a new metric by replacing the hyperbolic metric by the Hurwitz metric. We show that this is invariant under Möbius transformation. We establish the continuity of the Hurwitz metric on arbitrary proper sub-domains of the complex plane and it serves as a tool to prove that this new metric is a lower semi-continuous function in hyperbolic domains. Furthermore, we do establish a bi-Lipschitz equivalence of this new metric with the Hurwitz metric, the hyperbolic and the quasihyperbolic metrics. However, on hyperbolic domains with connected boundary, we give a sharper bi-Lipschitz constant for the first case.

According to [33], for any two sets X and Y, the *Hausdorff distance* $H(X,Y)$ between them is defined as

$$H(X,Y) := \inf_{r>0}\{X \subset N_r(Y) \text{ and } Y \subset N_r(X)\},$$

where $N_r(X)$ is defined as the collection of all points whose Euclidean distance from X is less than r. An equivalent definition of the Hausdorff distance is as follows:

$$H(X,Y) := \max\big(\sup_{w\in X} d(w,Y), \sup_{w\in Y} d(w,X)\big),$$

where $d(w,X) := \inf_{z\in X}\{|z-w|\}$. There are several convergence of the sequences of sets, for example Carathéodory convergence and convergence in boundary. For the sake of necessity in our results, the convergence in boundary is defined below.

Definition 5.1. [33] Let Ω_n be a sequence of domains and $\Omega \subset \mathbb{C}$ be a domain. Then Ω_n is said to *converge in boundary* to Ω if

1. $H(\partial\Omega, \partial\Omega_n) \to 0$ as $n \to \infty$; and
2. there exits $w_0 \in \Omega$ such that $w_0 \in \Omega_n$ for all but finitely many $n \in \mathbb{N}$.

5.1. Continuity of the Hurwitz metric

Recently the continuity of the Hurwitz metric in bounded planar domains is established by Sarkar and Verma (see [51, Theorem 1.4]) and in the same paper they made curvature calculation as well for arbitrary planar domains. The idea of scaling principle for planar domains is adopted for these studies from [11]. In this note, we provide an alternative and shorter proof of the continuity of the Hurwitz metric in arbitrary planar domains. The concept of the so-called convergence of sequence of domains in boundary is used in our investigation as a main tool (see the following lemma). However, this depends on the Hausdorff distance (see Definition 5.1).

Lemma 5.2. *Let Ω be a planar domain. If $w_n \in \Omega$ is a sequence of points converges to a point $w \in \Omega$, then the corresponding sequence of punctured domains $\Omega_{w_n} = \Omega \setminus \{w_n\}$ converges in boundary to $\Omega_w = \Omega \setminus \{w\}$.*

Proof. To prove our lemma, we first demonstrate that the Hausdorff distance $H(\partial\Omega_w, \partial\Omega_{w_n})$ approaches to zero as $n \to \infty$. Consider

$$\begin{aligned} H(\partial\Omega_w, \partial\Omega_{w_n}) &= \max\big(\sup_{w\in\partial\Omega_w} d(w, \partial\Omega_{w_n}), \sup_{w\in\partial\Omega_{w_n}} d(w, \partial\Omega_w)\big) \\ &= \max\big(|w - w_n|, |w_n - w|\big) \\ &= |w_n - w|, \end{aligned}$$

where the second equality follows from the simple observation that $\partial\Omega_{w_n} = \partial\Omega \cup \{w_n\}$ and $\partial\Omega_w = \partial\Omega \cup \{w\}$. Note that w_n converges to w, thus we conclude that the Hausdorff distance $H(\partial\Omega_w, \partial\Omega_{w_n})$ goes to zero as n tends to infinity.

Since Ω is a domain, we can choose a small ball $B(w, r) \subset \Omega$ and hence a point $w_0 \in B(w, r)$ such that $w_0 \neq w_n, w$ for $n \in \mathbb{N}$. As a consequence, w_0 belongs to Ω_w as well as Ω_{w_n}. Hence, by the definition, we conclude that Ω_{w_n} converges in boundary to Ω_w. $\square$

It is well known in literature that the classical hyperbolic metric [24], the Hahn metric [37], the capacity metric [49] are continuous functions. In this direction, as an application of Lemma 5.2, we now show that the Hurwitz metric is also a continuous function.

