PROPERTIES OF CERTAIN COMPLEX INTEGRAL OPERATORS

Ph.D. Thesis

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DEPARTMENT OF MATHEMATICS
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PROPERTIES OF CERTAIN COMPLEX INTEGRAL OPERATORS

A THESIS

Submitted in partial fulfillment of the
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by

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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **PROPERTIES OF CERTAIN COMPLEX INTEGRAL OPERATORS** in the partial fulfillment of the requirements for the award of the degree of **DOCTOR OF PHILOSOPHY** and submitted in the **DEPARTMENT OF MATHEMATICS, Indian Institute of Technology Indore,** is an authentic record of my own work carried out during the time period from December 2016 to May 2021 under the supervision of Dr. Swadesh Kumar Sahoo, Associate Professor, Indian Institute of Technology Indore.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other institute.

Shankey
26-05-2021
Signature of the student with date

(Shankey Kumar)

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

26/5/2021
Signature of Thesis Supervisor with date

(Dr. Swadesh Kumar Sahoo)

Shankey Kumar has successfully given his Ph.D. Oral Examination held on **October 08, 2021**.

8/10/2021
Signature of Thesis Supervisor with date

(Dr. Swadesh Kumar Sahoo)

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ABSTRACT

Keywords: Alexander operator, α-Bloch space, β-Cesàro operator, Bohr inequality, Boundary rotation, Bounded operator, Cesàro operator, Close-to-Convex functions, Compact operator, Convex functions, Essential norm, Hornich operations, Integral operator, Linear-invariant family, Locally univalent functions, Pre-Schwarzian norm, Radius of convexity, Separable space, Spectrum, Spirallike functions, Starlike and Univalent.

The work of the whole thesis is based on the study of certain types of complex integral operators over analytic function spaces. These operators are obtained through the well-known Hornich operations. These Hornich operations are frequently studied in Univalent function theory. In **Chapter 1** we will have a descriptive note on these operations. This chapter also provides basic definitions of function spaces, properties, and some results which are useful in later chapters. One of the operators, namely β-Cesàro operator, which we can obtain through the well-known Alexander operator and Hornich operations, which are studied over α-Bloch spaces in **Chapter 2**. In this chapter, we study the boundedness and compactness of β-Cesàro operators. Moreover, with the help of the compactness property, we found the complete spectrum of these operators. We also have the Taylor series expansion of β-Cesàro operators acting over bounded analytic functions. Therefore, we studied the Bohr phenomenon for the corresponding series representation of β-Cesàro operators in **Chapter 3** and similarly for other well-known integral operators. In **Chapter 4** we have remarked on an open problem related to the univalency of the Hornich operations. Further, we establish the univalence properties of β-Cesàro operators. Moreover, we calculated the Pre-Schwarzian norm of β-Cesàro operators over the class of univalent functions. In this sequence, in **Chapters 5** and **6** we study a more general operator which we obtain by the combination of Hornich operations and the Alexander operator. In **Chapters 5** we find a subdisk of the unit disk such that the image of the

subdisk under the integral operator is convex. In addition, we determine certain geometric properties such as convexity and close-to-convexity of the integral operators in **Chapter 6**.

TABLE OF CONTENTS

LIST OF TABLES

NOTATION

$\mathcal{A}$	class of normalized analytic functions in $\mathbb{D}$
$A \subset B$	A is a subset of B
$A \subsetneq B$	A is a proper subset of B
$\mathcal{B}$	class of bounded analytic functions
$\mathfrak{B}_\alpha$	α-Bloch space
$\mathbb{C}$	complex plane
$\mathcal{C}$	class of close-to-convex functions
$\mathbb{D}$	unit disk
$\mathbb{D}_r$	disk of radius r $(z : \lvert z\rvert < r,\ 0 < r \leq 1)$
$f \prec g$	f is subordinate to g
$\mathcal{H}$	class of analytic functions in $\mathbb{D}$
$\operatorname{Im} z$	imaginary part of z
$\mathcal{K}$	class of convex functions
$\mathcal{K}(\lambda)$	class of convex functions of order $\lambda,\ 0 \leq \lambda < 1$
$\operatorname{Log}(z)$	the principal value of the logarithmic function $\log z$ for $z \neq 0$
$\mathcal{S}$	class of univalent functions
$\mathcal{S}^*$	class of starlike functions
$\mathcal{S}^*(\lambda)$	class of starlike functions of order $\lambda,\ 0 \leq \lambda < 1$
$\mathcal{S}^*_\alpha(\lambda)$	class of α-spirallike functions of order $\lambda,\ 0 \leq \lambda < 1$
$\mathbb{N}$	set of natural numbers
$\mathbb{R}$	set of real numbers
$\operatorname{Re} z$	real part of z
$\mathcal{U}_\delta$	the universal linear-invariant family of order $\delta \geq 1$
$\mathcal{V}_k$	class of functions of bounded boundary rotation bounded by $k\pi$

Greek Symbols

Γ	gamma function

CHAPTER 1

INTRODUCTION

1.1. Motivation

The theory of integral operators is a source of all modern functional analysis. Integral operators are studied for various prospective, just name a few the boundedness, compactness and spectral properties of an integral operator are most important. Spectral theory has numerous applications in many parts of mathematics and physics including matrix theory, function theory, complex analysis, differential and integral equations, control theory and quantum physics. The eigenvalue problem for an integral equation is a special case of the spectral theory of linear operators. Also, the theory of integral equations is the origin of the theory of compact operators, where integral operators supply concrete examples of such operators. A typical Fredholm integral equation gives rise to a compact operator on function spaces. Compact operators are closely resembling the operators on finite-dimensional spaces. These operators are somewhat similar to the $n \times n$ complex matrices. That's why the study of the compactness of integral operators has been attracted to many researchers.

In 1960, Biernacki [19] proposed a conjecture that the Alexander operator maps a univalent function to a univalent function. Later, Krzyź and Lewandowski [57] disproved this conjecture by providing a counterexample. In the direction of proving the univalency of analytic functions, Nehari's theorem gives a sufficient condition in terms of the Schwarzian derivative. This theorem can also be applied to find a bound on scalars such that the Hornich scalar multiplication operator is univalent in that range. In [28], authors gave a bound $(\sqrt{5}-2)/3$ on scalars under which the Hornich scalar multiplication operator preserves the univalency. Becker [14] improved it to $1/6$. Further, Pfaltzgraff [79] made an improvement to $1/4$. In this sequence, Royster [96] found some scalars for which the Hornich scalar multiplication operator does not preserve the univalency. But still exact region of scalars which leads to the univalency of the Hornich scalar multiplication operator is unknown. These studies motivate many researchers to find the geometric properties of complex integral operators over various functions spaces.

The objective of this chapter includes the definitions of integral operators associated with the Hornich operations and analytic function spaces followed by the structure of this thesis. We begin with the definitions of some standard function spaces of analytic functions defined on the unit disk.

1.2. Locally univalent function spaces

Let $\mathcal{A}$ denote the class of all analytic functions f in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ with the normalization $f(0) = 0$ and $f'(0) = 1$. If an analytic function f is one-one in a domain then f is said to be *univalent* in that domain. The subclass of $\mathcal{A}$ consisting of all *univalent functions* is denoted by $\mathcal{S}$. An analytic function f is said to be *locally univalent* at a point z_0 if it is univalent in some neighborhood of z_0. It is well-known that if f is analytic on a domain, then $f'(z_0) \neq 0$ if and only if f is locally univalent at z_0. We denote by $\mathfrak{F}$ the class of all those functions $f \in \mathcal{A}$ satisfying $f'(z) \neq 0, z \in \mathbb{D}$.

A domain $D \subset \mathbb{C}$ is said to be *starlike* with respect to a point $z_0 \in D$ if the line segment joining z_0 to every other point $z \in D$ lies entirely in D. If $f(\mathbb{D})$, $f \in \mathcal{A}$, is a starlike domain with respect to origin then f said to be a *starlike function*. A natural generalization of starlike domain is a convex domain. If a domain D is starlike with respect to every point $z \in D$ then it becomes *convex domain*. Similiar to the definition of starlike function if $f(\mathbb{D})$, $f \in \mathcal{A}$, is a convex domain then f is a *convex function*. The notations $\mathcal{S}^*$ and $\mathcal{K}$ stand for the well-known classes of functions in $\mathcal{S}$ that are starlike (with respect to origin) and convex, respectively, see [27]. Analytically, both the classes $\mathcal{S}^*$ and $\mathcal{K}$ can be characterized as

$$\mathcal{S}^* = \left\{ f \in \mathcal{A} : \mathrm{Re}\, \frac{zf'(z)}{f(z)} > 0 \right\}$$

and

$$\mathcal{K} = \left\{ f \in \mathcal{A} : 1 + \mathrm{Re}\, \frac{zf''(z)}{f'(z)} > 0 \right\}.$$

Note that the Koebe functiion $z/(1-z)^2$ is starlike but not convex. Thus $\mathcal{K} \subset \mathcal{S}^*$. Moreover, the Alexander theorem gives that: a function $f \in \mathcal{K}$ if and only if $zf'(z) \in \mathcal{S}^*$.

A function $f \in \mathcal{S}$ is said to be *close-to-convex* if there is a function $g \in \mathcal{K}$ and a real number $\alpha \in (-\pi/2, \pi/2)$ such that

$$\mathrm{Re}\left(e^{i\alpha}\frac{f'(z)}{g'(z)}\right) > 0, \;\; z \in \mathbb{D},$$

see [32, Vol. 2, p. 2]. The notation $\mathcal{C}$ stands for the class of close-to-convex functions. By the definition, it is clear that $\mathcal{K} \subsetneq \mathcal{C}$. Indeed, each $f \in \mathcal{S}^*$ has the form $f(z) = zg'(z)$ for some $g \in \mathcal{K}$. It leads to the fact that every starlike function is close-to-convex. In 1952, Kaplan in [40] proved that a function $f \in \mathfrak{F}$ is close-to-convex if and only if

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\left(\frac{zf''(z)}{f'(z)} + 1\right) d\theta > -\pi, \quad z = re^{i\theta}, \tag{1.1}$$

for each $0 < r < 1$ and for each pair of real numbers θ_1 and θ_2 with $\theta_1 < \theta_2$; see [27,32,72] for more information. In this sequence, we have some other subclasses of $\mathcal{A}$, which were widely used by many authors for different prospective. The class of α-spirallike functions of order λ defined as follows:

$$\mathcal{S}_\alpha^*(\lambda) := \left\{ f \in \mathcal{A} : \mathrm{Re}\left(e^{i\alpha}\frac{zf'(z)}{f(z)}\right) > \lambda \cos\alpha \right\}, \tag{1.2}$$

and the class of convex functions of order λ is

$$\mathcal{K}(\lambda) := \left\{ f \in \mathcal{A} : \mathrm{Re}\,\frac{zf''(z)}{f'(z)} + 1 > \lambda \right\}, \tag{1.3}$$

where $\alpha \in (-\pi/2, \pi/2)$ and $\lambda < 1$. Recall that the class $\mathcal{S}_\alpha^*(\lambda)$, for $0 \le \lambda < 1$, is studied by several authors for different purposes (see, for instance, [32, p 93, Vol. 2] and [63,91]). Further, the class $\mathcal{K}(\lambda)$, $-1/2 \le \lambda < 1$, is introduced, for instance, in [62] and references therein. Originally, a slight modification of this class was first studied by Umezawa in 1952 [105] by characterizing with the class of functions convex in one direction. We can also easily observe that the class $\mathcal{K}(\lambda)$, $-1/2 \le \lambda < 1$, is contained in the class $\mathcal{C}$ that follows from Kaplan's Theorem, see [27, §2.6]. Note that $\mathcal{K}(\lambda)$, $0 \le \lambda < 1$, is the well-known class of normalized convex univalent functions.

Recall from the literature that

$$\mathcal{S}^*(\lambda) := \mathcal{S}_0^*(\lambda), \; \mathcal{S}_\alpha^* := \mathcal{S}_\alpha^*(0), \; \mathcal{S}^* := \mathcal{S}^*(0) \;\; \text{and} \;\; \mathcal{K} := \mathcal{K}(0).$$

Motivation to consider the class $\mathcal{S}^*(\lambda)$, $\lambda < 1$, comes, for instance, from the classes $\mathcal{S}^*(-1/2)$ and $\mathcal{K}(-1/2)$ already studied in the literature (see [72, p. 66] for some interesting results). Some of the main results in this thesis deal with certain generalizations of the classes $\mathcal{S}^*$, $\mathcal{K}$ and $\mathcal{C}$ in terms of subordination. So, we recall the definition of subordination here.

For f, $g \in \mathcal{A}$, we say that f is subordinate to g (symbolically we write $f \prec g$) if there is an analytic function $w : \mathbb{D} \to \mathbb{D}$ with $w(0) = 0$ such that $f = g \circ w$. Note that if g is univalent then the condition $f \prec g$ is equivalent to the conditions $f(0) = g(0)$ and $\{f(z) : |z| < r\} \subset \{g(z) : |z| < r\}$, $r \leq 1$. To know more about subordination, reader can refer to [27, 72].

For two distinct complex numbers A and B, we consider the classes defined by

$$\mathcal{S}^*(A, B) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz} \right\} \tag{1.4}$$

and

$$\mathcal{K}(A, B) = \left\{ f \in \mathcal{A} : zf'(z) \in \mathcal{S}^*(A, B) \right\}. \tag{1.5}$$

It is simple to check that the function $(1 + Az)/(1 + Bz)$ is univalent, for $A \neq B$. We can easily see that $f \in \mathcal{K}(A, B)$ if and only if $zf'(z) \in \mathcal{S}^*(A, B)$. For the real numbers A and B satisfying $-1 \leq B < A \leq 1$, these classes are widely used in the literature (see for instance [39, 85]). Note that, for $0 \leq \lambda < 1$ and $-\pi/2 < \alpha < \pi/2$, $\mathcal{S}^*_\alpha(\lambda) = \mathcal{S}^*((1 - \lambda)e^{2i\alpha} - \lambda, -1)$ and $\mathcal{K}(\lambda) = \mathcal{K}(1 - 2\lambda, -1)$ describe the classes of α-spirallike functions of order λ and convex functions of order λ, respectively. Moreover, for $\gamma > 0$, if we choose $A = 1 + \gamma$ and $B = 1$ in (1.5) then we have the following well-known class:

$$\mathcal{G}(\gamma) = \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} < 1 + \frac{\gamma}{2} \right\}.$$

The class $\mathcal{G} := \mathcal{G}(1)$ was first introduced by Ozaki [76] and proved the inclusion relation $\mathcal{G} \subset \mathcal{S}$. The Taylor coefficient problem for the class $\mathcal{G}(\gamma)$, $0 < \gamma \leq 1$, is discussed in [75]. Recently, the radius of convexity of the functions in the class $\mathcal{G}(\gamma)$, $\gamma > 0$, is obtained in [60]. More information about the class $\mathcal{G}(\gamma)$, $\gamma > 0$, can be found in [84, 87, 88].

1.3. α-Bloch space

For each $\alpha > 0$, the *α-Bloch space* [108] of $\mathbb{D}$, denoted by $\mathfrak{B}_\alpha$, consists of analytic functions f on $\mathbb{D}$ such that

$$\sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f'(z)| < +\infty.$$

The space $\mathfrak{B}_\alpha$ is a complex Banach space with the norm

$$\|f\| = |f(0)| + \|f\|_{\mathfrak{B}_\alpha}, \tag{1.6}$$

whereas $\|f\|_{\mathfrak{B}_\alpha} = \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f'(z)|$ represents a semi-norm. The proof of this follows from the proof of Proposition 2.5 of [35]. If we restrict this space with the condition $f(0) = 0$, for $f \in \mathfrak{B}_\alpha$, this restricted space is a subspace of $\mathfrak{B}_\alpha$, denoted by $\mathfrak{B}^0_\alpha$. The semi-norm $\|.\|_{\mathfrak{B}_\alpha}$ on $\mathfrak{B}_\alpha$ becomes norm on $\mathfrak{B}^0_\alpha$. We observe that $\mathfrak{B}^0_\alpha$ is a Banach space with norm $\|.\|_{\mathfrak{B}_\alpha}$ and proof of this is explained in Section 2.1. More on literature survey about the 1-Bloch space can be found in [109, 110].

1.4. Integral operators involving Hornich operations

The Hornich operations were originally introduced in [38]. The Hornich operations play a very important role in geometric function theory. The study of univalency of the Hornich operations is one of the main problems for many authors. We present here the Hornich operations and their associated integral operators that are under consideration in this thesis. The *Hornich addition operation* is denoted and defined as

$$(f \oplus g)(z) = \int_0^z f'(w)g'(w)dw,$$

for two given functions $f, g \in \mathfrak{F}$. The *Hornich scalar multiplication operation* for a function $g \in \mathfrak{F}$ is defined as

$$I_\gamma[g](z) := (\gamma \star g)(z) = \int_0^z \{g'(w)\}^\gamma dw, \tag{1.7}$$

where the branch of $\{g'(w)\}^\gamma = \exp(\gamma \log g'(z))$ is chosen so that $\{g'(0)\}^\gamma = 1$. It clearly follows that $I_\alpha I_\beta = I_{\alpha\beta}$. In the sequel, the following definition is due to Kim and Merkes [49] is used in our main results of **Chapter 4**:

$$A(\mathfrak{F}) = \{\gamma \in \mathbb{C} : I_\gamma(\mathfrak{F}) \subset \mathcal{S}\} \tag{1.8}$$

with $\mathfrak{F}$ as defined above. Here, the notation $I_\gamma(\mathfrak{F})$ is defined by

$$I_\gamma(\mathfrak{F}) = \{I_\gamma[f] : f \in \mathfrak{F}\}. \tag{1.9}$$

We say that a function $g \in I_\gamma(\mathfrak{F})$ if and only if $g = I_\gamma[f]$ for some $f \in \mathfrak{F}$. Recall that the inclusion $\{\gamma : |\gamma| \leq 1/2\} \subset A(\mathcal{K})$ was first proved by Singh and Chichra in [99] (see also [51] and [69]). Further, the inclusion $[0, 3/2] \subset A(\mathcal{K})$ was due to Nunokawa

[74]. In continuation to this analysis, in 1985, Merkes [69] proposed the conjecture that $\{\gamma \in \mathbb{C} : |\gamma - 1| \leq 1/2\} \subset A(\mathcal{K})$. However, Aksent'ev and Nezhmetdinov [8] disproved the conjecture of Merkes by showing that

$$A(\mathcal{K}) = \{\gamma \in \mathbb{C} : |\gamma| \leq 1/2\} \cup [1/2, 3/2] \tag{1.10}$$

(see also [51]).

An interesting relation between the classes $\mathcal{S}^*$ and $\mathcal{K}$ is the classical Alexander Theorem which states that $f \in \mathcal{S}^*$ if and only if $J[f] \in \mathcal{K}$, where $J[f]$ denotes the Alexander operator of $f \in \mathcal{A}$ defined as

$$J[f](z) = \int_0^z \frac{f(w)}{w} dw.$$

This operator is one of the main operators we consider in this thesis. We know that the class $\mathcal{S}$ does not preserve by the Alexander operator, see [27, §8.4]. This motivates us to study the classical classes of functions whose images lie on the class $\mathcal{S}$ under the Alexander and related operators considered in **Chapters 4**, **5** and **6**. We use the following notation concerning the Alexander operator J:

$$J(\mathfrak{F}) = \{J[f] : f \in \mathfrak{F}\}. \tag{1.11}$$

We say that a function $g \in J(\mathfrak{F})$ if and only if $g = J[f]$ for some $f \in \mathfrak{F}$.

In [54], Kim and Sugawa find a condition on α such that $J[f]$, $f \in \mathcal{S}_\alpha^*$, is univalent with the help of the problem of determining the set $A(J(\mathcal{S}_\alpha^*))$, where $J(\mathcal{S}_\alpha^*)$ is defined similar to the definition (1.11).

Next we observe that

$$(I_\gamma \circ J)[f](z) = \int_0^z \left(\frac{f(w)}{w}\right)^\gamma dw =: J_\gamma[f](z).$$

It is here appropriate to notice that $J_1[f] = J[f]$. Then by the definitions (1.8) and (1.9) we formulate

$$A\big(J(\mathfrak{F})\big) = \{\gamma \in \mathbb{C} : J_\gamma(\mathfrak{F}) \subset \mathcal{S}\} \ \text{ and } \ J_\gamma(\mathfrak{F}) = (I_\gamma \circ J)(\mathfrak{F}).$$

The operator $J_\gamma[f]$ was initially considered by Kim and Merkes in [49], and they showed that $J_\gamma(\mathcal{S}) \subset \mathcal{S}$ for $|\gamma| \leq 1/4$, i.e. $A(J(\mathcal{S})) = \{\gamma \in \mathbb{C} : |\gamma| \leq 1/4\}$. For the starlike family $\mathcal{S}^*$, Singh and Chichra in [99] proved that $A(J(\mathcal{S}^*)) \supset \{\gamma \in \mathbb{C} : |\gamma| \leq 1/2\}$. However, as noted in (1.10), the complete range of γ for $A(J(\mathcal{S}^*))$ was found by Aksent'ev and

Nezhmetdinov [8], since $J(\mathcal{S}^*) = \mathcal{K}$. More interestingly, for a given $\alpha > 0$, Kim et al. [52] could generate a subclass $\mathcal{F}$ of $\mathcal{A}$ such that $J_\gamma(\mathcal{F}) \subset \mathcal{S}$ for all $\gamma \in \mathbb{C}$ with $|\gamma| \leq \alpha$.

In 1974, Kim and Merkes [50] studied the operator

$$I_{\alpha,\beta}[f,g](z) := \big(I_\alpha[f] \oplus I_\beta[g]\big)(z) = \int_0^z (f'(w))^\alpha (g'(w))^\beta dw, \ \ \alpha, \beta \in \mathbb{R} \text{ and } |z| < 1,$$

defined for $f, g \in \mathfrak{F}$. By the definition of the operator $I_{\alpha,\beta}$ it is clear that this is a combination of the Hornich operations. One of the interesting results obtained in [50] for the operator $I_{\alpha,\beta}$ is the following:

Theorem A. *Let $f, g \in \mathcal{K}$. For the real numbers α and β, we have*

- **(i)** $I_{\alpha,\beta}[f,g] \in \mathcal{K}$ *if and only if* $\alpha \geq 0, \beta \geq 0, \alpha + \beta \leq 1$.
- **(ii)** $I_{\alpha,\beta}[f,g] \in \mathcal{C}$ *if and only if* $-1/2 \leq \alpha, \beta \leq 3/2, -1/2 \leq \alpha + \beta \leq 3/2$.

Theorem A(i) says that if there exist positive α and β satisfying $\alpha + \beta > 1$ or at least one of them is negative, then $I_{\alpha,\beta}[f,g]$ is no more in $\mathcal{K}$. This means that if we replace the term "*if and only if*" with "*if*" in Theorem A(i), then the result would be called sharp. Same concept is applied for similar other results in **Chapters 6**. Further, Theorem A has been extended in [11] by replacing $\mathcal{K}$ with $\mathcal{K}(\lambda)$, $-1/2 \leq \lambda < 1$, and $\mathcal{G}(\gamma)$, $0 < \gamma \leq 1$, separately.

In [50], authors also studied the operator, for $f, g \in \mathfrak{F}$,

$$J_{\alpha,\beta}[f,g](z) := \big(J_\alpha[f] \oplus J_\beta[g]\big)(z) = \int_0^z \left(\frac{f(w)}{w}\right)^\alpha \left(\frac{g(w)}{w}\right)^\beta dw, \ \ \alpha, \beta \in \mathbb{R} \text{ and } |z| < 1.$$

This can be easily generated with the help of the Alexander transformation and the Hornich operations. Corresponding to the operator $J_{\alpha,\beta}$ they have the following result:

Theorem B. *Let $f, g \in \mathcal{K}$. For the real quantities α and β, we have*

- **(i)** $J_{\alpha,\beta}[f,g] \in \mathcal{K}$ *if and only if* $\alpha \geq 0, \beta \geq 0, \alpha + \beta \leq 2$.
- **(ii)** $J_{\alpha,\beta}[f,g] \in \mathcal{C}$ *if and only if* $-1 \leq \alpha, \beta \leq 3, -1 \leq \alpha + \beta \leq 3$.

Similar to the operators $I_{\alpha,\beta}$ and $J_{\alpha,\beta}$, in **Chapters 5** and **6** we study a generalized integral operator which is denoted by $C_{\alpha,\beta}$ and defined as

$$C_{\alpha,\beta}[f,g](z) = \int_0^z \left(\frac{f(w)}{w}\right)^\alpha (g'(w))^\beta dw, \ \ \alpha, \beta \in \mathbb{R}, \ \ |z| < 1. \tag{1.12}$$

This can be obtained by replacing $f'(w)$ with $(J[f])'(w)$ in $I_{\alpha,\beta}[f,g]$ or $(J[g])'(w)$ with $g'(w)$ in $J_{\alpha,\beta}[f,g]$. Here, we choose branches of $(f(z)/z)^{\alpha}$ and $(g'(w))^{\beta}$ such that $(f'(0))^{\alpha} = 1 = (g'(0))^{\beta}$. In other words, the above operators are related by

$$C_{\alpha,0}[f,g] \equiv J_{\alpha,0}[f,g] \equiv C_{\alpha,\beta}[f,z] \equiv J_{\alpha,\beta}[f,z] \tag{1.13}$$

and

$$C_{0,\beta}[f,g] \equiv I_{0,\beta}[f,g] \equiv C_{\alpha,\beta}[z,g] \equiv I_{\alpha,\beta}[z,g]. \tag{1.14}$$

The operator $C_{\alpha,\beta}$ can easily be obtained by the Hornich sum of the operators J_{α} and I_{β} as

$$C_{\alpha,\beta}[f,g](z) = \big(J_{\alpha}[f] \oplus I_{\beta}[g]\big)(z).$$

The operator $C_{\alpha,\beta}$ contains several well-known operators, simultaneously. Also, we can obtain many known results with the help of this operator $C_{\alpha,\beta}$. For $f = g$, certain geometric properties of $C_{\alpha,\beta}$ have been studied in [26, 30–32].

The β-Cesàro operator is defined by

$$C_{\beta}[f](z) = \int_0^z \frac{f(w)}{w(1-w)^{\beta}}dw, \qquad \text{for } \beta \in \mathbb{R}, \tag{1.15}$$

where f is analytic in $\mathbb{D}$ and $f(0) = 0$. One can express the β-Cesàro operator in terms of the Hornich operations by writing

$$C_{\beta}[f](z) = \big(J[f] \oplus (\beta \star g)\big)(z),$$

where $g(z) = -\log(1-z) \in \mathcal{K}$. Note that the β-Cesàro operator reduces to the Alexander operator if we choose $\beta = 0$ and to the Cesàro operator [37] if we choose $\beta = 1$. We use the notation $C[f] := C_1[f]$ for the Cesàro operator. For more information about the β-Cesàro operator, see [59]. Here it is appropriate to recall that Hartmann and MacGregor in the same paper [37] provided examples of a univalent function and a starlike function whose images are not univalent and starlike, respectively, under the Cesàro operator. Recently, Ponnusamy et al. [86] studied the univalency of the Cesàro operator and even more general operators of functions of bounded boundary rotations. Moreover, boundedness of the Cesàro and related operators in various function spaces is studied in the literature; see [25, 71, 101, 106]. We can further generalize this operator, if we replace $(1-w)^{-\beta}$ by

$$g_{\beta}(w) = \sum_{j=1}^{k} \frac{a_j}{(1-b_j w)^{\beta}} + h(w) \tag{1.16}$$

in (1.15), where b_j, $1 \leq j \leq k$, are distinct points on the unit circle, $|a_j| > 0$, for all j, and h is bounded analytic function in $\mathbb{D}$. We call this operator as *generalized β-Cesàro operator* because for $b_j = a_j = k = 1$ and $h = 0$, we obtain the β-Cesàro operator. The generalized β-Cesàro operator denoted by C_{g_β}. For the choices $\beta = 0$ and $\beta = 1$ in (1.16), the generalized β- Cesàro operators are respectively called the *generalized Alexander operator* and the *generalized Cesàro operator*.

Many authors studied integral operators on analytic function spaces. For instance, Stevic studied the compactness and essential norm of the integral type operator

$$P_\varphi^g(f)(z) = \int_0^1 f(\varphi(tz))g(tz)\frac{dt}{t}$$

where g is an analytic function in $\mathbb{D}$, $g(0) = 0$ and φ is a holomorphic self-map of $\mathbb{D}$, acting on Bloch-type spaces, see [103, 104]. Secondly, in [6], boundedness of generalized Cesàro averaging operators on certain function spaces are investigated. These operators are very similar to our operator C_{g_β}, but they do not simultaneously include the Alexander operator as well as the Cesàro operator.

1.5. Bohr's Phenomenon

Let $\mathcal{H}$ be the class of all analytic functions defined on $\mathbb{D}$. We set $\mathcal{B} = \{f \in \mathcal{H} : |f(z)| \leq 1\}$. Let us first highlight a remarkable result of Bohr [20] that opens up a new type of research problems in geometric function theory, which states that "*If $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{B}$, then*

$$\sum_{n=0}^{\infty} |a_n| r^n \leq 1,$$

for $r \leq 1/3$ and the constant $1/3$ cannot be improved." As noted in the same paper, Bohr actually proved this result for the radius 1/6, later this result was improved in its final form by Riesz, Schhur and Wiener. The quantity 1/3 is known as the *Bohr radius* for the class $\mathcal{B}$. Moreover, for functions in $\mathcal{B}_0 := \{f \in \mathcal{B} \mid f(0) = 0\}$, Bombieri [22] found the Bohr radius, which is $1/\sqrt{2}$ (for more generalization of this result see [92]). These are not the only classes of the analytic functions where the Bohr radii are studied. Also, for many other classes of functions and for some integral operators the Bohr radii are studied. Some of those are highlighted below. In fact an interesting application of the Bohr radius problem for the class $\mathcal{B}$ can be found in [89].

In [10], Ali et al. brought into the notice of the problem of Bohr for odd analytic functions, which is settled by Kayumov and Ponnusamy in [44]. Also, Kayumov and Ponnusamy [46] generalized the problem of the Bohr radius for the odd analytic functions. Bhowmik and Das [16] studied the Bohr radius for families of certain analytic univalent (one-to-one) functions. In [15], the Bohr phenomenon is discussed for the functions in Hardy spaces. The study of the Bohr radius for the Bloch functions is discussed in [48]. The authors of [12, 90] studied the Bohr phenomenon for a quasi-subordination family of functions. Recently, Bhowmik and Das [18] studied the Bohr radius for derivatives of analytic functions. To find more achievements in this context, one may see the papers [1–4, 17, 45, 47, 65–68] and the references therein. Also, the survey article [5] and the references cited in it are useful in this direction.

A natural question arises *"can we find Bohr radius for certain complex integral operators defined either on the class $\mathcal{B}$ or $\mathcal{B}_0$?"*. This idea has been initiated first for the classical Cesáro operator in [42]. As our results of this thesis are motivated by [42], here first we recall the definition of the Cesáro operator followed by the statement of the result on absolute sum of the series representation of the operator. The Cesáro operator is studied in [36] (see, for more information, [100] and [101]) and defined as

$$T[f](z) := \int_0^1 \frac{f(tz)}{1-tz}dt = \sum_{n=0}^{\infty}\left(\frac{1}{n+1}\sum_{k=0}^{n} a_k\right) z^n, \tag{1.17}$$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic in $\mathbb{D}$. Also, a generalized form of the Cesáro operator is studied in [6].

As noted in [42],

$$|T[f](z)| \leq \frac{1}{r}\log\frac{1}{1-r}$$

for each $|z| = r < 1$. On the other hand, from (1.17), we also have the obvious estimate

$$|T[f](z)| \leq \sum_{n=0}^{\infty}\left(\frac{1}{n+1}\sum_{k=0}^{n}|a_k|\right)|z|^n,$$

the absolute sum of the series (1.17). However, if $|z| = r < 1$, Kayumov et al. [42] obtained the sharp radius r for which this absolute sum has the same upper bound $(1/r)\log(1/(1-r))$. This was important to study, as in general, a convergent series need not be absolutely convergent. Indeed, they established

Theorem C. *If $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{B}$, then*

$$\sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{k=0}^{n} |a_k| \right) r^n \leq \frac{1}{r} \log \frac{1}{1-r}$$

for $r \leq R = 0.5335\ldots$. Here the number R is the positive root of the equation

$$2x - 3(1-x) \log \frac{1}{1-x} = 0$$

that cannot be improved.

