A STUDY OF ROUGH SET APPROXIMATION OPERATORS BASED ON MODAL LOGIC

A THESIS

submitted in partial fulfillment of the
requirements for the award of the degree

of

DOCTOR OF PHILOSOPHY

by

Vineeta Singh Patel

DEPARTMENT OF MATHEMATICS

INDIAN INSTITUTE OF TECHNOLOGY INDORE

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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **A STUDY OF ROUGH SET APPROXIMATION OPERATORS BASED ON MODAL LOGIC** in the partial fulfillment of the requirements for the award of the degree of **DOCTOR OF PHILOSOPHY** and submitted in the **DEPARTMENT OF MATHEMATICS, Indian Institute of Technology Indore**, is an authentic record of my own work carried out during the time period from August 2016 to May 2021 under the supervision of Dr. Md. Aquil Khan, Associate Professor, Indian Institute of Technology Indore.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other institute.

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This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

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(**DR. MD. AQUIL KHAN**)

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To

My Parents

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ABBREVIATIONS

DIS	Deterministic Information System, Page 1
Wff	Well-formed formula, Page 5
MSAS	Multiple-source approximation system, Page 10
$MSAS^{D}$	Multiple-source approximation system with distributed knowledge, Page 11
IS	Information System, Page 13
IIS	Incomplete Information System, Page 13
NIS	Non-deterministic Information System, Page 14
PIS	Probabilistic Information System, Page 14
SAS	Subset Approximation Structure, Page 84
PWIS	Possible-worlds Information System, Page 118

NOTATION

$\mathbb{N}$	Set of positive integers, Page 10
N	Initial segment of $\mathbb{N}$, Page 10
$\wp(W)$	Powerset of W, Page 4
$R(z)$	The set $\{y \in W : (z, y) \in R\}$, Page 9
$\underline{Z}_R$, $\overline{Z}_R$	The lower and upper approximation of set Z, Page 9
$\mathrm{Ind}_B^{\mathcal{S}}$	The indiscernibility relation relative to set B of attributes in information system $\mathcal{S}$, Page 1
$\mathrm{Sim}_B^{\mathcal{S}}$	The similarity relation relative to set B of attributes in information system $\mathcal{S}$, Page 14
M	Class of all generalized MSASs, Page 22
$\mathsf{M_r}$	Class of all generalized MSASs having each R_i as reflexive relation, Page 22
$\mathsf{M_t}$	Class of all generalized MSASs having each R_i as transitive relation, Page 22
$\mathsf{M_s}$	Class of all generalized MSASs having each R_i as symmetric relation, Page 22
$\mathsf{M_{rs}}$	Class of all generalized MSASs having each R_i as reflexive and symmetric relation, Page 22
$\mathsf{M_{st}}$	Class of all generalized MSASs having each R_i as symmetric and transitive relation, Page 22
$\mathsf{M_{rt}}$	Class of all generalized MSASs having each R_i as reflexive and transitive relation, Page 22
$\mathsf{M_e}$	Class of all generalized MSASs having each R_i as an equivalence relation, Page 22

PV	The set of propositional variables, Page 24
$\mathcal{L}(\square, \triangle)$, $\mathcal{L}(\triangle)$	Page 24
$\mathcal{L}(\square)$	Page 24, 58, 91
$\underline{X}_{s_{\mathfrak{F}}}$, $\underline{X}_{w_{\mathfrak{F}}}$	The strong lower and weak lower approximations of X, Page 22
$\overline{X}_{s_{\mathfrak{F}}}$, $\overline{X}_{w_{\mathfrak{F}}}$	The strong upper and weak approximation of X, Page 22
$\mathcal{L}(L)$	Page 35
$\vdash_\Lambda \alpha$	α is a theorem of modal system Λ, Page 35, 97, 135
$[R]_S(Z)$, $\langle R\rangle_S(Z)$	The lower and upper approximation of Z relative to a set S, Page 85
A	Class of all SASs, Page 85
$\mathsf{A_r}$	Class of all SASs having each R_i as reflexive relation, Page 85
$\mathsf{A_t}$	Class of all SASs having each R_i as transitive relation, Page 85
$\mathsf{A_s}$	Class of all SASs having each R_i as symmetric relation, Page 85
$\mathsf{A_{rt}}$	Class of all SASs having each R_i as reflexive and transitive relation, Page 85
$\mathsf{A_{st}}$	Class of all SASs having each R_i as symmetric and transitive relation, Page 85
$\mathsf{A_{rs}}$	Class of all SASs having each R_i as reflexive and symmetric relation, Page 85
$L^n_{\mathfrak{F}}(X)$, $U^n_{\mathfrak{F}}(X)$	The necessity lower and upper approximation of X, Page 86
$L^p_{\mathfrak{F}}(X)$, $U^p_{\mathfrak{F}}(X)$	The possibility lower and upper approximation of X, Page 86
$\mathcal{L}(\square_1, \square)$, $\mathcal{L}(\square_1)$	Page 91
$E(\mathfrak{F})$	Page 91
$[\![\alpha]\!]_{\mathfrak{M},U}$, $[\![\alpha]\!]^*_{\mathfrak{M}}$, $[\![\alpha]\!]_{\mathfrak{M}}$	Page 92
$D(\mathfrak{F})$	Page 123
$\mathsf{F_e}$	The class of all PWISs where R is an equivalence relation, Page 121
$\mathsf{F_p}$	The class of all PWISs where R is reflexive and transitive relation, Page 121
$\mathsf{F_c}$	The class of all constant domain PWISs, Page 121

$\mathcal{D}$	The set of descriptors, Page 118
$\mathcal{D}_B$	The set of all B-basic wffs, Page 122
$\mathcal{L}, \mathcal{L}(\Box, \Box_\emptyset)$	Page 122
$[\![\alpha]\!]_{\mathfrak{M},s}$, $[\![\alpha]\!]_{\mathfrak{M}}$	Page 124
$th_{\mathcal{M}_i}(s)$	Page 159

CHAPTER 1

INTRODUCTION

In the 1980s, Pawlak proposed a mathematical approach, known as *rough set theory* (RST), for dealing with vague, uncertain, and incomplete data [73]. In his approach, the knowledge base about a non-empty set W of objects is given by an equivalence relation R. The pair (W, R) is called an *approximation space*. A set $A \subseteq W$ is approximated by its *lower approximation* $\underline{A}_R = \{x \in W \mid R(x) \subseteq A\}$ and *upper approximation* $\overline{A}_R = \{x \in W \mid R(x) \cap A \neq \emptyset\}$, where $R(x) = \{y \in W \mid (x, y) \in R\}$. The elements occurring in the set $\overline{A}_R \setminus \underline{A}_R$ are called *boundary elements* of A. The set of all the boundary elements of A is denoted by $Bd_R(A)$.

Applications of rough set theory are mostly based on an attribute-value representation model, called *(deterministic) information system*. These are essentially tables giving values taken for some attributes by objects of the domain – such as 'blue' for attribute 'colour', or 'round' for attribute 'shape'. The domain then gets partitioned into blocks of indiscernible objects, indiscernible as they match on all information available about them. Formally, the following is the mathematical representation of a deterministic information system and the induced indiscernibility among objects of the domain.

Definition 1.1. A *deterministic information system* (in brief, DIS) $\mathcal{S} := (W, \mathcal{A}, \bigcup_{a\in\mathcal{A}} \mathcal{V}_a, f)$ comprises a non-empty set W of objects, a non-empty finite set $\mathcal{A}$ of attributes, a non-empty finite set $\mathcal{V}_a$ of attribute-values for each $a \in \mathcal{A}$, and an assignment $f : W \times \mathcal{A} \to \bigcup_{a\in\mathcal{A}} \mathcal{V}_a$ such that $f(x, a) \in \mathcal{V}_a$.

Based on the information given by $\mathcal{S}$, each subset B of $\mathcal{A}$ induces an equivalence relation $\mathrm{Ind}_B^{\mathcal{S}}$ on the domain W, termed the *indiscernibility* relation induced by B, as follows:

$$(x, y) \in \mathrm{Ind}_B^{\mathcal{S}} \iff f(x, a) = f(y, a) \text{ for all } a \in B.$$

Thus, given a DIS $\mathcal{S}$ and a set B of attributes, we obtain an approximation space $(W, \mathrm{Ind}_B^{\mathcal{S}})$, and this, in turn, determines approximation operators. Note that $\mathrm{Ind}_{\emptyset}^{\mathcal{S}} = W \times W$.

There have been several generalizations of Pawlak's original notion of which one is generalizing the relation, that is, taking a relation with lesser constraints (cf. e.g. [32, 33, 44, 59, 95, 108]), and the other is taking a collection of relations instead of just one, known as *multigranulation rough set model* (cf. e.g. [39, 40, 64, 65, 69, 80, 83, 112]). The authors in [36, 37] also focus on a multigranulation rough set model with motivation from the multi-agent situation, although the more general term 'source' is used instead of 'agent'. They raised the important issues of counterparts of approximations of concepts in such a model and proposed the notions of *strong/weak* lower and upper approximations. Since the proposal of [36], the rough set model based on the collection of relations has caught a great deal of attention and interest among the researchers (cf. e.g. [29, 55, 57, 58, 80, 92, 103, 104, 112]).

In another direction of research, there have been extensive studies on logics with semantics based on structures inherited from rough set theory. The modal nature of the lower and upper approximations was evident from the beginning. Hence, it is no surprise that normal modal systems were focussed upon during investigations on logics for rough sets. In particular, in case of Pawlak rough sets, the two approximation operators clearly obey all the S5 laws. With the evolution of rough set theory with time, more expressive logics were required to be introduced to reason about generalized approximations (cf. e.g. [6,15]). Consequently, in literature, one can find complete formal systems for reasoning based on rough set theory in multi-agent systems (e.g. [14, 36, 38, 86]). The languages of most of these logics have 'agent-constants', and one or more binary operations are used to build the set of terms. The modal operators are indexed with these terms, which, in turn, are used to capture approximations relative to the knowledge base of individual or groups of agents. A binary relational symbol $\Rightarrow$ on the set of terms is also used in [86]. The expression $s \Rightarrow t$ reflects that "the classification ability of agent t is at least as good as that of agent s". In [36, 38], the first-order logic feature of quantification is also incorporated into the proposed logic. This enables the logic to capture quantification over the knowledge bases of the system. With this feature, the logics presented in [34, 36, 38] are able to capture the notions of strong/weak approximations proposed in [36]. We would like to remark here that the logic $\mathrm{LMSAS}^{\mathrm{D}}$ proposed in [38] has a close connection with the term-modal logics [24]. In $\mathrm{LMSAS}^{\mathrm{D}}$ as well as in term-modal logics, the modal operators are indexed with terms, and one can quantify over variables occurring in modal operators. However, there are differences as well. In term-modal logics, terms are used to point to the individual agents of the system. However, in LMSAS^{D}, terms point to the individual as well as the finite group of agents. Further, term-modal logics are based on predicate logic, but the language of LMSAS^{D} also contains propositional variables. It is to be noted that no axiomatization is provided for the logic proposed in [24]. However, a sound and complete modal system for LMSAS^{D} is obtained.