Theorem 5.3. *Let $\Omega \subsetneq \mathbb{C}$ be a domain, then the Hurwitz density η_Ω is a continuous function.*

Proof. Let $w \in \Omega$ be an arbitrary point and w_n be a sequence of points in Ω converging to w. Then we show that $\eta_\Omega(w_n) \to \eta_\Omega(w)$ as $n \to \infty$. By [35, Theorem 7.1], we have

$$|z - w_n| \log\left(\frac{2}{\eta_\Omega(w_n)|z - w_n|}\right) \lambda_{\Omega_{w_n}}(z) = 1 + O\left(\frac{|z - w_n|}{\log\left(\dfrac{2}{\eta_\Omega(w_n)|z - w_n|}\right)}\right).$$

To solve the above equation for η_{Ω_n}, we do the following calculations:

$$\begin{aligned}
\frac{1}{|z - w_n|\lambda_{\Omega_{w_n}}(z)} &= \frac{\log\left(\dfrac{2}{\eta_\Omega(w_n)|z - w_n|}\right)}{1 + O\left(\dfrac{|z - w_n|}{\log\left(\dfrac{2}{\eta_\Omega(w_n)|z - w_n|}\right)}\right)} \\
&= \log\left(\frac{2}{\eta_\Omega(w_n)|z - w_n|}\right)\left(1 - O\left(\frac{|z - w_n|}{\log\left(\dfrac{2}{\eta_\Omega(w_n)|z - w_n|}\right)}\right)\right) \\
&= \log\left(\frac{2}{\eta_\Omega(w_n)}\right) - \log(|z - w_n|) - O(|z - w_n|).
\end{aligned}$$

This implies that

$$\log\left(\frac{\eta_\Omega(w_n)}{2}\right) = \frac{1}{\log(|z - w_n|)} - \frac{1}{|z - w_n|\lambda_{\Omega_{w_n}}(z)} - O(|z - w_n|).$$

Let $f(z) = O(|z - w_n|)$ as $z \to w_n$. By definition of big O, there exist $M, \epsilon > 0$ such that for all z with $0 < |z - w_n| < \epsilon$ we have $|f(z)| \le M|z - w_n|$. Since $w_n \to w$, for $\delta > 0$ there exists a number $N \in \mathbb{N}$ such that $|w_n - w| < \delta$ for all $n \ge N$. Using the triangle inequality $|z - w| \le |z - w_n| + |w_n - w|$ and choosing $\delta = \epsilon$ we have $|f(z)| \le M|z - w|$ for all z with $0 < |z - w| < 2\epsilon$. Therefore, we conclude that $O(|z - w_n|) \to O(|z - w|)$ as $n \to \infty$.

Recall that the logarithm function and $1/|z-w_n|$ are continuous. In view of [33, Lemma 2], $\lambda_{\Omega_{w_n}}$ converges locally uniformly to λ_{Ω_w} whenever the sequence of domains Ω_{w_n} converges in boundary to Ω_w. These facts along with Lemma 5.2 give us

$$\lim_{n\to\infty} \log \frac{\eta_\Omega(w_n)}{2} = \log \frac{1}{|z-w|} - \frac{1}{|z-w|\lambda_{\Omega_w}(z)} - O(|z-w|).$$

Therefore, the continuity of the Hurwitz metric follows.

□

5.2. An invariant metric

This section is devoted to a new metric which in fact turned out to be bi-Lipschitz equivalent to the Hurwitz metric on the hyperbolic domains. We do prove the lower semi-continuity of this metric.

For a hyperbolic domain Ω and $w \in \Omega$, setting

$$\overline{\eta}_\Omega(w) = \sup \eta_{\mathbb{C}\setminus\{w_1,w_2\}}(w),$$

where the supremum is taken over all *distinct* pair of points $w_1, w_2 \in \Omega^c$. Note that, in the computation point of view, similar to the original Gardiner-Lakic metric and the Hurwitz metric itself, our new metric $\overline{\eta}_\Omega$ is difficult to compute. We are hopping to at least estimate $\overline{\eta}_\Omega$ in some specific domains.