1.6. Radius of convexity

Since all analytic univalent functions defined in $\mathbb{D}$ are not necessarily convex, it was important in the literature to find the largest subdisk $|z| \leq r$, of $\mathbb{D}$, whose image is a convex domain under analytic univalent functions (see [27, Theorem 2.13]). Here, the number r is called the *radius of convexity* for the univalent functions. Later, radius of convexity was studied for several other classical classes of functions different from the class of convex functions. For instance, the result on the radius of convexity for functions belonging to the linear invariant family was obtained by Pommerenke in 1964 (see [83]) whereas its proof was discussed in [33, Theorem 5.2.3]. Also, in 1969, Pinchuk [82] studied the radius of convexity for functions with bounded boundary rotations. Kargar in [41] finds the radius of convexity for a Volterra-type integral operator. The radii of convexity for certain integral operators are also studied by Najmadi et al. in [73]. Thus, the problem of finding the radius of convexity has been an important problem in geometric function theory and it attracted to many function theorists. Therefore, in **Chapter 5**, our objective in this prospective is to consider an integral operator, involving the classical Hornich operations, which generalizes even certain well-known integral operators having special attention in function theory. Moreover, we obtain the best radii of convexity in our problems under consideration although the operator looks complicated or more general in nature.

1.7. Structure of the thesis

Chapter 1 of this thesis includes the preliminaries and basic definitions, which help to understand the remaining chapters of this thesis. The whole work of research is contained in **Chapters 2-6**. In **Chapter 2** we study β-Cesàro operators as linear operators on α-Bloch space. Therefore, **Chapter 2** describes boundedness and compactness property of the β-Cesàro operators. With the help of the compactness property we have the complete spectrum of the β-Cesàro operators. We present here the following two results related to spectrum.

Theorem 1.1. *The spectrum of the generalized Alexander operator from $\mathfrak{B}_\alpha^0$ to $\mathfrak{B}_\alpha^0$ is*

$$\sigma(C_{g_0}) = \left\{\frac{g_0(0)}{n}, n \in \mathbb{N}\right\} \bigcup \left\{0\right\},$$

where 0 *is the approximate eigenvalue.*

Theorem 1.2. *The spectrum of the generalized β-Cesàro operator from $\mathfrak{B}_\alpha^0$ to $\mathfrak{B}_\alpha^0$, either for $0 < \beta \leq \alpha < 1$ or $\beta \leq 1 < \alpha$ or $\beta < \alpha = 1$, is*

$$\sigma(C_{g_\beta}) = \left\{\frac{g_\beta(0)}{n}, n \in \mathbb{N}\right\} \bigcup \left\{0\right\}.$$

As a motivation to Theorem C, in **Chapter 3** we study the Bohr radius problem for the *β-Cesáro operator* ($\beta > 0$) defined by

$$T_\beta[f](z) := \sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{k=0}^{n} \frac{\Gamma(n-k+\beta)}{\Gamma(n-k+1)\Gamma(\beta)} a_k\right) z^n = \int_0^1 \frac{f(tz)}{(1-tz)^\beta} dt, \ \ z \in \mathbb{D},$$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n$, and for the Bernardi operator defined as

$$L_\gamma[g](z) := \sum_{n=m}^{\infty} \frac{a_n}{n+\gamma} z^n = \int_0^1 g(zt) t^{\gamma-1} dt,$$

for $g(z) = \sum_{n=m}^{\infty} a_n z^n$ and $\gamma > -m$, here $m \geq 0$ is an integer. With the help of the Bernardi operator we also obtain the Bohr radii for some known operators. One of the results provided in **Chapter 3** is the following:

Theorem 1.3. *For $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{B}$ and $0 < \beta \neq 1$, we have*

$$\sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{k=0}^{n} \frac{\Gamma(n-k+\beta)}{\Gamma(n-k+1)\Gamma(\beta)} |a_k|\right) r^n \leq \frac{1}{r} \left[\frac{1-(1-r)^{1-\beta}}{1-\beta}\right],$$

for $r \leq R(\beta)$, where $R(\beta)$ is the positive root of the equation

$$\frac{3[1-(1-x)^{1-\beta}]}{1-\beta} - \frac{2[(1-x)^{-\beta}-1]}{\beta} = 0.$$

The radius $R(\beta)$ cannot be improved.

Finding the set $A(\mathcal{S})$ is one of the classical problems which has been investigated by many authors. In the sequel, **Chapter 4** gives a shape of the set $A(\mathcal{K}(\lambda))$, when $\lambda < 1$. The statement of this result is the following:

Theorem 1.4. *Let $\lambda < 1$. Then we have*

$$A(\mathcal{K}(\lambda)) = \left\{\gamma \in \mathbb{C} : \, |\gamma| \leq \frac{1}{2(1-\lambda)}\right\} \bigcup \left[\frac{1}{2(1-\lambda)}, \frac{3}{2(1-\lambda)}\right].$$

Also, this chapter includes the following relation between the Alexander operator and the Hornich scalar multiplication operator.

Lemma 1.5. *For $-\pi/2 < \alpha < \pi/2$ and $\lambda < 1$, we have*

$$J(\mathcal{S}_\alpha^*(\lambda)) = I_{e^{-i\alpha}\cos\alpha}(\mathcal{K}(\lambda)).$$

Next we investigate the univalency and preservation property of certain classes of functions under the β-Cesàro operator. Further, the following theorem says that univalency is not preserved under the β-Cesàro operator.

Theorem 1.6. *There exists a function $f \in \mathcal{S}$ such that $C_\beta[f]$ does not belong to $\mathcal{S}$ for $\beta \geq 0$.*

The well known Alexander theorem gives the transformation of the starlike domain into the convex domain and vice versa. But for every scalar, both of the operators J_α and I_β over various subclasses of the class $\mathcal{S}$ do not necessarily map $\mathbb{D}$ onto convex domains. So here, it is natural to ask a question that for what largest subdisk of the open unit disk in which the integral operators such as J_α and I_β of the class $\mathcal{A}$ map onto convex domains. Therefore, in **Chapter 5** we study the radius of convexity for a generalized integral operator $C_{\alpha,\beta}$, which includes both of the operators J_α and I_β. One of the main results we present here.

Theorem 1.7. *For two distinct complex numbers A and B with $|B| \leq 1$, let $f \in \mathcal{S}^*(A,B)$ and $g \in \mathcal{K}(A,B)$. Then $C_{\alpha,\beta}[f,g]$ is convex in $|z| < \min\{r_c(A,B,\alpha,\beta), 1\}$. Moreover, if $B \neq 0$ then the radius is obtained by*

$$r_c(A,B,\alpha,\beta) = \begin{cases} \dfrac{|\alpha+\beta||B-A| - |(\alpha+\beta)(B-A) - 2B|}{2\{(\alpha+\beta)\mathrm{Re}\,[(B-A)\overline{B}] - |B|^2\}}, & \text{if } \alpha\beta \geq 0, \\ \dfrac{|\alpha-\beta||B-A| - \sqrt{\xi}}{2\{(\alpha+\beta)\mathrm{Re}\,[(B-A)\overline{B}] - |B|^2\}}, & \text{if } \alpha\beta \leq 0, \end{cases}$$

with ξ *defined by*

$$\xi := |(\alpha+\beta)(B-A)-2B|^2 - 4\alpha\beta|B-A|^2, \tag{1.18}$$

for $(\alpha+\beta)\mathrm{Re}\,[(B-A)\overline{B}]-|B|^2 \neq 0$*, otherwise*

$$r_c(A,B,\alpha,\beta) = \begin{cases} \dfrac{1}{|\alpha+\beta||A-B|}, & \text{if } \alpha\beta \geq 0, \\ \dfrac{1}{|\alpha-\beta||A-B|}, & \text{if } \alpha\beta \leq 0, \end{cases}$$

and if $B=0$*, then the radius becomes*

$$r_c(A,\alpha,\beta) := r_c(A,0,\alpha,\beta) = \begin{cases} \dfrac{1}{|\alpha+\beta||A|}, & \text{if } \alpha\beta \geq 0 \text{ and } \beta \neq 0, \\ \dfrac{1}{|\alpha-\beta||A|}, & \text{if } \alpha\beta \leq 0 \text{ and } \alpha \neq 0. \end{cases}$$

These quantities are best possible for real numbers A *and* B*.*

Now similar to Theorems A and B, instead of finding subdisk $\mathbb{D}_r$, in **Chapter 6** we find the values of scalars α and β for which the operator $C_{\alpha,\beta}$, over various subclasses of the class $\mathfrak{F}$, has some nice geometric properties like convexity and close-to-convexity. One of the main results we obtain for the operator $C_{\alpha,\beta}$ is the following:

Theorem 1.8. *Let* $f, g \in \mathcal{K}$*. Then* $C_{\alpha,\beta}[f,g] \in \mathcal{K}$ *if and only if* $0 \leq \alpha, 2\beta, \alpha+2\beta \leq 2$*.*

At the end, **Chapter 7** gives the concluding remarks on the whole work done and also has future planning.

CHAPTER 2

β-CESÁRO OPERATORS ON α-BLOCH SPACE

The main motive of this chapter[1] is to study spectral properties of generalized β-Cesàro operators on $\mathfrak{B}_\alpha^0$. In this scenario, Section 2.1 contains the boundedness and unboundedness properties of the β-Cesàro operators. Compactness of these operators is studied in Section 2.2. In particular, the essential norm and spectrum are calculated in Section 2.3 and 2.4. Finally, Section 6 conclude this chapter with an application which assures that $\mathfrak{B}_\alpha^0$ is a separable space in the space $\mathfrak{B}_{\alpha+1}^0$, for each $\alpha > 0$. Throughout this chapter we consider $\alpha > 0$ unless it is specified.

2.1. Boundedness of the β-Cesàro operators

In this section, we discuss the boundedness and unboundedness of the β-Cesàro operators, defined by (1.15), on $\mathfrak{B}_\alpha^0$. At the end of this section, we provide some illustrative examples to show that for some β, the β-Cesàro operators are unbounded linear operators on $\mathfrak{B}_\alpha^0$. In Table 2.1, we discuss all restrictions on β for which the β-Cesàro operators are bounded and unbounded. In the sequel, first we describe the completeness property of $\mathfrak{B}_\alpha^0$ under $\|.\|_{\mathfrak{B}_\alpha}$.

Bounded	Unbounded
$\beta \le \alpha < 1$	$\beta > \alpha$ (Example 2.9)
$\alpha > 1 \ge \beta$	$\alpha = \beta \ge 1$ (Example 2.10)
$\beta < \alpha = 1$	$\alpha > \beta > 1$ (Example 2.12)

TABLE 2.1. Boundedness of β-Cesàro operator

Theorem 2.1. *For each $\alpha > 0$, $(\mathfrak{B}_\alpha^0, \|.\|_{\mathfrak{B}_\alpha})$ is a Banach space.*

[1]This chapter is prepared based on the paper: Kumar S., Sahoo S.K. (2020), *Properties of β-Cesàro operators on α-Bloch space*, Rocky Mountain J. Math., **50** (5), 1723–1743..

Proof. We know by [108, Proposition 1] that for each $\alpha > 0$, $\mathfrak{B}_\alpha$ is a Banach space. The only thing we need to show that this subspace is a closed subspace of $\mathfrak{B}_\alpha$. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $(\mathfrak{B}^0_\alpha, \|.\|_{\mathfrak{B}_\alpha})$. Then for $\epsilon > 0$, there exists a $P \in \mathbb{N}$ such that

$$\|f_n - f_m\|_{\mathfrak{B}_\alpha} < \epsilon \quad \text{for all } n, m \geq P.$$

As we know that

$$\|g\|_{\mathfrak{B}_\alpha} = \|g\|, \quad \text{for all } g \in \mathfrak{B}^0_\alpha$$

we obtain that

$$\|f_n - f_m\| < \epsilon, \quad \text{for all } n, m \geq P.$$

Then, there exist a function $f \in \mathfrak{B}_\alpha$ such that $(f_n)_{n\in\mathbb{N}}$ converges to f. Now, we need to show that $f \in \mathfrak{B}^0_\alpha$. By using equation (1.6), we have the property

$$\|.\|_{\mathfrak{B}_\alpha} \leq \|.\|.$$

This property implies that the sequence $(\|f_n\|_{\mathfrak{B}_\alpha})$ converges to $\|f\|_{\mathfrak{B}_\alpha}$; equivalently, from here, we can say that the sequence $(\|f_n\|)$ converges to $\|f\|_{\mathfrak{B}_\alpha}$, and also, this sequence converges to $\|f\|$. From the uniqueness of the limit of convergent sequence,

$$\|f\| = \|f\|_{\mathfrak{B}_\alpha}.$$

Since $f(0) = 0$, consequently we have $f \in \mathfrak{B}^0_\alpha$. $\square$

To obtain our desired results, we need the following lemma. The proof of this lemma plays a key role in most of the proofs of our main results. Therefore, we discuss the proof of this lemma in this section.

Lemma 2.2. *[93] For $\alpha > 0$, let $f \in \mathfrak{B}_\alpha$, we have the following basic properties:*

(i) *If $\alpha < 1$, then f is a bounded analytic function.*

(ii) *If $\alpha = 1$, then*

$$|f(z)| \leq |f(0)| + \frac{\|f\|_{\mathfrak{B}_1}}{2} \log\left(\frac{1+|z|}{1-|z|}\right).$$

(iii) *If $\alpha > 1$, then*

$$|f(z)| \leq |f(0)| + \frac{\|f\|_{\mathfrak{B}_\alpha}}{\alpha - 1}\left(\frac{1}{(1-|z|)^{\alpha-1}} - 1\right).$$

Proof. Suppose $f \in \mathfrak{B}_\alpha$ and $z \in \mathbb{D}$. Then

$$|f(z) - f(0)| = \left| z \int_0^1 f'(zt)dt \right| \leq |z| \int_0^1 |f'(zt)|dt.$$

By using the definition of α-Bloch space, we have

$$|f(z) - f(0)| \leq |z| \|f\|_{\mathfrak{B}_\alpha} \int_0^1 \frac{1}{(1 - |z|^2 t^2)^\alpha} dt. \tag{2.1}$$

Since $(1 + |z|t) \geq 1$, we obtain

$$|f(z) - f(0)| \leq |z| \|f\|_{\mathfrak{B}_\alpha} \int_0^1 \frac{1}{(1 - |z|t)^\alpha} dt \leq \|f\|_{\mathfrak{B}_\alpha} \frac{1}{1 - \alpha} \left(1 - \frac{1}{(1 - |z|)^{\alpha - 1}} \right). \tag{2.2}$$

We now complete the proofs of (i)-(iii) as described below.

(i) We notice that

$$|f(z)| \leq |f(0)| + \|f\|_{\mathfrak{B}_\alpha} \frac{1}{1 - \alpha} \big(1 - (1 - |z|)^{1 - \alpha} \big).$$

Since $1 - (1 - |z|)^{1-\alpha} \leq 1$, we obtain

$$|f(z)| \leq |f(0)| + \|f\|_{\mathfrak{B}_\alpha} \frac{1}{1 - \alpha}. \tag{2.3}$$

(ii) From (2.1), we estimate

$$|f(z)| \leq |f(0)| + \frac{\|f\|_{\mathfrak{B}_1}}{2} \log \left(\frac{1 + |z|}{1 - |z|} \right). \tag{2.4}$$

(iii) It easily follows that

$$|f(z) - f(0)| \leq \|f\|_{\mathfrak{B}_\alpha} \frac{1}{\alpha - 1} \left(\frac{1}{(1 - |z|)^{\alpha - 1}} - 1 \right).$$

By using triangle inequality, we finally obtain

$$|f(z)| \leq |f(0)| + \frac{\|f\|_{\mathfrak{B}_\alpha}}{\alpha - 1} \left(\frac{1}{(1 - |z|)^{\alpha - 1}} - 1 \right). \tag{2.5}$$

This completes the proof of our lemma. □

For $f \in \mathfrak{B}_\alpha^0$, $C_\beta(f)$ is an analytic function in $\mathbb{D}$ and $C_\beta(f)(0) = 0$. Now, we have three consecutive theorems, which describe the boundedness of β-Cesàro operators from $\mathfrak{B}_\alpha^0$ to $\mathfrak{B}_\alpha^0$ for three different restrictions on β.

Theorem 2.3. *For $\beta \leq \alpha < 1$, the β-Cesàro operator is a bounded linear operator from $\mathfrak{B}_\alpha^0$ to $\mathfrak{B}_\alpha^0$.*

Proof. Suppose that $f \in \mathfrak{B}^0_\alpha$, for $\alpha < 1$. From (2.2), we have

$$(1-|z|^2)^\alpha \left| \frac{f(z)}{z(1-z)^\beta} \right| \leq \frac{(1+|z|)^\alpha}{|z|} \frac{\|f\|_{\mathfrak{B}_\alpha}}{(1-\alpha)} \big[(1-|z|)^{\alpha-\beta} - (1-|z|)^{1-\beta}\big].$$

For $\beta \leq \alpha < 1$, this leads to

$$(1-|z|^2)^\alpha \left| \frac{f(z)}{z(1-z)^\beta} \right| \leq \frac{(1+|z|)^\alpha}{|z|} \frac{\|f\|_{\mathfrak{B}_\alpha}}{(1-\alpha)} (1-|z|)^{\alpha-\beta} \big[1 - (1-|z|)^{1-\alpha}\big].$$

Now for $\alpha < 1$, we have $1-(1-|z|)^{1-\alpha} \leq |z|$ and z is arbitrary point here, therefore

$$\|C_\beta(f)\|_{\mathfrak{B}_\alpha} \leq \sup \left\{ \frac{(1+|z|)^\alpha (1-|z|)^{\alpha-\beta}}{(1-\alpha)} : z \in \mathbb{D} \right\} \|f\|_{\mathfrak{B}_\alpha}.$$

This concludes the proof. □

Theorem 2.4. *For $\beta \leq 1 < \alpha$, the β-Cesàro operator is a bounded linear operator from $\mathfrak{B}^0_\alpha$ to $\mathfrak{B}^0_\alpha$.*

Proof. Suppose that $f \in \mathfrak{B}^0_\alpha$, for $\alpha > 1$. From (2.5), we have

$$(1-|z|^2)^\alpha \left| \frac{f(z)}{z(1-z)^\beta} \right| \leq \frac{(1+|z|)^\alpha}{|z|} \frac{\|f\|_{\mathfrak{B}_\alpha}}{(\alpha-1)} \big[(1-|z|)^{1-\beta} - (1-|z|)^{\alpha-\beta}\big].$$

If $\beta \leq 1 < \alpha$, it leads to

$$(1-|z|^2)^\alpha \left| \frac{f(z)}{z(1-z)^\beta} \right| \leq \frac{(1+|z|)^\alpha}{|z|} \frac{\|f\|_{\mathfrak{B}_\alpha}}{(\alpha-1)} (1-|z|)^{1-\beta} \big[1 - (1-|z|)^{\alpha-1}\big].$$

Since $1-(1-|z|)^{\alpha-1} \leq 1-(1-|z|)^{\lceil\alpha\rceil}$, where $\lceil . \rceil$ is the greatest integer function, we obtain

$$(1-|z|^2)^\alpha \left| \frac{f(z)}{z(1-z)^\beta} \right| \leq \frac{(1+|z|)^\alpha}{|z|} \frac{\|f\|_{\mathfrak{B}_\alpha}}{(\alpha-1)} (1-|z|)^{1-\beta} \big[1 - (1-|z|)^{\lceil\alpha\rceil}\big].$$

Note that

$$(1-|z|)^{\lceil\alpha\rceil} = \sum_{k=0}^{\lceil\alpha\rceil} \binom{\lceil\alpha\rceil}{k} (-|z|)^k.$$

Thus, we obtain

$$(1-|z|^2)^\alpha \left| \frac{f(z)}{z(1-z)^\beta} \right| \leq \frac{(1+|z|)^\alpha (1-|z|)^{1-\beta}}{(\alpha-1)} \sum_{k=1}^{\lceil\alpha\rceil} \binom{\lceil\alpha\rceil}{k} (-|z|)^{k-1} \|f\|_{\mathfrak{B}_\alpha}.$$

Since z is arbitrary point in $\mathbb{D}$, therefore we have

$$\|C_\beta(f)\|_{\mathfrak{B}_\alpha} \leq \sup \left\{ \frac{(1+|z|)^\alpha (1-|z|)^{1-\beta}}{(\alpha-1)} \sum_{k=1}^{\lceil\alpha\rceil} \binom{\lceil\alpha\rceil}{k} (-|z|)^{k-1} : z \in \mathbb{D} \right\} \|f\|_{\mathfrak{B}_\alpha},$$

which concludes the proof. □

For $\beta = 1$, the β-Cesàro operator is nothing but the Cesàro operator. Thus, Theorem 2.4 yields the following boundedness property of the Cesàro operator.

Corollary 2.5. *For $\alpha > 1$, the Cesàro operator is a bounded linear operator from $\mathfrak{B}_\alpha^0$ to $\mathfrak{B}_\alpha^0$.*

Theorem 2.6. *For $\beta < \alpha = 1$, the β-Cesàro operator is a bounded linear operator from $\mathfrak{B}_\alpha^0$ to $\mathfrak{B}_\alpha^0$.*

Proof. Suppose that $f \in \mathfrak{B}_\alpha^0$, for $\alpha = 1$. From (2.4), we obtain

$$(1-|z|^2)\left|\frac{f(z)}{z(1-z)^\beta}\right| \leq \frac{(1-|z|^2)}{|z|(1-|z|)^\beta}\frac{\|f\|_{\mathfrak{B}_1}}{2}\log\left(\frac{1+|z|}{1-|z|}\right).$$

For $\beta < \alpha = 1$, we have

$$(1-|z|^2)\left|\frac{f(z)}{z(1-z)^\beta}\right| \leq \frac{(1-|z|^2)}{2|z|(1-|z|)^\beta}\log\left(\frac{1+|z|}{1-|z|}\right)\|f\|_{\mathfrak{B}_1}.$$

Since z is an arbitrary point, we obtain that

$$\|C_\beta(f)\|_{\mathfrak{B}_1} \leq \sup\left\{\frac{(1-|z|)^{1-\beta}}{|z|}\log\left(\frac{1+|z|}{1-|z|}\right) : z \in \mathbb{D}\right\}\|f\|_{\mathfrak{B}_1},$$

completing the proof. □

As an immediate consequence of Theorems 2.3, 2.4 and 2.6, we can easily prove the following corollary.

Corollary 2.7. *The generalized β-Cesàro operator is a bounded linear operator from $\mathfrak{B}_\alpha^0$ to $\mathfrak{B}_\alpha^0$ either for $\beta \leq \alpha < 1$ or $\beta \leq 1 < \alpha$ or $\beta < \alpha = 1$.*

Proof. Let $f \in \mathfrak{B}_\alpha^0$ and consider the generalized β-Cesàro operator either for $\beta \leq \alpha < 1$ or $\beta \leq 1 < \alpha$ or $\beta < \alpha = 1$. Then from (1.16), we obtain

$$(1-|z|^2)^\alpha\left|\frac{f(z)g_\beta(z)}{z}\right| = (1-|z|^2)^\alpha\left|\frac{f(z)}{z}\left(\sum_{j=1}^{k}\frac{a_j}{(1-b_jz)^\beta} + h(z)\right)\right|.$$

By triangle inequality

$$\begin{aligned}(1-|z|^2)^\alpha\left|\frac{f(z)g_\beta(z)}{z}\right| &\leq \sum_{j=1}^{k}|a_k|(1-|z|^2)^\alpha\frac{|f(z)|}{|z|(1-|z|)^\beta} + \|h\|_\infty(1-|z|^2)^\alpha\frac{|f(z)|}{|z|}\\ &= (1-|z|^2)^\alpha\frac{|f(z)|}{|z|(1-|z|)^\beta}\sum_{j=1}^{k}|a_k| + \|h\|_\infty(1-|z|^2)^\alpha\frac{|f(z)|}{|z|},\end{aligned}$$

where $\|\cdot\|_\infty$ stands for the classical sup norm. From here we can further proceed as in the proof of Theorems 2.3, 2.4 and 2.6 according to the condition on α and β. $\square$

From the statement of Theorems 2.3, 2.4 and 2.6 we can conclude that the Alexander operator is a bounded linear operator from $\mathfrak{B}_\alpha^0$ to $\mathfrak{B}_\alpha^0$. But in addition, we discuss the exact operator norm for the Alexander operator from $\mathfrak{B}_\alpha^0$ to $\mathfrak{B}_\alpha^0$ in the following Theorem.

Theorem 2.8. *The Alexander operator is a bounded linear operator from $\mathfrak{B}_\alpha^0$ to $\mathfrak{B}_\alpha^0$, with operator norm* 1*.*

Proof. Suppose that $f \in \mathfrak{B}_\alpha^0$. Then we estimate

$$|f(z)| = \left| z \int_0^1 f'(zt)dt \right| \leq |z| \int_0^1 |f'(zt)|dt.$$

Multiplying by $(1-|z|^2)^\alpha$ on both sides, we have

$$(1-|z|^2)^\alpha |f(z)| \leq |z|(1-|z|^2)^\alpha \int_0^1 \frac{(1-|z|^2|t|^2)^\alpha}{(1-|z|^2|t|^2)^\alpha} |f'(zt)|dt.$$

By the definition of α-Bloch space, we estimate

$$(1-|z|^2)^\alpha |f(z)| \leq |z| \|f\|_{\mathfrak{B}_\alpha} \int_0^1 \frac{(1-|z|^2)^\alpha}{(1-|z|^2|t|^2)^\alpha} dt.$$

Since $(1-|z|^2) \leq (1-|z|^2|t|^2)$, we obtain

$$(1-|z|^2)^\alpha \left| \frac{d}{dz} \int_0^z \frac{f(t)}{t} dt \right| \leq \|f\|_{\mathfrak{B}_\alpha}.$$

Here z is an arbitrary point in $\mathbb{D}$, therefore

$$\|C_0(f)\|_{\mathfrak{B}_\alpha} \leq \|f\|_{\mathfrak{B}_\alpha}.$$

If we choose the identity function $f(z) = z$, then we obtain the exact operator norm 1; equivalently, we say that $\|C_0\|_{\mathfrak{B}_\alpha} = 1$. $\square$

Counterexamples

We just proved that for either of the cases $\beta \leq \alpha < 1$, $\beta \leq 1 < \alpha$ and $\beta < \alpha = 1$, the β-Cesàro operator is a bounded linear operator from $\mathfrak{B}_\alpha^0$ to $\mathfrak{B}_\alpha^0$. We now show that for the remaining cases: $\beta > \alpha$, $\beta = \alpha \geq 1$ and $1 < \beta < \alpha$, the β-Cesàro operators need not be bounded, as the following counterexamples show.

Example 2.9. *Let $f(z) = z$, then $f \in \mathfrak{B}^0_\alpha$, for $\beta > \alpha$ and we have*

$$(1-|z|^2)^\alpha \left| \frac{z}{z(1-z)^\beta} \right| = (1+|z|)^\alpha \frac{(1-|z|)^\alpha}{|(1-z)|^\beta}.$$

For $z = t \in (0,1)$, we obtain

$$(1-|z|^2)^\alpha \left| \frac{z}{z(1-z)^\beta} \right| = (1+t)^\alpha \frac{(1-t)^\alpha}{(1-t)^\beta} = \frac{(1+t)^\alpha}{(1-t)^{\beta-\alpha}}.$$

As t tends to 1, the right-hand side term tends to ∞. Therefore, for $\beta > \alpha$, the β-Cesàro operator is an unbounded linear operator from $\mathfrak{B}^0_\alpha$ to $\mathfrak{B}^0_\alpha$.

Example 2.10. *Let $f(z) = Log(1-z)$, where a principal value of the branch of logarithm is chosen. Then $f \in \mathfrak{B}^0_\alpha$ for $\alpha \geq 1$, and for $z = t \in (0,1)$ we have*

$$(1-|z|^2)^\alpha \left| \frac{\log(1-z)}{z(1-z)^\beta} \right| = (1-t^2)^\alpha \left| \frac{\log(1-t)}{t(1-t)^\beta} \right| = \frac{(1+t)^\alpha}{(1-t)^{\beta-\alpha}} \left| \frac{\log(1-t)}{t} \right|.$$

Then for $\beta \geq \alpha$, as t tends to 1, the right-hand side term diverges to ∞. Therefore, the β-Cesàro operator is an unbounded linear operator from $\mathfrak{B}^0_\alpha$ to $\mathfrak{B}^0_\alpha$, for $\beta \geq \alpha \geq 1$.

Remark 2.11. *By using the conclusion of Examples* 2.9 *and* 2.10*, we are able to conclude that the Cesàro operator is an unbounded linear operator from $\mathfrak{B}^0_\alpha$ to $\mathfrak{B}^0_\alpha$ for $\alpha \leq 1$.*

Example 2.12. *Let $f(z) = z/(1-z)^\alpha$ for $\alpha > 0$. Then $f \in \mathfrak{B}^0_{\alpha+1}$ and we have*

$$(1-|z|^2)^{\alpha+1} \left| \frac{z}{z(1-z)^{\alpha+\beta}} \right| = (1-|z|^2)^{\alpha+1} \frac{1}{|1-z|^{\alpha+\beta}}.$$

For $z = t \in (0,1)$, then it yields

$$(1-|z|^2)^{\alpha+1} \left| \frac{z}{z(1-z)^{\alpha+\beta}} \right| = (1+t)^{\alpha+1} \frac{(1-t)^{\alpha+1}}{(1-t)^{\alpha+\beta}} = \frac{(1+t)^{\alpha+1}}{(1-t)^{\beta-1}}.$$

Then for $\beta > 1$, as t tends to 1, the right-hand side term approaches ∞. Therefore, the β-Cesàro operator is an unbounded linear operator from $\mathfrak{B}^0_{\alpha+1}$ to $\mathfrak{B}^0_{\alpha+1}$ for $\beta > 1$.

2.2. Compactness of the β-Cesàro operators

In this section, we discuss compactness of the β-Cesàro operators, for $\beta < \alpha < 1$, $\beta < 1 < \alpha$ and $\beta < \alpha = 1$, and for its generalization with the help of Lemma 2.14. However, the same problem for the cases $\beta = \alpha < 1$ and $\beta = 1 < \alpha$ will be investigated in the next section. Before going to the equivalent condition for compactness of the generalized β-Cesàro operators (Lemma 2.14), we have the following Lemma, which is used to prove the subsequent lemma.

Lemma 2.13. *The generalized β-Cesàro operators mapping from $\mathfrak{B}_\alpha^0$ to $\mathfrak{B}_\alpha^0$ are continuous linear operators in the topology of uniform convergence on every compact subset of $\mathbb{D}$.*

Proof. Let f_m be a sequence in $\mathfrak{B}_\alpha^0$ which converges to f uniformly on every compact subset of $\mathbb{D}$. By the Weierstrass theorem for sequences, f_m' converges to f' uniformly on every compact subset of $\mathbb{D}$. On the other hand, we have

$$\left|\frac{f_m(z)-f(z)}{z}\right| \le \int_0^1 |f_m'(zt)-f'(zt)|dt. \tag{2.6}$$

Consequently, $f_m(z)/z$ converges to $f(z)/z$ uniformly on every compact subset of $\mathbb{D}$, which implies that $C_{g_\beta}(f_m)$ converges to $C_{g_\beta}(f)$ uniformly on every compact subset of $\mathbb{D}$. □

Lemma 2.14. *The generalized β-Cesàro operators mapping from $\mathfrak{B}_\alpha^0$ to $\mathfrak{B}_\alpha^0$ are compact if and only if for every bounded sequence (f_m) in $\mathfrak{B}_\alpha^0$ which converges to 0 uniformly on every compact subset of $\mathbb{D}$, we have $\lim_{m\to\infty}\|C_{g_\beta}f_m\|_{\mathfrak{B}_\alpha}=0$.*

Proof. The proof of this lemma follows from Lemma 2.13 and [102, Lemma 3]. □

With the help of Lemma 2.14, we prove the following theorems.