As mentioned above, the articles [36,38] proposed logics that can reason about strong/weak approximations. These logics are quantified propositional modal logics with excellent expressive power. This expressive power comes with a price - we do not have the decidability result for these logics. In this dissertation, although we are interested in a modal logic that can reason about the strong/weak approximations, we also want the logic to be based on the much simpler language - the basic modal language with two unary modal operators. The purpose is achieved by interpreting the two modal operators by strong and weak approximations. In Chapter 3, such logic is studied. Further, our study is not confined to collections of equivalence relations only, but other types of relations are also considered. This review is essential keeping in view the notions of generalized approximation spaces with relations other than the equivalence. Thus, we consider a generalized notion of *multiple-source approximation system* (MSAS) [36], called *generalized MSAS*. A generalized MSAS is a tuple consisting of a countable collections $\{R_i\}_{i \in N}$ of relations over the same domain, and where the accessibility relations R_i are of the same type, and maybe any binary relation, or have any of the properties of reflexivity, symmetry, transitivity or some combination thereof. The notions of strong/weak lower and upper approximations based on MSASs are extended readily to define these concepts for the generalized MSASs. The axiomatization problem of the proposed logic with respect to different classes of generalized MSASs based on various types of relations are explored. As a consequence of this study, we also get an insight into the axiomatic (abstract) characterizations of the strong/weak approximations based on different types of relations. Such a study on strong/weak approximations is still missing in the literature. A few results on invariance and definability related to the logic are also presented. Further, it is shown that the proposed logic can also be used to reason about the knowledge of a group of agents in the line of epistemic logic [20]. Moreover, with the motive to introduce a coalgebraic approach to the rough set community, we show that our study on generalized MSASs can be put under the framework of coalgebra [31, 76, 88, 101].

There is another interesting generalization of approximation space that takes a general covering of W instead of the partition due to the equivalence relation R (cf. e.g. [9, 10, 77, 115]). The first covering-based approach to rough set is by Żakowaski [18], but Pomykała [77] may be considered to be the first to proceed with this kind of approach in a systematic way. In subsequent years, covering-based definitions of approximations of a subset of the universe have emerged from various standpoints. For good surveys on this, we refer to [89–91, 106].

In [91], a comparison of the properties of lower and upper approximations emerging from different kinds of definitions of these operators has been studied. These are operators on $\wp(W)$, the power set of W. A covering with one pair of operators is called a *covering system*. In the same paper, seventeen covering systems are presented, and the possibility of study of modal logic systems corresponding to covering systems has been indicated. Subsequently, in [61], the first set of results in this direction is published. It is established that the modal system S4 may be endowed with covering semantics by C_2 and C_5 covering systems. Similarly, covering systems P_1 and C_4 give covering semantics to the modal system KTB. But S4 and KTB are the standard modal systems. In Chapter 4, a non-normal modal system has been introduced corresponding to covering systems P_3, C_1 and C_{Gr}. It is to be noted that the modality pair $(\Box, \Diamond)$ is interpreted by the (lower, upper) approximation pair relative to the corresponding covering systems.

The lower and upper approximation operators are extensively studied in rough set literature. Surprisingly, such a study is missing for the boundary operator that maps a set to the set of its boundary elements. In the second part of Chapter 4, we aim to fill this gap and present a study of the boundary operator through the modal logic approach. A pair of modalities $(\triangle, \bigtriangledown)$ different from the standard $(\Box, \Diamond)$, being borrowed from [22], has been incorporated in our language and given an interpretation in the rough set context. We first present modal systems for the boundary operators based on (generalized) approximation spaces or Kripke frames. Then the study is extended to the boundary operators based on covering systems. In this context, three modal systems are discussed, and their covering-based semantics are presented with respect to which the modal systems with $(\triangle, \bigtriangledown)$

modalities become sound and complete. Significance of the boundary operator approach in rough set theory is also discussed.

One common feature of the Pawlak's rough set model and the models discussed above is that all domain objects are taken into account while defining the set approximations. Nonetheless, in some practical cases, considering only specific subsets of the domain may be necessary. For example, we may wish to exclude certain objects due to the lack of information about them. Thus, in Chapter 5, we propose a generalized notion of approximation space called *subset approximation structure* (SAS). This structure is defined as a tuple $\mathfrak{F} := (W, \rho, R)$, where W is a non-empty set of objects, ρ is a non-empty collection of non-empty subsets of W, and $R \subseteq W \times W$. For $x \in W$, ρ_x is used to denote the collection $\{U \in \rho : x \in U\}$. The elements of ρ_x are called a *neighborhood* of x. The elements of ρ are used in determining the relative approximations of subsets of the domain. In Chapter 5, we aim to study the behaviour of rough sets under the framework of SAS. Approximation operators based on SAS is proposed, and some ensuing properties are discussed.

At this moment, it is apposite to tell that we have not seen any proposals of logics describing rough sets (including normal modal systems) that can capture the approximations of concepts discussed in this chapter. This is due to the fact that the proposed approximations are defined relative to elements of ρ. A modal logic for SAS is introduced that can be used for this purpose. The satisfiability of a well-formed formula (in brief, wff) is evaluated at an ordered pair whose first component corresponds to an object from the domain of discourse, and the second component corresponds to a set from ρ. The interpretation of proposed logic wffs within the framework of rough set theory is discussed. Sound and complete modal systems for different classes of SASs are presented. We also discuss a comparison of the proposed semantics with the well-known multi-modal logic semantics. This study also leads us to the fact that the problem of the decidability of the proposed logics is equivalent to that of some known multi-modal logics. This result is helpful in obtaining the decidability results for the logic. Some invariance results related to the presented logic are discussed. We also return to the issue of the expressibility power of the logic and provides a few classes of SASs that can be defined through wffs of the logic.

There is another family of modal logic under the umbrella of epistemic logic that provides a formal study of knowledge (cf. e.g. [20, 28, 99, 100]). The language of epistemic logic, apart from propositional variables and Boolean connectives, contains modal operators to capture, for instance, knowledge, belief, safe belief, plausibility (cf. e.g. [2, 98, 99]) etc. The semantics of the language is usually based on *possible-worlds* where one has a set of states, each representing a possible state of affairs. The propositional variables representing basic facts are assigned true or false value relative to the states. Besides the true state of affairs, an agent may consider many other states to be possible due to his/her partial knowledge. This possibility of the states is captured through binary relations between the states.

Chapter 6 aims to bring together the operators of epistemic logic and approximation operators of rough set theory by combining the ideas from epistemic logic and rough set logics. We extend the epistemic logic idea based on possible-worlds to consider a situation where each state carries information about a set of objects regarding attributes, besides information about basic facts represented by propositional variables. In other words, each state is assigned a deterministic information system (DIS), and thus we obtain a collection of DISs each indexed with a state. Since each constituent DIS of the collection generates corresponding approximation operators, it may happen that an object is in the lower/upper approximation of a set with respect to information in a state but may not be so with respect to information in some other state. Therefore, taking a cue from epistemic logic (cf. e.g. [20, 99, 100]), it becomes relevant to reason about statements involving basic statements like 'the object is in the lower approximation of the set in some states which are considered to be *at least as good as* the current state', or 'agent *knows/safely believes* that the object under consideration belongs to the upper approximation of the set'. This chapter presents a logic that can express such statements. To formally capture the situation described above, firstly the notion of a *possible-worlds information system* is proposed. A possible-worlds information system consists of a set of states where each state is assigned a DIS. It is shown that the situations captured by various types of information systems viz. incomplete, non-deterministic and probabilistic information systems can also be represented by possible-worlds information systems. A modal logic for possible-worlds information system is proposed that can be used to reason, relative to states, about attributes, attribute-vales of objects and the approximation operators with respect to indiscernibility relation. As expected, the language of the logic contains *descriptors* [74] to capture attributes, attribute-values of objects. Moreover, the proposed language has modal operators $\Box$ and $\Box_C$, for each subset C of the set of all attribute-constants. The modal operator $\Box$ is used to capture knowledge or plausibility, and hence the semantics of $\Box$ is defined using a relation on the set of states. On the other hand, $\Box_C$ captures the lower approximation operator relative to C, and thus its semantics is defined using indiscernibility relation. As the semantics of the modal operators $\Box$ and $\Box_C$ are defined using relations over two different sets, the proposed semantics is two dimensional, having the dimensions for states and objects. Modal systems for different classes of models are presented, and the corresponding soundness and completeness theorems are obtained. The step by step technique [7] of modal logic is adapted for the purpose. A comparison of the proposed semantics with the well-known multi-modal logic semantics is presented. A few decidability results related to the proposed logics are discussed.

The next chapter presents the needed preliminaries for the dissertation.

CHAPTER 2

PRELIMINARIES

This chapter is structured as follows. Section 2.1 provides some basic concepts related to the rough set theory. Section 2.2 lists a few generalizations of approximation space present in the literature. A few logics proposed for rough set theory, which are relevant to the dissertation, are presented in Section 2.3. In Section 2.4, we present some definitions related to category theory.

2.1. Approximation space

Let us recall the following notion of approximation space, which is fundamental to the rough set theory [73].

Definition 2.1. An *approximation space* is defined as a pair (W, R) consisting of a non-empty set W of objects and an equivalence relation R on W.

Objects being in the same equivalence class of R are indiscernible by means of knowledge provided by R. Based on this simple idea, any concept represented as a subset, say Z, of the partitioned domain W is approximated from 'within' and 'outside', by its *lower* and *upper approximations* given as follows.

Definition 2.2.

$$\underline{Z}_R := \{z \in W : R(z) \subseteq Z\},$$
$$\overline{Z}_R := \{z \in W : R(z) \cap Z \neq \emptyset\}.$$

Here, $R(z)$ denotes the set $\{y \in W : (z, y) \in R\}$. The function that maps a set to its lower (upper) approximation is called *lower (upper) approximation operator.*

The set-theoretic complement of a set $Z \subseteq W$ will be denoted by Z^c. Based on the information provided by the knowledge base R, for every set $Z \subseteq W$, the domain W gets partitioned into three disjoint sets viz. $\underline{Z}_R, \overline{Z}_R \setminus \underline{Z}_R, (\underline{Z}_R)^c$. The sets $\underline{Z}_R, \overline{Z}_R, (\overline{Z}_R)^c$, and

$\overline{Z}_R \setminus \underline{Z}_R$ are called the *positive region*, *possible region*, *negative region*, and *boundary region* of Z, respectively. Accordingly, elements of these regions are called *positive elements*, *possible elements*, *negative elements*, and *boundary elements*.

A set Z is called *definable* in (W, R) if there are no boundary elements of Z. We say a set Z is *rough* if it is not definable.

2.2. Generalizations of approximation space

Since Pawlak's proposal, the rough set model is generalized in many ways to extend its application in various practical problems. This section presents a few generalizations of approximation space extensively studied in the literature and relevant to the dissertation.

The most straightforward generalization of Pawlak's approximation space is obtained by relaxing the constraint on the relation [43, 44, 93, 95]. For instance, *tolerance approximation space*, where the relation is a *tolerance relation* (that is, reflexive and symmetric relation), is extensively studied in the literature [43, 93]. In this dissertation, following [105], by a *generalized approximation space*, we will mean a tuple (W, R), where R is a binary relation on the non-empty set W. Naturally, the notion of lower and upper approximations of a set $Z \subseteq W$ in such generalized approximation space (W, R) is defined as follows:

$$\underline{Z}_R := \{x \in W : R(x) \subseteq Z\},$$
$$\overline{Z}_R := \{x \in W : R(x) \cap Z \neq \emptyset\}.$$

2.2.1. Multigranulation rough set model

There is another demanding extension of approximation space that consists of more than one binary relation on the domain, known as a *multigranulation rough set model* (cf. e.g. [36, 39, 40, 64, 65, 69, 80, 83, 112]). The authors in [36] studied the multigranulation rough set model to capture a situation where information is obtained from different agents/sources regarding the same set of objects. The notion of *multiple-source approximation system* (MSAS) is considered for this purpose. Formally, it is defined as a tuple $(W, \{R_i\}_{i \in N})$ comprising a collection $\{R_i\}_{i \in N}$ of equivalence relations on domain W, where N is an initial segment of the set $\mathbb{N}$ of positive integers.

The following notion of strong/weak lower and upper approximations are proposed in [36]. Consider a MSAS $\mathfrak{F} = (W, \{R_i\}_{i \in N})$ and $Z \subseteq W$.