Remark 5.4. For a hyperbolic domain Ω and $w_1, w_2 \in \Omega^c$, it is easy to see that Ω is contained in $\mathbb{C} \setminus \{w_1, w_2\}$. By the domain monotonicity property of the Hurwitz metric, it follows that $\eta_{\mathbb{C}\setminus\{w_1,w_2\}}(w) \le \eta_\Omega(w)$ for all $w \in \Omega$ and hence on taking supremum over all $w_1, w_2 \in \Omega$, we have $\overline{\eta}_\Omega(w) \le \eta_\Omega(w)$ for all $w \in \Omega$.

Remark 5.5. Suppose $\Omega_1 \subset \Omega_2$ are hyperbolic domains, then $\Omega_2^c \subset \Omega_1^c$. Therefore, by the definition of $\overline{\eta}$, we have $\overline{\eta}_{\Omega_1} \ge \overline{\eta}_{\Omega_2}$. That is, $\overline{\eta}$ satisfies the domain monotonicity property.

In the definition of $\overline{\eta}_\Omega$, the supremum is attained for some pair of points in the complement of Ω. We now illustrate this in the form of the following lemma.

Lemma 5.6. *Let Ω be a hyperbolic domain. For every $w \in \Omega$ there exist distinct points $p, q \in \Omega^c$ for which supremum is attained for $\overline{\eta}_\Omega(w)$, that is $\overline{\eta}_\Omega(w) = \eta_{\mathbb{C}\setminus\{p,q\}}(w)$.*

Proof. Fix $w_0 \in \Omega$. Let $p_n \neq q_n \in \Omega^c$ be sequences such that $\eta_{\mathbb{C}\setminus\{p_n,q_n\}}(w_0) \to \overline{\eta}_\Omega(w_0)$. This is possible by the definition of supremum. Since Ω^c is a closed set, limit of the sequences p_n, q_n say p and q belong to Ω^c itself. For $w_0 \in \Omega$, let g_n and g be the Hurwitz covering maps from $\mathbb{D} \to \mathbb{C} \setminus \{p_n, q_n\}$ and $\mathbb{D} \to \mathbb{C} \setminus \{p, q\}$, respectively.

Note that the conformal map $T_n(w) = [(w-p)(q_n-p_n)/(q-p)] + p_n$ maps $\mathbb{C} \setminus \{p,q\}$ onto $\mathbb{C} \setminus \{p_n, q_n\}$. It is easy to see that the composition function $T_n \circ g$ from $\mathbb{D}$ onto $\mathbb{C} \setminus \{p_n, q_n\}$ is a Hurwitz covering map. By the uniqueness of the Hurwitz covering map, we compute

$$g_n(w_0) = (T_n \circ g)(w_0) = \left(\frac{g(w_0) - p}{q-p} \right) (q_n - p_n) + p_n. \tag{5.1}$$

Taking $n \to \infty$ in (5.1), the limit $g_n \to g$ follows. Also, the restricted holomorphic covering map $g_n : \mathbb{D} \setminus \{0\} \to \mathbb{C} \setminus \{p_n, q_n, w_0\}$ converges $g : \mathbb{D} \setminus \{0\} \to \mathbb{C} \setminus \{p, q, w_0\}$. Hence the points p, q are distinct; otherwise it converges locally uniformly to w_0 times the rotation map of $\mathbb{D} \setminus \{0\}$ [15]. Now, the result follows from the uniqueness of the limit. □

It is always interesting to know whether a metric is Möbius invariant. In literature many metrics such as the hyperbolic metric [24], the Hurwitz metric [35], the generalized Hurwitz metric, the Carathéodory density of the Hurwitz metric and the Gardiner-Lakic metric [12] are Möbius invariant. Precisely, Möbius invariance of the Kobayashi density of the Hurwitz metric follows by applying a Möbius map and its inverse in Theorem 2.15. We now demonstrate that $\overline{\eta}_\Omega$ is Möbius invariant.