Theorem 2.15. *The generalized Alexander operator is a compact linear operator from $\mathfrak{B}_\alpha^0$ to $\mathfrak{B}_\alpha^0$ for $\alpha>0$.*

Proof. Suppose (f_m) is a sequence in $\mathfrak{B}_\alpha^0$ which converges to 0 uniformly on every compact subset of $\mathbb{D}$ and is also bounded, s0 that there exists a constant $M\in\mathbb{N}$ with $\|f_m\|_{\mathfrak{B}_\alpha}\le M$. We need to show that $\lim_{m\to\infty}\|C_{g_0}f_m\|_{\mathfrak{B}_\alpha}=0$.

Let $(s_k)_{k\in\mathbb{N}}$ be a sequence which increasingly converges to 1. We have

$$\begin{aligned}\lim_{m\to\infty}\|C_{g_0}f_m\|_{\mathfrak{B}_\alpha} &= \lim_{m\to\infty}\sup_{|z|<1}(1-|z|^2)^\alpha\left|\frac{f_m(z)g_0(z)}{z}\right|\\ &\le \lim_{m\to\infty}\sup_{|z|\le s_k}(1-|z|^2)^\alpha\left|\frac{f_m(z)g_0(z)}{z}\right| + \lim_{m\to\infty}\sup_{s_k<|z|<1}(1-|z|^2)^\alpha\left|\frac{f_m(z)g_0(z)}{z}\right|.\end{aligned}$$

From (2.6), $f_m(z)g_0(z)/z$ converges to 0 uniformly on every compact subset of $\mathbb{D}$. Thus, we obtain

$$\lim_{m\to\infty}\sup_{|z|\le s_k}(1-|z|^2)^\alpha\left|\frac{f_m(z)g_0(z)}{z}\right|=0, \tag{2.7}$$

for each $k\in\mathbb{N}$.

We consider three cases here.

Case (i) Assume that $\alpha < 1$. Then from (2.3), we have

$$|f_m(z)| \leq \|f_m\|_{\mathfrak{B}_\alpha} \frac{1}{1-\alpha}.$$

This yields

$$\sup_{s_k<|z|<1} (1-|z|^2)^\alpha \left| \frac{f_m(z) g_0(z)}{z} \right| \leq \sup_{s_k<|z|<1} \frac{(1-|z|^2)^\alpha}{|z|(1-\alpha)} |g_0(z)| \|f_m\|_{\mathfrak{B}_\alpha}.$$

Equivalently, we obtain

$$\sup_{s_k<|z|<1} (1-|z|^2)^\alpha \left| \frac{f_m(z) g_0(z)}{z} \right| \leq M \frac{(1-s_k^2)^\alpha}{s_k(1-\alpha)} \|g_0\|_\infty, \tag{2.8}$$

which tends to 0 as $k \to \infty$.

Case (ii) Suppose that $\alpha = 1$. It follows from (2.4) that

$$|f_m(z)| \leq \frac{\|f_m\|_{\mathfrak{B}_1}}{2} \log\left(\frac{1+|z|}{1-|z|}\right).$$

Then we have

$$\sup_{s_k<|z|<1} (1-|z|^2) \left| \frac{f_m(z) g_0(z)}{z} \right| \leq M \sup_{s_k<|z|<1} \frac{(1-|z|^2)}{2|z|} \log\left(\frac{1+|z|}{1-|z|}\right) |g_0(z)|, \tag{2.9}$$

which tends to 0 as $k \to \infty$.

Case (iii) Assume that $\alpha > 1$. Then from (2.5), we have

$$|f_m(z)| \leq \frac{\|f_m\|_{\mathfrak{B}_\alpha}}{\alpha-1} \left(\frac{1}{(1-|z|)^{\alpha-1}} - 1\right).$$

This leads to

$$\sup_{s_k<|z|<1} (1-|z|^2)^\alpha \left| \frac{f_m(z) g_0(z)}{z} \right| \leq M \sup_{s_k<|z|<1} \frac{(1-|z|^2)^\alpha}{|z|(\alpha-1)} \left(\frac{1}{(1-|z|)^{\alpha-1}} - 1\right) |g_0(z)|. \tag{2.10}$$

The right-hand side quantity tends to 0 as $k \to \infty$. Thus, from (2.7), (2.8), (2.9) and (2.10), we conclude that

$$\lim_{m\to\infty} \|C_{g_0} f_m\|_{\mathfrak{B}_\alpha} = 0.$$

The proof of our theorem is complete. □

Theorem 2.16. *The β-Cesàro operator is a compact linear operator from $\mathfrak{B}_\alpha^0$ to $\mathfrak{B}_\alpha^0$, either for $\beta < \alpha < 1$ or $\beta < 1 < \alpha$ or $\beta < \alpha = 1$.*

Proof. Suppose (f_m) is a bounded sequence in $\mathfrak{B}_\alpha^0$ and also converges to 0 uniformly on compact subsets of $\mathbb{D}$. Let $(s_k)_{k\in\mathbb{N}}$ be a sequence which increasingly converges to 1. We compute

$$\begin{aligned}\lim_{m\to\infty}\|C_\beta f_m\|_{\mathfrak{B}_\alpha} &= \lim_{m\to\infty}\sup_{|z|<1}(1-|z|^2)^\alpha\left|\frac{f_m(z)}{z(1-z)^\beta}\right| \\ &\leq \lim_{m\to\infty}\sup_{|z|\leq s_k}\frac{(1-|z|^2)^\alpha}{(1-|z|)^\beta}\left|\frac{f_m(z)}{z}\right| + \lim_{m\to\infty}\sup_{s_k<|z|<1}\frac{(1-|z|^2)^\alpha}{(1-|z|)^\beta}\left|\frac{f_m(z)}{z}\right|.\end{aligned}$$

To compute the second term in right-hand side, we need to consider three cases on α.

Case (i) Consider $\beta<\alpha<1$. Then we have

$$\sup_{s_k<|z|<1}(1-|z|^2)^\alpha\left|\frac{f_m(z)}{z(1-z)^\beta}\right| \leq \sup_{s_k<|z|<1}\frac{(1-|z|^2)^\alpha}{|z|(1-|z|)^\beta(1-\alpha)}\|f_m\|_{\mathfrak{B}_\alpha},$$

since (2.3) gives $|f_m(z)|\leq(1-\alpha)^{-1}\|f_m\|_{\mathfrak{B}_\alpha}$. Further, we obtain

$$\sup_{s_k<|z|<1}(1-|z|^2)^\alpha\left|\frac{f_m(z)}{z(1-z)^\beta}\right| \leq 2^\alpha\frac{(1-s_k)^{\alpha-\beta}}{s_k(1-\alpha)}\|f_m\|_{\mathfrak{B}_\alpha}. \tag{2.11}$$

The above right-hand side term tends to 0 as $k\to\infty$.

Case (ii) Assume that $\beta<\alpha=1$. It follows that

$$\sup_{s_k<|z|<1}(1-|z|^2)\left|\frac{f_m(z)}{z(1-z)^\beta}\right| \leq \frac{\|f_m\|_{\mathfrak{B}_1}}{s_k}\sup_{s_k<|z|<1}(1-|z|)^{1-\beta}\log\left(\frac{1+|z|}{1-|z|}\right), \tag{2.12}$$

where we used the inequality

$$|f_m(z)|\leq\frac{\|f_m\|_{\mathfrak{B}_1}}{2}\log\left(\frac{1+|z|}{1-|z|}\right),$$

which is due to (2.4). Now, as $k\to\infty$

$$\sup_{s_k<|z|<1}(1-|z|)^{1-\beta}\log\left(\frac{1+|z|}{1-|z|}\right)$$

tends to 0.

Case (iii) Suppose that $\alpha>1>\beta$. Then from (2.5), we obtain

$$|f_m(z)|\leq\frac{\|f_m\|_{\mathfrak{B}_\alpha}}{\alpha-1}\left(\frac{1}{(1-|z|)^{\alpha-1}}-1\right).$$

It follows that

$$\sup_{s_k<|z|<1}(1-|z|^2)^\alpha\left|\frac{f_m(z)}{z(1-z)^\beta}\right| \leq \sup_{s_K<|z|<1}\frac{(1-|z|^2)^\alpha}{|z|(\alpha-1)}\left(\frac{1}{(1-|z|)^{\beta+\alpha-1}}-\frac{1}{(1-|z|)^\beta}\right)\|f_m\|_{\mathfrak{B}_\alpha}.$$

This equivalently gives

(2.13)
$$\sup_{s_k<|z|<1}(1-|z|^2)^\alpha\left|\frac{f_m(z)}{z(1-z)^\beta}\right| \leq \sup_{s_k<|z|<1}\frac{(1+|z|)^\alpha}{|z|(\alpha-1)}\Big((1-|z|)^{1-\beta}-(1-|z|)^{\alpha-\beta}\Big)\|f_m\|_{\mathfrak{B}_\alpha}.$$

The right-hand side quantity of (2.13) tends to 0 as $k \to \infty$.

By (2.6), $f_m(z)/z$ converges to 0 uniformly on compact subsets of $\mathbb{D}$. This leads to

$$\lim_{m\to\infty}\sup_{|z|\leq s_k}\frac{(1-|z|^2)^\alpha}{(1-|z|)^\beta}\left|\frac{f_m(z)}{z}\right| = 0, \tag{2.14}$$

for each $k \in \mathbb{N}$.

From (2.14), (2.11), (2.12) and (2.13), we obtain

$$\lim_{m\to\infty}\|C_\beta f_m\|_{\mathfrak{B}_\alpha} = 0$$

for $\beta < \alpha < 1$, $\beta < 1 < \alpha$ and $\beta < \alpha = 1$. This is what we wanted to show. □

Corollary 2.17. *The generalized β-Cesàro operator is a compact linear operator from $\mathfrak{B}^0_\alpha$ to $\mathfrak{B}^0_\alpha$ either for $\beta < \alpha < 1$ or $\beta < 1 < \alpha$ or $\beta < \alpha = 1$.*

Proof. The proof of Corollary 2.17 follows steps given in the proof of Corollary 2.7, and then we proceed as in the proofs of Theorem 2.15 and 2.16. □

2.3. Essential norm of the β-Cesáro operators

This section is devoted to obtaining the essential norm of the β-Cesàro operators. First we recall the concept of essential norm. Let X and Y be Banach spaces and $T : X \to Y$ be a bounded linear operator. The *essential norm* of the operator $T : X \to Y$, denoted by $\|T\|_e$, is defined as

$$\|T\|_e = \inf\{\|T+K\| : \text{K is a compact operator from X to Y}\}, \tag{2.15}$$

where $\|.\|$ denotes the operator norm. The following remark is a direct consequence of (2.15).

Remark 2.18. *It is well-known that the set of all compact operators from a normed linear space to a Banach space is a closed subset of the set of bounded operators. Using this fact together with* (2.15)*, one can easily show that an operator T is compact if and only if $\|T\|_e = 0$.*

The compactness of the generalized β-Cesàro operator is studied directly in the previous section; however, the situations when $\beta = \alpha < 1$ and $\beta = 1 < \alpha$ could not be handled directly. In this section, the concept of essential norm played a crucial role in handling these unsolved situations.

Theorem 2.19. *The essential norm of the generalized Alexander operator from $\mathfrak{B}_\alpha^0$ to $\mathfrak{B}_\alpha^0$ is* 0.

Proof. Consider the operator defined on $\mathfrak{B}_\alpha^0$ by

$$K_{g_0}^{s_k}(f)(z) = \int_0^z \frac{f(s_k t) g_0(t)}{t} dt$$

where $(s_k)_{k\in\mathbb{N}}$ is an increasing sequence converging to 1, and g_0 is a bounded analytic function in $\mathbb{D}$.

Suppose $(f_m)_{m\in\mathbb{N}}$ is a bounded sequence in $\mathfrak{B}_\alpha^0$ which converges to 0 uniformly on every compact subset of $\mathbb{D}$. Then we see that

$$\begin{aligned}\sup_{|z|<1}(1-|z|^2)^\alpha \left|\frac{f_m(s_k z) g_0(z)}{z}\right| &\le \sup_{|z|<1}(1-|s_k z|^2)^\alpha \left|\frac{f_m(s_k z) g_0(z)}{s_k z}\right| \\ &\le \|g_0\|_\infty \sup_{|z|\le s_k}(1-|z|^2)^\alpha \left|\frac{f_m(z)}{z}\right| \to 0\end{aligned}$$

as $m \to \infty$. Hence by Lemma 2.14, $K_{g_0}^{s_k}$ is compact for each $k \in \mathbb{N}$.

Let $\lambda \in (0,1)$ be fixed for the moment. Then we have

$$\begin{aligned}\|C_{g_0} - K_{g_0}^{s_k}\|_{\mathfrak{B}_\alpha} &= \sup_{\|f\|_{\mathfrak{B}_\alpha}\le 1} \sup_{|z|<1}(1-|z|^2)^\alpha \left|\frac{f(z)g_0(z)}{z} - \frac{f(s_k z) g_0(z)}{z}\right| \\ &\le \sup_{\|f\|_{\mathfrak{B}_\alpha}\le 1} \sup_{|z|\le\lambda}(1-|z|^2)^\alpha \left|\frac{f(z)g_0(z)}{z} - \frac{f(s_k z) g_0(z)}{z}\right| \\ &\quad + \sup_{\|f\|_{\mathfrak{B}_\alpha}\le 1} \sup_{\lambda<|z|<1}(1-|z|^2)^\alpha \left|\frac{f(z)g_0(z)}{z} - \frac{f(s_k z) g_0(z)}{z}\right|.\end{aligned}$$

By using the classical mean-value theorem and definition of α-Bloch space, we obtain

$$\begin{aligned}&\sup_{\|f\|_{\mathfrak{B}_\alpha}\le 1} \sup_{|z|\le\lambda}(1-|z|^2)^\alpha \left|\frac{f(z)g_0(z)}{z} - \frac{f(s_k z) g_0(z)}{z}\right| \\ &\qquad \le \sup_{\|f\|_{\mathfrak{B}_\alpha}\le 1} \sup_{|z|\le\lambda}(1-|z|^2)^\alpha (1-s_k)|g_0(z)| \sup_{|w|\le\lambda}|f'(w)|.\end{aligned}$$

It follows that

$$\sup_{\|f\|_{\mathfrak{B}_\alpha}\le 1}\sup_{|z|\le\lambda}(1-|z|^2)^\alpha\left|\frac{f(z)g_0(z)}{z}-\frac{f(s_kz)g_0(z)}{z}\right| \le \frac{(1-s_k)}{(1-\lambda^2)^\alpha}\|g_0\|_\infty\sup_{\|f\|_{\mathfrak{B}_\alpha}\le 1}\|f\|_{\mathfrak{B}_\alpha}, \tag{2.16}$$

which tends to 0 as $k\to\infty$.

We consider the following cases to complete our proof.

Case (i) Assume that $\alpha<1$. Then from (2.3), we have

$$|f(z)|\le\|f\|_{\mathfrak{B}_\alpha}\frac{1}{1-\alpha}.$$

It follows that

$$\sup_{\|f\|_{\mathfrak{B}_\alpha}\le 1}\sup_{\lambda<|z|<1}(1-|z|^2)^\alpha\left|\frac{f(z)g_0(z)}{z}-\frac{f(s_kz)g_0(z)}{z}\right|\le 2\sup_{\lambda<|z|<1}\frac{(1-|z|^2)^\alpha}{|z|(1-\alpha)}|g_0(z)|.$$

Thus, we obtain

$$\sup_{\|f\|_{\mathfrak{B}_\alpha}\le 1}\sup_{\lambda<|z|<1}(1-|z|^2)^\alpha\left|\frac{f(z)g_0(z)}{z}-\frac{f(s_kz)g_0(z)}{z}\right|\le 2\frac{(1-\lambda^2)^\alpha}{\lambda(1-\alpha)}\|g_0\|_\infty. \tag{2.17}$$

The right-hand side of (2.17) tends to 0 as $\lambda\to 1$.

Case (ii) Consider $\alpha=1$. From (2.4), we obtain

$$|f(z)|\le\frac{\|f\|_{\mathfrak{B}_1}}{2}\log\left(\frac{1+|z|}{1-|z|}\right).$$

This simplifies to

$$\sup_{\|f\|_{\mathfrak{B}_1}\le 1}\sup_{\lambda<|z|<1}(1-|z|^2)\left|\frac{f(z)g_0(z)}{z}-\frac{f(s_kz)g_0(z)}{z}\right| \le \sup_{\lambda<|z|<1}\frac{(1-|z|^2)}{|z|}\log\left(\frac{1+|z|}{1-|z|}\right)|g_0(z)|, \tag{2.18}$$

which tends to 0 as $\lambda\to 1$.

Case (iii) Suppose that $\alpha>1$. Then (2.5) obtains

$$|f(z)|\le\frac{\|f\|_{\mathfrak{B}_\alpha}}{\alpha-1}\left(\frac{1}{(1-|z|)^{\alpha-1}}-1\right).$$

It leads to

(2.19)

$$\begin{aligned}\sup_{\|f\|_{\mathfrak{B}_\alpha}\le 1}\ \sup_{\lambda<|z|<1}\ &(1-|z|^2)^\alpha\left|\frac{f(z)g_0(z)}{z}-\frac{f(s_kz)g_0(z)}{z}\right|\\ &\le 2\sup_{\lambda<|z|<1}\frac{(1-|z|^2)^\alpha}{|z|(\alpha-1)}\left(\frac{1}{(1-|z|)^{\alpha-1}}-1\right)|g_0(z)|.\end{aligned}$$

The right-hand side quantity of (2.19) tends to 0 as $\lambda\to 1$. Thus, from (2.16), (2.17), (2.18) and (2.19), we obtain

$$\lim_{k\to\infty}\|C_{g_0}-K_{g_0}^{s_k}\|_{\mathfrak{B}_\alpha}=0,$$

and the conclusion follows from the definition of essential norm. $\square$

Theorem 2.20. *The essential norm of the β-Cesàro operator from $\mathfrak{B}_\alpha^0$ to $\mathfrak{B}_\alpha^0$ is 0 either for $\beta<\alpha<1$ or $\beta<1<\alpha$ or $\beta<\alpha=1$.*

Proof. Consider the operator defined on $\mathfrak{B}_\alpha^0$ by

$$K_{s_k}(f)(z)=\int_0^z\frac{f(s_kt)}{t(1-t)^\beta}dt,$$

where $(s_k)_{k\in\mathbb{N}}$ is an increasing sequence, which converges to 1.

Suppose $(f_m)_{m\in\mathbb{N}}$ is a bounded sequence in $\mathfrak{B}_\alpha^0$, which converges to 0 uniformly on compact subsets of $\mathbb{D}$ and $\beta\le\alpha$. Then we obtain

$$\begin{aligned}\sup_{|z|<1}(1-|z|^2)^\alpha\left|\frac{f_m(s_kz)}{z(1-z)^\beta}\right| &\le 2^\alpha\sup_{|z|<1}(1-|z|)^{\alpha-\beta}\left|\frac{f_m(s_kz)}{s_kz}\right|\\ &\le 2^\alpha\sup_{|z|\le s_k}(1-|z|)^{\alpha-\beta}\left|\frac{f_m(z)}{z}\right|,\end{aligned}$$

which tends to 0 as $m\to\infty$. Hence by Lemma 2.14, K_{s_k} is a compact operator for each $k\in\mathbb{N}$.

Let $\lambda\in(0,1)$ be fixed for the moment. We compute

$$\begin{aligned}\|C_\beta-K_{s_k}\|_{\mathfrak{B}_\alpha}&=\sup_{\|f\|_{\mathfrak{B}_\alpha}\le 1}\sup_{|z|<1}(1-|z|^2)^\alpha\left|\frac{f(z)}{z(1-z)^\beta}-\frac{f(s_kz)}{z(1-z)^\beta}\right|\\ &\le\sup_{\|f\|_{\mathfrak{B}_\alpha}\le 1}\sup_{|z|\le\lambda}(1-|z|^2)^\alpha\left|\frac{f(z)}{z(1-z)^\beta}-\frac{f(s_kz)}{z(1-z)^\beta}\right|\\ &\quad+\sup_{\|f\|_{\mathfrak{B}_\alpha}\le 1}\sup_{\lambda<|z|<1}(1-|z|^2)^\alpha\left|\frac{f(z)}{z(1-z)^\beta}-\frac{f(s_kz)}{z(1-z)^\beta}\right|.\end{aligned}$$

By the mean-value theorem and definition of α-Bloch space, it follows that

$$\sup_{\|f\|_{\mathfrak{B}_\alpha}\le 1}\sup_{|z|\le\lambda}(1-|z|^2)^\alpha\left|\frac{f(z)}{z(1-z)^\beta}-\frac{f(s_kz)}{z(1-z)^\beta}\right|$$
$$\le \sup_{\|f\|_{\mathfrak{B}_\alpha}\le 1}\sup_{|z|\le\lambda}\frac{(1-|z|^2)^\alpha}{(1-|z|)^\beta}(1-s_k)\sup_{|w|\le\lambda}|f'(w)|.$$

This obtains

$$\sup_{\|f\|_{\mathfrak{B}_\alpha}\le 1}\sup_{|z|\le\lambda}(1-|z|^2)^\alpha\left|\frac{f(z)}{z(1-z)^\beta}-\frac{f(s_kz)}{z(1-z)^\beta}\right| \le \frac{(1-s_k)}{(1-\lambda^2)^\alpha(1-\lambda)^\beta}\sup_{\|f\|_{\mathfrak{B}_\alpha}\le 1}\|f\|_{\mathfrak{B}_\alpha}, \tag{2.20}$$

which approaches to 0 as $k\to\infty$.

To pursue our goal, we will go through the following cases for α and β.

Case (i) Consider $\beta<\alpha<1$. Then from (2.3), $|f(z)|\le(1-\alpha)^{-1}\|f\|_{\mathfrak{B}_\alpha}$, it follows that

$$\sup_{\|f\|_{\mathfrak{B}_\alpha}\le 1}\sup_{\lambda<|z|<1}(1-|z|^2)^\alpha\left|\frac{f(z)}{z(1-z)^\beta}-\frac{f(s_kz)}{z(1-z)^\beta}\right|\le 2\sup_{\lambda<|z|<1}\frac{(1-|z|^2)^\alpha}{|z|(1-|z|)^\beta(1-\alpha)} \le 2^{\alpha+1}\frac{(1-\lambda)^{\alpha-\beta}}{\lambda(1-\alpha)}, \tag{2.21}$$

which tends to 0 as $\lambda\to 1$.

Case (ii) Assume that $\beta<\alpha=1$. First we see that

$$\sup_{\|f\|_{\mathfrak{B}_1}\le 1}\sup_{\lambda<|z|<1}(1-|z|^2)\left|\frac{f(z)}{z(1-z)^\beta}-\frac{f(s_kz)}{z(1-z)^\beta}\right|\le\sup_{\lambda<|z|<1}\frac{(1-|z|^2)}{|z|(1-|z|)^\beta}\log\left(\frac{1+|z|}{1-|z|}\right).$$

The above inequality easily follows from (2.4). This simplifies to

$$\sup_{\|f\|_{\mathfrak{B}_1}\le 1}\sup_{\lambda<|z|<1}(1-|z|^2)\left|\frac{f(z)}{z(1-z)^\beta}-\frac{f(s_kz)}{z(1-z)^\beta}\right| \le\frac{2}{\lambda}\sup_{\lambda<|z|<1}(1-|z|)^{1-\beta}\log\left(\frac{1+|z|}{1-|z|}\right). \tag{2.22}$$

The right-hand side quantity of (2.22) tends to 0 as $\lambda\to 1$.

Case (iii) Suppose that $\alpha>1>\beta$. From (2.5), we have

$$|f(z)|\le\frac{\|f\|_{\mathfrak{B}_\alpha}}{\alpha-1}\left(\frac{1}{(1-|z|)^{\alpha-1}}-1\right).$$

Then we obtain

$$\sup_{\|f\|_{\mathfrak{B}_\alpha}\le 1}\sup_{\lambda<|z|<1}(1-|z|^2)^\alpha\left|\frac{f(z)}{z(1-z)^\beta}-\frac{f(s_kz)}{z(1-z)^\beta}\right|$$
$$\le 2\sup_{\lambda<|z|<1}\frac{(1+|z|)^\alpha}{|z|(\alpha-1)}\Big((1-|z|)^{1-\beta}-(1-|z|)^{\alpha-\beta}\Big), \tag{2.23}$$

which tends to 0 as $\lambda\to 1$.

From (2.20), (2.21), (2.22) and (2.23), we thus obtain

$$\lim_{k\to\infty}\|C_\beta-K_{s_k}\|_{\mathfrak{B}_\alpha}=0$$

for $\beta<\alpha<1$, $\beta<1<\alpha$ and $\beta<\alpha=1$. This completes the proof. □

Corollary 2.21. *The essential norm of the generalized β-Cesàro operator from $\mathfrak{B}^0_\alpha$ to $\mathfrak{B}^0_\alpha$ is 0 either for $\beta<\alpha<1$ or $\beta<1<\alpha$ or $\beta<\alpha=1$.*

Proof. As in the proof of Theorem 2.20, we similarly define

$$K_{s_k}(f)(z)=\int_0^z\frac{f(s_kt)g_\beta(t)}{t}dt$$

where $(s_k)_{k\in\mathbb{N}}$ is a increasing sequence, which converges to 1.

To complete the proof, we follow the steps given in the proof of Corollary 2.7 and follow the proofs of Theorem 2.19 and Theorem 2.20. □

Theorem 2.22. *The essential norm of the β-Cesàro operator from $\mathfrak{B}^0_\alpha$ to $\mathfrak{B}^0_\alpha$ is 0 for $\beta=\alpha<1$.*

Proof. Let $\beta=\alpha-\frac{1}{n}$, then by Theorem 2.20, the β-Cesàro operator from $\mathfrak{B}^0_\alpha$ to $\mathfrak{B}^0_\alpha$, for $\alpha<1$, is a compact linear operator for each $n\in\mathbb{N}$. Note that

$$\|C_\alpha-C_{\alpha-\frac{1}{n}}\|_{\mathfrak{B}_\alpha}=\sup_{\|f\|_{\mathfrak{B}_\alpha}\le 1}\sup_{|z|<1}(1-|z|^2)^\alpha\left|\frac{f(z)}{z(1-z)^\alpha}-\frac{f(z)}{z(1-z)^{\alpha-\frac{1}{n}}}\right|.$$

For $f\in\mathfrak{B}^0_\alpha$, $\alpha<1$, it is noted in Lemma 2.2 that f is a bounded analytic function. Then we see that

$$\|C_\alpha-C_{\alpha-\frac{1}{n}}\|_{\mathfrak{B}_\alpha}\le\left\|\frac{f(z)}{z}\right\|_\infty\sup_{|z|<1}\frac{(1-|z|^2)^\alpha}{(1-|z|)^\alpha}|1-(1-z)^{\frac{1}{n}}|,$$

which is equivalent to

$$\|C_\alpha-C_{\alpha-\frac{1}{n}}\|_{\mathfrak{B}_\alpha}\le 2^\alpha\left\|\frac{f(z)}{z}\right\|_\infty\sup_{|z|<1}|1-(1-z)^{\frac{1}{n}}|. \tag{2.24}$$

For $z \in \mathbb{D}$ and b on the unit circle, we compute

$$\begin{aligned}
|1-(1-bz)^{\frac{1}{n}}| &= \left|1-\exp\left(\frac{\mathrm{Log}(1-bz)}{n}\right)\right| \\
&= \left|1-\exp\left(\frac{\ln|1-bz|}{n}+\frac{i\mathrm{Arg}(1-bz)}{n}\right)\right| \\
&= \left|1-\exp\left(\frac{\ln|1-bz|}{n}\right)+\exp\left(\frac{\ln|1-bz|}{n}\right)\right. \\
&\qquad \left. -\exp\left(\frac{\ln|1-bz|}{n}+\frac{i\mathrm{Arg}(1-bz)}{n}\right)\right|.
\end{aligned}$$

By the triangle inequality, we have

$$\begin{aligned}
|1-(1-bz)^{\frac{1}{n}}| &\le \left|1-\exp\left(\frac{\ln|1-bz|}{n}\right)\right|+\exp\left(\frac{\ln|1-bz|}{n}\right)\left|1-\exp\left(\frac{i\mathrm{Arg}(1-bz)}{n}\right)\right| \\
&\le \left|1-\exp\left(\frac{\ln 2}{n}\right)\right|+\exp\left(\frac{\ln 2}{n}\right)\left|1-\exp\left(\frac{i\mathrm{Arg}(1-bz)}{n}\right)\right|.
\end{aligned}$$

But

$$\left|1-\exp\left(\frac{i\mathrm{Arg}(1-bz)}{n}\right)\right| = \Big(\big(\cos(\mathrm{Arg}(1-bz)/n)-1\big)^2+\sin^2\big(\mathrm{Arg}(1-bz)/n\big)\Big)^{1/2}.$$

For $z \in \mathbb{D}$, it is clear that $-\pi/2 \le \mathrm{Arg}(1-bz) \le \pi/2$. It follows that

$$\left|1-\exp\left(\frac{i\mathrm{Arg}(1-bz)}{n}\right)\right| \le \Big(\big(\cos(\pi/2n)-1\big)^2+\sin^2\big(\pi/2n\big)\Big)^{1/2},$$

and hence

$$|1-(1-bz)^{\frac{1}{n}}| \le \left|1-\exp\left(\frac{\ln 2}{n}\right)\right|+\exp\left(\frac{\ln 2}{n}\right)\Big(\big(\cos(\pi/2n)-1\big)^2+\sin^2\big(\pi/2n\big)\Big)^{1/2}.$$

The right-hand side quantity is independent of z and tends to 0 as $n \to \infty$. Then we obtain

$$\lim_{n\to\infty}\sup_{|z|<1}|1-(1-bz)^{\frac{1}{n}}| = 0. \tag{2.25}$$

From (2.24) and (2.25), it follows that

$$\lim_{n\to\infty}\|C_\alpha - C_{\alpha-\frac{1}{n}}\|_{\mathfrak{B}_\alpha} = 0,$$

completing the proof. □

Corollary 2.23. *For $\beta = \alpha < 1$, the essential norm of the generalized β-Cesàro operator from $\mathfrak{B}^0_\alpha$ to $\mathfrak{B}^0_\alpha$ is 0.*

Proof. To obtain the essential norm of the generalized β-Cesàro operator for $\beta = \alpha < 1$, we compute

$$\|C_{g_\alpha} - C_{g_{\alpha-\frac{1}{n}}}\|_{\mathfrak{B}_\alpha} = \sup_{\|f\|_{\mathfrak{B}_\alpha}\le 1} \sup_{|z|<1} (1-|z|^2)^\alpha \left| \frac{f(z)g_\alpha(z)}{z} - \frac{f(z)g_{\alpha-\frac{1}{n}}(z)}{z} \right|.$$

To proceed further, we follow the steps given in the proofs of Corollary 2.7 and Theorem 2.22. $\square$

Theorem 2.24. *The essential norm of the β-Cesàro operator from $\mathfrak{B}^0_\alpha$ to $\mathfrak{B}^0_\alpha$ is 0, for $\beta = 1 < \alpha$.*

Proof. Let $\beta = 1 - 1/n$, then by Theorem 2.20, the β-Cesàro operator from $\mathfrak{B}^0_\alpha$ to $\mathfrak{B}^0_\alpha$, for $\alpha > 1$, is a compact linear operator for each $n \in \mathbb{N}$.