Definition 2.3. The *strong lower approximation* $\underline{Z}_s$, *weak lower approximation* $\underline{Z}_w$, *strong upper approximation* $\overline{Z}_s$, and *weak upper approximation* $\overline{Z}_w$ of Z, are defined as follows:

$$\underline{Z}_s := \bigcap_{i \in N} \underline{Z}_{R_i}; \quad \underline{Z}_w := \bigcup_{i \in N} \underline{Z}_{R_i};$$
$$\overline{Z}_s := \bigcap_{i \in N} \overline{Z}_{R_i}; \quad \overline{Z}_w := \bigcup_{i \in N} \overline{Z}_{R_i}.$$

It is worth mentioning here that, in literature, the above notions of strong and weak approximations are also known as *pessimistic and optimistic approximations*, respectively (cf. [80]).

In a MSAS $(W, \{R_i\}_{i \in N})$, R_i represents the knowledge base of the i^{th} source of the system. In order to bring into the picture the knowledge base corresponding to groups of sources of the system, the notion of MSAS is extended to define *multiple-source approximation system with distributed knowledge* (MSAS^{D}) in [38]. A MSAS^{D} is a tuple $\mathfrak{F} := (W, \{R_G\}_{G \subseteq N})$, where W is a non-empty set of objects, N denotes the initial segment of the set $\mathbb{N}$ of positive integers, and for every $G \subseteq N$, R_G is a binary relation on W satisfying the following properties:

- R_G is an equivalence relation;
- $R_G = \bigcap_{i \in G} R_i$, where $G \neq \emptyset$;
- $R_\emptyset = W \times W$.

In the definition of MSAS^{D}, for $G \subseteq N$, R_G denotes the knowledge base of the group G of sources. The notions of strong/weak lower and upper approximations are naturally defined for MSAS^{D}. We refer to [38] for a detailed study on MSAS^{D}.

2.2.2. Covering based rough set model

Another natural generalization of Pawlak's approximation space, known as *covering space*, is obtained by considering a covering of the domain instead of a partition. It is a pair $(W, \mathcal{C})$, where W is a non-empty set, and $\mathcal{C} = \{C_i \subseteq W : i \in I\}$ is a collection of subsets of W satisfying $\bigcup_{i \in I} C_i = W$, I being an index set. Here $\mathcal{C}$ is called a *covering* of W.

In 1983, Żakowaski [18] generalized the notion of Pawlak's approximation operators by covering based rough set approximation operators. In this generalization, the lower and upper approximation operators do not remain dual to each other. Later Pomykała [77] modified his approach and studied two pairs of dual approximation operators. Further, he suggested some additional pairs of dual approximation operators based on some mathematical structures induced by covering [79]. With time, various approximation operators based on covering space are proposed and studied in the literature (cf. e.g. [54,78,94,109,114,116]). We present a few of these approximation operators here and refer to [91] for a detailed study on rough set models based on covering. Let $(W, \mathcal{C})$ be a covering space where $\mathcal{C} = \{C_i\}_{i\in I}$. Consider the functions $N_{\mathcal{C}} : W \to \wp(W)$ and $F_{\mathcal{C}} : W \to \wp(W)$ defined as follows:

$$N_{\mathcal{C}}(x) := \bigcap\{C_i \in \mathcal{C} \mid x \in C_i\};$$
$$F_{\mathcal{C}}(x) := \bigcup\{C_i \in \mathcal{C} \mid x \in C_i\}.$$

Consider the following notions of lower and upper approximations based on covering space $(W, \mathcal{C})$. Let $A \subseteq W$.

$$\underline{P_3}(A) := \bigcup\{C_i \in \mathcal{C} : C_i \subseteq A\};$$
$$\overline{P_3}(A) := \{x \in W : C_i \cap A \neq \emptyset \text{ for all } C_i \in \mathcal{C} \text{ with } x \in C_i\};$$

$$\underline{C_1}(A) := \bigcup\{C_i \in \mathcal{C} : C_i \subseteq A\};$$
$$\overline{C_1}(A) := (\underline{C_1}(A^c))^c = \bigcap\{{C^c}_i : C_i \cap A = \emptyset\};$$

$$\underline{C_2}(A) := \{x \in W : N_{\mathcal{C}}(x) \subseteq A\};$$
$$\overline{C_2}(A) := \{x \in W : N_{\mathcal{C}}(x) \cap A \neq \emptyset\};$$

$$\underline{C_4}(A) := \{x \in W : N_{\mathcal{C}}(y) \subseteq A \text{ for all } y \text{ with } x \in N_{\mathcal{C}}(y)\};$$
$$\overline{C_4}(A) := \bigcup\{N_{\mathcal{C}}(x) : N_{\mathcal{C}}(x) \cap A \neq \emptyset\};$$

$$\underline{C_5}(A) := \{x \in W : y \in A \text{ for all } y \text{ with } x \in N_{\mathcal{C}}(y)\};$$
$$\overline{C_5}(A) := \bigcup\{N_{\mathcal{C}}(x) : x \in A\};$$

$$\underline{P_1}(A) := \{x \in W : F_{\mathcal{C}}(x) \subseteq A\};$$
$$\overline{P_1}(A) := \bigcup\{C_i \in \mathcal{C} : C_i \cap A \neq \emptyset\}.$$

The following pair of lower and upper approximation operators is considered in [91, 94].

$$\underline{C_{\mathrm{Gr}}}(A) := \bigcup\{C_i : C_i \subseteq A\},$$
$$\overline{C_{\mathrm{Gr}}}(A) = \mathrm{Gr}^*(A) \setminus \mathrm{NEG}_{\mathrm{Gr}}(A),$$

where $\mathrm{Gr}^*(A) := \bigcup\{C_i \in \mathcal{C} : C_i \cap A \neq \emptyset\}$ and $\mathrm{NEG}_{\mathrm{Gr}}(A) := \underline{C_{\mathrm{Gr}}}(A^c)$. Note that

$$\underline{P_3}(A) = \underline{C_1}(A) = \underline{C_{\mathrm{Gr}}}(A).$$

Further, as noted in [91],

$$\overline{P_3}(A) = \overline{C_1}(A) = \overline{C_{\mathrm{Gr}}}(A).$$

The naming in [96] is retained for the above covering systems. The properties of these lower and upper approximation operators are well studied in [96]. In Chapter 4, we will return to the covering based rough set models and will discuss modal systems for such models.

2.2.3. Extensions of information system

Initially, rough set theory was applied to deterministic information systems, which are complete in the sense that each object takes precisely one value for each attribute. However, due to imperfect/partial knowledge about the objects, this may not always be the case. Thus, the notion of deterministic information system has been generalized in many ways to consider different practical situations. For instance, information about some objects regarding some attributes may not be available. A distinguished attribute-value $*$ is used to depict such a situation. Thus, we have the following generalization of DIS.

Definition 2.4. A tuple $\mathcal{S} := (W, \mathcal{A}, \bigcup_{a \in \mathcal{A}} \mathcal{V}_a, f)$ is called an *information system* (IS), where $W, \mathcal{A}, \mathcal{V}_a, f$ are as in Definition 1.1 and $* \in \bigcap_{a \in \mathcal{A}} \mathcal{V}_a$. An information system which satisfies $f(x,a) = *$ for some $x \in W$ and $a \in \mathcal{A}$ will be called an *incomplete information system* (IIS).

Observe that a deterministic information system can be identified with the information system $\mathcal{S} := (W, \mathcal{A}, \bigcup_{a\in\mathcal{A}} \mathcal{V}_a, f)$, where $f(x,a) \neq *$ for all $x \in W$ and $a \in \mathcal{A}$.

In [45,46], instead of an indiscernibility relation, a *similarity* relation (defined below) is considered as the distinguishability relation in the context of an incomplete information system. The assumption here is that the value of all those attributes for which information is not known comes from the attribute-value domain.

$$(x,y) \in \mathrm{Sim}_B^{\mathcal{S}} \text{ if and only if, } f(x,a) = f(y,a) \text{ or } f(x,a) = *, \text{ or } f(y,a) = *, \text{ for all } a \in B.$$

One could easily verify that $\mathrm{Sim}_B^{\mathcal{S}}$ is a tolerance relation, and thus an IIS $\mathcal{S}$ and an attribute set B give rise to a tolerance approximation space $(W, \mathrm{Sim}_B^{\mathcal{S}})$.

Non-deterministic information system [66] is another generalization of DIS where objects take a set of attribute-values for each attribute.

Definition 2.5. A tuple $\mathcal{S} := (W, \mathcal{A}, \bigcup_{a\in\mathcal{A}} \mathcal{V}_a, f)$ is called a *non-deterministic information system* (NIS), where $W, \mathcal{A}, \mathcal{V}_a$ are as in Definition 1.1 and $f : W \times \mathcal{A} \to \bigcup_{a\in\mathcal{A}} \mathcal{V}_a$ such that $f(x,a) \subseteq \mathcal{V}_a$.

Note that an indiscernibility relation $\mathrm{Ind}_B^{\mathcal{S}}$ for NISs can be defined in the same way as it is done for DISs.

One may attach different interpretations with '$f(x,a) = V$', for $V \subseteq \mathcal{V}_a$. For instance, one could interpret $f(x,a) = V$ as object x takes precisely one attribute-value from V, and under this interpretation, the following similarity relation is found to be useful.

$$(x,y) \in \mathrm{Sim}_B^{\mathcal{S}} \text{ if and only if } f(x,a) \cap f(y,a) \neq \emptyset \text{ for all } a \in B.$$

The notion of *probabilistic information system* is proposed in [35] to capture situations where information regarding attributes of objects is not precise, but given in terms of probability.

Definition 2.6. A *probabilistic information system* (PIS) is defined as a tuple $\mathcal{S} := (W, \mathcal{A}, \bigcup_{a\in\mathcal{A}} \mathcal{V}_a, F)$, where $W, \mathcal{A}, \mathcal{V}_a$ are as in Definition 1.1, and F is a function from the set $\{(x,a,V) : x \in W, a \in \mathcal{A}, \ \& \ V \subseteq \mathcal{V}_a\}$ to $[0,1] \cap \mathbb{Q}$ satisfying the following:

- $F(x, a, \mathcal{V}_a) = 1$, (unit measure)
- $F(x, a, U) + F(x, a, V) = F(x, a, U \cup V)$ for disjoint U and V. (additivity)

The function F, called an *information function*, provides the probability of objects of the domain W to take the attribute-values from the set $\mathcal{V}_a$ for each $a \in \mathcal{A}$.

In [35], approximations operators based on PISs are proposed, and it is shown that the approximation operators based on deterministic, non-deterministic and incomplete information systems could be studied in the framework of PISs. We refer to [35] for a detailed study on PIS.

We end this section with a brief discussion on dependency and data reduction, which are essential issues in rough set theory (cf. e.g. [45, 74]). These involve removing all 'superfluous' attributes in a deterministic information system, that is, those attributes that do not affect the partition of the domain, and consequently set approximations. Thus, we have the following definition.

Definition 2.7. Let $\mathcal{S} := (W, \mathcal{A}, \cup_{a\in\mathcal{A}}\mathcal{V}_a, f)$ be a deterministic information system, and $B, C \subseteq \mathcal{A}$.

- $a \in B$ is said to be *dispensable* in B if $\mathrm{Ind}^{\mathcal{S}}_B = \mathrm{Ind}^{\mathcal{S}}_{B\setminus\{a\}}$; otherwise a is *indispensable* in B.
- B is said to be *independent* if each $a \in B$ is indispensable in B; otherwise B is *dependent*.
- $C \subseteq B$ is a *reduct* of B if C is independent and $\mathrm{Ind}^{\mathcal{S}}_B = \mathrm{Ind}^{\mathcal{S}}_C$.

For a better understanding of the above notions, let us see an example.

Example 2.8. Consider the deterministic information system $\mathcal{S}$ given by Table 2.1, where W denotes the set of objects, a, b, c, d are the attributes, and $\mathcal{V}_a = \{v_1, v_2\}$, $\mathcal{V}_b = \{v_3, v_4\}$, $\mathcal{V}_c = \{v_5, v_6\}$, and $\mathcal{V}_d = \{v_7, v_8\}$ are attribute-value domains for the attributes a, b, c and d, respectively.