Lemma 5.7. *The metric $\overline{\eta}_\Omega$ is invariant under Möbius transformations which fix infinity.*

Proof. Suppose that T is a Möbius transformation and $T(\Omega_1) = \Omega_2$, where Ω_1 is a hyperbolic domain. In the definition of $\overline{\eta}_\Omega$, since $w_1, w_2 \in \Omega_1^c$, their images under T belong to Ω_2^c, that is $T(w_1) = s_1, T(w_2) = s_2 \in \Omega_2^c$ and $s_1 \neq s_2$. In order to prove that $\overline{\eta}_\Omega$ is Möbius invariant, it is enough to show that

$$\overline{\eta}_{\Omega_2}(T(w))|T'(w)| = \overline{\eta}_{\Omega_1}(w).$$

As a consequence of distance decreasing property of the Hurwitz density we have

$$\eta_{\mathbb{C}\setminus\{s_1,s_2\}}(T(w))|T'(w)| = \eta_{\mathbb{C}\setminus\{w_1,w_2\}}(w).$$

By the definition of $\overline{\eta}_{\Omega_2}$, it follows that

$$\overline{\eta}_{\Omega_2}(T(w))|T'(w)| \geq \eta_{\mathbb{C}\setminus\{w_1,w_2\}}(w).$$

Therefore, we have $\overline{\eta}_{\Omega_2}(T(w))|T'(w)| \geq \overline{\eta}_{\Omega_1}(w)$. The reverse inequality is guaranteed by applying the same argument to T^{-1}. $\square$

In Theorem 5.3, we proved the continuity of the Hurwitz metric. Our next result demonstrates the lower semi-continuity of the $\overline{\eta}_{\Omega}$ metric by using the fact that the Hurwitz metric is continuous. We are looking forward to establish the continuity of $\overline{\eta}_{\Omega}$, however, at present we do not have a proof for it.

Theorem 5.8. *For a hyperbolic domain Ω, the density function $\overline{\eta}_{\Omega}$ is a lower semi-continuous function.*

Proof. Let $w_0 \in \Omega$ be a fixed point. To prove the lower semi-continuity, we shall demonstrate that

$$\overline{\eta}_{\Omega}(w_0) \leq \liminf_{w\to w_0} \overline{\eta}_{\Omega}(w).$$

Choose a sequence $w_n \in \Omega$ converging to the point w_0 such that $\overline{\eta}_{\Omega}(w_n) \to \liminf_{w\to w_0} \overline{\eta}_{\Omega}(w)$. From Lemma 5.6, there exit points $p, q \in \Omega^c$ with $\overline{\eta}_{\Omega}(w_0) = \eta_{\mathbb{C}\setminus\{p,q\}}(w_0)$. By the definition of $\overline{\eta}_{\Omega}$ it follows that $\eta_{\mathbb{C}\setminus\{p,q\}}(w_n) \leq \overline{\eta}_{\Omega}(w_n)$ for all $n \in \mathbb{N}$. Thus, we have

$$\overline{\eta}_{\Omega}(w_0) = \eta_{\mathbb{C}\setminus\{p,q\}}(w_0) = \lim_{n\to\infty} \eta_{\mathbb{C}\setminus\{p,q\}}(w_n) \leq \liminf_{n\to\infty} \overline{\eta}_{\Omega}(w_n) \leq \liminf_{w\to w_0} \overline{\eta}_{\Omega}(w),$$

where the second equality follows from Theorem 5.3. $\square$

The following result provides a bi-Lipschitz equivalence of the $\overline{\eta}_{\Omega}$ and the quasihyperbolic densities in the hyperbolic domain Ω.

Theorem 5.9. *Let Ω be a hyperbolic domain, then the quasi-hyperbolic density δ_{Ω} and the density $\overline{\eta}_{\Omega}$ are bi-Lipschitz equivalent, that is*

$$\frac{1}{8\delta_{\Omega}(w)} \leq \overline{\eta}_{\Omega}(w) \leq \frac{2}{\delta_{\Omega}(w)},$$

for every $w \in \Omega$.