Let $\lambda \in (0, 1)$ be fixed for the moment. Then we have

$$\begin{aligned} \|C_1 - C_{1-\frac{1}{n}}\|_{\mathfrak{B}_\alpha} &= \sup_{\|f\|_{\mathfrak{B}_\alpha}\le 1} \sup_{|z|<1} (1-|z|^2)^\alpha \left| \frac{f(z)}{z(1-z)} - \frac{f(z)}{z(1-z)^{1-\frac{1}{n}}} \right| \\ &\le \sup_{\|f\|_{\mathfrak{B}_\alpha}\le 1} \sup_{|z|\le\lambda} (1-|z|^2)^\alpha \left| \frac{f(z)}{z(1-z)} - \frac{f(z)}{z(1-z)^{1-\frac{1}{n}}} \right| \\ &\quad + \sup_{\|f\|_{\mathfrak{B}_\alpha}\le 1} \sup_{|z|>\lambda} (1-|z|^2)^\alpha \left| \frac{f(z)}{z(1-z)} - \frac{f(z)}{z(1-z)^{1-\frac{1}{n}}} \right|. \end{aligned} \tag{2.26}$$

For $\alpha > 1$, from (2.5), we have

$$|f(z)| \le \frac{\|f\|_{\mathfrak{B}_\alpha}}{\alpha-1}\left(\frac{1}{(1-|z|)^{\alpha-1}} - 1\right).$$

We obtain

$$\begin{aligned} &\sup_{\|f\|_{\mathfrak{B}_\alpha}\le 1} \sup_{|z|>\lambda} (1-|z|^2)^\alpha \left| \frac{f(z)}{z(1-z)} - \frac{f(z)}{z(1-z)^{1-\frac{1}{n}}} \right| \\ &\le \sup_{\|f\|_{\mathfrak{B}_\alpha}\le 1} \sup_{|z|>\lambda} \frac{(1-|z|^2)^\alpha}{|z|(1-|z|)} |1-(1-z)^{\frac{1}{n}}||f(z)| \\ &\le \frac{2^\alpha}{\lambda} \sup_{|z|>\lambda} |1-(1-z)^{\frac{1}{n}}|\left(1-(1-|z|)^{\alpha-1}\right) \\ &\le \frac{2^\alpha}{\lambda} \sup_{|z|>\lambda} |1-(1-z)^{\frac{1}{n}}|, \end{aligned}$$

which tends to 0 as $n \to \infty$ due to (2.25). The first term of the right-hand quantity of (2.26) also tends to 0 as $n \to \infty$. It yields

$$\lim_{n\to\infty} \|C_1 - C_{1-\frac{1}{n}}\|_{\mathfrak{B}_\alpha} = 0,$$

which concludes the proof. ☐

Corollary 2.25. *For $\beta = 1 < \alpha$, the essential norm of the generalized β-Cesàro operator from $\mathfrak{B}^0_\alpha$ to $\mathfrak{B}^0_\alpha$ is* 0.

Proof. The proof follows the steps as in the proofs of Corollary 2.23 and Theorem 2.24. ☐

2.4. Spectral Properties of the β-Cesàro operators

In this section, we compute the (point) spectrum of the generalized β-Cesàro operators mapping from $\mathfrak{B}^0_\alpha$ to $\mathfrak{B}^0_\alpha$ for $\alpha > 0$. For the concept of spectral analysis, we refer to [21, 56, 64].

Theorem 2.26. *For $\alpha \geq 1$, the point spectrum of the generalized Cesàro operator from $\mathfrak{B}^0_\alpha$ to $\mathfrak{B}^0_\alpha$ is*

$$\sigma_P(C_{g_1}) = \left\{ \frac{g_1(0)}{n}, n \in \mathbb{N} : \operatorname{Re}\left(\frac{a_j}{g_1(0)}\right) \leq 0, 1 \leq j \leq k \right\}.$$

Proof. We adopt the idea of the proof partially from the work of Albrecht, Miller, and Neumann [9]. Suppose that $f \in \mathfrak{B}^0_\alpha \setminus \{0\}$. Write $f = z^n\psi$, where $n \geq 1$ and ψ is an analytic function in $\mathbb{D}$ with $\psi(0) \neq 0$. We desire to show that $\psi \in \mathfrak{B}_\alpha$.

Suppose $(\lambda I - C_{g_\beta})f = 0$. We obtain that

$$\lambda f(z) - \int_0^z \frac{f(w)}{w} g_\beta(w)dw = 0,$$

which is equivalent to

$$\lambda z^n \psi(z) - \int_0^z \frac{w^n\psi(w)}{w} g_\beta(w)dw = 0.$$

Differentiating both sides with respect to nonzero $z \in \mathbb{D}$, we obtain

$$(\lambda n z^{n-1}\psi(z) + \lambda z^n \psi'(z)) - z^{n-1}\psi(z)g_\beta(z) = 0,$$

which is equivalent to

$$(\lambda n \psi(z) + \lambda z \psi'(z)) - \psi(z)g_\beta(z) = 0. \tag{2.27}$$

By the continuity, (2.27) also holds at 0. Then at $z = 0$ we have

$$\lambda n \psi(0) - \psi(0)g_\beta(0) = 0.$$

On simplification, we obtain

$$\lambda = \frac{g_\beta(0)}{n}.$$

Then we have the possible point spectrum

$$\sigma_P(C_{g_\beta}) \subseteq \left\{ \frac{g_\beta(0)}{n} : n \in \mathbb{N} \right\}.$$

If $g_\beta(0) = 0$, then $\sigma_P(C_{g_\beta}) = \emptyset$.

We further need to obtain the condition under which the generalized Cesàro operator has eigenvector corresponding to $g_1(0)/n$. If $g_1(0) \neq 0$, then with $g_1(0)/n$, (2.27) has a solution $\psi = c\psi_n$, where

$$\begin{aligned}\psi_n(z) &= \exp\left(\frac{n}{g_1(0)} \int_0^z \frac{g_1(w) - g_1(0)}{w} dw \right) \\ &= \exp\left(\frac{n}{g_1(0)} \int_0^z \frac{\sum_{j=1}^k \frac{a_j}{1-b_j w} + h(w) - \sum_{j=1}^k a_j - h(0)}{w} dw \right).\end{aligned}$$

This simplifies to

$$\psi_n(z) = \exp\left(\frac{n}{g_1(0)} \sum_{j=1}^k \int_0^z \frac{a_j b_j}{1 - b_j w} dw \right) \exp\left(\frac{n}{g_1(0)} \int_0^z \frac{h(w) - h(0)}{w} dw \right). \tag{2.28}$$

Set

$$\phi(z) = \exp\left(\frac{n}{g_1(0)} \sum_{j=1}^k \int_0^z \frac{a_j b_j}{1 - b_j w} dw \right)$$

and

$$\eta(z) = \exp\left(\frac{n}{g_1(0)} \int_0^z \frac{h(w) - h(0)}{w} dw \right).$$

Then we have, by Schwarz Lemma

$$\exp\left(-2\left| \frac{n}{g_1(0)} \right| \|h\|_\infty \right) \leq |\eta(z)| \leq \exp\left(2\left| \frac{n}{g_1(0)} \right| \|h\|_\infty \right). \tag{2.29}$$

Next we show that ϕ is bounded. For this, we compute

$$\begin{aligned}|\phi(z)| &= \left| \exp\left(\frac{n}{g_1(0)} \sum_{j=1}^k \int_0^z \frac{a_j b_j}{1 - b_j w} dw \right) \right| \\ &= \left| \exp\left(\sum_{j=1}^k -n\left(\frac{a_j}{g_1(0)} \right) \log(1 - b_j z) \right) \right|.\end{aligned}$$

This is equivalent to

$$|\phi(z)| = \left| \prod_{j=1}^k (1 - b_j z)^{-n(a_j/g_1(0))} \right|.$$

To prove the boundedness of $|\phi(z)|$, it is sufficient to show that for each j, $1 \le j \le k$, the quantity $(1-b_jz)^{-n(a_j/g_1(0))}$ is bounded in $\mathbb{D}$. For this purpose, we see that

$$\begin{aligned}(1-b_jz)^{-n(a_j/g_1(0))} &= \exp\Big(-n\frac{a_j}{g_1(0)}\log(1-b_jz)\Big)\\ &= \exp\Big(-n\frac{a_j}{g_1(0)}\big(\ln|1-b_jz|+i\arg(1-b_jz)\big)\Big)\\ &= \exp\Big(-n\frac{a_j}{g_1(0)}\big(\ln|1-b_jz|\big)\Big)\exp\Big(-n\frac{a_j}{g_1(0)}i\arg(1-b_jz)\Big).\end{aligned}$$

Now,

$$\begin{aligned}&\Big|\exp\Big(-n\frac{a_j}{g_1(0)}\big(\ln|1-b_jz|\big)\Big)\Big|\\ &\quad=\Big|\exp\Big(\big(-n\ln|1-b_jz|\big)\Big(\mathrm{Re}\Big(\frac{a_j}{g_1(0)}\Big)+i\ \mathrm{Im}\Big(\frac{a_j}{g_1(0)}\Big)\Big)\Big)\Big|\\ &\quad=\Big|\exp\Big(\big(-n\ln|1-b_jz|\big)\mathrm{Re}\Big(\frac{a_j}{g_1(0)}\Big)\Big)\Big|.\end{aligned}$$

If $\mathrm{Re}\,(a_j/g_1(0)) \le 0$, then

$$\Big|\exp\Big(\big(-n\ln|1-b_jz|\big)\mathrm{Re}\Big(\frac{a_j}{g_1(0)}\Big)\Big)\Big|$$

is bounded. This implies that ϕ is bounded analytic function in $\mathbb{D}$.

Differentiating $\psi_n(z)$ with respect to z, we obtain

$$\begin{aligned}\psi_n'(z) =&\frac{n}{g_1(0)}\left(\frac{\sum_{j=1}^k \frac{a_j}{1-b_jz}+h(z)-\sum_{j=1}^k a_j - h(0)}{z}\right)\psi_n(z)\\ =&\frac{n}{g_1(0)}\Big(\sum_{j=1}^k \frac{a_jb_j}{1-b_jz}+\frac{h(z)-h(0)}{z}\Big)\psi_n(z).\end{aligned}$$

Write

$$\rho(z) = \frac{n}{g_1(0)}\Big(\sum_{j=1}^k \frac{a_jb_j}{1-b_jz}+\frac{h(z)-h(0)}{z}\Big).$$

Now we need to check under what condition we have $\psi_n(z) \in \mathfrak{B}_\alpha$. For this purpose, we need to calculate $(1-|z|^2)^\alpha|\psi_n'(z)|$. Now

$$\begin{aligned}(1-|z|^2)^\alpha|\psi_n'(z)| =&(1-|z|^2)^\alpha|\rho(z)\psi_n(z)|\\ =&(1-|z|^2)^\alpha|\rho(z)\phi(z)\eta(z)|\\ =&(1-|z|^2)^\alpha|\rho(z)||\phi(z)||\eta(z)|.\end{aligned}$$

For $\alpha \geq 1$,

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|\rho(z)| < \infty$$

and for $\alpha < 1$, this is unbounded as can be seen from Example 2.9. As we have already seen that ϕ and η are bounded analytic functions in $\mathbb{D}$, we obtain

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|\psi(z)| < \infty,$$

as desired, to have $f \in \mathfrak{B}^0_\alpha$, $\alpha \geq 1$. $\square$

If $g_1(w) = h(w)$ for each $w \in \mathbb{D}$ as in (1.16), then by (2.28) and (2.29) we establish

Theorem 2.27. *The point spectrum of the generalized Alexander operator from $\mathfrak{B}^0_\alpha$ to $\mathfrak{B}^0_\alpha$ is*

$$\sigma_P(C_{g_0}) = \left\{\frac{g_0(0)}{n}, n \in \mathbb{N}\right\}.$$

Example 2.28. *Let $f_n(z) = z^n/n$ for $n \in \mathbb{N}$. Clearly for each $n \in \mathbb{N}$, $f_n \in \mathfrak{B}^0_\alpha$. Define*

$$h_n(z) = \frac{z^n}{n\|f_n\|_{\mathfrak{B}_\alpha}}.$$

We can easily obtain that $\|h_n\|_{\mathfrak{B}_\alpha} = 1$. We estimate

$$\|C_{g_0}(h_n)(z)\|_{\mathfrak{B}_\alpha} = \left\|\int_0^z \frac{h_n(t)g_0(t)}{t}dt\right\|_{\mathfrak{B}_\alpha} \leq \frac{\|g_0\|_\infty}{n\|f_n\|_{\mathfrak{B}_\alpha}}\left\|\int_0^z t^{n-1}dt\right\|_{\mathfrak{B}_\alpha} = \frac{\|g_0\|_\infty}{n},$$

which tends to 0 as n tends to ∞. This implies that (h_n) is an approximate eigenvector with eigenvalue 0. The definition of approximate eigenvector is found in [21, chapter 12]. In other words, we also say that 0 is the approximate eigenvalue of C_{g_0}.

Using Theorem 2.15 or Theorem 2.19 together with Theorem 2.27, we can compute the spectrum of the generalized Alexander operator on $\mathfrak{B}^0_\alpha$, which is stated in Theorem 1.1.

Remark 2.29. *The approximate eigenvalue stated in Theorem 1.1 is also discussed in Example* 2.28.

The following theorem provides us the point spectrum of the β-Cesàro operator from $\mathfrak{B}^0_\alpha$ to itself for various choices of positive β. The non-positive values of β turns the operator into the generalized Alexander operator which is already covered in Theorem 1.1.

Theorem 2.30. *The point spectrum of the generalized β-Cesàro operator from $\mathfrak{B}^0_\alpha$ to $\mathfrak{B}^0_\alpha$, either for $0<\beta\leq\alpha<1$ or $0<\beta<1<\alpha$ or $0<\beta<\alpha=1$, is*

$$\sigma_P(C_{g_\beta})=\left\{\frac{g_\beta(0)}{n},n\in\mathbb{N}\right\}.$$

Proof. If $g_\beta(0)\neq 0$, then with $g_\beta(0)/n$, (2.27) has a solution $\psi=c\psi_n$, where

$$\begin{aligned}\psi_n(z)&=\exp\left(\frac{n}{g_\beta(0)}\int_0^z\frac{g_\beta(w)-g_\beta(0)}{w}dw\right)\\&=\exp\left(\frac{n}{g_\beta(0)}\int_0^z\frac{\sum_{j=1}^k\frac{a_j}{(1-b_jw)^\beta}+h(w)-\sum_{j=1}^k a_j-h(0)}{w}dw\right).\end{aligned}$$

This is equivalent to

$$\psi_n(z)=\exp\left(\frac{n}{g_\beta(0)}\sum_{j=1}^k\int_0^z\frac{\frac{a_j}{(1-b_jw)^\beta}-a_j}{w}dw\right)\exp\left(\frac{n}{g_\beta(0)}\int_0^z\frac{h(w)-h(0)}{w}dw\right).\tag{2.30}$$

We have already proved in Theorem 2.26 that the second factor is bounded. Hence, it remains to consider only the first factor here. Recall that

$$\frac{1}{(1-b_jw)^\beta}=\sum_{n=0}^\infty\frac{\Gamma(n+\beta)}{\Gamma(n+1)\Gamma(\beta)}(b_jw)^n,$$

where Γ is the classical Euler gamma function. We compute the integral

$$\begin{aligned}\int_0^z\frac{\frac{a_j}{(1-b_jw)^\beta}-a_j}{w}dw&=a_j\int_0^z\sum_{n=1}^\infty\frac{\Gamma(n+\beta)}{\Gamma(n+1)\Gamma(\beta)}b_j^nw^{n-1}dw\\&=a_j\sum_{n=1}^\infty\frac{\Gamma(n+\beta)}{\Gamma(n+1)\Gamma(\beta)}\frac{(b_jz)^n}{n}.\end{aligned}$$

We know that $\Gamma(n+\beta)/\Gamma(\beta)=(\beta)_n$ for $\beta\geq 0$ and $n\geq 0$, where $(\beta)_n$ denotes the shifted factorial defined by

$$(a)_n=a(a+1)\dots(a+n-1)$$

for $n>0$, and $(a)_0=1$ for a complex number a. Then

$$\int_0^z\frac{\frac{a_j}{(1-b_jw)^\beta}-a_j}{w}dw=a_j\sum_{n=1}^\infty\frac{(\beta)_n}{n}\frac{(b_jz)^n}{n!}.$$

This implies that

$$\left|\int_0^z\frac{\frac{a_j}{(1-b_jw)^\beta}-a_j}{w}dw\right|\leq|a_j|\sum_{n=1}^\infty\frac{(\beta)_n}{n}\frac{|b_j|^n}{n!}$$

on the circle of convergence $|z| = 1$. From now onward assume that $\beta > 0$. We set $\delta = \beta/m$, $m \in \mathbb{N}$. For $|z| = 1$, comparing the terms of above series with the corresponding terms of the convergent series

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+\delta}},$$

and to use the limit comparison test, we compute

$$\lim_{n\to\infty} \left| \frac{n^{\delta}(\beta)_n}{n!} \right| = \lim_{n\to\infty} \left| \frac{(\beta)_n}{(n-1)!n^{\beta}} \frac{(n-1)!n^{\delta+\beta}}{n!} \right|.$$

Since

$$\frac{1}{\Gamma(\beta)} = \lim_{n\to\infty} \frac{(\beta)_n}{(n-1)!n^{\beta}}.$$

We obtain

$$\lim_{n\to\infty} \left| \frac{n^{\delta}(\beta)_n}{n!} \right| = \left| \frac{1}{\Gamma(\beta)} \right| \lim_{n\to\infty} \left| \frac{(n-1)!n^{\delta+\beta}}{n!} \right| = \left| \frac{1}{\Gamma(\beta)} \right| \lim_{n\to\infty} \left| \frac{1}{n^{1-\delta-\beta}} \right|,$$

which tends to 0 as $n \to \infty$, if $1 - \delta - \beta > 0$, i.e., if $\beta < m/(m+1) < 1$, since $\delta = \beta/m$. This implies that the series is absolutely convergent for $|z| = 1$. So $\psi_n(z)$ is a bounded analytic function in $\mathbb{D}$.

Differentiating $\psi_n(z)$ with respect to z, we obtain

$$\begin{aligned} \psi_n'(z) =& \frac{n}{g_\beta(0)} \left(\frac{\sum_{j=1}^{k} \frac{a_j}{(1-b_j z)^{\beta}} + h(z) - \sum_{j=1}^{k} a_j - h(0)}{z} \right) \psi_n(z) \\ =& \frac{n}{g_\beta(0)} \left(\sum_{j=1}^{k} \frac{\frac{a_j}{(1-b_j z)^{\beta}} - a_j}{z} + \frac{h(z) - h(0)}{z} \right) \psi_n(z). \end{aligned}$$

Since $\beta \le \alpha$, $h(z)$ and $\psi_n(z)$ are bounded analytic functions in $\mathbb{D}$, it follows that

$$\sup_{z\in\mathbb{D}} (1 - |z|^2)^{\alpha} |\psi_n'(z)| < \infty,$$

as desired to have $\psi \in \mathfrak{B}_\alpha$, and consequently $f \in \mathfrak{B}_\alpha^0$ either for $0 < \beta \le \alpha < 1$ or $0 < \beta < 1 < \alpha$ or $0 < \beta < \alpha = 1$. $\square$

Remark 2.31. *Define* $\chi_n(z) = \psi_n(z)/\|\psi_n\|_{\mathfrak{B}_\alpha}$, *where* ψ_n *is defined as in* (2.28) *and* (2.30) *according to the value of* α *and* β. *We know that*

$$C_{g_\beta}(\chi_n(z)) = \frac{g_\beta(0)}{n} \chi_n(z)$$

either for $0 < \beta \le \alpha < 1$ *or* $\beta \le 1 < \alpha$ *or* $\beta < \alpha = 1$. *Then we obtain*

$$\|C_{g_\beta}(\chi_n)\|_{\mathfrak{B}_\alpha} = \left|\frac{g_\beta(0)}{n}\right|.$$

The right-hand side approaches to 0 *as* n *tends to* ∞ *i.e.,* 0 *is an approximate eigenvalue of the generalized* β*-Cesàro operator with the approximate eigenvector* $\chi_n(z)$.

By using the compactness properties (see Sections 2.2 and 2.3) of the β-Cesàro operators from $\mathfrak{B}^0_\alpha$ to $\mathfrak{B}^0_\alpha$, for $0 < \beta \le \alpha < 1$, $\beta \le 1 < \alpha$ and $\beta < \alpha = 1$, we establish Theorem 1.2.

2.5. An Application: separability of the space $\mathfrak{B}^0_\alpha$

Theorem 2.24 says that the Cesàro operator is a compact linear operator on $\mathfrak{B}^0_{\alpha+1}$, for $\alpha > 0$. Then by [56, Theorem 8.2-3], the range $\mathfrak{R}(C_1)$ is separable. Therefore, we obtain the following property of $\mathfrak{B}^0_\alpha$, with the help of the Cesàro operator.

Theorem 2.32. $\mathfrak{B}^0_\alpha$ *is a separable space in the space* $\mathfrak{B}^0_{\alpha+1}$, *for* $\alpha > 0$.

Proof. To prove this theorem, we need to show that the range $\mathfrak{R}(C_1)$ contains $\mathfrak{B}^0_\alpha$, for $\alpha > 0$, equivalently for $g \in \mathfrak{B}^0_\alpha$ there exists an $f \in \mathfrak{B}^0_{\alpha+1}$ such that $C_1(f) = g$.

Let $g \in \mathfrak{B}^0_\alpha$ and define $f(z) = z(1-z)g'(z)$. Then $g'(z) = f(z)/z(1-z)$. Taking the line integral from 0 to z, we get $g(z) = \int_0^z f(t)/t(1-t)dt$. Now we estimate

$$\begin{aligned}
\sup_{z\in\mathbb{D}}(1-|z|^2)^{\alpha+1}|f'(z)| &= \sup_{z\in\mathbb{D}}(1-|z|^2)^{\alpha+1}|(1-z)g'(z) - zg'(z) + z(1-z)g''(z)| \\
&\le 3\sup_{z\in\mathbb{D}}(1-|z|^2)^{\alpha+1}|g'(z)| + 2\sup_{z\in\mathbb{D}}(1-|z|^2)^{\alpha+1}|g''(z)| \\
&\le 3\sup_{z\in\mathbb{D}}(1-|z|^2)^{\alpha}|g'(z)| + 2\sup_{z\in\mathbb{D}}(1-|z|^2)^{\alpha+1}|g''(z)|.
\end{aligned}$$

By [108, Proposition 8], we obtain

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^{\alpha+1}|g''(z)| < \infty,$$

which says that $f \in \mathfrak{B}^0_{\alpha+1}$. □

CHAPTER 3

BOHR INEQUALITIES

In this chapter[1], we determine sharp Bohr-type radii for certain complex integral operators defined on a set of bounded analytic functions in the unit disk.

3.1. Results on the β-Cesáro and Bernardi operators

Note that the β-Cesáro operator T_β ($\beta > 0$) is a natural generalization of the Cesáro operator T defined by (1.17) and indeed, we have $T_1 = T$. For $f \in \mathcal{B}$ and $\beta > 0$, an elementary estimation of the integral in absolute value gives us the sharp inequality

$$|T_\beta[g](z)| \leq \begin{cases} \dfrac{1}{r}\left[\dfrac{1-(1-r)^{1-\beta}}{1-\beta}\right], & \text{if } \beta \neq 1, \\[2ex] \dfrac{1}{r}\log\dfrac{1}{1-r}, & \text{if } \beta = 1, \end{cases}$$

for each $|z| = r < 1$. In this line, similar to Theorem C, our first main result is Theorem 1.3.

Here, it is easy to observe that if we take the limit $\beta \to 1$ in Theorem 1.3 then we can obtain Theorem C.

Remark 3.1. *Another form of the β-Cesáro operator of a normalized analytic function $g(z) = \sum_{n=1}^{\infty} b_n z^n$ in $\mathbb{D}$ has been studied in the literature (see [59]):*

$$C_\beta[g](z) = \int_0^1 \frac{g(tz)}{t(1-tz)^\beta}dt = \sum_{n=0}^{\infty}\left(\frac{1}{n+1}\sum_{k=0}^{n}\frac{\Gamma(n-k+\beta)}{\Gamma(n-k+1)\Gamma(\beta)}b_{k+1}\right)z^{n+1},\ z \in \mathbb{D},$$

for $\beta > 0$. This version of the β-Cesáro operator was initially considered to study its boundedness, compactness, and spectral properties, and more recently its univalency properties were investigated in [58]. To study its Bohr radius problem, it is necessary for us

[1]This chapter is based on the paper: Kumar S., Sahoo S.K., *Bohr inequalities for certain integral operators*, Accepted in Mediterr. J. Math., `arXiv:2008.00468`.

to assume that $g(z) = \sum_{n=1}^{\infty} b_n z^n \in \mathcal{B}_0$. *An easy calculation gives us the sharp inequality, for* $g \in \mathcal{B}_0$ *and* $\beta > 0$,

$$|C_\beta[g](z)| \leq \begin{cases} \dfrac{1-(1-r)^{1-\beta}}{1-\beta}, & \text{if } \beta \neq 1, \\ \log \dfrac{1}{1-r}, & \text{if } \beta = 1, \end{cases}$$

for each $|z| = r < 1$. *It is well-known by the Schwarz lemma that if* $g(z) = \sum_{n=1}^{\infty} b_n z^n \in \mathcal{B}_0$ *then we can write* $g(z) = zh(z)$ *for* $h(z) = \sum_{n=0}^{\infty} b_{n+1} z^n \in \mathcal{B}$. *So, we have*

$$C_\beta[g](z) = \sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{k=0}^{n} \frac{\Gamma(n-k+\beta)}{\Gamma(n-k+1)\Gamma(\beta)} b_{k+1} \right) z^{n+1} = zT_\beta[h](z).$$

Now, by using Theorem 1.3 we obtain

$$\sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{k=0}^{n} \frac{\Gamma(n-k+\beta)}{\Gamma(n-k+1)\Gamma(\beta)} |b_{k+1}| \right) r^{n+1} \leq \frac{1-(1-r)^{1-\beta}}{1-\beta}, \ 0 < \beta \neq 1,$$

for $r \leq R(\beta)$. *Here* $R(\beta)$ *is the positive root of the equation*

$$\frac{3[1-(1-x)^{1-\beta}]}{1-\beta} - \frac{2[(1-x)^{-\beta}-1]}{\beta} = 0$$

that cannot be improved. Recall that the operator C_1 *has been considered in [37, 58, 59, 86] for various aspects. Moreover, in the limit* $\beta \to 1$, *we can indeed obtain the Bohr radius problem:* If $g(z) = \sum_{n=1}^{\infty} b_n z^n \in \mathcal{B}_0$ then

$$\sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{k=0}^{n} |b_{k+1}| \right) r^{n+1} \leq \log \frac{1}{1-r}$$

for $r \leq R = 0.5335\ldots$. The number R is the positive root of the equation

$$2x - 3(1-x)\log \frac{1}{1-x} = 0$$

that cannot be improved. *This remark observes that the Bohr radii for the operators* T_β *and* C_β *are exactly the same.* □

Similar to the Bohr-type radius problem for the operator T_β, $\beta > 0$, we also study the Bohr radius of the absolute series of the Bernardi operator [72, P. 11] (see also [85]) defined by

$$L_\gamma[f](z) := \sum_{n=m}^{\infty} \frac{a_n}{n+\gamma} z^n = \int_0^1 f(zt) t^{\gamma-1} dt,$$

for $f(z) = \sum_{n=m}^{\infty} a_n z^n$ and $\gamma > -m$, here $m \geq 0$ is an integer. The function $L_\gamma[f]$ is analytic in $\mathbb{D}$ and several properties of $L_\gamma[f]$ when $m = 1$ (with a normalization) are well-known (see, for instance [72, 78, 85]).

It is easy to calculate the following sharp bound

$$|L_\gamma[f](z)| \leq \frac{1}{m+\gamma} r^m, \ |z| = r < 1$$

for $f(z) = \sum_{n=m}^{\infty} a_n z^n$. Corresponding to the above inequality, we obtain the following result.

Theorem 3.2. *Let $\gamma > -m$. If $f(z) = \sum_{n=m}^{\infty} a_n z^n \in \mathcal{B}$, then*

$$\sum_{n=m}^{\infty} \frac{|a_n|}{n+\gamma} r^n \leq \frac{1}{m+\gamma} r^m$$

for $r \leq R(\gamma)$. Here, $R(\gamma)$ is the positive root of the equation

$$\frac{x^m}{m+\gamma} - 2 \sum_{n=m+1}^{\infty} \frac{x^n}{n+\gamma} = 0$$

that cannot be improved.

Letting $\gamma = 1$ and $m = 0$ in the Bernardi operator L_γ, we obtain the well-known Libera operator [72, 85] defined as

$$L[f](z) := \int_0^1 f(zt)\, dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} z^n.$$

The multiplication of z in the Libera operator L gives the integral

$$I[f](z) := \sum_{n=0}^{\infty} \frac{a_n}{n+1} z^{n+1} = \int_0^z f(w) dw, \ |z| < 1.$$

It is easy to check that

$$|L[f](z)| \leq 1 \text{ and } |I[f](z)| \leq r, \ |z| = r.$$

As a special case of Theorem 3.2 ($\gamma = 1$ and $m = 0$), we get the Bohr radius for the Libera operator as well as for the operator I as follows.

Corollary 3.3. *If $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{B}$, then*

$$\sum_{n=0}^{\infty} \frac{|a_n|}{n+1} r^n \leq 1,$$

for $r \leq R$ with $R = 0.5828\ldots$, the positive root of the equation $3x + 2\log(1-x) = 0$. Here, R is the best possible.

Also, the Alexander operator [27, 58, 59, 72]

$$J[g](z) := \int_0^1 \frac{g(zt)}{t} dt = \sum_{n=1}^{\infty} \frac{b_n}{n} z^n,$$

for $g(z) = \sum_{n=1}^{\infty} b_n z^n$, extensively studied in the univalent function theory. We have sharp bound

$$|J[g](z)| \leq r$$

for each $|z| = r < 1$, since $|g(zt)/t| \leq 1$ here. Then from the observation of the Schwarz lemma, for every $g \in \mathcal{B}_0$ we can obtain an element $h \in \mathcal{B}$ such that $g(z) = zh(z)$. So, we have the following result as a consequence of Corollary 3.3.

Corollary 3.4. *If $g(z) = \sum_{n=1}^{\infty} b_n z^n \in \mathcal{B}_0$, then*

$$\sum_{n=1}^{\infty} \frac{|b_n|}{n} r^n \leq r,$$

for $r \leq R$. Here, $R = 0.5828\cdots$ is the positive root of the equation $3x + 2\log(1-x) = 0$ that cannot be improved.

In the next section, we discuss the proofs of Theorems 1.3 and 3.2.