$W/\mathrm{Ind}^{\mathcal{S}}_{\{a\}} = \{\{w_1, w_2, w_4, w_5\}, \{w_3, w_6\}\}$;
$W/\mathrm{Ind}^{\mathcal{S}}_{\{b\}} = \{\{w_1, w_3, w_4, w_5\}, \{w_2, w_6\}\}$;
$W/\mathrm{Ind}^{\mathcal{S}}_{\{c\}} = \{\{w_1, w_3, w_5, w_6\}, \{w_2, w_4\}\}$;
$W/\mathrm{Ind}^{\mathcal{S}}_{\{d\}} = \{\{w_1, w_2, w_4, w_5, w_6\}, \{w_3\}\}$;
$W/\mathrm{Ind}^{\mathcal{S}}_{\{a,b\}} = \{\{w_1, w_4, w_5\}, \{w_2\}, \{w_3\}, \{w_6\}\}$;

W	a	b	c	d
w_1	v_1	v_3	v_6	v_7
w_2	v_1	v_4	v_5	v_7
w_3	v_2	v_3	v_6	v_8
w_4	v_1	v_3	v_5	v_7
w_5	v_1	v_3	v_6	v_7
w_6	v_2	v_4	v_6	v_7

Table 2.1. Deterministic information system $\mathcal{S}$

$$W/\text{Ind}^{\mathcal{S}}_{\{a,b,c\}} = \{\{w_1, w_5\}, \{w_2\}, \{w_3\}, \{w_4\}, \{w_6\}\};$$
$$W/\text{Ind}^{\mathcal{S}}_{\{a,b,d\}} = \{\{w_1, w_4, w_5\}, \{w_2\}, \{w_3\}, \{w_6\}\}.$$

Let $B = \{a, b\}$ and $C = \{a, b, d\}$. We have $\text{Ind}^{\mathcal{S}}_B \neq \text{Ind}^{\mathcal{S}}_{B\setminus\{a\}}$ and $\text{Ind}^{\mathcal{S}}_B \neq \text{Ind}^{\mathcal{S}}_{B\setminus\{b\}}$. Thus a and b are indispensable in B and B is independent.

Further, $\text{Ind}^{\mathcal{S}}_C = \text{Ind}^{\mathcal{S}}_{C\setminus\{d\}}$, and hence d is dispensable in C.

Since B is independent and $\text{Ind}^{\mathcal{S}}_B = \text{Ind}^{\mathcal{S}}_C$, we obtain $B \subseteq C$ as a reduct of C.

2.3. Logics for rough set theory

It is apparent that the notions of Kripke frame with an equivalence relation [7] and Pawlak's approximation space are mathematically the same, although the motivations for these structures are very different. Thus, it becomes evident at the very beginning of the development of RST that the *necessity* and *possibility* operators of modal logic capture the lower and upper approximations, respectively. To put it formally, consider the basic modal language with one unary modal operator $\Box$ and its dual operator $\Diamond$. Recall the notion of satisfiability of a wff α in a *Kripke model* $\mathfrak{M} = (W, R, m)$, where R is an equivalence relation on W, and m is a valuation function from the set of propositional variables to $\wp(W)$. Let us denote the *truth set* of a wff α in a model $\mathfrak{M}$ by $[\![\alpha]\!]_{\mathfrak{M}}$, that is, $[\![\alpha]\!]_{\mathfrak{M}} = \{w \in W : \mathfrak{M}, w \models \alpha\}$. Then the following hold.

$$[\![\Box\alpha]\!]_{\mathfrak{M}} = \underline{[\![\alpha]\!]_{\mathfrak{M}}}_R, \text{ and } [\![\Diamond\alpha]\!]_{\mathfrak{M}} = \overline{[\![\alpha]\!]_{\mathfrak{M}}}^R.$$

We have similar results for generalized approximation space. Further, the relationship between modal logics and rough sets have been studied extensively by many authors [82, 84, 85, 96, 97].

With time Pawlak's rough set theory has seen many generalizations, and consequently, various logics are proposed to capture approximation operators based on these generalizations. For instance, a quantified modal logic LMSAS (logic for MSAS) is proposed in [36] to study the strong/weak approximations based on MSAS. In the language of LMSAS, the set T of *terms* are formed by using a non-empty countable set Var of variables and a non-empty countable set Con of constants. Along with the usual Boolean connectives $\neg$(negation), $\wedge$(conjunction), unary modal connectives $\langle t \rangle$ for each term $t \in T$, and the universal quantifier $\forall$ are present in the language. The wffs are obtained by the following schema

$$\top \mid \bot \mid p \mid \neg\alpha \mid \alpha \wedge \beta \mid \langle t \rangle\alpha \mid \forall x\alpha,$$

where $\top, \bot$ are propositional constants, p is a propositional variable, $t \in T, x \in Var$ and α, β are wffs. For a set Γ of wffs of LMSAS, let us denote the set of terms and constants present in wffs of Γ by $Term(\Gamma)$ and $Con(\Gamma)$.

The *interpretation* for Γ is defined as a tuple $\mathscr{M} := (\mathfrak{F}, m, I)$, where $\mathfrak{F} = (W, \{R_i\}_{i \in N})$ is a MSAS, m is a *valuation map* assigning objects to propositional variables, and $I : Con(\Gamma) \to N$. An *assignment* for interpretation $\mathscr{M}$ is a function $v : Terms(\Gamma) \to N$ such that for constants it coincides with the map I. As in classical first-order logic, we say that two assignments v, v' for interpretation $\mathscr{M}$ are *x-equivalent* for variable x if v and v' agree on every variable other than x. The satisfiability of a wff α of Γ for an interpretation $\mathscr{M}$, under an assignment v and at object $w \in W$, denoted by $\mathscr{M}, v, w \models \alpha$, is defined inductively as follows. We have the standard definition for Boolean cases. For the modal operators, we have the following definition.

$\mathscr{M}, v, w \models \langle t \rangle\alpha$ if and only if there exists a $w' \in W$ such that $(w, w') \in R_{v(t)}$, and $\mathscr{M}, v, w' \models \alpha$.

$\mathscr{M}, v, w \models \forall x\alpha$ if and only if for every v' x-equivalent to $v, \mathscr{M}, v', w \models \alpha$.

The wffs $\forall x[x]\alpha$ and $\exists x[x]\alpha$ capture the strong lower and weak lower approximation operators, whereas their duals $\forall x\langle x \rangle\alpha$ and $\exists x\langle x \rangle\alpha$ capture the strong upper and weak upper approximation operators, respectively. A sound and complete modal system is presented in [36]. We refer to [36, 38] for a detailed study on this line.

2.4. Some basic concepts related to category theory

In this section, we present some basic definitions related to category theory. These are required to follow Section 3.6 of Chapter 3. Let us begin with the definition of category [87].

Definition 2.9. A *category* C consists of the following.

(1) (**Objects**) A class $Obj(\mathsf{C})$ whose elements are called the *Objects*. It is customary to write $A \in \mathsf{C}$ in place of $A \in Obj(\mathsf{C})$.

(2) **(Morphisms)** For each (not necessarily distinct) pair of objects $A, B \in \mathsf{C}$, a set $hom_{\mathsf{C}}(A, B)$, called the *hom-set* for the pair (A, B). The elements of $hom_{\mathsf{C}}(A, B)$ are called *morphisms, maps* or *arrows* from A to B. If $f \in hom_{\mathsf{C}}(A, B)$, we also write $f : A \to B$ or f_{AB}. The object A is called the *domain* of f and the object B is called the *codomain* of f.

(3) Distinct hom-sets are disjoint, that is, $hom_{\mathsf{C}}(A, B)$ and $hom_{\mathsf{C}}(C, D)$ are disjoint unless $A = C$ and $B = D$.

(4) **(Composition)** For $f \in hom_{\mathsf{C}}(A, B)$ and $g \in hom_{\mathsf{C}}(B, C)$, there is a morphism $g \circ f \in hom_{\mathsf{C}}(A, C)$, called the *composition* of g with f. Moreover, composition is associative:

$$f \circ (g \circ h) = (f \circ g) \circ h$$

whenever the compositions are defined.

(5) **(Identity morphisms)** For each object $A \in \mathsf{C}$ there is a morphism $1_A \in hom_{\mathsf{C}}(A, A)$, called the *identity morphism* for A, with the property that if $f_{AB} \in hom_{\mathsf{C}}(A, B)$ then

$$1_B \circ f_{AB} = f_{AB} \text{ and } f_{AB} \circ 1_A = f_{AB}.$$

The class of all morphisms of C is denoted by $Mor(\mathsf{C})$.

Example 2.10. Let us consider the following examples of category.

(1) The category **Set** of sets

Obj is the class of all sets.

hom(A, B) is the set of all functions from A to B.

(2) The category **Grp** of groups

Obj is the class of all groups.

hom(A, B) is the set of all group homomorphisms from A to B.

(3) The Category **Rel** of relations

Obj is the class of all sets.

hom(A, B) is the set of all binary relations from A to B, that is, subsets of the cartesian product $A \times B$.

Definition 2.11. Let C be a category. A *subcategory* D of C is a category which consists of a non-empty subclass $Obj(\mathsf{D})$ of $Obj(\mathsf{C})$ and a non-empty subclass $Mor(\mathsf{D})$ of $Mor(\mathsf{C})$ with the following properties:

(1) $Obj(\mathsf{D}) \subseteq Obj(\mathsf{C})$, as classes.

(2) For every $A, B \in \mathsf{D}$,

$$hom_{\mathsf{D}}(A, B) \subseteq hom_{\mathsf{C}}(A, B)$$

and the identity map $1_A \in \mathsf{D}$ is the identity map 1_A in C, that is,

$$(1_A)_{\mathsf{D}} = (1_A)_{\mathsf{C}}$$

(3) Composition in D is the composition from C, that is, if

$$f : A \to B \text{ and } g : B \to C$$

are morphisms in D, then the C-composite $g \circ f$ is the D-composite $g \circ f$.

Example 2.12. The category **AbGrp** of abelian groups is a subcategory of the category Grp, since the definition of group morphism is independent of whether or not the groups involved are abelian. Put another way, a group homomorphism between abelian groups is just a group homomorphism.

Structure-preserving maps between categories are called *functors*. Since the structure of a category consists of both its objects and its morphisms, a functor should map objects to objects and morphisms to morphisms.

Definition 2.13. Let C and D be categories. A functor $\mathcal{H} : \mathsf{C} \Rightarrow \mathsf{D}$ is a pair of functions (as is customary, we use the same symbol $\mathcal{H}$ for both functions):

(1) The *object part* of the functor $\mathcal{H} : Obj(\mathsf{C}) \to Obj(\mathsf{D})$ maps objects in C to objects in D.

(2) The *arrow part* $\mathcal{H} : Mor(\mathsf{C}) \to Mor(\mathsf{D})$ maps morphisms in C to morphisms in D as follows:

   – $\mathcal{H} : hom_{\mathsf{C}}(A, B) \to hom_{\mathsf{D}}(\mathcal{H}A, \mathcal{H}B)$ for all $A, B \in \mathsf{C}$, that is, $\mathcal{H}$ maps a morphism $f : A \to B$ in C to a morphism $\mathcal{H}f : \mathcal{H}A \to \mathcal{H}B$ in D.

(3) Identity and composition are preserved, that is,

$$\mathcal{H}1_A = 1_{\mathcal{H}A}, \text{ and } \mathcal{H}(g \circ f) = \mathcal{H}g \circ \mathcal{H}f$$

whenever all compositions are defined.

A functor $\mathcal{H}$ from category C to itself is called *endofunctor on* C.

Example 2.14. The *power set functor* $\mathcal{P}$: **Set** $\Rightarrow$ **Set** sends a set A to its power set $\mathcal{P}(A)$ and sends each set function $f : A \to B$ to the induced function $f : \mathcal{P}(A) \to \mathcal{P}(B)$ that sends X to $f(X)$.