Proof. Let $p \in \partial\Omega$ be the closest point from w in the boundary of Ω, that is $|w-p| = \delta_{\Omega}(w)$. Since Ω is a hyperbolic domain, we can find a point $q \in \Omega^c$ with $q \neq p$. It is easy to see that $(\mathbb{C}\setminus\{p,q\})\subset(\mathbb{C}\setminus\{p\})$. By the domain monotonicity property of the Hurwitz density, we have

$$\eta_{\mathbb{C}\setminus\{p,q\}}(w)\geq\eta_{\mathbb{C}\setminus\{p\}}(w) \tag{5.2}$$

for every $w\in\Omega$. Taking the supremum over $p,q\in\Omega^c$ in (5.2) and using the fact that $\eta_{\mathbb{C}\setminus\{p\}}(w)=1/(8|w-p|)$, we obtain

$$\begin{aligned}\overline{\eta}_\Omega(w)=\sup_{w_1,w_2\in\Omega^c}\eta_{\mathbb{C}\setminus\{w_1,w_2\}}(w)\geq\sup_{w_1\in\Omega^c}\eta_{\mathbb{C}\setminus\{w_1\}}(w)&=\sup_{w_1\in\Omega^c}\frac{1}{8|w-w_1|}\\&=\frac{1}{\inf_{w_1\in\Omega^c}8|w-w_1|}=\frac{1}{8\delta_\Omega(w)}.\end{aligned}$$

The left inequality follows.

For the right inequality, we consider the open ball

$$B(w,\delta_\Omega(w))=\{z\in\Omega:|w-z|<\delta_\Omega(w)\}\subset\mathbb{C}\setminus\{a,b\}$$

for every $a,b\in\Omega^c$. Then it is easy to see by the domain monotonicity property of the Hurwitz density that for $w\in\Omega$

$$\overline{\eta}_\Omega(w)\leq\eta_{B(w,\delta(w))}(w)=\frac{2}{\delta_\Omega(w)}, \tag{5.3}$$

where last equality follows from the fact that the Hurwitz and the hyperbolic densities coincide on simply connected domains. $\square$

In the next two corollaries, the bi-Lipschitz equivalence of the density $\overline{\eta}_\Omega$ with the Hurwitz density followed by with the hyperbolic density are provided.

Corollary 5.10. *The Hurwitz density η_Ω and the density $\overline{\eta}_\Omega$ are bi-Lipschitz equivalent in the hyperbolic domain Ω. Precisely, we have*

$$\frac{\eta_\Omega(w)}{16}\leq\overline{\eta}_\Omega(w)\leq\eta_\Omega(w),$$

where $w\in\Omega$. Moreover, the second inequality is sharp.

Proof. The first inequality directly follows from [35, Theorem 6.4] and Theorem 5.9, while the last one always holds. It is easy to see that the equality in the second inequality holds when Ω is a twice punctured complex plane. $\square$

A hyperbolic domain Ω is said to be *uniformly perfect* if there exists a constant $b > 0$ such that $\lambda_\Omega(w) \geq b/\delta(w)$ for all $w \in \Omega$. However, from (5.1) and the domain monotonicity property of the hyperbolic density we have the reverse inequality $\lambda_\Omega(w) \leq 2/\delta(w)$. This leads to the second corollary as follows.

Corollary 5.11. *Let Ω be a uniformly perfect domain in $\mathbb{C}$. Then the hyperbolic density λ_Ω and the density $\overline{\eta}_\Omega$ are bi-Lipschitz equivalent in Ω.*

Observe that Corollary 5.10 establishes a bi-Lipschitz equivalence of the Hurwitz density η_Ω and the density $\overline{\eta}_\Omega$ on hyperbolic domains. However, in our next theorem we provide a sharper bound whenever domain is hyperbolic with connected boundary.

As an application to the Riemann mapping theorem, it is easy to see that simply connected proper domains of $\mathbb{C}$ have connected boundaries. However, there exist a non-simply connected hyperbolic domain with connected boundary. For instance the exterior of the unit disk $\Omega = \{w \in \mathbb{C} : |w| > 1\}$ is a non-simply connected hyperbolic domain whose boundary $\partial\Omega = \partial\mathbb{D} = \{w \in \mathbb{C} : |w| = 1\}$ is a connected set.