3.2. Proofs of the results

Proof of Theorem 1.3

First we define

$$T_\beta^f(r) := \sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{k=0}^{n} \frac{\Gamma(n-k+\beta)}{\Gamma(n-k+1)\Gamma(\beta)} |a_k| \right) r^n, \tag{3.1}$$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{B}$, $0 < \beta \neq 1$ and $r = |z| < 1$. We set $|a_0| := a$ and let $a < 1$. By Wiener's estimate we know that $|a_n| \leq 1 - a^2$ for $n \geq 1$. This yields

$$T_\beta^f(r) \leq a \sum_{n=0}^{\infty} \left(\frac{1}{n+1} \frac{\Gamma(n+\beta)}{\Gamma(n+1)\Gamma(\beta)} \right) r^n + (1-a^2) \sum_{n=1}^{\infty} \left(\frac{1}{n+1} \sum_{k=1}^{n} \frac{\Gamma(n-k+\beta)}{\Gamma(n-k+1)\Gamma(\beta)} \right) r^n.$$

The above inequality is equivalent to

$$T_\beta^f(r) \le \frac{a}{r}\int_0^r \frac{1}{(1-t)^\beta}\,dt + \frac{(1-a^2)}{r}\int_0^r \frac{t}{(1-t)^{\beta+1}}\,dt$$
$$= \frac{(a^2+a-1)}{r}\int_0^r \frac{1}{(1-t)^\beta}\,dt + \frac{(1-a^2)}{r}\int_0^r \frac{1}{(1-t)^{\beta+1}}\,dt.$$

It follows that

$$T_\beta^f(r) \le \frac{1}{r}\left[\frac{(a^2+a-1)[1-(1-r)^{1-\beta}]}{1-\beta} + \frac{(1-a^2)[(1-r)^{-\beta}-1]}{\beta}\right] := \phi(a).$$

Differentiation of the function ϕ with respect to a gives us

$$\phi'(a) = \frac{1}{r}\left[\frac{(2a+1)[1-(1-r)^{1-\beta}]}{1-\beta} - \frac{2a[(1-r)^{-\beta}-1]}{\beta}\right]$$

and so

$$\phi''(a) = \frac{1}{r}\left[\frac{2[1-(1-r)^{1-\beta}]}{1-\beta} - \frac{2[(1-r)^{-\beta}-1]}{\beta}\right].$$

It is easy to see that $\phi''(a) \le 0$ for every $a \in [0,1)$ and $r \in [0,1)$. This provides that $\phi'(a) \ge \phi'(1)$. Here

$$\phi'(1) = \frac{1}{r}\left[\frac{3[1-(1-r)^{1-\beta}]}{1-\beta} - \frac{2[(1-r)^{-\beta}-1]}{\beta}\right] \ge 0$$

holds for $r \le R(\beta)$, where $R(\beta)$ is the positive root of the equation

$$\frac{3[1-(1-x)^{1-\beta}]}{1-\beta} - \frac{2[(1-x)^{-\beta}-1]}{\beta} = 0.$$

Then $\phi(a)$ is an increasing function of a, for $r \le R(\beta)$. It implies that

$$\phi(a) \le \phi(1) = \frac{1}{r}\left[\frac{1-(1-r)^{1-\beta}}{1-\beta}\right],$$

for $r \le R(\beta)$. It is easy to observe that $R(\beta) < 1$. This completes the first part of the theorem.

To conclude the final part, we consider the function

$$\phi_a(z) = \frac{z-a}{1-az} = -a + (1-a^2)\sum_{n=1}^{\infty} a^{n-1}z^n,$$

where $z \in \mathbb{D}$ and $a \in [0,1)$. By using (3.1), we obtain the sum

$$T_\beta^{\phi_a}(r) = \frac{a}{r}\left[\frac{1-(1-r)^{1-\beta}}{1-\beta}\right] + (1-a^2)\sum_{n=1}^{\infty}\left(\frac{1}{n+1}\sum_{k=1}^{n}\frac{\Gamma(n-k+\beta)}{\Gamma(n-k+1)\Gamma(\beta)}a^{k-1}\right)r^n$$
$$= \frac{a}{r}\left[\frac{1-(1-r)^{1-\beta}}{1-\beta}\right] + \frac{(1-a^2)}{r}\int_0^r \frac{t}{(1-at)(1-t)^\beta}dt.$$

We can rewrite the last expression as

(3.2)
$$T_\beta^{\phi_a}(r) = \frac{1}{r}\left[\frac{1-(1-r)^{1-\beta}}{1-\beta}\right] - \frac{(1-a)}{r}\left[\frac{3[1-(1-r)^{1-\beta}]}{1-\beta} - \frac{2[(1-r)^{-\beta}-1]}{\beta}\right] + N_a(r),$$

where

$$N_a(r) = \frac{2(1-a)}{r}\left[\frac{[1-(1-r)^{1-\beta}]}{1-\beta} - \frac{[(1-r)^{-\beta}-1]}{\beta}\right] + \frac{(1-a^2)}{r}\int_0^r \frac{t}{(1-at)(1-t)^\beta}dt.$$

Using the series representation of $N_a(r)$, we have

$$N_a(r) = \sum_{n=0}^{\infty}\frac{1}{n+1}\Bigg(-\frac{(1-a)^2}{a}\frac{\Gamma(n+\beta)}{\Gamma(n+1)\Gamma(\beta)} - 2(1-a)\frac{\Gamma(n+\beta+1)}{\Gamma(n+1)\Gamma(\beta+1)} + \frac{(1-a^2)}{a}\sum_{m=0}^{n}\frac{\Gamma(n-m+\beta)}{\Gamma(n-m+1)\Gamma(\beta)}a^m\Bigg)r^n.$$

By using the identity

$$\sum_{m=0}^{n}\frac{\Gamma(n-m+\beta)}{\Gamma(n-m+1)\Gamma(\beta)} = \frac{\Gamma(n+\beta+1)}{\Gamma(n+1)\Gamma(\beta+1)},$$

we can get that $N_a(r) = O((1-a)^2)$, as a tends to 1. Further, a simple computation shows that for $r > R(\beta)$ the quantity

$$\frac{3[1-(1-r)^{1-\beta}]}{1-\beta} - \frac{2[(1-r)^{-\beta}-1]}{\beta} < 0.$$

After using these observations in (3.2) we conclude that $R(\beta)$ cannot be improved. This completes the proof. $\square$

Proof of Theorem 3.2

Given that $f(z) = \sum_{n=m}^{\infty} a_n z^n \in \mathcal{B}$. We set the notation

$$L_f(r) := \sum_{n=m}^{\infty}\frac{|a_n|}{n+\gamma}r^n. \tag{3.3}$$

It is evident that $f(z) = z^m h(z)$, where $h(z) = \sum_{n=m}^{\infty} a_n z^{n-m}$. Denoting by $a := |a_m| < 1$ and using the Wiener estimate $|a_n| \leq (1-a^2)$ for $n \geq m+1$ in (3.3), we obtain the following inequality

$$L_f(r) \leq \frac{a}{m+\gamma}r^m + (1-a^2)\sum_{n=m+1}^{\infty}\frac{1}{n+\gamma}r^n := \psi(a).$$

It is easy to see that

$$\psi''(a) = -2\sum_{n=m+1}^{\infty}\frac{1}{n+\gamma}r^n \leq 0.$$

Thus,

$$\psi'(a) \geq \psi'(1) = \frac{1}{m+\gamma}r^m - 2\sum_{n=m+1}^{\infty}\frac{1}{n+\gamma}r^n \geq 0,$$

for $r \leq R(\gamma)$, where $R(\gamma)$ is the positive root of the equation

$$\frac{1}{m+\gamma}r^m - 2\sum_{n=m+1}^{\infty}\frac{1}{n+\gamma}r^n = 0.$$

Hence, $\psi(a)$ is an increasing function of a for $r \leq R(\gamma)$. This gives

$$\sum_{n=m}^{\infty}\frac{|a_n|}{n+\gamma}r^n \leq \frac{1}{m+\gamma}r^m, \quad \text{for } r \leq R(\gamma).$$

Also, a simple observation gives $R(\gamma) < 1$.

To prove $R(\gamma)$ to be the best possible bound, we consider the function

$$\psi_a(z) = z^m\frac{z-a}{1-az} = -az^m + (1-a^2)\sum_{n=1}^{\infty}a^{n-1}z^{n+m},$$

where $z \in \mathbb{D}$ and $a \in [0,1)$. We obtain the following equality

$$L_{\psi_a}(r) = \frac{a}{m+\gamma}r^m + (1-a^2)\sum_{n=m+1}^{\infty}\frac{a^{n-1}}{n+\gamma}r^n$$

with the help of (3.3), which is equivalent to

$$L_{\psi_a}(r) = \frac{1}{m+\gamma}r^m - (1-a)\left(\frac{1}{m+\gamma}r^m - 2\sum_{n=m+1}^{\infty}\frac{1}{n+\gamma}r^n\right) + M_a(r), \tag{3.4}$$

where

$$M_a(r) = 2(a-1)\sum_{n=m+1}^{\infty}\frac{1}{n+\gamma}r^n + (1-a^2)\sum_{n=m+1}^{\infty}\frac{a^{n-1}}{n+\gamma}r^n.$$

Letting $a \to 1$, we obtain

$$M_a(r) = \sum_{n=m+1}^{\infty}\frac{2(a-1)+(1-a^2)a^{n-1}}{n+\gamma}r^n = O((1-a)^2).$$

Further, the quantity

$$\frac{1}{m+\gamma}r^m - 2\sum_{n=m+1}^{\infty}\frac{1}{n+\gamma}r^n < 0$$

whenever $r > R(\gamma)$. These facts in (3.4) give that $R(\gamma)$ cannot be improved and the proof is complete. $\square$

CHAPTER 4

PRESERVING PROPERTIES AND PRE-SCHWARZIAN NORMS

We organize the structure of this chapter[1] as follows: throughout this chapter we assume $\alpha \in (-\pi/2, \pi/2)$ and $\lambda < 1$. First we study the univalency of the Hornich scalar multiplication operator on the class $\mathcal{K}(\lambda)$. By setting $\mathcal{S}(\lambda) := \bigcup_\alpha \mathcal{S}_\alpha^*(\lambda)$, we next compute the sets $A\big(J(\mathcal{S}_\alpha^*(\lambda))\big)$ and $A\big(J(\mathcal{S}(\lambda))\big)$. Also, we find the values of β for which $C_\beta(\mathcal{S}^*(\lambda)) = \{C_\beta[f] : f \in \mathcal{S}^*(\lambda)\} \subset \mathcal{S}$, $C_\beta(\mathcal{S}^*(\lambda)) \subset \mathcal{S}^*$, $C_\beta(\mathcal{K}) = \{C_\beta[f] : f \in \mathcal{K}\} \subset \mathcal{K}$ and $C_\beta(\mathcal{C}) = \{C_\beta[f] : f \in \mathcal{C}\} \subset \mathcal{C}$. We set $C(\mathcal{S}^*(\lambda)) = C_1(\mathcal{S}^*(\lambda))$ when we talk about the classical Cesàro transform $C[f]$. In this context, we also have an example of univalent function whose image is not univalent under the β-Cesàro transform. Finally, we deal with pre-Schwarzian norm of some of the above integral transforms and as a result we could find an alternate way to show that the class $\mathcal{S}^*(\lambda)$ is not contained in $\mathcal{S}$ for $\lambda < 0$.

4.1. Preserving Properties

It is here appropriate to recall that, in one hand, due to J. A. Pfaltzgraff as shown in [79, Corollary 1] $I_\gamma(\mathcal{S}) \subset \mathcal{S}$ for $|\gamma| \leq 1/4$. On the other hand, W. C. Royster proved in [96, Theorem 2] that for each number $\gamma \neq 1$ with $|\gamma| > 1/3$, there exists a function $f \in \mathcal{S}$ such that $I_\gamma[f] \notin \mathcal{S}$ (see also [11, 52, 53]). Also, recall from (1.10) that $A(\mathcal{K}) = \{\gamma \in \mathbb{C} : |\gamma| \leq 1/2\} \cup [1/2, 3/2]$. However, as a result of our first main result stated in Theorem 1.4 which generalizes the set $A(\mathcal{K})$ to the set $A(\mathcal{K}(\lambda))$, $\lambda < 1$, we have a larger class of functions $\mathcal{K}(-1/2)$ than $\mathcal{K}$ for which

$$A(\mathcal{K}(-1/2)) = \left\{\gamma \in \mathbb{C} : |\gamma| \leq \frac{1}{3}\right\} \bigcup \left[\frac{1}{3}, 1\right].$$

Note that the description of the whole set $A(\mathcal{S})$ is still open. The proof of Theorem 1.4 is the following.

[1]Contents of this chapter are published in: Kumar S., Sahoo S.K. (2020), *Preserving properties and pre-Schwarzian norms of nonlinear integral transforms*, Acta Math. Hungar., **162** (1), 84–97.

Proof of Theorem 1.4. As observed in [55], $\mathcal{K}(\lambda)$ can be expanded in terms of the Hornich scalar multiplication: $(1-\lambda)\star\mathcal{K} = \{(1-\lambda)\star f : f \in \mathcal{K}\}$. Then, for $f \in \mathcal{K}(\lambda)$, there exists a function $g \in \mathcal{K}$ such that $f(z) = ((1-\lambda)\star g)(z)$. This relation gives that $I_\gamma[f] = I_{(1-\lambda)\gamma}[g]$ for a function $g \in \mathcal{K}$. This concludes the proof by the help of the set $A(\mathcal{K})$. $\square$

We now collect an important Lemma 1.5, which is a generalization of a result of Y. C. Kim and T. Sugawa (see [54, Lemma 4]), to conclude our next main result and its consequences. Following is the proof of Lemma 1.5.

Proof of Lemma 1.5. Let $f \in J(\mathcal{S}_\alpha^*(\lambda))$. We write

$$\frac{1}{\cos\alpha}\left[e^{i\alpha}\left(\frac{zf''(z)}{f'(z)}+1\right)-i\sin\alpha\right] = p(z),$$

where p is an analytic function in $|z|<1$. Clearly, $p(0)=1$ and $\operatorname{Re} p(z) > \lambda$.

If we take $k \in \mathcal{K}(\lambda)$ such that $1 + zk''(z)/k'(z) = p(z)$ then we obtain

$$\frac{f''(z)}{f'(z)} = e^{-i\alpha}\cos\alpha\frac{k''(z)}{k'(z)},$$

which yields $f = I_{e^{-i\alpha}\cos\alpha}[k]$. This implies that $J(\mathcal{S}_\alpha^*(\lambda)) \subset I_{e^{-i\alpha}\cos\alpha}(\mathcal{K}(\lambda))$. If we take the backward process, then we obtain the reverse inclusion $J(\mathcal{S}_\alpha^*(\lambda)) \supset I_{e^{-i\alpha}\cos\alpha}(\mathcal{K}(\lambda))$. The desired result is thus obtained. $\square$

For $z, w \in \mathbb{C}$, we denote by $[z,w]$ for the line segment joining z and w. An immediate consequence of Theorem 1.4 and Lemma 1.5 leads to the following theorem:

Theorem 4.1. *For $-\pi/2 < \alpha < \pi/2$ and $\lambda < 1$, we have*

$$A\big(J(\mathcal{S}_\alpha^*(\lambda))\big) = \left\{\gamma \in \mathbb{C} : |\gamma| \le \frac{1}{2(1-\lambda)\cos\alpha}\right\} \bigcup \left[\frac{e^{i\alpha}}{2(1-\lambda)\cos\alpha}, \frac{3e^{i\alpha}}{2(1-\lambda)\cos\alpha}\right].$$

Proof. By using Lemma 1.5 and the property $I_aI_b = I_{ab}$, for $a, b \in \mathbb{C}$, we have

$$I_\gamma\big(J(\mathcal{S}_\alpha^*(\lambda))\big) = I_\gamma I_{e^{-i\alpha}\cos\alpha}(\mathcal{K}(\lambda)) = I_{\gamma e^{-i\alpha}\cos\alpha}(\mathcal{K}(\lambda)).$$

Therefore, $\gamma \in A\big(J(\mathcal{S}_\alpha^*(\lambda))\big)$ if and only if $\gamma e^{-i\alpha}\cos\alpha \in A\big(\mathcal{K}(\lambda)\big)$. Now we are able to conclude the proof by Theorem 1.4. $\square$

We remark that the special choice $\lambda = 0$ takes Theorem 4.1 to [54, Theorem 3].

By the definition of $\mathcal{S}(\lambda)$, we have

$$A\big(J(\mathcal{S}(\lambda))\big) = \bigcap_\alpha A\big(J\big(\mathcal{S}_\alpha^*(\lambda)\big)\big).$$

Using Theorem 4.1, we now conclude the following theorem.

Theorem 4.2. *For $\lambda < 1$, we have*

$$A\big(J(\mathcal{S}(\lambda))\big) = \left\{ |\gamma| \le \frac{1}{2(1-\lambda)} \right\}.$$

For the special case $\lambda = 0$, this theorem was considered in [54].

In the next theorem, we have the inclusion of the image set $J(\mathcal{S}_\alpha^*(\lambda))$ in the class $\mathcal{S}$ for some restrictions on α. However, the case $\lambda = 0$ has also been considered in [54].

Theorem 4.3. *If $\lambda < 1$, then the relation*

$$J(\mathcal{S}_\alpha^*(\lambda)) \subset \mathcal{S}$$

holds precisely for $\cos\alpha \le 1/2(1-\lambda)$. However, if $-1/2 \le \lambda < 1$, then the same inclusion follows for $\alpha = 0$.

Proof. If $\alpha = 0$, the result is trivial to prove. Indeed, in this case, we have $J[f] \in \mathcal{C} \subset \mathcal{S}$ for $f \in \mathcal{S}^*(\lambda)$, $-1/2 \le \lambda < 1$.

Thus, we assume that $\alpha \neq 0$. We have $J\big(\mathcal{S}_\alpha^*(\lambda)\big) \subset \mathcal{S}$ if and only if $1 \in A\big(J\big(\mathcal{S}_\alpha^*(\lambda)\big)\big)$. This gives that $\cos\alpha \le 1/2\big(1-\lambda\big)$, completing the proof. $\square$

The following lemma gives a relation of the β-Cesàro transform of with the transform J_γ for $\gamma = e^{-i\alpha}\sec\alpha$.

Lemma 4.4. *Let $0 \le \beta$ and $-\pi/2 < \alpha < \pi/2$. Let $f \in \mathcal{A}$ be such that*

$$\left[\frac{g(z)}{z(1-z)^\beta}\right]^{e^{i\alpha}\cos\alpha} = \frac{f(z)}{z}$$

for some $g \in \mathcal{S}^*(\lambda)$, then $f \in \mathcal{S}_\alpha^*(\lambda - \beta/2)$ for $\lambda < 1$.

Proof. Suppose that $g \in \mathcal{S}^*(\lambda)$ and

$$\left[\frac{g(z)}{z(1-z)^\beta}\right]^{e^{i\alpha}\cos\alpha} = \frac{f(z)}{z}.$$

The logarithm derivative obtains

$$e^{i\alpha}\left[\frac{zf'(z)}{f(z)} - 1\right] = \cos\alpha\left[\frac{zg'(z)}{g(z)} - 1 + \frac{\beta z}{1-z}\right],$$

which implies that

$$\operatorname{Re}\left[e^{i\alpha}\frac{zf'(z)}{f(z)}\right] = \operatorname{Re}\left[\cos\alpha\frac{zg'(z)}{g(z)}\right] + \operatorname{Re}\left[\cos\alpha\frac{\beta z}{1-z}\right].$$

Since $g \in \mathcal{S}^*(\lambda)$ and $\operatorname{Re}(z/(1-z)) > -1/2$ for $|z| < 1$, it follows that

$$\operatorname{Re}\left[e^{i\alpha}\frac{zf'(z)}{f(z)}\right] > \left(\lambda - \frac{\beta}{2}\right)\cos\alpha.$$

Thus, $f \in \mathcal{S}_\alpha^*(\lambda - \beta/2)$ for $\lambda < 1$ follows by the definition (1.2), completing the proof. $\square$

As an application of Theorem 4.1 and Lemma 4.4, we next find restriction on β for which $C_\beta(\mathcal{S}^*(\lambda))$ is contained in $\mathcal{S}$.

Theorem 4.5. *For $-1/2 \le \lambda < 1$ and $0 \le \beta \le 2\lambda+1$, the relation $C_\beta(\mathcal{S}^*(\lambda)) \subset \mathcal{S}$ holds.*

Proof. Substituting $\alpha = 0$ in Lemma 4.4, for a given function $g \in \mathcal{S}^*(\lambda)$, we can find another function $f \in \mathcal{S}^*(\lambda - \beta/2)$ satisfying

$$\int_0^z \frac{g(w)}{w(1-w)^\beta}dw = \int_0^z \frac{f(w)}{w}dw.$$

Secondly, Theorem 4.1 gives that $J_\gamma(\mathcal{S}^*(\lambda - \beta/2)) \subset \mathcal{S}$ whenever γ lies either in $\{\gamma \in \mathbb{C} : |\gamma| \le 1/2(1-\lambda+\beta/2)\}$ or in $[1/2(1-\lambda+\beta/2), 3/2(1-\lambda+\beta/2)]$. It follows that

$$C_\beta(\mathcal{S}^*(\lambda)) \subset \mathcal{S} \quad \text{for } 1 \le \frac{3}{2(1-\lambda+\beta/2)},$$

that is for $\beta \le 2\lambda + 1$. This completes the proof. $\square$

We remark that Theorem 4.5 can be proved alternatively by using the classical theorem of Kaplan ([27, §2.6]) which states that $f \in \mathcal{C}$ if and only if

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right)d\theta > -\pi,$$

whenever $0 \le \theta_1 < \theta_2 \le 2\pi$. Using this, we obtain

$$\begin{aligned}\int_{\theta_1}^{\theta_2} \operatorname{Re}\left(1 + \frac{zC_\beta[f]''(z)}{C_\beta[f]'(z)}\right)d\theta &= \int_{\theta_1}^{\theta_2} \operatorname{Re}\left(\frac{zf'(z)}{f(z)} + \frac{\beta z}{1-z}\right)d\theta \\ &> \lambda(\theta_2-\theta_1) - \frac{\beta}{2}(\theta_2-\theta_1) \ge -(\beta - 2\lambda)\pi.\end{aligned}$$

This gives that $C_\beta[f] \in \mathcal{C} \subset \mathcal{S}$, for $\beta \le 2\lambda + 1$.

In the following example, we show that the quantity $2\lambda + 1$ can not be replaced by any bigger number in Theorem 4.5.

Example 4.6. *For $-1/2 \leq \lambda < 1$, let $f(z) = z/(1-z)^{2-2\lambda}$. Recall that this f is an element of the class $\mathcal{S}^*(\lambda)$. From the definition of $C_\beta[f]$ we obtain*

$$C_\beta[f](z) = \int_0^z \frac{1}{(1-w)^{\beta-2\lambda+2}} dw = \frac{1}{(\beta-2\lambda+1)}\left[\frac{1}{(1-z)^{(\beta-2\lambda+1)}} - 1\right]. \tag{4.1}$$

Note that $C_\beta[f]$ is univalent if and only if $g(z) = (1-z)^{1-\beta-2+2\lambda}$ is univalent. However, by the lemma of W. C. Royster [96, p. 386], we obtain that $g(z)$ is univalent if and only if $2\lambda - 3 \leq \beta \leq 2\lambda + 1$. It follows that if $\beta > 2\lambda + 1$, then $C_\beta[f]$ does not belong to the class $\mathcal{S}$.

If we choose $\beta = 1$ in Theorem 4.5, it produces the following well-known result [37] concerning the Cesàro transform $C[f]$ on the class $\mathcal{S}^*(\lambda)$, $0 \leq \lambda < 1$:

Corollary 4.7. *For $0 \leq \lambda < 1$, the relation $C(\mathcal{S}^*(\lambda)) \subset \mathcal{S}$ holds.*

In the statement of Theorem 4.5, if we replace $\mathcal{S}$ by $\mathcal{S}^*$ then we have new restriction on β, which is described in the following theorem:

Theorem 4.8. *For $0 \leq \lambda < 1$ and $0 \leq \beta \leq 2\lambda$, the inclusion relation $C_\beta(\mathcal{S}^*(\lambda)) \subset \mathcal{S}^*$ holds.*

Proof. If $0 \leq \lambda < 1$ and $0 \leq \beta \leq 2\lambda$, then we notice that the restrictions on the parameters in the hypothesis of Lemma 4.4 are satisfied. Thus, it gives

$$C_\beta(\mathcal{S}^*(\lambda)) \subset J(\mathcal{S}^*(\lambda - \beta/2)).$$

Also, if we choose $\alpha = 0$ in Lemma 1.5 we obtain $J(\mathcal{S}^*(\lambda)) = \mathcal{K}(\lambda)$. Combination of the above two relations clearly yields

$$C_\beta(\mathcal{S}^*(\lambda)) \subset \mathcal{K}(\lambda - \beta/2),$$

which is valid since $\lambda - \beta/2 \geq 0$ and we complete the proof. $\square$

The following example shows that, for $\beta > 2\lambda$, the image of $\mathcal{S}^*(\lambda)$ under the β-Cesàro transform does not lie in the starlike family.

Example 4.9. *Consider the function $f(z) = z/(1-z)^{2-2\lambda}$ for $0 \leq \lambda < 1$. The β-Cesàro transform thus takes to the form (4.1). It is easy to calculate that*

$$\mathrm{Re}\left(1 + \frac{zC_\beta[f]''(z)}{C_\beta[f]'(z)}\right) = \mathrm{Re}\left(1 + \frac{(\beta-2\lambda+2)z}{1-z}\right) > 1 - \frac{\beta-2\lambda+2}{2} = \lambda - \frac{\beta}{2},$$

for $\beta > 2\lambda$.

On the other hand, for $2\lambda < \beta$, J. A. Pfaltzgraff, M. O. Reade, and T. Umezawa in [80] showed that there exists a point $z_0 \in \mathbb{D}$ such that

$$\mathrm{Re}\left(\frac{z_0 C_\beta[f]'(z_0)}{C_\beta[f](z_0)}\right) < 0$$

(see also [72, pp. 44-45]). Hence $C_\beta[f]$ is not a starlike function.

Remark 4.10. *Recall from [37] that the Cesàro transform does not preserve the starlikeness. More generally, here Example 4.9 shows that the β-Cesàro transform also does not preserve the starlikeness, for any $\beta > 0$.*

We already know that the Alexander transform and the Cesàro transform preserve the class $\mathcal{K}$. In the following, we determine the values of β for which the β-Cesàro transform preserve the class $\mathcal{K}$.

Theorem 4.11. *For $0 \leq \beta \leq 1$, the inclusion relation*

$$C_\beta(\mathcal{K}) \subset \mathcal{K}((1-\beta)/2)$$

holds. In particular, we have $C_\beta(\mathcal{K}) \subset \mathcal{K}$.

Proof. For $f \in \mathcal{K}$, it is easy to see that

$$\mathrm{Re}\left(1 + \frac{zC_\beta[f]''(z)}{C_\beta[f]'(z)}\right) = \mathrm{Re}\left(\frac{zf'(z)}{f(z)} + \frac{\beta z}{1-z}\right) > \frac{1-\beta}{2},$$

since $\mathcal{K} \subset \mathcal{S}^*(1/2)$ and $\mathrm{Re}\,(z/(1-z)) > -1/2$ for $|z| < 1$. By the definition (1.3), it follows that $C_\beta[f] \in \mathcal{K}((1-\beta)/2)$, for $0 \leq \beta \leq 1$. Hence proved. $\square$

For $\beta > 1$, we have the following counterexample to show that $C_\beta[f]$ need not be convex though $f \in \mathcal{K}$.

Example 4.12. *Let $f(z) = z/(1-z)$, $z \in \mathbb{D}$. It is well known that $f \in \mathcal{K}$. We obtain*

$$C_\beta[f](z) = \int_0^z \frac{1}{(1-w)^{\beta+1}} dw.$$

It is easy to calculate that

$$\mathrm{Re}\left(1 + \frac{zC_\beta[f]''(z)}{C_\beta[f]'(z)}\right) = 1 + \mathrm{Re}\left(\frac{(\beta+1)z}{1-z}\right).$$

For $\beta > 1$, there is a sequence of points $z_n = -1 + 1/n \in \mathbb{D}$ such that

$$1 + \mathrm{Re}\left(\frac{(\beta+1)z_n}{1-z_n}\right) = \frac{n(1-\beta)+\beta}{2n-1} < 0$$

for $n > \beta/(\beta-1)$. Therefore, $C_\beta[f]$ need not be a convex function, for $\beta > 1$.

The following lemma is due to E. P. Merkes and J. Wright [70] which gives a refinement of Theorem 4.11 to the close-to-convex family.

Lemma 4.13. *Let $f(z) = \sum_{n=1}^{\infty} a_n z^n$ be analytic and $g(z) = \sum_{n=1}^{\infty} b_n z^n$ be an analytic univalent starlike function in $\mathbb{D}$. If H denotes the convex hull of the image of $\mathbb{D}$ under the mapping $e^{i\alpha}(f'/g')$ for all $\alpha \in \mathbb{R}$, then $e^{i\alpha}(f/g) \in H$ in $\mathbb{D}$.*

Now we prove the refinement of Theorem 4.11 as indicated above.

Theorem 4.14. *For $0 \leq \beta \leq 1$, the inclusion relation $C_\beta(\mathcal{C}) \subset \mathcal{C}$ holds.*

Proof. Since $f \in \mathcal{C}$, by its definition there exists a function $\psi \in \mathcal{K}$ and $\alpha \in (-\pi/2, \pi/2)$ such that $\mathrm{Re}\,(e^{i\alpha} f'/\psi') > 0$, for $z \in \mathbb{D}$. If $\beta \in [0,1]$, we set

$$g(z) = \int_0^z \frac{\psi(w)}{w(1-w)^\beta} dw.$$

Then in view of Theorem 4.11, g is convex for $0 \leq \beta \leq 1$. Now we compute and see by using Lemma 4.13 that

$$\mathrm{Re}\left\{e^{i\alpha}\frac{C_\beta[f]'(z)}{g'(z)}\right\} = \mathrm{Re}\left\{e^{i\alpha}\frac{f(z)}{\psi(z)}\right\} > 0$$

for $z \in \mathbb{D}$. This gives that $C_\beta[f] \in \mathcal{C}$, completing the proof. $\square$

Remark 4.15. *If we choose $\beta > 1$ in Theorem 4.14, then the result may not hold as can be seen from Example 4.6 that the β-Cesàro transform C_β of the Koebe function is not univalent in $\mathbb{D}$ and hence not close-to-convex.*

Theorem 1.6 shows that there is a function $f \in \mathcal{S}$ such that its β-Cesàro transform is not univalent in $\mathbb{D}$. Here is the proof of Theorem 1.6.

Proof of Theorem 1.6. Consider the function $f(z) = z(1-z)^{i-1}$ for $z \in \mathbb{D}$. We can rewrite f in the composition form $f = (-i)(g \circ h)$ with

$$g(z) = z(1-iz)^{i-1} \quad \text{and} \quad h(z) = -iz$$

for $z \in \mathbb{D}$. As shown in [27, p. 257], g is univalent in $\mathbb{D}$. Since composition of two univalent functions is univalent f is univalent in $\mathbb{D}$.

Now, if we calculate $C_\beta[f](z)$, for $f(z) = z(1-z)^{i-1}$, then we obtain

$$C_\beta[f](z) = \int_0^z (1-w)^{i-1-\beta} dw = \frac{1}{i-\beta}[1-(1-z)^{i-\beta}].$$

W. C. Royster [96] proved that the function $g(z) = \exp[\mu \log(1-z)]$ is univalent in $\mathbb{D}$ if and only if $0 \neq \mu$ lies in either of the closed disks $|\mu+1| \leq 1$, $|\mu-1| \leq 1$. Using this fact, we show that $(1-z)^{i-\beta}$ is not univalent in $\mathbb{D}$ for $\beta \neq 1$. It thus follows that $C_\beta[f]$ is not univalent in $\mathbb{D}$ for $\beta \neq 1$. The remaining case $\beta = 1$ is already handled by F. W. Hartman and T. H. MacGregor [37]. $\square$

4.2. Pr-Schwarzian Norms

Recall the definition of the per-Schwarzian norm of a function $f \in \mathfrak{F}$:

$$\|f\| = \sup_{z \in \mathbb{D}} (1-|z|^2) \left| \frac{f''(z)}{f'(z)} \right|.$$

It is well-known that $\|f\| \leq 6$ for $f \in \mathcal{S}$ as well as for $f \in \mathcal{S}^*$. The sharp estimation $\|f\| \leq 4$, for $f \in \mathcal{K}$, was later generalized by S. Yamagata to the class $\mathcal{K}(\lambda)$, $0 \leq \lambda < 1$ (see [107]). Recently, in [11], S. Yamashita's result has been further extended to $\mathcal{K}(\lambda)$, $-1/2 \leq \lambda < 1$. However, here we prove that the result of Yamagata holds true for all $\lambda < 1$.

Theorem 4.16. *For $\lambda < 1$, if $f \in \mathcal{K}(\lambda)$ then $\|f\| \leq 4(1-\lambda)$ and the bound is sharp.*

Proof. Recall the relation

$$\mathcal{K}(\lambda) = (1-\lambda) \star \mathcal{K} = \{(1-\lambda) \star f : f \in \mathcal{K}\}.$$

Thus, if $f \in \mathcal{K}(\lambda)$, then there exists a function $g \in \mathcal{K}$ such that $f(z) = (1-\lambda) \star g(z)$. It follows that

$$\|f\| = (1-\lambda)\|g\| \leq 4(1-\lambda),$$

completing the proof. $\square$

Our purpose in this section is to obtain the per-Schwarzian norm of the elements in $J(\mathcal{S}_\alpha^*(\lambda))$, $C_\beta(\mathcal{S}^*(\lambda))$ and in $C_\beta(\mathcal{S})$ leading to certain observation highlighted at the end of Section 1.