The *identity functor* $\mathcal{I} : \mathsf{C} \Rightarrow \mathsf{C}$ acts as identity on both the object part and arrow part of C.

Definition 2.15 (Composition of functors)**.** Functors can be composed in the "obvious" way. Specifically, if $\mathcal{H} : \mathsf{C} \Rightarrow \mathsf{D}$ and $\mathcal{G} : \mathsf{D} \Rightarrow \mathsf{E}$ are functors, then $\mathcal{G} \circ \mathcal{H} : \mathsf{C} \Rightarrow \mathsf{E}$ is defined by

$$(\mathcal{G} \circ \mathcal{H})(A) = \mathcal{G}(\mathcal{H}A)$$

for $A \in \mathsf{C}$, and

$$(\mathcal{G} \circ \mathcal{H})(f) = G(\mathcal{H}f)$$

for $f \in hom_{\mathsf{C}}(A, B)$. We will often write the composition $\mathcal{G} \circ \mathcal{H}$ as $\mathcal{GH}$.

Definition 2.16. A functor $\mathcal{H} : \mathsf{C} \Rightarrow \mathsf{D}$ is an *isomorphism* if there exists a functor $\mathcal{G} : \mathsf{D} \Rightarrow \mathsf{C}$ such that $\mathcal{GH} = 1_{\mathsf{C}}$ and $\mathcal{HG} = 1_D$. Two categories C and D are *isomorphic*, written $\mathsf{C} \cong \mathsf{D}$ if there exists an isomorphism from C to D.

CHAPTER 3

MODAL SYSTEMS FOR MULTIGRANULATION ROUGH SET MODEL

As mentioned in Chapter 1, the quantified modal logic LMSAS is proposed in [36] to reason about the strong/weak approximations of concepts. In fact, this logic can capture the quantification over relations, and hence it can express statements that even cannot be expressed through first-order logic. This expressive power comes with a price. The decidability of the logic is still not known. In this chapter, we consider the modal language with two unary modal operators $\Box$ and $\triangle$, and we propose a semantics based on MSASs for this language. The modal operators $\Box$ and $\triangle$ capture strong lower and weak lower approximation, whereas their duals $\Diamond$ and $\bigtriangledown$ capture strong upper and weak upper approximation, respectively. In addition, our study is not restricted to collections of equivalence relations only, but other types of binary relations are also taken into account. Further, with the aim to introduce a coalgebraic approach to the rough set community, we have also tried to illustrate how our study on generalized MSASs can be put under the framework of coalgebra.

3.1. Generalized multiple-source approximation systems

The notion of *multiple-source approximation systems* (MSASs) was used in [36] to study rough set theory under a situation where information arrives from multiple sources. These are collections of the form $(W, \{R_i\}_{i\in N})$, where W is a non-empty set, N is an initial segment of the set $\mathbb{N}$ of positive integers, and R_i's are equivalence relations on W. N is called the *cardinality* of the MSAS. For each $i \in N$, R_i represents the knowledge base with respect to the i^{th} source of the system.

It is important to note that relations other than equivalences can be quite relevant while dealing with approximations of concepts in rough set theory (cf. e.g. [43, 56, 66, 69, 93]). Therefore, in this chapter, we consider a generalized notion of MSAS, called

Class of generalized MSASs	Defining condition	Class of generalized MSASs	Defining condition
M	Class of all generalized MSASs	M_{rs}	$\mathsf{M}_{\mathsf{r}} \cap \mathsf{M}_{\mathsf{s}}$
M_{r}	Each R_i is reflexive	M_{rt}	$\mathsf{M}_{\mathsf{r}} \cap \mathsf{M}_{\mathsf{t}}$
M_{t}	Each R_i is transitive	M_{st}	$\mathsf{M}_{\mathsf{s}} \cap \mathsf{M}_{\mathsf{t}}$
M_{s}	Each R_i is symmetric	M_{e}	$\mathsf{M}_{\mathsf{rs}} \cap \mathsf{M}_{\mathsf{t}}$

Table 3.1. Classes of generalized MSASs

generalized MSAS, that contains a countable collections $\{R_i\}_{i \in N}$ of relations over the same domain. Subclasses consisting of generalized MSASs are also considered where the accessibility relations R_i are of the same type, and maybe any binary relation, or have any of the properties reflexivity, symmetry, transitivity or some combination thereof. Table 3.1 gives different classes of generalized MSASs.

The notions of approximations based on MSASs are proposed in [36], which can be naturally extended to generalized MSASs as follows. Let $\mathfrak{F} := (W, \{R_i\}_{i \in N})$ be a generalized MSAS and $X \subseteq W$.

Definition 3.1. The *strong lower approximation* $\underline{X}_{s_{\mathfrak{F}}}$, *weak lower approximation* $\underline{X}_{w_{\mathfrak{F}}}$, *strong upper approximation* $\overline{X}_{s_{\mathfrak{F}}}$, and *weak upper approximation* $\overline{X}_{w_{\mathfrak{F}}}$ of X are defined as follows.

$$\underline{X}_{s_{\mathfrak{F}}} := \bigcap_{i \in N} \underline{X}_{R_i};\ \ \underline{X}_{w_{\mathfrak{F}}} := \bigcup_{i \in N} \underline{X}_{R_i}.$$
$$\overline{X}_{s_{\mathfrak{F}}} := \bigcap_{i \in N} \overline{X}_{R_i};\ \ \overline{X}_{w_{\mathfrak{F}}} := \bigcup_{i \in N} \overline{X}_{R_i}.$$

If there is no confusion, we shall omit $\mathfrak{F}$ as the subscript in the above definition.

The following proposition lists some of the properties of strong lower and weak upper approximations based on MSAS.

Proposition 3.2. *Let* $\mathfrak{F} := (W, \{R_i\}_{i \in N}) \in \mathsf{M}_{\mathsf{e}}$. *Then we have the following.*

(Dual): $\underline{X}_s = (\overline{X^c}_w)^c$.

(U1): $\overline{\emptyset}_w = \emptyset$.

(U2): $\overline{X \cup Y}_w = \overline{X}_w \cup \overline{Y}_w$.

(U3): $X \subseteq \overline{X}_w$.

(U4): $X \subseteq \underline{\overline{X}_w}_w$.

(L1): $\underline{W}_s = W$.

(L2): $\underline{X \cap Y}_s = \underline{X}_s \cap \underline{Y}_s$.

(L3): $\underline{X}_s \subseteq X$.

(L4): $\overline{\underline{X}_s}_w \subseteq X$.

Here, we claim that the properties (U1)-(U4) listed above are the characterizing properties of weak upper approximation based on MSAS. That is, given a function $f : \wp(W) \to \wp(W)$ satisfying these properties, there exists a MSAS $\mathfrak{F} := (W, \{R_i\}_{i \in N})$ such that $f(X) = \overline{X}_{w_{\mathfrak{F}}}$, for all $X \subseteq W$. Thus, the properties of weak upper approximation and its dual strong lower approximation are precisely that of upper and lower approximations defined on tolerance approximation spaces (cf. [43,93]). The above characterization result also naturally leads us to the question of whether we can identify weak/strong approximations based on different classes of generalized MSASs with approximations based on a suitable type of relation, just as we can do for strong lower and weak upper approximations based on MSASs. The answer to this question and the proof of the above characterization theorem will be obtained as a consequence of our study in Section 3.5 on the axiomatization of the logic proposed in this chapter.

The rest of the chapter is arranged as follows. Section 3.2 presents the syntax and semantics of the logic. Section 3.3.1 illustrates how the wffs of the proposed logic can express the properties of strong/weak approximations. Further, Section 3.3.2 shows that the logic can be used to reason about the knowledge of a group of agents in the line of epistemic logic [20]. Section 3.4 discusses the invariance results related to the proposed logic and presents a few limiting results of the logic associated with the definability of some subclasses of the class of all generalized MSASs. In Section 3.5, we discuss the axiomatization concerning different subclasses of the class of all generalized MSASs. This study also gives some insight into the nature of strong/weak approximations. Section 3.6 presents a coalgebraic perspective of our research of generalized MSASs. Section 3.7 concludes the chapter.

The work presented in this chapter is based on the article [41].

3.2. A modal logic for generalized MSASs

This section presents a modal logic that can be used to reason about the properties of approximations based on generalized MSASs.

3.2.1. Syntax

The language $\mathcal{L}(\Box, \triangle)$ of the logic consists of a set PV of propositional variables, a propositional constant $\top$ and unary modalities $\Box, \triangle$. The modal operators $\Box$ and $\triangle$ are intended to capture strong and weak lower approximations, respectively. Using the Boolean logical connectives $\neg$ (negation) and $\wedge$ (conjunction), wffs of $\mathcal{L}(\Box, \triangle)$ are then defined recursively as

$$\top \mid p \in PV \ \mid \ \neg\alpha \ \mid \ \alpha \wedge \beta \ \mid \ \Box\alpha \ \mid \ \triangle\alpha.$$

Apart from the usual derived connectives $\bot, \vee, \rightarrow, \leftrightarrow$, we have the connectives $\Diamond$ and $\triangledown$ defined as $\Diamond\alpha := \neg\Box\neg\alpha$ and $\triangledown\alpha := \neg \triangle \neg\alpha$.
We will make use of the same symbol $\mathcal{L}(\Box, \triangle)$ to denote the set of all wffs of the language $\mathcal{L}(\Box, \triangle)$. Let $\mathcal{L}(\Box)$ be the fragment of $\mathcal{L}(\Box, \triangle)$ obtained by taking the wffs not involving the modal operator $\triangle$. Similarly, $\mathcal{L}(\triangle)$ is the fragment of $\mathcal{L}(\Box, \triangle)$ obtained by considering the wffs not involving the modal operator $\Box$.

3.2.2. Semantics

The semantics of $\mathcal{L}(\Box, \triangle)$ is based on generalized MSAS equipped with a valuation function for the propositional variables. Formally, we have the following.

Definition 3.3 (Models)**.** By a *model*, we mean a tuple $\mathfrak{M} := (\mathfrak{F}, m)$, where

- $\mathfrak{F} := (W, \{R_i\}_{i\in N})$ is a generalized MSAS;
- $m : PV \rightarrow \wp(W)$.

A model $\mathfrak{M}$ is said to be reflexive if the constituent generalized MSAS $\mathfrak{F} \in \mathsf{M_r}$. Similarly, we have other classes of models depending on the type of the constituent generalized MSAS (cf. Table 3.1).

Definition 3.4 (Satisfiability)**.** Let $\mathfrak{M} := (\mathfrak{F}, m)$ be a model based on a generalized MSAS $(W, \{R_i\}_{i\in N})$. The *satisfiability* of a wff α in $\mathfrak{M}$ at $w \in W$, denoted as $\mathfrak{M}, w \models \alpha$, is defined inductively:

$$\mathfrak{M}, w \models \top \qquad \text{always.}$$

$$\mathfrak{M}, w \models p \iff w \in m(p), \text{ for } p \in PV.$$

$$\mathfrak{M}, w \models \neg\alpha \iff \mathfrak{M}, w \not\models \alpha.$$

$$\mathfrak{M}, w \models \alpha \wedge \beta \iff \mathfrak{M}, w \models \alpha \text{ and } \mathfrak{M}, w \models \beta.$$

$$\mathfrak{M}, w \models \Box\alpha \iff \mathfrak{M}, w' \models \alpha \text{ for all } w' \in \cup_{i \in N} R_i(w).$$

$$\mathfrak{M}, w \models \triangle\alpha \iff \text{there exists an } i \in N \text{ such that } \mathfrak{M}, w' \models \alpha \text{ for all } w' \text{ with } w' \in R_i(w).$$

Conditions of satisfiability of the derived connectives $\Diamond$ and $\triangledown$ are then obtained as follows:

Proposition 3.5.

$$\mathfrak{M}, w \models \Diamond\alpha \iff \mathfrak{M}, w' \models \alpha \textit{ for some } w' \in \cup_{i \in N} R_i(w).$$

$$\mathfrak{M}, w \models \triangledown\alpha \iff \textit{for each } i \in N \textit{ there exists } w' \in R_i(w) \textit{ such that } \mathfrak{M}, w' \models \alpha.$$

Let us use $[\![\alpha]\!]_{\mathfrak{M}}$ to denote the set $\{x \in W : \mathfrak{M}, x \models \alpha\}$, the *truth set* of the wff α relative to the model $\mathfrak{M}$.