Theorem 5.12. *Let Ω be a hyperbolic domain with connected boundary, then for every $w \in \Omega$ we have*

$$\overline{\eta}_\Omega(w) \leq \eta_\Omega(w) \leq \frac{K}{4}\overline{\eta}_\Omega(w),$$

where $K = 1/(2\lambda_{\mathbb{C}\setminus\{0,1\}}(-1)) \approx 4.3859$.

Proof. Choose a point $w_1 \in \partial\Omega$ such that the Euclidean distance of w to the boundary of Ω is attained for the point w_1. Consider the circle $\Gamma = \{z \in \mathbb{C} : |z - w_1| = |w - w_1|\}$. Since boundary of Ω is connected, there exists a point $w_2 \in \partial\Omega \cap \Gamma$. Since the function $h(w) = (w - w_1)/(w_2 - w_1)$ from $\mathbb{C} \setminus \{w_1, w_2\}$ onto $\mathbb{C} \setminus \{0, 1\}$ is a Möbius transformation, it is an infinitesimal isometry of the $\overline{\eta}$ density. As a consequence, we have

$$\overline{\eta}_\Omega(w) = |h'(w)|\overline{\eta}_{h(\Omega)}(h(w)) = \frac{1}{|w_2 - w_1|}\overline{\eta}_{h(\Omega)}\left(\frac{w - w_1}{w_2 - w_1}\right).$$

Since on hyperbolic domains, the Hurwitz density exceeds over the hyperbolic density, we have

$$\begin{aligned}
\overline{\eta}_\Omega(w) &\geq \frac{1}{|w_2 - w_1|}\overline{\lambda}_{h(\Omega)}\left(\frac{w - w_1}{w_2 - w_1}\right) \\
&\geq \frac{1}{|w_2 - w_1|}\frac{1}{\left(2\left|\dfrac{w - w_1}{w_2 - w_1}\log\left|\dfrac{w - w_1}{w_2 - w_1}\right|\right| + 2K\left|\dfrac{w - w_1}{w_2 - w_1}\right|\right)},
\end{aligned}$$

where the last inequality follows from [16, Theorem 2]. Since $w_2 \in \Gamma$, it follows that

$$\overline{\eta}_\Omega(w) \geq \frac{1}{2K|w - w_1|}.$$

The desired result follows from the inequality $1/8|w - w_1| \geq \eta_\Omega$. $\square$

CHAPTER 6

SUMMARY AND FUTURE DIRECTIONS

Chapter 1 provides some introductory discussion along with some fundamental definitions in hyperbolic geometry. We studied the Hurwitz metric in the sense of Kobayashi and the Carathéodory densities in Chapter 2 and Chapter 3. Various important properties like distance decreasing property, infinitesimal isometry, comparison with other well known metrics etc. are discussed. We also characterised the domains where these density functions agree with the classical Hurwitz density. In Chapter 4, we obtained the lower and upper bounds for the quotients of the Hurwitz metrics on hyperbolically covered domains. Further, we establish the local uniform convergence of the Hurwitz metric whenever the corresponding sequence of hyperbolically covered domains converges to a hyperbolically covered domains in the sense of Carathéodory. We discussed the continuity of the Hurwitz metric, Gardiner-Lakic version of the Hurwitz metric in the fifth chapter.

Note that the explicit formula of the Hurwitz metric is only known for the punctured plane and some simply connected domains. Due to non-trivial nature of the Hurwitz covering map for most of the domains, it is difficult to compute the Hurwitz metric explicitly. Therefore, to know its geodesics is also challenging in its own way. We are further looking forward to obtain the isometries of the distance function induced from the Hurwitz metric. The Apollonian metric [3, 18], the Seittenranta metric [41, 48], the triangular ratio metric [22, 47], the visual angle metric [29, 53], the Cassinian metric [19, 23, 30] are some well known metrics in $\mathbb{R}^n$. We expect that these metrics may have connection with the Hurwitz metric on plane domains.

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