First, as a consequence of Theorem 4.16, we obtain a sharp estimate of $\|J[f]\|$ for $f \in \mathcal{S}_\alpha^*(\lambda)$. This can be rewritten in the following form.

Theorem 4.17. *For each $\alpha \in (-\pi/2, \pi/2)$ and each $\lambda < 1$, the sharp inequality $\|f\| \leq 4(1-\lambda)\cos\alpha$ holds for $f \in J(\mathcal{S}_\alpha^*(\lambda))$.*

Proof. It is easy to calculate that $\|I_\gamma(f)\| = |\gamma|\|f\|$. Secondly, By Lemma 1.5 for $f \in J(\mathcal{S}_\alpha^*(\lambda))$ there exists a function $k \in \mathcal{K}(\lambda)$ such that $f = I_{e^{-i\alpha}\cos\alpha}[k]$. It concludes that $\|f\| = |\cos\alpha|\|k\| \leq 4(1-\lambda)\cos\alpha$.

It is evident that the equality holds for the function

$$h(z) = \frac{1}{(1-2\lambda)}\left[\frac{1}{(1-z)^{(1-2\lambda)}} - 1\right].$$

belonging to the class $\mathcal{K}(\lambda)$. Indeed, we have

$$\lim_{t\to 1^-}(1-t^2)\left|\frac{h''(t)}{h'(t)}\right| = \lim_{t\to 1^-}[2(1+t)(1-\lambda)] = 4(1-\lambda).$$

It is easy to compute that the function $g_\alpha \in \mathcal{S}_\alpha^*(\lambda)$ corresponding to the function $h(z)$ is given by

$$g_\alpha(z) = z(1-z)^{2(\lambda-1)e^{-i\alpha}\cos\alpha}. \tag{4.2}$$

This completes the proof. □

We remark that if we choose $\lambda = 0$ in Theorem 4.17, then it reduces to [54, Proposition 6].

Remark 4.18. *It is well-known that for each $\lambda < 0$, the class $\mathcal{S}^*(\lambda)$ is not contained in the class $\mathcal{S}$ (see for instance [72, p. 66]). However, here we provide an alternate method to show this. As we computed above, $\|J[g_0]\| = 4(1-\lambda)$ for $g_0 \in \mathcal{S}^*(\lambda)$ defined by (4.2). For $\lambda < 0$, it is clear that $4 < \|J[g_0]\|$. Thus, by [51, Theorem 1.1], we conclude that $g_0 \notin \mathcal{S}$.*

The next theorem obtains the per-Schwarzian norm estimate of the elements in the image set of the β-Cesàro transform of functions from the class $\mathcal{S}_\alpha^*(\lambda)$.

Theorem 4.19. *Let $-\pi/2 < \alpha < \pi/2$, $\beta \geq 0$ and $\lambda < 1$. If $f \in C_\beta(\mathcal{S}_\alpha^*(\lambda))$, then the sharp inequality $\|f\| \leq 4(1-\lambda)\cos\alpha + 2\beta$ holds.*

Proof. We observe from the definition of the β-Ceàro transform that the inequality $\|C_\beta[f]\| \leq \|J[f]\| + 2\beta$ holds for any $f \in \mathfrak{F}$. To complete the proof, we recall from Theorem 4.17 that $\|J[f]\| \leq 4(1-\lambda)\cos\alpha$ for $f \in \mathcal{S}^*_\alpha(\lambda)$. It thus concludes that $\|C_\beta[f]\| \leq 4(1-\lambda)\cos\alpha + 2\beta$ for $f \in \mathcal{S}^*_\alpha(\lambda)$.

For the sharpness, let us consider the function g_α defined by (4.2). We see that

$$\lim_{t\to 1^-}(1-t^2)\left|\frac{C_\beta[g_\alpha]''(t)}{C_\beta[g_\alpha]'(t)}\right| = \lim_{t\to 1^-}(1+t)[2(1-\lambda)\cos\alpha + \beta] = 4(1-\lambda)\cos\alpha + 2\beta,$$

completing the proof. □

A similar technique that is adopted in Theorem 4.19 further leads to the norm estimate of the β-Ceàro transform of functions $f \in \mathcal{S}$, which is presented below.

Theorem 4.20. *The sharp inequality $\|f\| \leq 4 + 2\beta$ holds for $f \in C_\beta(\mathcal{S})$.*

Proof. As explained in the proof of Theorem 4.19, we have $\|C_\beta[f]\| \leq \|J[f]\| + 2\beta$ for any $f \in \mathfrak{F}$. Also, we recall from [51, Theorem 1.1] that $\|J[f]\| \leq 4$ for $f \in \mathcal{S}$. Then we conclude that $\|C_\beta[f]\| \leq 4 + 2\beta$ for $f \in \mathcal{S}$.

This is sharp as we can see from the sharpness part of the proof of Theorem 4.19 by considering cases $\lambda = 0$ and $\alpha = 0$ (i.e. by considering the Kobe function). □

CHAPTER 5

RADIUS OF CONVEXITY

We start this chapter[1] by presenting some of the subclasses of $\mathfrak{F}$ which are given special attention in the literature for different prospective, followed by main results and their consequences associated with those subclasses.

5.1. The class $\mathcal{S}^*(A,B)$

Let f be a member of a subclass $\mathcal{F} \subset \mathfrak{F}$. We say r_0, the *radius of convexity for* $\mathcal{F}$ if

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>0$$

for all $|z|\leq r_0$ and for all $f\in\mathcal{F}$, where r_0 is the largest number for which this holds.

We now begin to present the discussion on the radius of convexity of $C_{\alpha,\beta}[f,g]$ for $f\in\mathcal{S}^*(A,B)$ and $g\in\mathcal{K}(A,B)$, where $A\neq B$ and $|B|\leq 1$, with the help of the subordination. For $A\neq B$, if $f\in\mathcal{S}^*(A,B)$ and $g\in\mathcal{K}(A,B)$ then by the subordination principle, we obtain

$$\left|\frac{zf'(z)}{f(z)}-\frac{1-A\overline{B}r^2}{1-|B|^2r^2}\right|\leq\frac{|B-A|r}{1-|B|^2r^2},\ |z|=r \tag{5.1}$$

and

$$\left|\frac{zg''(z)}{g'(z)}-\frac{(|B|^2-A\overline{B})r^2}{1-|B|^2r^2}\right|\leq\frac{|B-A|r}{1-|B|^2r^2},\ |z|=r. \tag{5.2}$$

We are now ready to prove our first main result, stated in Theorem 1.7, concerning radius of convexity for $C_{\alpha,\beta}[f,g]$ when $f\in\mathcal{S}^*(A,B)$ and $g\in\mathcal{K}(A,B)$. However, for the sharpness, we could handle the situation when A and B are real. Note that now onward we will denote the radius of convexity by the symbol r_c, where c is not a parameter, but it refers to convexity. Following is the proof of Theorem 1.7.

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[1]The results of this chapter are published in: Kumar S., Sahoo S.K. (2021), *Radius of convexity for integral operators involving Hornich operations*, J. Math. Anal. Appl., **502** (2), 125265.

Proof of Theorem 1.7. It is easy to compute that

$$u := \mathrm{Re}\left\{1+\frac{z(C_{\alpha,\beta}[f,g])''(z)}{(C_{\alpha,\beta}[f,g])'(z)}\right\} = \alpha\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\}+\beta\,\mathrm{Re}\left\{\frac{zg''(z)}{g'(z)}\right\}+1-\alpha. \tag{5.3}$$

Description of our proof has three parts.

Computation of radii. Here, two cases arise on α and β.

Case (i): $\alpha\beta\geq 0$.

Using the inequalities (5.1) and (5.2), we obtain that (5.3) has the lower bound

$$u \geq \frac{1-|\alpha+\beta||B-A||z|-\{(\alpha+\beta)\mathrm{Re}\,(A\overline{B})+|B|^2-(\alpha+\beta)|B|^2\}|z|^2}{1-|B|^2|z|^2}. \tag{5.4}$$

The right hand side of the above inequality is strictly greater than 0 provided that

$$\phi_1(r) := 1-|\alpha+\beta||B-A|r-\{(\alpha+\beta)\mathrm{Re}\,(A\overline{B})+|B|^2-(\alpha+\beta)|B|^2\}r^2>0.$$

If $\beta\neq 0$ and $B=0$ then $\phi_1(r)>0$ for $r<1/|\alpha+\beta||A|$.

Further, assume $B\neq 0$ then $(\alpha+\beta)\mathrm{Re}\,(A\overline{B})+|B|^2-(\alpha+\beta)|B|^2=0$, we obtain $\phi_1(r)>0$ for $r<1/|\alpha+\beta||A-B|$, otherwise $\phi_1(r)>0$ either for

$$r<\frac{|\alpha+\beta||B-A|-|(\alpha+\beta-2)B-(\alpha+\beta)A|}{2\{(\alpha+\beta)|B|^2-(\alpha+\beta)\mathrm{Re}\,(A\overline{B})-|B|^2\}}=r_c(A,B,\alpha,\beta)$$

or for

$$r>\frac{|\alpha+\beta||B-A|+|(\alpha+\beta-2)B-(\alpha+\beta)A|}{2\{(\alpha+\beta)|B|^2-(\alpha+\beta)\mathrm{Re}\,(A\overline{B})-|B|^2\}}.$$

Case (ii): $\alpha\beta\leq 0$.

The inequality

$$u \geq \frac{1-|\alpha-\beta||B-A||z|-\{(\alpha+\beta)\mathrm{Re}\,(A\overline{B})+|B|^2-(\alpha+\beta)|B|^2\}|z|^2}{1-|B|^2|z|^2}$$

is obtained from (5.3) using some elementary estimates generated through (5.1) and (5.2). This is positive if

$$\phi_2(r) := 1-|\alpha-\beta||B-A|r-\{(\alpha+\beta)\mathrm{Re}\,(A\overline{B})+|B|^2-(\alpha+\beta)|B|^2\}r^2>0.$$

If $\alpha\neq 0$ and $B=0$ then $\phi_2(r)>0$ for $r<1/|\alpha-\beta||A|$.

Now, consider $B\neq 0$ then for $(\alpha+\beta)\mathrm{Re}\,(A\overline{B})+|B|^2-(\alpha+\beta)|B|^2=0$, we obtain $\phi_1(r)>0$ for $r<1/|\alpha-\beta||A-B|$, otherwise $\phi_2(r)>0$ either for

$$r<\frac{|\alpha-\beta||B-A|-\sqrt{\xi}}{2\{(\alpha+\beta)|B|^2-(\alpha+\beta)\mathrm{Re}\,(A\overline{B})-|B|^2\}}=r_c(A,B,\alpha,\beta)$$

or for

$$r > \frac{|\alpha-\beta||B-A|+\sqrt{\xi}}{2\{(\alpha+\beta)|B|^2-(\alpha+\beta)\mathrm{Re}\,(A\overline{B})-|B|^2\}},$$

where

$$\begin{aligned}\xi &= (\alpha-\beta)^2|B-A|^2+4\{(\alpha+\beta)[\mathrm{Re}\,(A\overline{B})-|B|^2]+|B|^2\}\\ &= [(\alpha-\beta)^2-(\alpha+\beta)^2]|B-A|^2+(\alpha+\beta)^2|B-A|^2+4(\alpha+\beta)\mathrm{Re}\,[(A-B)\overline{B}]+4|B|^2\\ &= |(\alpha+\beta)(B-A)-2B|^2-4\alpha\beta|B-A|^2,\end{aligned}$$

which is nothing but (1.18). Hence ξ is non-negative.

The calculation of the radius for each case is thus complete.

Positivity of the radius. We use the method of contradiction to verify that the quantity $r_c(A,B,\alpha,\beta)$ is non-negative. If we compare $\phi_1(r)=0$ with $ar^2-br+1=0$ and use the Sridharacharya formula to finding the roots then we obtain

$$r_c(A,B,\alpha,\beta)=\frac{b-\sqrt{b^2-4a}}{2a}.$$

Let us deal the situation first for $a>0$. On contrary if $r_c(A,B,\alpha,\beta)<0$ then $b-\sqrt{b^2-4a}<0$, equivalently, $b^2<b^2-4a$ which is impossible. Also we can follow the similar steps for the situation $a<0$. The situation $a=0$ is trivial to handle.

Now, we present the sharpness of the radius in each case for real numbers A and B.

Sharpness in Case (i). Let $\alpha\geq 0$. For $B\neq 0$, we choose $f(z)=z(1+Bz)^{A/B-1}\in\mathcal{S}^*(A,B)$ and $g\in\mathcal{K}(A,B)$ with $g'(z)=(1+Bz)^{A/B-1}$. Then the expression

$$1+\frac{z(C_{\alpha,\beta}[f,g])''(z)}{(C_{\alpha,\beta}[f,g])'(z)}=\frac{1-(\alpha+\beta)(B-A)z-\{(\alpha+\beta)AB+B^2-(\alpha+\beta)B^2\}z^2}{1-B^2z^2}$$

shows that the radius is best possible.

For $B=0$, we consider the functions $f(z)=ze^{Az}\in\mathcal{S}^*(A,0)$ and $g(z)=e^{Az}\in\mathcal{K}(A,0)$. The computation

$$1+\frac{z(C_{\alpha,\beta}[f,g])''(z)}{(C_{\alpha,\beta}[f,g])'(z)}=1+(\alpha+\beta)Az$$

thus gives the sharpness.

Let $\alpha<0$. If $B\neq 0$, the functions $f(z)=z(1-Bz)^{A/B-1}\in\mathcal{S}^*(A,B)$ and $g\in\mathcal{K}(A,B)$ with $g'(z)=(1-Bz)^{A/B-1}$ yield

$$1+\frac{z(C_{\alpha,\beta}[f,g])''(z)}{(C_{\alpha,\beta}[f,g])'(z)}=\frac{1+(\alpha+\beta)(B-A)z-\{(\alpha+\beta)AB+B^2-(\alpha+\beta)B^2\}z^2}{1-B^2z^2},$$

which gives the sharpness.

If $B = 0$, we choose $f(z) = ze^{-Az} \in \mathcal{S}^*(A,B)$ and $g(z) = e^{-Az} \in \mathcal{K}(A,B)$. Then

$$1 + \frac{z(C_{\alpha,\beta}[f,g])''(z)}{(C_{\alpha,\beta}[f,g])'(z)} = 1 - (\alpha+\beta)Az$$

gives the sharpness in this case.

Sharpness in Case (ii). Let $\alpha \geq 0$. If $B \neq 0$, the functions $f(z) = z(1+Bz)^{A/B-1} \in \mathcal{S}^*(A,B)$ and $g \in \mathcal{K}(A,B)$ with $g'(z) = (1-Bz)^{A/B-1}$, for which

$$1 + \frac{z(C_{\alpha,\beta}[f,g])''(z)}{(C_{\alpha,\beta}[f,g])'(z)} = \frac{1 - (\alpha-\beta)(B-A)z - \{(\alpha+\beta)AB + B^2 - (\alpha+\beta)B^2\}z^2}{1 - B^2z^2},$$

provide the sharpness in this case.

If $B = 0$, the choices $f(z) = ze^{Az} \in \mathcal{S}^*(A,B)$ and $g(z) = e^{-Az} \in \mathcal{K}(A,B)$ give

$$1 + \frac{z(C_{\alpha,\beta}[f,g])''(z)}{(C_{\alpha,\beta}[f,g])'(z)} = 1 + (\alpha-\beta)Az,$$

which is sufficient to show the sharpness.

Let $\alpha < 0$. If $B \neq 0$, the sharpness is obtain by considering the functions $f(z) = z(1-Bz)^{A/B-1} \in \mathcal{S}^*(A,B)$ and $g \in \mathcal{K}(A,B)$ with $g'(z) = (1+Bz)^{A/B-1}$ because they lead to

$$1 + \frac{z(C_{\alpha,\beta}[f,g])''(z)}{(C_{\alpha,\beta}[f,g])'(z)} = \frac{1 + (\alpha-\beta)(B-A)z - \{(\alpha+\beta)AB + B^2 - (\alpha+\beta)B^2\}z^2}{1 - B^2z^2}.$$

If $B = 0$, for the functions $f(z) = ze^{-Az} \in \mathcal{S}^*(A,B)$ and $g(z) = e^{Az} \in \mathcal{K}(A,B)$, we have

$$1 + \frac{z(C_{\alpha,\beta}[f,g])''(z)}{(C_{\alpha,\beta}[f,g])'(z)} = 1 + (\beta-\alpha)Az.$$

This provides the sharpness.

The proof of our theorem is thus complete. □

A direct consequence of Theorem 1.7 gives the following radius of convexity for $I_\beta[g]$ when $g \in \mathcal{K}(A,B)$, $A \neq B$.

Corollary 5.1. *Let $g \in \mathcal{K}(A,B)$, where A and B are two complex numbers with $A \neq B$ and $|B| \leq 1$. Then $I_\beta[g]$ is convex in $|z| < \min\{r_c(A,B,\beta),1\}$, where for $B \neq 0$ we have*

$$r_c(A,B,\beta) = \begin{cases} \dfrac{|\beta||B-A| - |\beta(B-A) - 2B|}{2\{\beta\mathrm{Re}\,[(B-A)\overline{B}] - |B|^2\}}, & \text{if } \beta\,\mathrm{Re}\,[(B-A)\overline{B}] \neq |B|^2, \\ \dfrac{1}{|\beta||A-B|}, & \text{if } \beta\,\mathrm{Re}\,[(B-A)\overline{B}] = |B|^2, \end{cases}$$

and for $B=0$ the radius becomes

$$r_c(A,\beta)=\frac{1}{|\beta||A|},\quad \textit{if } \beta\neq 0.$$

These radii are best possible for the real numbers A and B.

The next corollary provides the radius of convexity for $J_\alpha[f] = C_{\alpha,0}[f]$ when $f \in S^*(A,B)$, $A\neq B$. This can also be obtained directly by replacing β with α in Corollary 5.1, since $J_\alpha = I_\alpha(J)$.

Corollary 5.2. *Let $f\in\mathcal{S}^*(A,B)$, where A and B are two complex numbers with $A\neq B$ and $|B|\le 1$. Then $J_\alpha[f]$ is convex in $|z|<\min\{r_c(A,B,\alpha),1\}$, where for $B\neq 0$ we have*

$$r_c(A,B,\alpha)=\begin{cases}\dfrac{|\alpha||B-A|-|\alpha(B-A)-2B|}{2\{\alpha\mathrm{Re}\,[(B-A)\overline{B}]-|B|^2\}}, & \textit{if } \alpha\,\mathrm{Re}\,[(B-A)\overline{B}]\neq|B|^2,\\[2ex] \dfrac{1}{|\alpha||A-B|}, & \textit{if } \alpha\,\mathrm{Re}\,[(B-A)\overline{B}]=|B|^2,\end{cases}$$

and for $B=0$ the radius becomes

$$r_c(A,\alpha)=\frac{1}{|\alpha||A|},\quad \textit{if } \alpha\neq 0.$$

These quantities are best possible for the real numbers A and B.

The substitution $\alpha=1$ in Corollary 5.2 gives us the radius of convexity for the Alexander operator over the class $\mathcal{S}^*(A,B)$, $A\neq B$. Indeed, we have

Corollary 5.3. *Let $f\in\mathcal{S}^*(A,B)$, where A and B are two complex numbers with $A\neq B$ and $|B|\le 1$. Then $J[f]$ is convex in $|z|<\min\{r_c(A,B),1\}$, where for $B\neq 0$ we have*

$$r_c(A,B)=\begin{cases}\dfrac{|A+B|-|B-A|}{2\mathrm{Re}\,[A\overline{B}]}, & \textit{if } \mathrm{Re}\,[A\overline{B}]\neq 0,\\[2ex] \dfrac{1}{|A-B|}, & \textit{otherwise},\end{cases}$$

and for $B=0$ the radius becomes

$$r_c(A)=\frac{1}{|A|}.$$

These quantities are best possible for the real numbers A and B.

If we consider $A = (1-\gamma)e^{2i\theta} - \gamma$ and $B = -1$, with $\gamma \in [0,1)$ and $\theta \in (-\pi/2, \pi/2)$, then the classes $\mathcal{S}^*(A,B)$ and $\mathcal{K}(A,B)$ reduce to the well-known classes of θ-spirallike functions of order γ and θ-convex functions of order γ, respectively. For the simplicity, we use the notations $\mathcal{S}_\theta(\gamma) := \mathcal{S}^*((1-\gamma)e^{2i\theta} - \gamma, -1)$ and $\mathcal{K}_\theta(\gamma) := \mathcal{K}((1-\gamma)e^{2i\theta} - \gamma, -1)$. Each function in $\mathcal{S}_\theta(\gamma)$ is univalent in $\mathbb{D}$ (see [63]). More literature on spirallike functions can be found in [7, 63]. Further, the class $\mathcal{K}_0(\gamma)$, $-1/2 \leq \gamma < 1$, is studied, for instance, in [62] and references therein.

In the following corollary, we obtain the radius of convexity for $C_{\alpha,\beta}[f,g]$ if $f \in \mathcal{S}_\theta(\gamma)$ and $g \in \mathcal{K}_\theta(\gamma)$ with $\gamma \in [0,1)$ and $\theta \in (-\pi/2, \pi/2)$.

Corollary 5.4. *For $\gamma \in [0,1)$ and $\theta \in (-\pi/2, \pi/2)$, let $f \in \mathcal{S}_\theta(\gamma)$ and $g \in \mathcal{K}_\theta(\gamma)$. Then $C_{\alpha,\beta}[f,g]$ is convex in $|z| < \min\{r_c(\gamma,\theta,\alpha,\beta), 1\}$, where, for $(\alpha+\beta)(1-\gamma)(1+\cos 2\theta) \neq 1$, we have*

$$r_c(\gamma,\theta,\alpha,\beta) = \begin{cases} \dfrac{2\cos\theta(1-\gamma)|\alpha+\beta| - \zeta}{2\{(\alpha+\beta)(1-\gamma)(1+\cos 2\theta) - 1\}}, & \textit{if } \alpha\beta \geq 0, \\ \dfrac{2\cos\theta(1-\gamma)|\alpha-\beta| - \sqrt{\xi}}{2\{(\alpha+\beta)(1-\gamma)(1+\cos 2\theta) - 1\}}, & \textit{if } \alpha\beta \leq 0, \end{cases}$$

with $\zeta = |(\alpha+\beta)(\gamma-1)(1+e^{2i\theta}) + 2|$ and $\xi = \zeta^2 - 16\alpha\beta\cos^2\theta(1-\gamma)^2$, and otherwise

$$r_c(\gamma,\theta,\alpha,\beta) == \begin{cases} \dfrac{1}{(1-\gamma)|\alpha+\beta||e^{2i\theta}+1|}, & \textit{if } \alpha\beta \geq 0, \\ \dfrac{1}{(1-\gamma)|\alpha-\beta||e^{2i\theta}+1|}, & \textit{if } \alpha\beta \leq 0. \end{cases}$$

In [105], Umezawa studied the class $\mathcal{C}_\beta$ of functions $f \in \mathcal{A}$ satisfying

$$\frac{-\beta}{2\beta-3} < \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < \beta, \quad z \in \mathbb{D},$$

for some $\beta \geq 3/2$ and showed that this class contains the class of functions convex in one direction. This class, with special choices of the parameter β, has also been considered by many researchers in the literature for different purposes; see for instance [13, 84, 85, 98]. In particular, it has been proved in [98] that the class $\mathcal{C}_\infty$ contains the close-to-convex functions and in [84] that the class $\mathcal{C}_{3/2} = \mathcal{K}(2,1)$ contains the starlike functions. As a consequence of Theorem 1.7, we now proceed for a corollary which obtains the radius of convexity for $C_{\alpha,\beta}[f,g]$ when $f \in \mathcal{S}^*(2,1)$ and $g \in \mathcal{K}(2,1)$.

Corollary 5.5. *Let $f \in \mathcal{S}^*(2,1)$ and $g \in \mathcal{K}(2,1)$ then $C_{\alpha,\beta}[f,g]$ is convex in $|z| < r_c(\alpha,\beta)$, where, for $\alpha+\beta+1 \neq 0$, we have*

$$r_c(\alpha,\beta) = \begin{cases} \dfrac{|\alpha+\beta+2| - |\alpha+\beta|}{2(\alpha+\beta+1)}, & \text{if } \alpha\beta \geq 0, \\ \dfrac{\sqrt{(\alpha+\beta+2)^2 - 4\alpha\beta} - |\alpha-\beta|}{2(\alpha+\beta+1)}, & \text{if } \alpha\beta \leq 0, \end{cases}$$

and otherwise

$$r_c(\alpha,\beta) == \begin{cases} \dfrac{1}{|\alpha+\beta|}, & \text{if } \alpha\beta \geq 0, \\ \dfrac{1}{|\alpha-\beta|}, & \text{if } \alpha\beta \leq 0. \end{cases}$$

The radii $r_c(\alpha,\beta) \leq 1$ are best possible.

Proof. Here we observe that $r_c(\alpha,\beta) \leq 1$, which can be verified by analyzing the function ϕ_1 and ϕ_2 given in the proof of Theorem 1.7. Indeed, we have

$$\phi_1(r) = 1 - |\alpha+\beta|r - (1+\alpha+\beta)r^2$$

and observe on the one side that $\phi_1(0) = 1 > 0$ and on the other side that $\phi_1(1) \leq 0$. This means that $\phi_1(r)$ has a positive real root less than or equal to 1. Similarly, we can see that $\phi_2(r)$ has a positive real root less than or equal to 1. $\square$

By adopting the proof technique from Corollary 5.5, we can find that the radius of convexity is less than or equal to 1 in the corollaries presented below.

The particular values $A = 2$ and $B = -1$ give the well known classes $\mathcal{S}^*(2,-1)$ and $\mathcal{K}(2,-1)$. The classes $\mathcal{S}^*(2,-1)$ and $\mathcal{K}(2,-1)$ already studied in the literature (see [72, p. 66] for some interesting results). For this, we have the following consequence of Theorem 1.7.

Corollary 5.6. *Let $f \in \mathcal{S}^*(2,-1)$ and $g \in \mathcal{K}(2,-1)$ then $C_{\alpha,\beta}[f,g]$ is convex in the disk $|z| < r_c(\alpha,\beta)$, where, for $3(\alpha+\beta)-1 \neq 0$ we have*

$$r_c(\alpha,\beta) = \begin{cases} \dfrac{3|\alpha+\beta| - |3(\alpha+\beta)-2|}{2\{3(\alpha+\beta)-1\}}, & \text{if } \alpha\beta \geq 0, \\ \dfrac{3|\alpha-\beta| - \sqrt{[3(\alpha+\beta)-2]^2 - 36\alpha\beta}}{2\{3(\alpha+\beta)-1\}}, & \text{if } \alpha\beta \leq 0, \end{cases}$$

and otherwise

$$r_c(\alpha,\beta)=\begin{cases}\dfrac{1}{3|\alpha+\beta|}, & \text{if } \alpha\beta\geq 0,\\[2mm] \dfrac{1}{3|\alpha-\beta|}, & \text{if } \alpha\beta\leq 0.\end{cases}$$

The radii $r_c(\alpha,\beta)\leq 1$ *are best possible.*

The radius of convexity for $C_{\alpha,\beta}[f,g]$ when $f\in\mathcal{S}^*(\lambda)$ and $g\in\mathcal{K}(\lambda)$, $\lambda<1$, is provided in the following corollary.

Corollary 5.7. *For* $\lambda<1$*, let* $f\in\mathcal{S}^*(\lambda)$ *and* $g\in\mathcal{K}(\lambda)$*. Then* $C_{\alpha,\beta}[f,g]$ *is convex in* $|z|<r_c(\alpha,\beta,\lambda)$*, where, for* $2(1-\lambda)(\alpha+\beta)\neq 1$*, we have*

$$r_c(\alpha,\beta,\lambda)=\begin{cases}\dfrac{(1-\lambda)|\alpha+\beta|-|(1-\lambda)(\alpha+\beta)-1|}{2(1-\lambda)(\alpha+\beta)-1}, & \text{if } \alpha\beta\geq 0,\\[2mm] \dfrac{(1-\lambda)|\alpha-\beta|-\sqrt{[(1-\lambda)(\alpha+\beta)-1]^2-4\alpha\beta(1-\lambda)^2}}{2(1-\lambda)(\alpha+\beta)-1}, & \text{if } \alpha\beta\leq 0,\end{cases}$$

and otherwise

$$r_c(\alpha,\beta,\lambda)=\begin{cases}\dfrac{1}{2(1-\lambda)|\alpha+\beta|}, & \text{if } \alpha\beta\geq 0,\\[2mm] \dfrac{1}{2(1-\lambda)|\alpha-\beta|}, & \text{if } \alpha\beta\leq 0.\end{cases}$$

The quantities $r_c(\alpha,\beta,\lambda)\leq 1$ *are best possible.*

One of the consequences of Corollary 5.7, for the operator J_α, is provided below.

Corollary 5.8. *Let* $\lambda<1$*. If* $f\in\mathcal{S}^*(\lambda)$ *then* $J_\alpha[f]$ *is convex in* $|z|<r_c(\alpha,\lambda)$*, where*

$$r_c(\alpha,\lambda)=\begin{cases}\dfrac{(1-\lambda)|\alpha|-|(1-\lambda)\alpha-1|}{2(1-\lambda)\alpha-1}, & \text{for } \alpha\in\mathbb{R}\setminus\{1/[2(1-\lambda)]\},\\[2mm] 1, & \text{for } \alpha=1/[2(1-\lambda)].\end{cases}$$

The radii $r_c(\alpha,\lambda)\leq 1$ *are best possible.*

If we substitute $\alpha=1$ in Corollary 5.8, the Alexander Theorem immediately follows. Indeed, the case $\lambda=0$ leads to the classical Alexander Theorem.

Corollary 5.9. *For* $0\leq\lambda<1$*. Let* $f\in\mathcal{S}^*(\lambda)$ *then* $J[f]\in\mathcal{K}$*.*

For $f \in \mathcal{K}$, it is obtained in [50] that the operator $I_\beta[f] \in \mathcal{K}$ if and only if $0 \le \beta \le 1$. Also, the operator I_β over the generalized class $\mathcal{K}(\lambda)$ of the class $\mathcal{K}$ is studied in [11], in which the authors showed that $I_\beta[f] \in \mathcal{K}(\lambda)$ if and only if $0 \le \beta \le 1/(1-\lambda)$. Also, for $f \in \mathcal{K}(\lambda)$, $\lambda < 1$, we have $I_\beta[f] \in \mathcal{S}$ if and only if

$$\left\{\beta \in \mathbb{C} : \, |\beta| \le \frac{1}{2(1-\lambda)}\right\} \bigcup \left[\frac{1}{2(1-\lambda)}, \frac{3}{2(1-\lambda)}\right]$$

see [58, Theorem 2.1]).

However, the next result provides the radius of convexity for $I_\beta[g]$, if $g \in \mathcal{K}(\lambda)$, for every $\beta \in \mathbb{R}$.

Corollary 5.10. *For $\lambda < 1$, let $g \in \mathcal{K}(\lambda)$. Then $I_\beta[g]$ is convex in $|z| < r_c(\beta, \lambda)$, where*

$$r_c(\beta,\lambda) = \begin{cases} \dfrac{(1-\lambda)|\beta| - |(1-\lambda)\beta - 1|}{2(1-\lambda)\beta - 1}, & \text{for } \beta \in \mathbb{R} \setminus \{1/[2(1-\lambda)]\}, \\ 1, & \text{for } \beta = 1/[2(1-\lambda)]. \end{cases}$$

The radii $r_c(\beta, \lambda) \le 1$ are best possible.