A wff α is said to be *valid* in $\mathfrak{M}$, notation: $\mathfrak{M} \models \alpha$, if $[\![\alpha]\!]_{\mathfrak{M}} = W$. A wff α is said to be *valid* in a generalized MSAS $\mathfrak{F}$, denoted as $\mathfrak{F} \models \alpha$, if $\mathfrak{M} \models \alpha$ for all models $\mathfrak{M}$ based on $\mathfrak{F}$. A wff α is *valid* in a given class $\mathfrak{G}$ of generalized MSASs, notation: $\mathfrak{G} \models \alpha$, if α is valid in every generalized MSAS $\mathfrak{F}$ in $\mathfrak{G}$. For a class $\mathfrak{G}$ of generalized MSASs, and a set Γ of wffs, we will write $\mathfrak{G} \models \Gamma$ if $\mathfrak{G} \models \alpha$ for all $\alpha \in \Gamma$.
A wff α is said to be *satisfiable* in a model $\mathfrak{M}$ if $[\![\alpha]\!]_{\mathfrak{M}} \neq \emptyset$. α is said to be *satisfiable* in a given class $\mathfrak{G}$ of generalized MSASs if α is satisfiable in a model $\mathfrak{M} := (\mathfrak{F}, m)$ based on a generalized MSAS $\mathfrak{F}$ belonging to the class $\mathfrak{G}$.

Let us talk a little about the valid wffs in the class $\mathsf{M_e}$ of the generalized MSASs based on equivalence relations. It may appear that the S5 axioms for the modal operators $\Box$ and $\triangle$ will be valid in the class $\mathsf{M_e}$. Nevertheless, interestingly, it is not the case. One can verify that $\mathsf{M_e} \not\models \Box p \to \Box\Box p$ and $\mathsf{M_e} \not\models p \to \triangle \triangledown p$. This observation hints that the axiomatization problem for the proposed semantics will behave differently compared to that of the standard modal logic. In Section 3.5, we will discuss the axiomatization problem related to the proposed semantics of the modal operators $\Box$ and $\triangle$ relative to different classes of generalized MSASs.

We end this section with the remark that the syntax and semantics proposed above are strong enough to capture the notions of strong/weak approximations defined on generalized MSASs, and will be illustrated in Section 3.3.1. Further, the operators $\Box$ and $\triangle$ have natural interpretations in epistemic logic (cf. e.g. [20]), as discussed in the next section.

3.3. Interpretation

3.3.1. Interpretation in rough set theory

We first note that the operators $\Box, \triangle, \Diamond$ and $\bigtriangledown$ have interpretations in terms of strong/weak approximations as shown by the following proposition.

Proposition 3.6. *Let $\mathfrak{F} := (W, \{R_i\}_{i\in N})$ be a generalized MSAS. Consider a model $\mathfrak{M} := (\mathfrak{F}, m)$ based on $\mathfrak{F}$. Then the following hold:*

$$[\![\Box\alpha]\!]_{\mathfrak{M}} = \underline{[\![\alpha]\!]_{\mathfrak{M}}}_{s_{\mathfrak{F}}}; \qquad [\![\triangle\alpha]\!]_{\mathfrak{M}} = \underline{[\![\alpha]\!]_{\mathfrak{M}}}_{w_{\mathfrak{F}}};$$

$$[\![\Diamond\alpha]\!]_{\mathfrak{M}} = \overline{[\![\alpha]\!]_{\mathfrak{M}}}_{w_{\mathfrak{F}}}; \qquad [\![\bigtriangledown\alpha]\!]_{\mathfrak{M}} = \overline{[\![\alpha]\!]_{\mathfrak{M}}}_{s_{\mathfrak{F}}}.$$

Proof.

$$\begin{aligned} w \in [\![\Box\alpha]\!]_{\mathfrak{M}} &\iff \cup_{i\in N}\{u \in W : (w,u) \in R_i\} \subseteq [\![\alpha]\!]_{\mathfrak{M}} \\ &\iff w \in \bigcap_{i\in N} \underline{[\![\alpha]\!]_{\mathfrak{M}}}_{R_i} \\ &\iff w \in \underline{[\![\alpha]\!]_{\mathfrak{M}}}_{s_{\mathfrak{F}}}. \end{aligned}$$

Thus, we have shown $[\![\Box\alpha]\!]_{\mathfrak{M}} = \underline{[\![\alpha]\!]_{\mathfrak{M}}}_{s_{\mathfrak{F}}}$. Similarly, we can prove the remaining equalities. $\Box$

Proposition 3.6 establishes that the operators $\Box$ and $\triangle$ capture the strong and weak lower approximations, respectively. Moreover, dual operators $\Diamond$ and $\bigtriangledown$ capture weak and strong upper approximations. Thus, we can use the wffs of the language $\mathcal{L}(\Box, \triangle)$ to express the properties of strong/weak approximations. For instance, the properties of weak upper approximation listed in Proposition 3.2 turn into wffs valid in the class $\mathsf{M_e}$ as we see in the next proposition.

Proposition 3.7. *The following wffs are valid in the class* $\mathsf{M_e}$.

- $\Diamond\bot \leftrightarrow \bot$.
- $\Diamond(\alpha \vee \beta) \leftrightarrow \Diamond\alpha \vee \Diamond\beta$.
- $\alpha \to \triangledown\alpha$.
- $\alpha \to \triangle\Diamond\alpha$.

3.3.2. Interpretation in epistemic logic

Let us choose an arbitrary finite initial segment N of $\mathbb{N}$, and fix it. Consider a modal language $\mathfrak{L}$ with PV as the set of propositional variables, and unary modal operators K_i, $i \in N$, where $K_i\alpha$ denotes 'agent i knows α'. Consider the translation f_N from $\mathcal{L}(\Box, \triangle)$ to $\mathfrak{L}$ defined as follows:

$$f_N(p) := p,\ p \in PV;$$

$$f_N(\neg\alpha) := \neg f_N(\alpha);$$

$$f_N(\alpha \wedge \beta) := f_N(\alpha) \wedge f_N(\beta);$$

$$f_N(\Box\alpha) := \bigwedge_{i\in N} K_i f_N(\alpha);$$

$$f_N(\triangle\alpha) := \bigvee_{i\in N} K_i f_N(\alpha).$$

Let $\mathfrak{M} := (\mathfrak{F}, m)$ be a model based on a MSAS $\mathfrak{F}$, that is, $\mathfrak{F} \in \mathsf{M_e}$. Note that $\mathfrak{M}$ is also an epistemic model. Moreover, we have the following.

Proposition 3.8. *For each* $\alpha \in \mathcal{L}(\Box, \triangle)$, *we have*

$$[\![\alpha]\!]_{\mathfrak{M}} = [\![f_N(\alpha)]\!]_{\mathfrak{M}}.$$

Proof is by induction on the complexity of the wff α, and we omit it.

From Proposition 3.8, and the translation of the operators $\Box$ and $\triangle$ in the language $\mathfrak{L}$, it is evident that one can read $\Box\alpha$ as 'all the agents of the system know α', and $\triangle\alpha$ as 'some agents of the system know α'. Let us examine a few properties for these two knowledge operators using the syntax and semantics of the language $\mathcal{L}(\Box, \triangle)$.

Let us first consider the distribution axiom

$$(\Box\alpha \wedge \Box(\alpha \to \beta)) \to \Box\beta. \qquad (\mathrm{A}(\Box))$$

It says that if all the agents know α and all the agents know that α implies β, then all the agents must know β. The wff A($\Box$) is valid in the class $\mathsf{M_e}$ of generalized MSASs. However,

$$(\triangle\alpha \wedge \triangle(\alpha \to \beta)) \to \triangle\beta. \qquad (\mathrm{A}(\triangle))$$

is not valid in the class $\mathsf{M_e}$. The wff (K($\triangle$)) says that if some agents know α and some agents know that α implies β, then some agents must know β. This is, of course, not a valid property as the same agent may not know both α, and α implies β.
Next, consider these three wffs:

$$\neg\alpha \to \Box\neg\Box\alpha. \qquad (\mathrm{B(a)})$$

$$\neg\alpha \to \Box\neg \triangle \alpha. \qquad (\mathrm{B(b)})$$

$$\neg\alpha \to \triangle\neg \triangle \alpha. \qquad (\mathrm{B(c)})$$

The wff B(a) is valid in $\mathsf{M_e}$, but it is not the case with wffs B(b) and B(c). In other words, if a statement α is not true, then we can conclude (i) but not (ii) and (iii), where (i)-(iii) are given as follows:

(i): *All* agents know that it is not the case that *all* agents know α;

(ii): *All* agents know that it is not the case that *some* agents know α;

(iii): *Some* agents know that it is not the case that *some* agents know α.

The next two wffs we consider are variants of *positive introspection axiom.* If an agent knows α, then he/she knows that he/she knows α, but he/she may not know whether the other agents of the system also know α. Thus, if some agents know α, then we can conclude that some agents know that some agents know α, but we cannot conclude that some agents know that all the agents know α. Formally, the wff $\triangle\alpha \to \triangle \triangle \alpha$ is valid in $\mathsf{M_e}$, but $\triangle\alpha \to \triangle\Box\alpha$ is not.

3.4. Invariance and definability

This section discusses the issues of invariance and definability related to the proposed logic.

3.4.1. Invariance

Let us first study when two objects in distinct models are indistinguishable by the language $\mathcal{L}(\Box, \triangle)$, in the sense of satisfying the same wffs.

Definition 3.9. Let $\mathfrak{M} := (W, \{R_i\}_{i \in N}, m)$, and $\mathfrak{M}' := (W', \{R'_i\}_{i \in N'}, m')$ be two models, and let $w \in W$ and $w' \in W'$. Let L be one of the languages $\mathcal{L}(\Box, \triangle)$, $\mathcal{L}(\Box)$, or $\mathcal{L}(\triangle)$. The *L-theory* of w is defined as the set $\{\alpha \in L : \mathfrak{M}, w \models \alpha\}$. We say that w and w' are *L-equivalent*, notation: $\mathfrak{M}, w \leftrightsquigarrow_L \mathfrak{M}', w'$, if they have the same L-theories.

Our first observation of the section is that the proposed semantics is invariant under *bounded morphism*. To put it more formally, recall the following notion of bounded morphism [7].

Definition 3.10. Let $\mathfrak{F} := (W, \{R_i\}_{i \in N})$, and $\mathfrak{F}' := (W', \{R'_i\}_{i \in N})$ be two generalized MSASs with the same cardinality N. A mapping $f : \mathfrak{F} \to \mathfrak{F}'$ is a *bounded morphism* if it satisfies the following conditions for all $i \in N$.

1. If $(w, v) \in R_i$, then $(f(w), f(v)) \in R'_i$.
2. If $(f(w), v') \in R'_i$, then there exists a v with $f(v) = v'$, and $(w, v) \in R_i$.

Theorem 3.11. *Let $f : \mathfrak{F}_1 \to \mathfrak{F}_2$ be a bounded morphism from the generalized MSAS $\mathfrak{F}_1$ to the generalized MSAS $\mathfrak{F}_2$. Let $\mathfrak{M}_1 := (\mathfrak{F}_1, m_1)$, and $\mathfrak{M}_2 := (\mathfrak{F}_2, m_2)$ be such that for all $p \in PV$, $w \in m_1(p)$ if and only if $f(w) \in m_2(p)$. Then $\mathfrak{M}_1, w \leftrightsquigarrow_{\mathcal{L}(\Box, \triangle)} \mathfrak{M}_2, f(w)$.*

The proof of Theorem 3.11 is very standard, and we omit the same. Note that Theorem 3.11 does not talk about invariance between models based on generalized MSASs with different cardinality. Moreover, it is evident that bounded morphism ensures that whenever it is possible to make a transition in one model using a relation with index $i \in N$, it is possible to make a matching transition in the other model using the relation index with the same i. This correspondence between the relations with the same index turns out to be very strong as far as invariance of wffs is concerned, and we consider the following weaker notion.

Definition 3.12. Let $\mathfrak{F} := (W, \{R_i\}_{i\in N})$ and $\mathfrak{F}' := (W', \{R'_i\}_{i\in N'})$ be two generalized MSASs. Let $Z \subseteq W \times W'$ be non-empty.

- Z is called a $\mathcal{L}(\Box)$-*invariance relation* between $\mathfrak{F}$ and $\mathfrak{F}'$, if the following conditions are satisfied.
  - **(B1):** If $(w, w') \in Z$ and $v' \in R'_i(w')$ for some $i \in N'$, then there exists a $j \in N$ and a $v \in R_j(w)$ such that $(v, v') \in Z$ (the back condition).
  - **(F1):** If $(w, w') \in Z$ and $v \in R_i(w)$ for some $i \in N$, then there exists a $j \in N'$, and a $v' \in R'_j(w')$ such that $(v, v') \in Z$ (the forth condition).
- Z is called $\mathcal{L}(\triangle)$-*invariance relation* between $\mathfrak{F}$ and $\mathfrak{F}'$ if the following conditions are satisfied.
  - **(B2):** For each $i \in N$, there exists a $j \in N'$ such that $(w, w') \in Z$ and $v' \in R'_j(w')$ imply $(v, v') \in Z$ and $v \in R_i(w)$ for some $v \in W$ (the back condition).
  - **(F2):** For each $i \in N'$, there exists a $j \in N$ such that $(w, w') \in Z$ and $v \in R_j(w)$ imply $(v, v') \in Z$ and $v' \in R'_i(w')$ for some $v' \in W'$ (the forth condition).
- Z is called $\mathcal{L}(\Box, \triangle)$-*invariance relation* between $\mathfrak{F}$ and $\mathfrak{F}'$ if Z is both $\mathcal{L}(\Box)$-invariance and $\mathcal{L}(\triangle)$-invariance relation between $\mathfrak{F}$ and $\mathfrak{F}'$.

We will write $Z : \mathfrak{F}, w \underline{\longleftrightarrow}_L \mathfrak{F}', w'$, L being one of the languages $\mathcal{L}(\Box, \triangle), \mathcal{L}(\Box)$, or $\mathcal{L}(\triangle)$, if Z is an L-invariance relation between $\mathfrak{F}$ and $\mathfrak{F}'$ with $(w, w') \in Z$.

It is pertinent to note that the above notion of $\mathcal{L}(\Box, \triangle)$-invariance relation is a generalization of the idea of *bisimulation* [7] between Kripke frames. In fact, one can view Kripke frames as generalized MSASs with cardinality 1. Moreover, for generalized MSASs with cardinality 1, the back and forth conditions of Definition 3.12 reduce to the back and forth conditions in the definition of bisimulation between Kripke frames.

The following theorem shows that the notion of invariance relation is weaker than the notion of bounded morphism.

Proposition 3.13. *Let $f : \mathfrak{F} \to \mathfrak{F}'$ be a bounded morphism between two generalized MSASs $\mathfrak{F}$ and $\mathfrak{F}'$. Then $Z : \mathfrak{F}, w \underline{\longleftrightarrow}_{\mathcal{L}(\Box,\triangle)} \mathfrak{F}', f(w)$, where $Z := \{(w, f(w)) : w \in W\}$, W being the domain of $\mathfrak{F}$.*

Theorem 3.14 (Invariance Theorem)**.** *Let $\mathfrak{M} := (\mathfrak{F}, m)$, and $\mathfrak{M}' = (\mathfrak{F}', m')$ be two models based on the generalized MSAS $\mathfrak{F} := (W, \{R_i\}_{i\in N})$ and $\mathfrak{F}' := (W', \{R'_i\}_{i\in N'})$. Let*

$\emptyset \neq Z \subseteq W \times W'$ be such that for $(w, w') \in Z$, $w \in m(p)$ if and only if $w' \in m'(p)$ for all $p \in PV$. Then the following hold.

1. *If $Z : \mathfrak{F}, w \underline{\longleftrightarrow}_{\mathcal{L}(\Box)} \mathfrak{F}', w'$, then $\mathfrak{M}, w \leftsquigarrow\!\!\!\rightsquigarrow_{\mathcal{L}(\Box)} \mathfrak{M}', w'$.*
2. *If $Z : \mathfrak{F}, w \underline{\longleftrightarrow}_{\mathcal{L}(\triangle)} \mathfrak{F}', w'$, then $\mathfrak{M}, w \leftsquigarrow\!\!\!\rightsquigarrow_{\mathcal{L}(\triangle)} \mathfrak{M}', w'$.*
3. *If $Z : \mathfrak{F}, w \underline{\longleftrightarrow}_{\mathcal{L}(\Box,\triangle)} \mathfrak{F}', w'$, then $\mathfrak{M}, w \leftsquigarrow\!\!\!\rightsquigarrow_{\mathcal{L}(\Box,\triangle)} \mathfrak{M}', w'$.*

Proof. Item 3 follows directly from Items 1 and 2. The proofs of Items 1 and 2 are by induction on the complexity of the wffs. We provide the arguments for the cases when wff is of the form $\Box\alpha$ and $\triangle\alpha$ in Items 1 and 2, respectively.

(1): Let us assume

$$\mathfrak{M}, w \models \Box\alpha, \tag{3.1}$$

and we prove $\mathfrak{M}', w' \models \Box\alpha$. Let $v' \in \cup_{i \in N'} R'_i(w')$, that is, $v' \in R'_i(w')$ for some $i \in N'$. To complete the proof, it is enough to show that $\mathfrak{M}', v' \models \alpha$.

From the back condition (B1) of Definition 3.12, we obtain a $j \in N$ and a $v \in R_j(w)$ such that $(v, v') \in Z$. Thus, from (3.1), we get $\mathfrak{M}, v \models \alpha$. Therefore, by induction hypothesis, and the fact that $(v, v') \in Z$, we obtain $\mathfrak{M}', v' \models \alpha$.

Converse can be proved in the same way using forth condition (F1).

(2): Let us assume

$$\mathfrak{M}, w \models \triangle\alpha, \tag{3.2}$$

and we prove $\mathfrak{M}', w' \models \triangle\alpha$. From (3.2), we obtain an $i \in N$ such that

$$\mathfrak{M}, v \models \alpha \text{ for all } v \in R_i(w). \tag{3.3}$$

From back condition (B2) of Definition 3.12, we obtain a $j \in N'$ such that the following holds:

$$(u, u') \in Z \;\&\; t' \in R'_j(u') \implies \text{there exists } t \in R_i(u) \text{ with } (t, t') \in Z. \tag{3.4}$$

To complete the proof, it is enough to show that $\mathfrak{M}', v' \models \alpha$ for all $v' \in R'_j(w')$. So let us take an arbitrary $v' \in R'_j(w')$. Then from (3.4), we obtain a $v \in R_i(w)$ with $(v, v') \in Z$. Therefore, from (3.3), we get $\mathfrak{M}, v \models \alpha$. Finally, by induction hypothesis, and the fact that $(v, v') \in Z$, we obtain $\mathfrak{M}', v' \models \alpha$.

Giving similar arguments and using forth condition (F2), we obtain the converse. $\Box$

We have the following converse of Theorem 3.14(1) in a restricted case. A model $\mathfrak{M} := (\mathfrak{F}, m)$ based on the generalized MSAS $\mathfrak{F} := (W, \{R_i\}_{i \in N})$ will be called *image finite* if for each i and $w \in W$, $R_i(w)$ is finite.

Theorem 3.15. *Let $\mathfrak{M} := (\mathfrak{F}, m)$, and $\mathfrak{M}' = (\mathfrak{F}', m')$ be two models based on the generalized MSAS $\mathfrak{F} := (W, \{R_i\}_{i \in N})$ and $\mathfrak{F}' := (W', \{R'_i\}_{i \in N'})$, where N and N' are finite. Then $\mathfrak{M}, w \leftrightsquigarrow_{\mathcal{L}(\Box)} \mathfrak{M}', w'$ implies $Z : \mathfrak{F}, w \underline{\longleftrightarrow}_{\mathcal{L}(\Box)} \mathfrak{F}', w'$ for some Z.*

Proof of Theorem 3.15 is very much in the line of Hennessy-Milner Theorem [7], and we omit it. It is to be noted that we do not have Theorem 3.15(2) for the language $\mathcal{L}(\triangle)$, as shown by the following example.

Example 3.16. Let us consider the models $\mathfrak{M} := (\mathfrak{F}, m)$, and $\mathfrak{M}' := (\mathfrak{F}', m')$ based on the generalized MSASs $\mathfrak{F} := (W, \{R_1, R_2, R_3, R_4\})$ and $\mathfrak{F}' := (W', \{R'_1, R'_2\})$ given by Figures 3.1 and 3.2, respectively. The valuation functions m, m' are such that $w_i \in m(p)$ if and only if $u_i \in m'(p)$, $i = 1, 2, 3$. One can verify that $\mathfrak{M}, w_1 \leftrightsquigarrow_{\mathcal{L}(\triangle)} \mathfrak{M}', u_1$, but there does

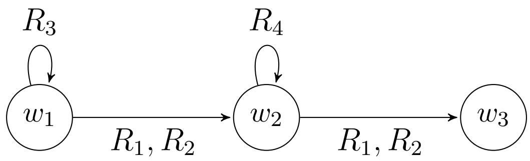


Figure 3.1. Generalized MSAS $\mathfrak{F}$

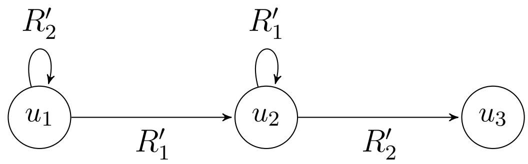


Figure 3.2. Generalized MSAS $\mathfrak{F}'$

not exists a Z such that $Z : \mathfrak{F}, w_1 \underline{\longleftrightarrow}_{\mathcal{L}(\triangle)} \mathfrak{F}', u_1$.

3.4.2. Definability

We now present a few limiting results related to the expressibility power of the proposed logic. Consider the following notion of definability.

Definition 3.17 (Definability)**.** A wff α is said to *define* a class $\mathfrak{G}$ of generalized MSASs if for all generalized MSASs $\mathfrak{F}$, $\mathfrak{F}$ is in $\mathfrak{G}$ if and only if $\mathfrak{F} \models \alpha$. Similarly, a set Γ of wffs is said to *define* a class $\mathfrak{G}$ of generalized MSASs if for all generalized MSASs $\mathfrak{F}$, $\mathfrak{F}$ is in $\mathfrak{G}$ if and only if $\mathfrak{F} \models \Gamma$.

A class of generalized MSASs is said to be *definable* if there is some set of wffs that defines it.