By putting $\beta = 1$ in Corollary 5.10, we obtain a nice consequence that provides the radius of convexity for the functions in the class $\mathcal{K}(\lambda)$, $\lambda < 0$.

Corollary 5.11. *For $\lambda < 0$, let $g \in \mathcal{K}(\lambda)$. Then g is convex in $|z| < r_c(\lambda)$, where*

$$r_c(\lambda) = \frac{1}{1-2\lambda}$$

and the radii $r_c(\lambda) < 1$ are best possible.

From now onward, we find the radius of convexity of $C_{\alpha,\beta}[f, g]$ for $f \in \mathcal{S}^*(\lambda)$, $0 \le \lambda < 1$, and g from different subclasses of $\mathfrak{F}$.

5.2. The class $\mathcal{G}(\gamma)$

If $f \in \mathcal{K}(1+\gamma, 1)$, $\gamma \in (0, 1]$, then from (1.5) we have

$$\text{(5.5)} \qquad \mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < 1 + \frac{\gamma}{2}, \quad z \in \mathbb{D}.$$

This motivates us to consider the following class of functions, for $\gamma \in (0, 1]$:

$$\text{(5.6)} \qquad \mathcal{G}(\gamma) := \left\{f \in \mathfrak{F} : \mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < 1 + \frac{\gamma}{2}, \quad z \in \mathbb{D}\right\}.$$

We notice that $\mathcal{C}_{1+\frac{\gamma}{2}} \subset \mathcal{G}(\gamma)$ holds with $\mathcal{G}(1) = \mathcal{C}_{3/2} \subset \mathcal{S}^* \subset \mathcal{S}$ (see the previous section). Note that the class $\mathcal{G}(\gamma)$ has been studied for different purposes in [13, 87, 88, 98].

The following result is due to Obradović et al. [75].

Lemma 5.12. *For $\gamma \in (0,1]$, if $g \in \mathcal{G}(\gamma)$ then*

$$\left|\frac{zg''(z)}{g'(z)}\right| \leq \frac{\gamma|z|}{1-|z|}, \quad z \in \mathbb{D}.$$

For $0 \leq \lambda < 1$, if $f \in \mathcal{S}^*(\lambda)$ then from (5.1) we get

$$\frac{1-(2\lambda-1)|z|^2-(2-2\lambda)|z|}{1-|z|^2} \leq \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \leq \frac{1-(2\lambda-1)|z|^2+(2-2\lambda)|z|}{1-|z|^2}. \tag{5.7}$$

The following theorem establishes the radius of convexity for $C_{\alpha,\beta}[f,g]$ when $f \in \mathcal{S}^*(\lambda)$, $0 \leq \lambda < 1$, and $g \in \mathcal{G}(\gamma)$, $0 < \gamma \leq 1$.

Theorem 5.13. *Let $0 \leq \lambda < 1$ and $0 < \gamma \leq 1$. If $f \in \mathcal{S}^*(\lambda)$ and $g \in \mathcal{G}(\gamma)$ then $C_{\alpha,\beta}[f,g]$ is convex in $|z| < r_c(\alpha,\beta,\lambda,\gamma)$, where, for $\alpha \geq 0$ we have*

$$r_c(\alpha,\beta,\lambda,\gamma) = \begin{cases} \dfrac{1}{2\alpha(1-\lambda)+\gamma|\beta|}, & \text{if } 2\alpha(1-\lambda)-\gamma|\beta| = 1, \\[2ex] \dfrac{2\alpha(1-\lambda)+\gamma|\beta|-\sqrt{\xi}}{2\{2\alpha(1-\lambda)-\gamma|\beta|-1\}}, & \text{if } 2\alpha(1-\lambda)-\gamma|\beta| \neq 1, \end{cases}$$

here $\xi = (2\lambda\alpha - \gamma|\beta| - 2\alpha + 2)^2 + 8\gamma|\beta|$ and for $\alpha < 0$

$$r_c(\alpha,\beta,\lambda,\gamma) = \frac{1}{1+\gamma|\beta|-2\alpha(1-\lambda)}.$$

The quantities $r_c(\alpha,\beta,\lambda,\gamma) \leq 1$ are best possible.

Proof. For $g \in \mathcal{G}(\gamma)$, Lemma 5.12 gives that

$$\frac{-\gamma|z|^2-\gamma|z|}{1-|z|^2} \leq \mathrm{Re}\left\{\frac{zg''(z)}{g'(z)}\right\} \leq \frac{\gamma|z|^2+\gamma|z|}{1-|z|^2}. \tag{5.8}$$

We present our proof by considering the following two cases on α.

Case (i): $\alpha \geq 0$.

Using the inequalities (5.7) and (5.8), we first obtain from (5.3) that

$$\mathrm{Re}\left\{1+\frac{z(C_{\alpha,\beta}[f,g])''(z)}{(C_{\alpha,\beta}[f,g])'(z)}\right\} \geq \frac{1-(2\alpha-2\lambda\alpha+\gamma|\beta|)|z|-(2\lambda\alpha+\gamma|\beta|+1-2\alpha)|z|^2}{1-|z|^2}.$$

Define $\psi_1(r) := 1-(2\alpha-2\lambda\alpha+\gamma|\beta|)r-(2\lambda\alpha+\gamma|\beta|+1-2\alpha)r^2$. If $2\alpha(1-\lambda)-\gamma|\beta|=1$, then $\psi_1(r)>0$ for

$$r<\frac{1}{2\alpha(1-\lambda)+\gamma|\beta|},$$

otherwise $\psi_1(r)>0$ either for

$$r<\frac{2\alpha-2\lambda\alpha+\gamma|\beta|-\sqrt{(2\lambda\alpha-\gamma|\beta|-2\alpha+2)^2+8\gamma|\beta|}}{-2(2\lambda\alpha+\gamma|\beta|+1-2\alpha)}=r_c(\alpha,\beta,\lambda,\gamma)$$

or for

$$r>\frac{2\alpha-2\lambda\alpha+\gamma|\beta|+\sqrt{(2\lambda\alpha-\gamma|\beta|-2\alpha+2)^2+8\gamma|\beta|}}{-2(2\lambda\alpha+\gamma|\beta|+1-2\alpha)}.$$

It is simple to calculate that $\psi_1(0)=1$ and $\psi_1(1)=-2\gamma|\beta|<0$. Then continuity of $\psi_1(r)$ ensures that $\psi_1(r)$ has at least one positive root not more than 1. This gives $0\le r_c(\alpha,\beta,\lambda,\gamma)\le 1$.

Now we discuss the sharpness by considering two situations on β.

For $\beta\ge 0$, we choose

$$f(z)=\frac{z}{(1+z)^{2-2\lambda}}\in\mathcal{S}^*(\lambda)\quad\text{and}\quad g(z)=\frac{1-(1-z)^{\gamma+1}}{\gamma+1}\in\mathcal{G}(\gamma).$$

An elementary computation gives us

$$1+\frac{z(C_{\alpha,\beta}[f,g])''(z)}{(C_{\alpha,\beta}[f,g])'(z)}=\frac{1-(2\alpha-2\lambda\alpha+\gamma\beta)z-(2\lambda\alpha+\gamma\beta+1-2\alpha)z^2}{1-z^2},$$

which clearly justifies the sharpness for this case.

For $\beta<0$, the functions

$$f(z)=\frac{z}{(1+z)^{2-2\lambda}}\in\mathcal{S}^*(\lambda)\quad\text{and}\quad g(z)=\frac{(1+z)^{\gamma+1}-1}{\gamma+1}\in\mathcal{G}(\gamma)$$

yield

$$1+\frac{z(C_{\alpha,\beta}[f,g])''(z)}{(C_{\alpha,\beta}[f,g])'(z)}=\frac{1-(2\alpha-2\lambda\alpha+\gamma\beta)z-(2\lambda\alpha+\gamma\beta+1-2\alpha)z^2}{1-z^2}.$$

This gives the sharpness.

Case (ii): $\alpha<0$.

The inequality

$$\operatorname{Re}\left\{1+\frac{zC_{\alpha,\beta}[f,g]''(z)}{C_{\alpha,\beta}[f,g]'(z)}\right\}\ge\frac{1-(2\lambda\alpha+\gamma|\beta|-2\alpha)|z|-(2\lambda\alpha+\gamma|\beta|+1-2\alpha)|z|^2}{1-|z|^2}$$

holds by (5.3) using the inequalities (5.7) and (5.8).

Define

$$\psi_2(r) := 1 - (2\lambda\alpha + \gamma|\beta| - 2\alpha)r - (2\lambda\alpha + \gamma|\beta| + 1 - 2\alpha)r^2.$$

Then $\psi_2(r) > 0$ either for

$$r < \frac{1}{(2\lambda\alpha + \gamma|\beta| + 1 - 2\alpha)} = r_c(\alpha, \beta, \lambda, \gamma)$$

or for $r > -1$. Now, we can verify the bound $0 \leq r_c(\alpha, \beta, \lambda, \gamma) \leq 1$ in a similar fashion as given in Case (i).

Sharpness part follows as below.

For $\beta \geq 0$, the functions

$$f(z) = \frac{z}{(1-z)^{2-2\lambda}} \in \mathcal{S}^*(\lambda) \quad \text{and} \quad g(z) = \frac{1-(1-z)^{\gamma+1}}{\gamma+1} \in \mathcal{G}(\gamma)$$

give

$$1 + \frac{z(C_{\alpha,\beta}[f,g])''(z)}{(C_{\alpha,\beta}[f,g])'(z)} = \frac{1 - (2\lambda\alpha + \gamma\beta - 2\alpha)z - (2\lambda\alpha + \gamma\beta + 1 - 2\alpha)z^2}{1 - z^2},$$

which is enough to justify for the sharpness.

For $\beta < 0$, the choices

$$f(z) = \frac{z}{(1-z)^{2-2\lambda}} \in \mathcal{S}^*(\lambda) \quad \text{and} \quad g(z) = \frac{(1+z)^{\gamma+1} - 1}{\gamma+1} \in \mathcal{G}(\gamma)$$

provide

$$1 + \frac{z(C_{\alpha,\beta}[f,g])''(z)}{(C_{\alpha,\beta}[f,g])'(z)} = \frac{1 - (2\lambda\alpha - \gamma\beta - 2\alpha)z - (2\lambda\alpha - \gamma\beta + 1 - 2\alpha)z^2}{1 - z^2}.$$

This concludes the sharpness in this case as well, and it completes our proof. □

In 2018, Ali and Vasudevarao [11] show that for a given function $g \in \mathcal{G}(\gamma)$, the operator $I_\beta[g]$ maps $\mathbb{D}$ onto a convex domain only for $-2/\gamma \leq \beta \leq 0$. However, our next result covers all ranges for $\beta \in \mathbb{R}$. Indeed, we determine a subdisk $\mathbb{D}_r$, $0 \leq r < 1$ depending upon β and γ, such that $I_\beta[g]$ maps $\mathbb{D}_r$ onto a convex domain.

Corollary 5.14. *Let $0 < \gamma \leq 1$. If $g \in \mathcal{G}(\gamma)$ then $I_\beta[g]$ is convex in $|z| < r_c(\beta, \gamma)$ with*

$$r_c(\beta, \gamma) = \frac{1}{1 + \gamma|\beta|}.$$

These quantities are best possible.

Remark 5.15. *If we choose* $\beta = 1$ *in Corollary 5.14 then we have the following radius of convexity in* $\mathcal{G}(\gamma)$, $0 < \gamma \leq 1$*:*

$$r_c(\gamma) = \frac{1}{1+\gamma}.$$

It is easy to obtain that the value $\gamma = 0$ *is the only solution of*

$$\frac{1}{1+\gamma} = 1,$$

which guarantees that $\mathcal{G}(\gamma)$, *for* $0 < \gamma \leq 1$, *is not contained in the class* $\mathcal{K}$. *Indeed, we have* $g(z) = [1-(1-z)^{\gamma+1}]/(\gamma+1) \in \mathcal{G}(\gamma) \setminus \mathcal{K}$.

5.3. The class $\mathcal{V}_k$

The class $\mathcal{V}_k,\ k \geq 2$, consists of all functions $f \in \mathfrak{F}$ satisfying

$$\int_0^{2\pi} \left| \mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\}\right| d\theta \leq k\pi, \quad z = re^{i\theta},\ 0 \leq r < 1.$$

It is clear from the definition of $\mathcal{V}_k$ that the range set of its elements have bounded boundary rotation bounded by $k\pi$. The functions of bounded boundary rotation were introduced by Loewner in 1917. In 1931, Paatero [77] showed that, for $2 \leq k \leq 4$, the class $\mathcal{V}_k$ lies in the class $\mathcal{S}$. Later on, in 1969, Pinchuk [81] examined that, for $2 \leq k \leq 4$, all elements of $\mathcal{V}_k$ are close-to-convex in $\mathbb{D}$ but the class $\mathcal{V}_k$, $k > 4$, contains non-univalent functions (see also [23, 34, 86]).

The following useful lemma was introduced by Robertson [94].

Lemma 5.16. *For* $2 \leq k < \infty$, *the following inequality*

$$\left| \frac{zg''(z)}{g'(z)} - \frac{2|z|^2}{1-|z^2|} \right| \leq \frac{k|z|}{1-|z|^2}, \quad z \in \mathbb{D}$$

holds for each $g \in \mathcal{V}_k$.

In the next result, we have the radius of convexity for $C_{\alpha,\beta}[f,g]$, when $f \in \mathcal{S}^*(\lambda)$, $0 \leq \lambda < 1$, and $g \in \mathcal{V}_k$, $k \geq 2$.

Theorem 5.17. *For $0 \le \lambda < 1$ and $k \ge 2$, let $f \in \mathcal{S}^*(\lambda)$ and $g \in \mathcal{V}_k$. Then $C_{\alpha,\beta}[f,g]$ is convex in $|z| < r_c(\alpha,\beta,\lambda,k)$, where, for $2\alpha(1-\lambda)+2\beta \ne 1$,*

$$r_c(\alpha,\beta,\lambda,k) = \begin{cases} \dfrac{2\alpha(1-\lambda)+k|\beta|-\sqrt{\xi}}{2\{2\alpha(1-\lambda)+2\beta-1\}}, & \text{if } \alpha \ge 0, \\ \dfrac{2\alpha(\lambda-1)+k|\beta|-\sqrt{\xi-8\alpha|\beta|k(1-\lambda)}}{2\{2\alpha(1-\lambda)+2\beta-1\}}, & \text{if } \alpha < 0, \end{cases}$$

with $\xi = [2\alpha(1-\lambda)+k|\beta|-2]^2+4|\beta|k-8\beta$, and for $2\alpha(1-\lambda)+2\beta=1$,

$$r_c(\alpha,\beta,\lambda,k) = \frac{1}{2|\alpha|(1-\lambda)+k|\beta|}.$$

The quantities $r_c(\alpha,\beta,\lambda,k) \le 1$ are best possible.

Proof. Suppose $g \in \mathcal{V}_k$, where $k \ge 2$. From Lemma 5.16 we obtain

$$\frac{2|z|^2-k|z|}{1-|z|^2} \le \operatorname{Re}\left\{\frac{zg''(z)}{g'(z)}\right\} \le \frac{2|z|^2+k|z|}{1-|z|^2}. \tag{5.9}$$

We prove our theorem by considering the following two cases on α.

Case(i): $\alpha \ge 0$.

Our hypothesis provides (5.7) and (5.9). Hence, (5.3) leads to the inequality

$$\operatorname{Re}\left\{1+\frac{z(C_{\alpha,\beta}[f,g])''(z)}{(C_{\alpha,\beta}[f,g])'(z)}\right\} \ge \frac{(2\alpha-2\lambda\alpha+2\beta-1)|z|^2-(2\alpha-2\lambda\alpha+k|\beta|)|z|+1}{1-|z|^2}.$$

Let

$$\xi_1(r) := (2\alpha-2\lambda\alpha+2\beta-1)r^2-(2\alpha-2\lambda\alpha+k|\beta|)r+1.$$

If $2\alpha-2\lambda\alpha+2\beta=1$, then $\xi_1(r)>0$ for

$$r < \frac{1}{2\alpha(1-\lambda)+k|\beta|},$$

otherwise $\xi_1(r)>0$ either for

$$r < \frac{2\alpha-2\lambda\alpha+k\beta-\sqrt{(2\lambda\alpha-k|\beta|-2\alpha+2)^2+4k|\beta|-8\beta}}{2(2\alpha-2\lambda\alpha+2\beta-1)} = r_c(\alpha,\beta,\lambda,k)$$

or for

$$r > \frac{2\alpha-2\lambda\alpha+k\beta+\sqrt{(2\lambda\alpha-k|\beta|-2\alpha+2)^2+4k|\beta|-8\beta}}{2(2\alpha-2\lambda\alpha+2\beta-1)}.$$

It is easy to calculate that $\xi_1(0)=1$ and $\xi_1(1)=2\beta-k|\beta|\le 0$. Then $\xi_1(r)$ has at least one positive root which is less than or equal to 1 by the property of continuity. It gives $0 \le r_c(\alpha,\beta,\lambda,k) \le 1$.

We now discuss the sharpness part.

For $\beta \geq 0$, if we choose

$$f(z) = \frac{z}{(1-z)^{2-2\lambda}} \in \mathcal{S}^*(\lambda) \quad \text{and} \quad g(z) = \frac{((1+z)/(1-z))^{k/2} - 1}{k} \in \mathcal{V}_k$$

then a simple calculation leads to

$$1 + \frac{z(C_{\alpha,\beta}[f,g])''(z)}{(C_{\alpha,\beta}[f,g])'(z)} = \frac{(2\alpha - 2\lambda\alpha + 2\beta - 1)z^2 - (2\alpha - 2\lambda\alpha + k\beta)z + 1}{1 - z^2}.$$

Sharpness for the radius of convexity of $C_{\alpha,\beta}[f,g]$ clearly follows in this case.

For $\beta < 0$, a simple computation by considering the functions

$$f(z) = \frac{z}{(1-z)^{2-2\lambda}} \in \mathcal{S}^*(\lambda) \quad \text{and} \quad g(z) = \frac{1 - ((1-z)/(1+z))^{k/2}}{k} \in \mathcal{V}_k$$

leads to

$$1 + \frac{z(C_{\alpha,\beta}[f,g])''(z)}{(C_{\alpha,\beta}[f,g])'(z)} = \frac{(2\alpha - 2\lambda\alpha + 2\beta - 1)z^2 - (2\alpha - 2\lambda\alpha - k\beta)z + 1}{1 - z^2}.$$

Sharpness for the radius of convexity in this case trivially follows from here.

Case(ii): $\alpha < 0$.

Given conditions provide (5.7) and (5.9). Then with (5.3) we have

$$\operatorname{Re}\left\{1 + \frac{z(C_{\alpha,\beta}[f,g])''(z)}{(C_{\alpha,\beta}[f,g])'(z)}\right\} \geq \frac{(2\alpha - 2\lambda\alpha + 2\beta - 1)|z|^2 - (2\lambda\alpha - 2\alpha + k|\beta|)|z| + 1}{1 - |z|^2}.$$

Let

$$\xi_2(r) := (2\alpha - 2\lambda\alpha + 2\beta - 1)r^2 - (2\lambda\alpha - 2\alpha + k|\beta|)r + 1.$$

If $2\alpha - 2\lambda\alpha + 2\beta = 1$, then $\xi_2(r) > 0$ for

$$r < \frac{1}{-2\alpha(1-\lambda) + k|\beta|},$$

otherwise $\xi_2(r) > 0$ either for

$$r < \frac{2\alpha\lambda - 2\alpha + k|\beta| - \sqrt{(2\lambda\alpha + k|\beta| - 2\alpha + 2)^2 - 4k|\beta| - 8\beta}}{2(2\alpha - 2\lambda\alpha + 2\beta - 1)} = r_c(\alpha, \beta, \lambda, k)$$

or for

$$r > \frac{2\alpha\lambda - 2\alpha + k|\beta| + \sqrt{(2\lambda\alpha + k|\beta| - 2\alpha + 2)^2 - 4k|\beta| - 8\beta}}{2(2\alpha - 2\lambda\alpha + 2\beta - 1)}.$$

A similar method as given in Case (i) shows that $0 \leq r_c(\alpha, \beta, \lambda, k) \leq 1$.

For the sharpness, we choose

$$f(z)=\frac{z}{(1+z)^{2-2\lambda}}\in\mathcal{S}^*(\lambda)\quad\text{and}\quad g(z)=\frac{((1+z)/(1-z))^{k/2}-1}{k}\in\mathcal{V}_k$$

if $\beta\geq 0$. Conclusion for the sharpness follows from the computation

$$1+\frac{z(C_{\alpha,\beta}[f,g])''(z)}{(C_{\alpha,\beta}[f,g])'(z)}=\frac{(2\alpha-2\lambda\alpha+2\beta-1)z^2-(2\lambda\alpha-2\alpha+k\beta)z+1}{1-z^2}.$$

Secondly, if $\beta<0$, then the choices

$$f(z)=\frac{z}{(1+z)^{2-2\lambda}}\in\mathcal{S}^*(\lambda)\quad\text{and}\quad g(z)=\frac{1-((1-z)/(1+z))^{k/2}}{k}\in\mathcal{V}_k$$

give us

$$1+\frac{z(C_{\alpha,\beta}[f,g])''(z)}{(C_{\alpha,\beta}[f,g])'(z)}=\frac{(2\alpha-2\lambda\alpha+2\beta-1)z^2-(2\alpha-2\lambda\alpha-k\beta)z+1}{1-z^2}.$$

The sharpness follows in this case as well. Thus, we conclude that $C_{\alpha,\beta}[f,g]$ is convex in $|z|<r_c(\alpha,\beta,\lambda,k)\leq 1$, completing the proof. $\square$

In 2019, Ponnusamy et al. [86] studied preserving properties of the class $\mathcal{V}_k$ under the operator I_β. Here, we obtain the radius of convexity for $I_\beta[g]$, $g\in\mathcal{V}_k$, as a consequence of Theorem 5.17.

Corollary 5.18. *Let $g\in\mathcal{V}_k$ for $k\geq 2$. Then $I_\beta[g]$ is convex in $|z|<r_c(\beta,k)$ with*

$$r_c(\beta,k)=\begin{cases}\dfrac{k|\beta|-\sqrt{(k|\beta|-2)^2+4|\beta|k-8\beta}}{2(2\beta-1)}, & \text{if } \beta\in\mathbb{R}\setminus\{1/2\},\\[2ex] \dfrac{2}{k}, & \text{if } \beta=1/2.\end{cases}$$

The radius $r_c(\beta,k)\leq 1$ is best possible.

Remark 5.19. *The value $\beta=1$ in Corollary 5.18 gives the radius of convexity for a function $g\in\mathcal{V}_k$, $k\geq 2$, which is $(k-\sqrt{k^2-4})/2$. This radius is obtained by Pinchuk in [82]. Now, if*

$$\frac{k-\sqrt{k^2-4}}{2}=1$$

then we obtain $k=2$, which shows that $\mathcal{V}_2$ is contained in $\mathcal{K}$.

5.4. The class $\mathcal{U}_\delta$

A family $\mathcal{F}$ of $\mathcal{A}$ is said to be *linear-invariant family* (L.I.F.), if $f \in \mathcal{F} \subseteq \mathfrak{F}$ be such that

$$\frac{f(\varphi(z)) - f(\varphi(0))}{f'(\varphi(0))\varphi'(0)} \in \mathcal{F},$$

where $\varphi \in Aut(\mathbb{D})$, i.e. φ has the representation

$$\varphi(z) = e^{i\theta}\frac{z+a}{1+\overline{a}z}, \quad z, a \in \mathbb{D} \text{ and } \theta \in \mathbb{R}.$$

For example, the classes $\mathcal{S}$ and $\mathcal{K}$ are linear-invariant families, whereas, the class $\mathcal{S}^*$ is not linear-invariant. For more example(s), it is referred to [33, Chapter 5].

The order of the L.I.F. was introduced by Pommerenke in [83] and it is defined as

$$\operatorname{ord}\mathcal{F} := \sup\left\{\left|\frac{f''(0)}{2}\right| : f \in \mathcal{F}\right\}.$$

He also proved that

$$\operatorname{ord}\mathcal{F} = \sup_{g\in\mathcal{F}}\sup_{|z|<1}\left| -\overline{z} + \frac{(1-|z|^2)}{2}\frac{g''(z)}{g'(z)}\right|, \quad z \in \mathbb{D}.$$

It is easy to see that if $\mathcal{F}$ is a compact family then $\operatorname{ord}\mathcal{F} < \infty$. Note that $\mathfrak{F}$ is a L.I.F. of infinite order (as we can see by considering the function $f(z) = [e^{kz}-1]/k, z \in \mathbb{D}$, see [33, Chapter 5]). The universal linear-invariant family of order $\delta \geq 1$ is described as

$$\mathcal{U}_\delta := \bigcup_{\operatorname{ord}\mathcal{F}\leq\delta} \mathcal{F}.$$

In 1971, Campbell et al. [24] defined the order of a function $g \in \mathfrak{F}$ as

$$\operatorname{ord} g = \sup_{|z|<1}\left| -\overline{z} + \frac{(1-|z|^2)}{2}\frac{g''(z)}{g'(z)}\right|, \quad z \in \mathbb{D}. \tag{5.10}$$

Then $\mathcal{U}_\delta$ is a L.I.F. of order δ, and we can also write

$$\mathcal{U}_\delta = \{g \in \mathfrak{F} : \operatorname{ord} g \leq \delta\}.$$

This is well known that $\mathcal{U}_1$ and $\mathcal{S}$ are subfamilies of $\mathcal{K}$ and $\mathcal{U}_2$, respectively.

With the help of (5.10), Ebadian and Kargar in [29] proved the following lemma.

Lemma 5.20. *For $\delta \geq 1$, if $g \in \mathcal{U}_\delta$ then*

$$\left|\frac{zg''(z)}{g'(z)} - \frac{2|z|^2}{1-|z|^2}\right| \leq \frac{2\delta|z|}{1-|z|^2}, \quad z \in \mathbb{D}.$$

The following theorem contains information about the radius of convexity for $C_{\alpha,\beta}[f,g]$ when $f \in \mathcal{S}^*(\lambda)$, $0 \le \lambda < 1$, and $g \in \mathcal{U}_\delta$, $\delta \ge 1$, which can be easily obtained by replacing k with 2δ in Theorem 5.17.

Theorem 5.21. *Let $0 \le \lambda < 1$ and $\delta \ge 1$. If $f \in \mathcal{S}^*(\lambda)$ and $g \in \mathcal{U}_\delta$ then $C_{\alpha,\beta}[f,g]$ is convex in $|z| < r_c(\alpha,\beta,\lambda,\delta)$, where, for $2\alpha(1-\lambda)+2\beta \neq 1$,*

$$r_c(\alpha,\beta,\lambda,\delta) = \begin{cases} \dfrac{\alpha(1-\lambda)+|\beta|\delta-\sqrt{\xi}}{2\alpha(1-\lambda)+2\beta-1}, & \text{if } \alpha \ge 0; \\ \dfrac{\alpha(\lambda-1)+|\beta|\delta-\sqrt{\xi-4\alpha|\beta|\delta(1-\lambda)}}{2\alpha(1-\lambda)+2\beta-1}, & \text{if } \alpha < 0, \end{cases}$$

with $\xi = [\alpha(1-\lambda)+|\beta|\delta-1]^2+2|\beta|\delta-2\beta$, and for $2\alpha(1-\lambda)+2\beta = 1$

$$r_c(\alpha,\beta,\lambda,\delta) = \frac{1}{2|\alpha|(1-\lambda)+2\delta|\beta|}.$$

The radii $r_c(\alpha,\beta,\lambda,\delta) \le 1$ are best possible.

The choice $\alpha = 0$ in Theorem 5.21 leads to the following corollary which gives the radius of convexity for $I_\beta[g]$, when $g \in \mathcal{U}_\delta$, $\delta \ge 1$.

Corollary 5.22. *Let $\delta \ge 1$. If $g \in \mathcal{U}_\delta$ then $I_\beta[g]$ is convex in $|z| < r_c(\beta,\delta)$, where*

$$r_c(\beta,\delta) = \begin{cases} \dfrac{|\beta|\delta-\sqrt{(|\beta|\delta-1)^2+2|\beta|\delta-2\beta}}{2\beta-1}, & \text{if } \beta \in \mathbb{R}\backslash\{1/2\}, \\ \dfrac{1}{\delta}, & \text{if } \beta = 1/2. \end{cases}$$

The quantity $r_c(\beta,\delta)$ is best possible.

Remark 5.23. *We can obtain radius of convexity for the functions in the class $\mathcal{U}_\delta$, $\delta \ge 1$, which is $\delta-\sqrt{\delta^2-1}$ by substituting the value $\beta = 1$ in Corollary 5.22. Also, we can obtain $\delta-\sqrt{\delta^2-1} = 1$ for $\delta = 1$, which gives that $\mathcal{U}_1 \subset \mathcal{K}$.*

5.5. The class $\mathcal{S}$

We have already discussed that the class $\mathcal{S}$ contains the normalized analytic univalent functions. Many authors studied the range of functions for the class $\mathcal{S}$. The covering theorem (also known as the Koebe One-Quarter theorem) says that the range of every function for the class $\mathcal{S}$ contains the disk of radius $1/4$ (proof is given in [27, p. 31]). The radius of convexity for functions in the class $\mathcal{S}$ is known as $2-\sqrt{3}$ (see [27, Page 44]), which also comes as one of the consequences of our results in this section.

Recall the basic estimate [27, Theorem 2.4]:

Lemma 5.24. *For each $g \in \mathcal{S}$, we have*

$$\left|\frac{zg''(z)}{g'(z)} - \frac{2|z|^2}{1-|z|^2}\right| \leq \frac{4|z|}{1-|z|^2}, \quad z \in \mathbb{D}.$$

The inequality is sharp for a suitable rotation of the Koebe function.

The value $k=4$ in Theorem 5.17 or $\delta = 2$ in Theorem 5.21 gives the following theorem in which we study the radius of convexity for $C_{\alpha,\beta}[f,g]$, if $g \in \mathcal{S}$ and $f \in \mathcal{S}^*(\lambda)$.

Theorem 5.25. *For $0 \leq \lambda < 1$, let $f \in \mathcal{S}^*(\lambda)$ and $g \in \mathcal{S}$. Then $C_{\alpha,\beta}[f,g]$ is convex in $|z| < r_c(\alpha,\beta,\lambda,\delta)$, where, for $2\alpha(1-\lambda)+2\beta \neq 1$,*

$$r_c(\alpha,\beta,\lambda,\delta) = \begin{cases} \dfrac{\alpha(1-\lambda)+2|\beta|-\sqrt{\xi}}{2\alpha(1-\lambda)+2\beta-1}, & \text{if } \alpha \geq 0; \\[2ex] \dfrac{\alpha(\lambda-1)+2|\beta|-\sqrt{\xi-8\alpha|\beta|(1-\lambda)}}{2\alpha(1-\lambda)+2\beta-1}, & \text{if } \alpha < 0, \end{cases}$$

with $\xi = [\alpha(1-\lambda)+2|\beta|-1]^2 + 4|\beta| - 2\beta$, and for $2\alpha(1-\lambda)+2\beta = 1$,

$$r_c(\alpha,\beta,\lambda) = \frac{1}{2|\alpha|(1-\lambda)+4|\beta|}.$$

These radii bounded above by 1 are best possible.

In 1975, Pfaltzgraff [79, Corollary 1] observed that $I_\beta[g] \in \mathcal{S}$ when $g \in \mathcal{S}$ and $|\beta| \leq 1/4$. A remarkable result is also added by Royster in [96, Theorem 2] that for each number $\beta \neq 1$ with $|\beta| > 1/3$, there exists a function $f \in \mathcal{S}$ such that $I_\beta[f] \notin \mathcal{S}$ (see also [11,52,53]). Finding the exact region of β is still an open problem for which $I_\beta[g] \in \mathcal{S}$ when $g \in \mathcal{S}$. For more about this operator, we refer to [51,74].

The next corollary provides the radius of convexity for the Hornich scalar multiplication operator over the class $\mathcal{S}$.

Corollary 5.26. *Let $g \in \mathcal{S}$. Then $I_\beta[g]$ is convex in $|z| < r_c(\beta)$, where*

$$r_c(\beta) = \begin{cases} \dfrac{2|\beta| - \sqrt{(2|\beta|-1)^2 + 4|\beta| - 2\beta}}{2\beta - 1}, & \textit{if } \beta \in \mathbb{R}\backslash\{1/2\}, \\ \dfrac{1}{2}, & \textit{if } \beta = 1/2, \end{cases}$$

are the best possible radii.

Remark 5.27. *As a consequence, the choice $\beta = 1$ in Corollary 5.26 provides the radius of convexity for functions in the class $\mathcal{S}$, which is known as $2 - \sqrt{3}$ in [27].*

CHAPTER 6

CERTAIN GEOMETRIC PROPERTIES

This chapter[1] contains several results concerning the preserving properties of the operator $C_{\alpha,\beta}$ and their important consequences.

6.1. Results on the operator $C_{\alpha,\beta}$

The following theorem characterizes the set in $\alpha\beta$-plane for which the operators $C_{\alpha,\beta}[f,g]$ either belong to $\mathcal{K}(\lambda)$ or $\mathcal{G}(\gamma)$ or $\mathcal{C}$, whenever $f,g \in \mathfrak{F}$.

Theorem 6.1. *Let H be a set in $\mathbb{R}^2$. For $\lambda < 1$, $\gamma > 0$ and $(\alpha,\beta) \in H$, if $C_{\alpha,\beta}[f,g] \in \mathcal{K}(\lambda)$ or $\mathcal{G}(\gamma)$ or $\mathcal{C}$, then H is a convex set.*

The relation (1.13) with Theorem 1.8 obtains the following result:

Corollary 6.2. *Let $f \in \mathcal{K}$. Then $J_\alpha[f] \in \mathcal{K}$ if and only if $0 \le \alpha \le 2$.*

The relation (1.14) with Theorem 1.8 leads to the result of Kim and Srivastava [53, Theorem 2] also stated as follows:

Corollary 6.3. *Let $g \in \mathcal{K}$. Then $I_\beta[g] \in \mathcal{K}$ if and only if $0 \le \beta \le 1$.*

In the next theorem we obtain the region in $\alpha\beta$-plane in which $C_{\alpha,\beta}[f,g] \in \mathcal{C}$ whenever $f,g \in \mathcal{K}$.

Theorem 6.4. *Let $f,g \in \mathcal{K}$. Then $C_{\alpha,\beta}[f,g] \in \mathcal{C}$ if and only if $-1 \le \alpha, 2\beta, \alpha+2\beta \le 3$.*

Due to the relation (1.13), we have the following consequence of either Theorem 6.4 or Theorem B:

Corollary 6.5. *[70, Theorem 2] Let $f \in \mathcal{K}$. Then $J_\alpha[f] \in \mathcal{C}$ if and only if $-1 \le \alpha \le 3$.*

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[1]This chapter is based on the manuscript: Kumar S., *Geometric properties of certain integral operators involving Hornich operations*, Submitted.

The relation (1.14) produces the following corollary as a consequence of either Theorem 6.4 or Theorem A:

Corollary 6.6. *Let $g \in \mathcal{K}$. Then $I_\beta[g] \in \mathcal{C}$ if $-1/2 \le \beta \le 3/2$. This result is sharp.*

It is appropriate to remark here that Corollary 6.6 can also be deduced from [70, Theorem 1] with the help of the classical Alexander theorem.

The following two theorems are natural generalizations of Theorem 1.8 and Theorem 6.4.

Theorem 6.7. *For $\lambda < 1$, let $f \in \mathcal{K}$ and $g \in \mathcal{K}(\lambda)$ then we have*

- **(i)** $C_{\alpha,\beta}[f,g] \in \mathcal{K}$ *if and only if* $0 \le \alpha, 2\beta(1-\lambda), \alpha + 2\beta(1-\lambda) \le 2$.
- **(ii)** $C_{\alpha,\beta}[f,g] \in \mathcal{K}(\lambda)$ *if and only if* $0 \le \alpha, 2\beta(1-\lambda), \alpha + 2\beta(1-\lambda) \le 2(1-\lambda)$.
- **(iii)** $C_{\alpha,\beta}[f,g] \in \mathcal{G}(\gamma)$, $\gamma > 0$, *if and only if* $-\gamma \le \alpha, 2\beta(1-\lambda), \alpha + 2\beta(1-\lambda) \le 0$.
- **(iv)** $C_{\alpha,\beta}[f,g] \in \mathcal{C}$ *if and only if* $-1 \le \alpha, 2\beta(1-\lambda), \alpha + 2\beta(1-\lambda) \le 3$.

Theorem 6.8. *For $\gamma > 0$, let $f \in \mathcal{K}$ and $g \in \mathcal{G}(\gamma)$ then we have*

- **(i)** $C_{\alpha,\beta}[f,g] \in \mathcal{K}$, $\lambda < 1$, *if and only if* $0 \le \alpha, -\beta\gamma, \alpha - \beta\gamma \le 2$.
- **(ii)** $C_{\alpha,\beta}[f,g] \in \mathcal{K}(\lambda)$ *if and only if* $0 \le \alpha, -\beta\gamma, \alpha - \beta\gamma \le 2(1-\lambda)$.
- **(iii)** $C_{\alpha,\beta}[f,g] \in \mathcal{G}(\gamma)$ *if and only if* $-\gamma \le \alpha, -\beta\gamma, \alpha - \beta\gamma \le 0$.
- **(iv)** $C_{\alpha,\beta}[f,g] \in \mathcal{C}$ *if and only if* $-1 \le \alpha, -\beta\gamma, \alpha - \beta\gamma \le 3$.

In theorem 1.8 if we choose $f \in \mathcal{G}$ with the remaining conditions unchanged, then we obtain the following result:

Theorem 6.9. *Let $f \in \mathcal{G}$ and $g \in \mathcal{K}$. Then $C_{\alpha,\beta}[f,g] \in \mathcal{K}$ if and only if $0 \le \beta \le 1$, $\alpha + \beta \le 1$ and $3\beta - \alpha \le 3$.*

The following two consecutive theorems are the consequences of Theorem 6.9.

Theorem 6.10. *For $\lambda < 1$, let $f \in \mathcal{G}$ and $g \in \mathcal{K}(\lambda)$ then we have*

- **(i)** $C_{\alpha,\beta}[f,g] \in \mathcal{K}$ *if and only if* $-3 \le \alpha \le 1$, $0 \le \beta(1-\lambda) \le 1$, $\alpha + \beta(1-\lambda) \le 1$ *and* $3\beta(1-\lambda) - \alpha \le 3$.
- **(ii)** $C_{\alpha,\beta}[f,g] \in \mathcal{K}(\lambda)$ *if and only if* $-3(1-\lambda) \le \alpha \le (1-\lambda)$, $0 \le \beta \le 1$, $\alpha + \beta(1-\lambda) \le (1-\lambda)$ *and* $3\beta(1-\lambda) - \alpha \le 3(1-\lambda)$.

(iii) $C_{\alpha,\beta}[f,g] \in \mathcal{G}$ *if and only if* $-1/2 \leq \alpha \leq 3/2$, $-1/2 \leq \beta(1-\lambda) \leq 0$, $\alpha+\beta(1-\lambda) \geq -1/2$ *and* $3\beta(1-\lambda)-\alpha \geq -1/2$.

(iv) $C_{\alpha,\beta}[f,g] \in \mathcal{G}(\gamma)$, $\gamma > 0$, *if and only if* $-\gamma/2 \leq \alpha \leq 3\gamma/2$, $-\gamma/2 \leq \beta(1-\lambda) \leq 0$, $\alpha+\beta(1-\lambda) \geq -\gamma/2$ *and* $3\beta(1-\lambda)-\alpha \geq -\gamma/2$.

Theorem 6.11. *For* $\gamma > 0$, *let* $f \in \mathcal{G}$ *and* $g \in \mathcal{G}(\gamma)$ *then we have*

(i) $C_{\alpha,\beta}[f,g] \in \mathcal{K}$ *if and only if* $-3 \leq \alpha \leq 1$, $-2/\gamma \leq \beta \leq 0$, $\alpha - \beta\gamma/2 \leq 1$ *and* $-3\beta\gamma/2 - \alpha \leq 3$.

(ii) $C_{\alpha,\beta}[f,g] \in \mathcal{K}(\lambda)$, $\lambda < 1$ *if and only if* $-3(1-\lambda) \leq \alpha \leq (1-\lambda)$, $-2(1-\lambda)/\gamma \leq \beta \leq 0$, $\alpha - \beta\gamma/2 \leq (1-\lambda)$ *and* $-3\beta\gamma/2 - \alpha \leq 3(1-\lambda)$.

(iii) $C_{\alpha,\beta}[f,g] \in \mathcal{G}$ *if and only if* $-1/2 \leq \alpha \leq 3/2$, $0 \leq \beta \leq 1$, $-2\alpha + \beta \leq 1$ *and* $3\beta + 2\alpha \leq 3$.

(iv) $C_{\alpha,\beta}[f,g] \in \mathcal{G}(\gamma)$ *if and only if* $-\gamma/2 \leq \alpha \leq 3\gamma/2$, $0 \leq \beta \leq 1$, $-2\alpha/\gamma + \beta \leq 1$ *and* $3\beta + 2\alpha/\gamma \leq 3$.

In view of the relation (1.13), Theorem 6.10(ii) or Theorem 6.11(ii) obtain the following corollary which may be of independent interest.

Corollary 6.12. *For* $\alpha \in \mathbb{R}$ *and* $f \in \mathcal{G}$, $J_\alpha[f] \in \mathcal{K}(\lambda)$ *if and only if* $-3(1-\lambda) \leq \alpha \leq (1-\lambda)$.

With the help of (1.13), Theorem 6.10(iv) or in Theorem 6.11(iv) leads to the following corollary.

Corollary 6.13. *For* $\alpha \in \mathbb{R}$ *and* $f \in \mathcal{G}$, $J_\alpha[f] \in \mathcal{G}(\gamma)$ *if and only if* $-\gamma/2 \leq \alpha \leq 3\gamma/2$.

The substitution $\gamma = 1$ in Corollary 6.13 takes to the following immediate consequence:

Corollary 6.14. *For* $\alpha \in \mathbb{R}$ *and* $f \in \mathcal{G}$, $J_\alpha[f] \in \mathcal{G}$ *if and only if* $-1/2 \leq \alpha \leq 3/2$.

As we can see that in compare to Theorem A and B, Theorems 1.8, 6.4 and 6.9 give the information of J_α and I_β, simultaneously. In the next section, we set the proof of our main results.

6.2. Proof of the results

This section begins with the proof of Theorem 6.1.

Proof of Theorem 6.1

Suppose that (α, β) is a locus of point on the line segment $[(\alpha_1, \beta_1), (\alpha_2, \beta_2)]$ joining (α_1, β_1) and (α_2, β_2). Then it is easy to obtain that

$$C_{\alpha,\beta}[f,g](z) = (t \star C_{\alpha_1,\beta_1}[f,g] \oplus (1-t) \star C_{\alpha_2,\beta_2}[f,g])(z), \quad \text{for } 0 \le t \le 1.$$

A simple computation shows that

$$1 + \frac{zC_{\alpha,\beta}[f,g]''(z)}{C_{\alpha,\beta}[f,g]'(z)} = t\left\{1 + \frac{zC_{\alpha_1,\beta_1}[f,g]''(z)}{C_{\alpha_1,\beta_1}[f,g]'(z)}\right\} + (1-t)\left\{1 + \frac{zC_{\alpha_2,\beta_2}[f,g]''(z)}{C_{\alpha_2,\beta_2}[f,g]'(z)}\right\}. \tag{6.1}$$

The conditions $C_{\alpha_i,\beta_i}[f,g] \in \mathcal{K}(\lambda)$, $i = 1, 2$, in (6.1) give

$$\text{Re}\left\{1 + \frac{zC_{\alpha,\beta}[f,g]''(z)}{C_{\alpha,\beta}[f,g]'(z)}\right\} > \lambda.$$

so that $C_{\alpha,\beta}[f,g] \in \mathcal{K}(\lambda)$.

Similarly, the assumptions $C_{\alpha_i,\beta_i}[f,g] \in \mathcal{G}(\gamma)$, $i = 1, 2$, in (6.1) provide

$$\text{Re}\left\{1 + \frac{zC_{\alpha,\beta}[f,g]''(z)}{C_{\alpha,\beta}[f,g]'(z)}\right\} < 1 + \frac{\gamma}{2},$$

which implies that $C_{\alpha,\beta}[f,g] \in \mathcal{G}(\gamma)$.

Finally, if $C_{\alpha_i,\beta_i}[f,g] \in \mathcal{C}$, $i = 1, 2$, then from (6.1) we get that

$$\int_{\theta_1}^{\theta_2} \text{Re}\left\{1 + \frac{zC_{\alpha,\beta}[f,g]''(z)}{C_{\alpha,\beta}[f,g]'(z)}\right\} d\theta > -\pi, \quad z = re^{i\theta} \text{ and } 0 \le \theta_1 < \theta_2 \le 2\pi.$$

The Kaplan's theorem gives that $C_{\alpha,\beta}[f,g] \in \mathcal{C}$. Hence concludes the proof. $\square$

The following useful lemma is due to Kim and Merkes which is proved in [50].

Lemma 6.15. *Let $\alpha \in \mathbb{R}$. The function $b_\alpha(z) = \int_0^z (1+t)^\alpha dt \in \mathcal{C}$ (or $\mathcal{K}$) if and only if $-3 \le \alpha \le 1$ (if and only if $-2 \le \alpha \le 0$).*

The proof of the first main result is the following.

Proof of Theorem 1.8

It is easy to calculate that

$$\text{Re}\left\{1 + \frac{zC_{\alpha,\beta}[f,g]''(z)}{C_{\alpha,\beta}[f,g]'(z)}\right\} = \alpha\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} + \beta\,\text{Re}\left\{\frac{zg''(z)}{g'(z)} + 1\right\} + (1 - \alpha - \beta). \tag{6.2}$$

It is known that

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \frac{1}{2}$$

for $f \in \mathcal{K}$. Then, for $\alpha \geq 0$ and $\beta \geq 0$

$$\operatorname{Re}\left\{1 + \frac{zC_{\alpha,\beta}[f,g]''(z)}{C_{\alpha,\beta}[f,g]'(z)}\right\} > 0$$

if $\alpha + 2\beta \leq 2$.

For the only if part, we take $f(z) = z$ and $g(z) = z/(1+z)$. Then by Lemma 6.15 we have $C_{\alpha,\beta}[f,g] \in \mathcal{K}$ if and only if $0 \leq \beta \leq 1$. For $g(z) = z$ and $f(z) = z/(1+z)$, $C_{\alpha,\beta}[f,g] \in \mathcal{K}$ if and only if $0 \leq \alpha \leq 2$, which can easily be verified by using Lemma 6.15. Finally, if we choose $f(z) = z/(1+z)$ and $g(z) = z/(1+z)$ then $C_{\alpha,\beta}[f,g] \in \mathcal{K}$ if and only if $0 \leq \alpha + 2\beta \leq 2$, which also follows by using Lemma 6.15. This completes the proof. □

The following lemma is discussed in [50].

Lemma 6.16. *If $f \in \mathcal{K}$, then for $0 \leq r < 1$, $0 \leq \theta_1 < \theta_2 \leq 2\pi$, we have*

$$\frac{\theta_2 - \theta_1}{2} < \int_{\theta_1}^{\theta_2} \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) d\theta \leq \pi + \frac{\theta_2 - \theta_1}{2},$$

and

$$0 < \int_{\theta_1}^{\theta_2} \operatorname{Re}\left(\frac{zf''(z)}{f'(z)} + 1\right) d\theta \leq 2\pi,$$

where $z = re^{i\theta}$.

With the help of Lemma 6.16, the proof of the second main result is provided below.

Proof of Theorem 6.4

To obtain the region in the $\alpha\beta$-plane for which $C_{\alpha,\beta}[f,g] \in \mathcal{C}$, whenever $f, g \in \mathcal{K}$, we need to consider four cases on α and β.

Case (i) $\alpha \geq 0$ **and** $\beta \geq 0$**:**

Given that $f, g \in \mathcal{K}$. Then by Lemma 6.16 together with (6.2) we obtain that

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{1 + \frac{zC_{\alpha,\beta}[f,g]''(z)}{C_{\alpha,\beta}[f,g]'(z)}\right\} d\theta > (1 - \frac{\alpha}{2} - \beta)(\theta_2 - \theta_1).$$

Then by the Kaplan's theorem $C_{\alpha,\beta}[f,g] \in \mathcal{C}$ if $\alpha + 2\beta \leq 3$.

Case (ii) $\alpha \geq 0$ **and** $\beta < 0$**:**

For $f, g \in \mathcal{K}$, Lemma 6.16 gives that

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{1 + \frac{zC_{\alpha,\beta}[f,g]''(z)}{C_{\alpha,\beta}[f,g]'(z)}\right\} d\theta > (1 - \frac{\alpha}{2} - \beta)(\theta_2 - \theta_1) + 2\pi\beta.$$

Then Kaplan's theorem concludes that $C_{\alpha,\beta}[f,g] \in \mathcal{C}$ if $\alpha \leq 3$ and $\beta \geq -1/2$.

Case (iii) $\alpha < 0$ **and** $\beta \geq 0$**:**

By the assumption on $f, g \in \mathcal{K}$ we estimate

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\left\{1 + \frac{zC_{\alpha,\beta}[f,g]''(z)}{C_{\alpha,\beta}[f,g]'(z)}\right\} d\theta > \alpha\pi + (1 - \frac{\alpha}{2} - \beta)(\theta_2 - \theta_1)$$

with the help of Lemma 6.16. Then $C_{\alpha,\beta}[f,g] \in \mathcal{C}$ if $\alpha \geq -1$ and $\beta \leq 3/2$ by using the Kaplan's theorem.

Case (iv) $\alpha < 0$ **and** $\beta < 0$**:**

We derive the inequality

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\left\{1 + \frac{zC_{\alpha,\beta}[f,g]''(z)}{C_{\alpha,\beta}[f,g]'(z)}\right\} d\theta > (\alpha + 2\beta)\pi$$

by using Lemma 6.16. Now, the Kaplan's theorem gives that $C_{\alpha,\beta}[f,g] \in \mathcal{C}$ if $\alpha + 2\beta \geq -1$.

To prove the sharpness of the result, on the one side we consider the functions $f(z) = z$ and $g(z) = z/(1+z)$. Now by Lemma 6.15 we have $C_{\alpha,\beta}[f,g] \in \mathcal{C}$ if and only if $-1/2 \leq \beta \leq 3/2$. On the other side, we choose $g(z) = z$ and $f(z) = z/(1+z)$ to verify that $C_{\alpha,\beta}[f,g] \in \mathcal{C}$ if and only if $-1 \leq \alpha \leq 3$ which is due to Lemma 6.15. Finally, if we choose $f(z) = z/(1+z)$ and $g(z) = z/(1+z)$ then $C_{\alpha,\beta}[f,g] \in \mathcal{C}$ if and only if $-1 \leq \alpha + 2\beta \leq 3$, which follows from Lemma 6.15. This concludes the proof. □

The next lemma provides that every function in the class $\mathcal{G}(\gamma)$, $0 < \gamma$, can be recovered from a function in the class $\mathcal{K}$ by the Hornich scalar multiplication operation, which is already studied in [11] for the limited range $0 < \gamma \leq 1$.

Lemma 6.17. *For every* $\gamma > 0$*, we have* $\mathcal{G}(\gamma) = (-\gamma/2) \star \mathcal{K}$*.*

Proof. Let the mapping $\xi : \mathfrak{F} \longrightarrow \mathfrak{F}$ be defined as $\xi(f) = (-\gamma/2) \star f$, where $\gamma > 0$. Suppose $\xi(f) = g$ then it is easy to compute that $g' = (f')^{(-\gamma/2)}$. Thus we obtain

$$\frac{zg''(z)}{g'(z)} + 1 = -\frac{\gamma}{2}\left[\frac{zf''(z)}{f'(z)} + 1\right] + \frac{\gamma}{2} + 1.$$

It follows that $f \in \mathcal{K}$ if and only if $g \in \mathcal{G}(\gamma)$, which leads to the fact that $\xi(\mathcal{K}) = \mathcal{G}(\gamma)$. □

The following lemma is observed by Koepf [55].

Lemma 6.18. *For all* $\lambda < 1$*, we have* $\mathcal{K}(\lambda) = (1 - \lambda) \star \mathcal{K}$*.*

Now, we are ready to prove Theorems 6.7 and 6.8.

Proof of Theorem 6.7

(i) Given that $g \in \mathcal{K}(\lambda)$. Then by Lemma 6.18 there exists a function $h \in \mathcal{K}$ such that $g(z) = ((1-\lambda) \star h)(z) = I_{1-\lambda}[h](z)$, which implies that $C_{\alpha,\beta}[f,g](z) = C_{\alpha,\beta(1-\lambda)}[f,h](z)$. Now conclusion follows from Theorem 1.8.

(ii) From part (i) and by using Lemma 6.18, we have

$$C_{\alpha,\beta}[f,g] = C_{\alpha,\beta(1-\lambda)}[f,h] \in \mathcal{K}(\lambda) = (1-\lambda) \star \mathcal{K}.$$

It is easy to obtain that $(1/(1-\lambda)) \star C_{\alpha,\beta(1-\lambda)}[f,h] = C_{\alpha/(1-\lambda),\beta}[f,h] \in \mathcal{K}$. Remaining work can be completed by using Theorem 1.8.

(iii) From part (i) we obtain $C_{\alpha,\beta}[f,g](z) = C_{\alpha,\beta(1-\lambda)}[f,h](z)$. By using Lemma 6.17, we observe that $C_{\alpha,\beta(1-\lambda)}[f,h] \in \mathcal{G}(\gamma) = (-\gamma/2) \star \mathcal{K}$. A simple computation provides us $(-2/\gamma) \star C_{\alpha,\beta(1-\lambda)}[f,h] = C_{-2\alpha/\gamma,-2\beta(1-\lambda)/\gamma}[f,h] \in \mathcal{K}$. Now, one can find the desired restrictions on α and β by using Theorem 1.8.

(iv) As we know from part (i) that $C_{\alpha,\beta}[f,g](z) = C_{\alpha,\beta(1-\lambda)}[f,h](z)$, the rest of the steps of the proof follow from Theorem 6.4. This completes the proof. $\square$

Proof of Theorem 6.8

The proof of part (i), (ii), (iii) and (iv) follows from the proof of corresponding part of Theorem 6.7 by taking $g \in \mathcal{G}(\gamma)$ instead of $\mathcal{K}(\lambda)$ and using Lemma 6.17. $\square$

The proof of Theorem 6.9 is based on the result [84, Example 1, Equation (16)] of Ponnusamy and Rajasekaran, in which they observed that

$$0 < \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) < \frac{4}{3} \tag{6.3}$$

for $f \in \mathcal{G}$. We are using this result to prove the following theorem.

Proof of Theorem 6.9

The given hypothesis $g \in \mathcal{K}$ along with the relation (6.2) provides us

$$\mathrm{Re}\left(1 + \frac{zC_{\alpha,\beta}[f,g][f]''(z)}{C_{\alpha,\beta}[f,g]'(z)}\right) > \alpha \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) + 1 - \alpha - \beta, \tag{6.4}$$

for $\beta \geq 0$.

If $\alpha \geq 0$ then from (6.3) and (6.4) we have

$$\mathrm{Re}\left(1 + \frac{zC_{\alpha,\beta}[f,g]''(z)}{C_{\alpha,\beta}[f,g]'(z)}\right) > 1 - \alpha - \beta.$$

This provides us $C_{\alpha,\beta}[f,g] \in \mathcal{K}$ for $\alpha+\beta \le 1$.

Now, if $\alpha < 0$ then again from (6.3) and (6.4) we obtain

$$\operatorname{Re}\left(1+\frac{zC_{\alpha,\beta}[f,g]''(z)}{C_{\alpha,\beta}[f,g]'(z)}\right) > \frac{4}{3}\alpha+1-\alpha-\beta = \frac{1}{3}\alpha+1-\beta.$$

This gives that $C_{\alpha,\beta}[f,g] \in \mathcal{K}$ for $3\beta-\alpha \le 3$.

Now, we show the sharpness of the result. For the choices $f(z) = z \in \mathcal{G}$ and $g(z) = z/(1+z) \in \mathcal{K}$, we have $C_{\alpha,\beta}[f,g](z) = \int_0^z (1+t)^{-2\beta} \in \mathcal{K}$ if and only if $0 \le \beta \le 1$ by using Lemma 6.15.

Further, we consider $f(z) = [1-(1-z)^2]/2 \in \mathcal{G}$ and $g(z) = z/(1+z)$ then we obtain

$$1+\frac{zC_{\alpha,\beta}[f,g]''(z)}{C_{\alpha,\beta}[f,g]'(z)} = \frac{2-(1+\alpha)z}{2-z} - \frac{2\beta z}{1+z}.$$

For $\alpha+\beta > 1$, it is easy to see that $0 < 2/(\alpha+\beta+1) < 1$. So if we choose $z = 2/(\alpha+\beta+1)$ then we have

$$1+\frac{zC_{\alpha,\beta}[f,g]''(z)}{C_{\alpha,\beta}[f,g]'(z)} = \frac{\beta}{\alpha+\beta} - \frac{4\beta}{\alpha+\beta+3} = \frac{-3\beta(\alpha+\beta-1)}{(\alpha+\beta)(\alpha+\beta+3)}.$$

The above calculation shows that

$$\operatorname{Re}\left(1+\frac{zC_{\alpha,\beta}[f,g]''(z)}{C_{\alpha,\beta}[f,g]'(z)}\right) < 0,$$

for $z = 2/(\alpha+\beta+1)$. This implies that $C_{\alpha,\beta}[f,g] \notin \mathcal{K}$ for $\alpha+\beta > 1$.

To complete our proof, we choose $f(z) = [(1+z)^2-1]/2 \in \mathcal{G}$ and $g(z) = z/(1+z) \in \mathcal{K}$. Then we get

$$1+\frac{zC_{\alpha,\beta}[f,g]''(z)}{C_{\alpha,\beta}[f,g]'(z)} = \frac{2+(1+\alpha)z}{2+z} - \frac{2\beta z}{1+z}.$$

For $z = 2/(3\beta-\alpha-1)$, we obtain

$$\operatorname{Re}\left(1+\frac{zC_{\alpha,\beta}[f,g]''(z)}{C_{\alpha,\beta}[f,g]'(z)}\right) = \frac{3\beta}{3\beta-\alpha} - \frac{4\beta}{3\beta-\alpha+1} = \frac{\beta(\alpha-3\beta+3)}{(3\beta-\alpha)(3\beta-\alpha+1)} < 0,$$

for $3\beta-\alpha-3 > 0$, $0 < z < 1$. It concludes that $C_{\alpha,\beta}[f,g] \notin \mathcal{K}$ for $3\beta-\alpha-3 > 0$, completing the proof. $\square$

Proof of Theorem 6.10 and Theorem 6.11

We can obtain the proof of these theorems by a similar process, which we are using in Theorem 6.7 with the help of Theorem 6.9. $\square$

CHAPTER 7

CONCLUSION AND FUTURE DIRECTIONS

In the history of geometric function theory, several integral operators play a very important role in developing the theory, for more information see **Chapter 1**. For example, the study of the univalency of the Alexander and Cesàro operators has been attracted by many authors. This is the first time, we study the spectrum property of these operators in **Chapter 2**, even for more general operators. We know that under Hornich operations the class of locally univalent functions $\mathfrak{F}$ forms a vector space. Also, it is easy to see that β-Cesàro operator is a nonlinear operator over the class $\mathfrak{F}$. Thus, it is natural to ask the following problem:

Problem 1: What are the spectrum of the β-Cesàro operator over some subclasses of $\mathfrak{F}$?

In **Chapter 3**, we obtained the Bohr radius for β-Cesàro operator and for some familiar integral operators. Similar to the Bohr radius, in [95], the concept of the Rogosinski radius is defined as follows: if $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{B}$ then $|S_M(z)| = |\sum_{n=0}^{M-1} a_n z^n| < 1$ for $|z| < 1/2$, here $1/2$ is the best possible quantity (see also [61, 97]). In [43], Kayumov and Ponnusamy studied the sum

$$R_N^f(z) := |f(z)|^p + \sum_{k=N}^{\infty} |a_k| r^k, \ |z| = r, \ \text{and } N \in \mathbb{N},$$

namely, the *Bohr-Rogosinski sum* of f for $p \in \{1, 2\}$. If we choose $N = 1$ and $f(0)$ instead of $f(z)$ in the sum then it is easy to see that the Bohr-Rogosinski sum is closely related to the classical Bohr sum. Moreover, the Bohr-Rogosinski radius is the largest number $r > 0$ such that $R_N^f(z) \leq 1$, known as the *Bohr-Rogosinski inequality*, for $|z| \leq r$. The Bohr-Rogosinski radius for the Cesàro operator is obtained in [43]. Then one can ask the following problem:

Problem 2: What is the Bohr-Rogosinski radius for the β-Cesàro operator?

As we know that Pfaltzgraff [79] and Royster [96] gave remarkable results about the univalency of the operator I_γ, $\gamma \in \mathbb{C}$, over the class $\mathcal{S}$. Also, we gave a result about the univalency of this operator over the class $\mathcal{K}(\lambda)$, when $\lambda < 1$, in **Chapter 4**. But still univalency of this operator over some subclasses of the class $\mathcal{S}$ even on the class $\mathcal{S}$ is not completely studied. It is well-known that $\mathcal{K}(\lambda) \subset \mathcal{C} \subset \mathcal{S}$, for $-1/2 \leq \lambda < 1$. Therefore, one can try the following problem:

Problem 3: It would be interesting to find the values of $\gamma \in \mathbb{C}$ such that $I_\gamma(\mathcal{C}) \subseteq \mathcal{S}$ or to find the set $A(\mathcal{C})$?

Also, I will try to solve the following open problem:

Problem 4: How is the complete structure of the set $A(\mathcal{S})$?

The difficulty of determining the set $A(\mathcal{S})$ seems to come from the fact that we have only few functions f for which the shapes of $A(\mathcal{S})$ are completely determined. Moreover, as a generalization of Alexander's theorem, Lemma 1.5 gives an equivalent relation between the Alexander operator and the Hornich scalar multiplication operator. As an application, this lemma provides the univalency of the Alexander operator. Further, we set the univalency of the β-Cesàro operator in the same chapter. Also, we find that image of the β-Cesàro operator over the class $\mathcal{S}$ is not contained in the class $\mathcal{S}$. Finally, **Chapter 4** provides the pre-Schwarzian norm of the β-Cesàro operator over some subclasses of the class $\mathfrak{F}$.

The Koebe function $z/(1-z)^2$, $z \in \mathbb{D}$, is univalent but it is not a convex function. But image of subdisk $\mathbb{D}_r$, for $r \leq 2-\sqrt{3}$, under the Koebe function is convex, see [27] or **Chapter 5**. Recall that this r is called the radius of convexity for the Koebe function. From **Chapter 4** we know that operators J_α and I_β over subclasses of the class $\mathfrak{F}$ are not univalent for all values of α and β then they are not convex. Therefore, to deal with both of the operators, we gave radius of convexity for the operator $C_{\alpha,\beta}$ over some subclasses of the class $\mathfrak{F}$ in **Chapter 5**. Similar to the radius of convexity, in future, I can think about the following problem related to the radius of starlikeness and univalency:

Problem 5: What are the radius of starlikeness and univalency of $C_{\alpha,\beta}$ over certain subclasses of the class $\mathfrak{F}$?

Further, in **Chapter 6** we got the coordinates (α, β) such that $C_{\alpha,\beta}$ over some subclasses of the class $\mathfrak{F}$ satisfy the convexity, close-to-convexity and other univalent conditions. Also, one can investigated the following problem in future.

Problem 6: What will be the coordinates (α, β) such that $C_{\alpha,\beta} \in \mathcal{S}^*$ or $\mathcal{S}$ over some classical subclasses of the class $\mathfrak{F}$?

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