Proposition 3.18. *The wff $\triangle\alpha \to \alpha$ defines the class $\mathsf{M_r}$ of generalized MSASs.*

Proof. It is not difficult to show that $\triangle\alpha \to \alpha$ is valid in the class $\mathsf{M_r}$ of generalized MSASs. So, let $\mathfrak{F} := (W, \{R_i\}_{i\in N}) \notin \mathsf{M_r}$, and we show that $\mathfrak{F} \not\models \triangle\alpha \to \alpha$. Since $\mathfrak{F} \notin \mathsf{M_r}$, there exists an $i \in N$ such that $(w, w) \notin R_i$ for some $w \in W$. Let m be a valuation such that $m(p) := R_i(w)$. We note that $w \notin m(p)$ and hence $\mathfrak{M}, w \not\models \triangle p \to p$, where $\mathfrak{M} := (\mathfrak{F}, m)$. □

Next, we present two classes of generalized MSASs that cannot be defined through the language $\mathcal{L}(\Box, \triangle)$. Our proof technique is based on the following simple idea. In order to show that a class, say, $\mathfrak{C}$ of generalized MSASs is not definable, we find two generalized MSASs $\mathfrak{F}$ and $\mathfrak{F}'$ such that (i) $\mathfrak{F} \in \mathfrak{C}$, but $\mathfrak{F}' \notin \mathfrak{C}$, (ii) $\mathfrak{F}$ and $\mathfrak{F}'$ are based on the same domain and (iii) the identity relation becomes $\mathcal{L}(\Box, \triangle)$-invariance relation between $\mathfrak{F}$ and $\mathfrak{F}'$. Once we have (i)-(iii), we obtained the desired result by the direct application of the Invariance Theorem 3.14. It is worth noting here that the usual way of using bisimulations to prove undefinability is at the model level. However, in this section, we will be able to show the indefinability of classes of generalized MSASs (frames) due to the conditions (ii) and (iii).

Proposition 3.19. *The class $\mathsf{M_t}$ of generalized MSASs is not definable.*

Proof. Consider the generalized MSASs $\mathfrak{F} := (W, \{R_i\}_{i\in N})$ and $\mathfrak{F}' := (W, \{R'_i\}_{i\in N})$, $N = \{1, 2, 3\}$, given by Figure 3.3. If possible, let a set Γ of wffs defines the class $\mathsf{M_t}$. Then, since $\mathfrak{F}' \in \mathsf{M_t}$ and $\mathfrak{F} \notin \mathsf{M_t}$, we will obtain $\mathfrak{F}' \models \Gamma$ and $\mathfrak{F} \not\models \Gamma$. But this is not possible as the identity relation $Z := \{(w, w), (u, u), (v, v)\}$ is an $\mathcal{L}(\Box, \triangle)$-invariance relation between $\mathfrak{F}$ and $\mathfrak{F}'$. □

Proposition 3.20. *The class $\mathsf{M_s}$ of generalized MSASs is not definable.*

Proof. Consider the generalized MSASs $\mathfrak{F} := (W, \{R_i\}_{i\in N})$ and $\mathfrak{F}' := (W, \{R'_i\}_{i\in N})$, $N := \{1, 2, 3\}$, given by Figure 3.4. If possible, let the set Γ of wffs defines the class $\mathsf{M_s}$. Then,

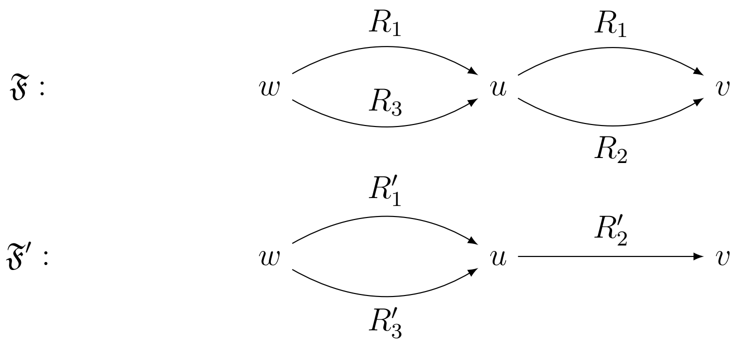


Figure 3.3

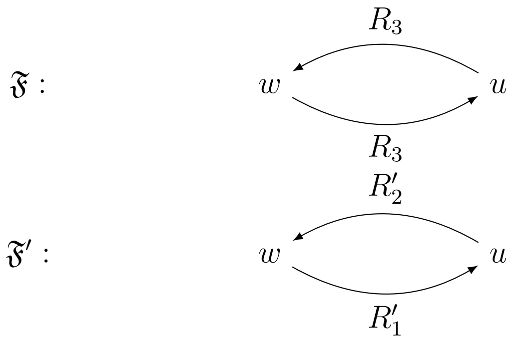


Figure 3.4

since $\mathfrak{F} \in \mathsf{M_s}$ and $\mathfrak{F}' \notin \mathsf{M_s}$, we will obtain $\mathfrak{F} \models \Gamma$ and $\mathfrak{F}' \not\models \Gamma$. But this is not possible as the relation $Z := \{(w, w), (u, u)\}$ is an $\mathcal{L}(\Box, \triangle)$-invariance relation between $\mathfrak{F}$ and $\mathfrak{F}'$. Note that the conditions (B2) and (F2) are direct consequence of the fact that $3 \in N$ and $1 \in N$, respectively. □

3.5. Axiomatization

This section deals with the axiomatization problem of the fragments $\mathcal{L}(\triangle)$ and $\mathcal{L}(\Box)$ concerning different classes of generalized MSASs (cf. Table 3.1). This study will also give some insight into the nature of strong/weak approximations. We will see that logics for different classes of generalized MSASs are obtained as known normal modal systems as well as non-normal modal systems (cf. [7]). Moreover, some classes also correspond to logics that lies between known modal systems. Here, it is interesting to note that, although the proposed semantics is based on a relational structure, we will find that none of the logics on the language $\mathcal{L}(\triangle)$ obtained in the process of our study contains axiom K. On the contrary, all the obtained logics on the language $\mathcal{L}(\Box)$ contains axiom K.

Consider the following axioms and inference rules. Let $\blacksquare \in \{\Box, \triangle\}$, and $\blacklozenge$ be the dual of $\blacksquare$.

$$\text{All axioms of classical propositional logic,} \tag{Taut}$$

$$\blacksquare(\alpha \wedge \beta) \to \blacksquare\alpha \wedge \blacksquare\beta, \tag{M($\blacksquare$)}$$

$$\blacksquare\top, \tag{N($\blacksquare$)}$$

$$\blacksquare\alpha \to \alpha, \tag{T($\blacksquare$)}$$

$$\blacksquare\alpha \to \blacksquare\blacksquare\alpha, \tag{4($\blacksquare$)}$$

$$\blacksquare\alpha \to \blacksquare(\alpha \wedge \blacksquare\alpha), \tag{4^0}$$

$$\alpha \to \blacksquare\blacklozenge\alpha, \tag{B($\blacksquare$)}$$

$$\blacksquare\alpha \wedge \blacksquare\beta \to \blacksquare(\alpha \wedge \beta), \tag{C($\blacksquare$)}$$

$$\text{From } \alpha \text{ and } \alpha \to \beta, \text{ infer } \beta, \tag{MP}$$

$$\text{From } \alpha \leftrightarrow \beta, \text{ infer } \blacksquare\alpha \leftrightarrow \blacksquare\beta. \tag{RE($\blacksquare$)}$$

The axioms and inference rules of a few modal systems are given in the Table 3.2.

Notion of theorem is defined in the standard way. In this section, we will mainly work with the languages $\mathcal{L}(\Box), \mathcal{L}(\triangle)$ and the basic modal language $\mathcal{L}(L)$ with PV as the set of propositional variables, and 'necessity' modal operator L. For a modal system Λ, language $\mathfrak{L} \in \{\mathcal{L}(\Box), \mathcal{L}(\triangle), \mathcal{L}(L)\}$ and a wff α of the language $\mathfrak{L}$, we will write $\vdash_\Lambda \alpha$ to mean that α is a theorem of the system Λ in the language $\mathfrak{L}$. Recall the following well-known axioms and inference rules of modal logic.

$$\blacksquare(\alpha \to \beta) \to (\blacksquare\alpha \to \blacksquare\beta). \tag{K($\blacksquare$)}$$

$$\text{From } \alpha \to \beta, \text{ infer } \blacksquare\alpha \to \blacksquare\beta. \tag{RM($\blacksquare$)}$$

$$\text{From } \alpha, \text{ infer } \blacksquare\alpha. \tag{Nec($\blacksquare$)}$$

Let us note the following facts.

Proposition 3.21. *1. The* RM($\blacksquare$) *rule is derivable in modal system* EM.

2. The necessitation rule Nec($\blacksquare$) *rule is derivable in modal system* EMN.

Modal systems	Axioms and Inference Rules	Modal systems	Axioms and Inference Rules
EM	Taut, M(■), MP, RE(■)	K	EMN+ axiom C(■)
EMN	EM+ axiom N(■)	B	K+ axiom B(■)
EMNT	EMN+ axiom T(■)	T	K+ axiom T(■)
EMN4	EMN+ axiom 4(■)	KTB	B+ axiom T(■)
EMNT4	EMNT+ axiom 4(■)	KB4	B+ axiom 4(■)
EMN4^0	EMN+ axiom 4^0	S5	KTB+ axiom 4(■)

Table 3.2. A few modal systems

3. *The axiom* K(■) *is derivable in modal system* K.

We refer to [67] for the proof of this proposition.

Proposition 3.22.

1. $\vdash_{\text{EMN4}^0} \blacksquare\alpha \rightarrow \blacksquare\blacksquare\alpha$.
2. $\vdash_{\text{EMNT4}} \blacksquare\alpha \rightarrow \blacksquare(\alpha \wedge \blacksquare\alpha)$.
3. $\not\vdash_{\text{EMN4}} \blacksquare\alpha \rightarrow \blacksquare(\alpha \wedge \blacksquare\alpha)$.

Items 1 and 2 can easily be proved. Moreover, Item 3 can be proved using Soundness Theorem for the modal system MN4' given in [27].

For later development of the section, we will also require the following definition.

Definition 3.23. Let * be the translation from $\mathcal{L}(\Box, \triangle)$ to the wffs of the language $\mathcal{L}(L)$, which just replaces the occurrences of $\Box$ and $\triangle$ with L.

Note that for each $\alpha \in \mathcal{L}(L)$, there exists a $\beta_1 \in \mathcal{L}(\Box)$ and a $\beta_2 \in \mathcal{L}(\triangle)$ such that $\beta_1^\star$ and $\beta_2^\star$ are α.

Corresponding to the class $\mathsf{M_r}$ of generalized MSASs, we define the set of wffs

$$\Lambda_{\mathrm{r}}(\triangle) := \{\alpha^\star : \alpha \in \mathcal{L}(\triangle) \text{ is valid in } \mathsf{M_r}\}, \text{ and}$$

$$\Lambda_{\mathrm{r}}(\Box) := \{\alpha^\star : \alpha \in \mathcal{L}(\Box) \text{ is valid in } \mathsf{M_r}\}.$$

Similar sets are defined for each of the classes of generalized MSASs given in Table 3.1.

The following is obvious.

Proposition 3.24. *Let Λ be one of the modal systems listed in Table 3.2.*

- *For each $\alpha \in \mathcal{L}(\Box)$, $\vdash_\Lambda \alpha$ if and only if $\vdash_\Lambda \alpha^\star$.*
- *For each $\alpha \in \mathcal{L}(\triangle)$, $\vdash_\Lambda \alpha$ if and only if $\vdash_\Lambda \alpha^\star$.*

3.5.1. Axiomatization of the fragment $\mathcal{L}(\triangle)$

The following theorem can be proved in the standard way.

Theorem 3.25 (Lindenbaum's Lemma)**.** *For $\Lambda \in \{\mathrm{EMN}, \mathrm{EMNT}, \mathrm{EMN4}, \mathrm{EMNT4}\}$, every Λ-consistent set can be extended to a Λ-maximal consistent set.*

Note that $\mathcal{L}(\triangle)$ is countably infinite. Let $f : \mathbb{N} \to \mathcal{L}(\triangle)$ be a bijection.

3.5.1.1. *Axiomatization relative to the classes* M *and* $\mathsf{M_r}$. We first note the following soundness theorem.

Theorem 3.26 (Soundness Theorem)**.** *For $\alpha \in \mathcal{L}(\triangle)$, the following hold.*

- *If $\vdash_{\mathrm{EMN}} \alpha$, then $\mathsf{M} \models \alpha$.*
- *If $\vdash_{\mathrm{EMNT}} \alpha$, then $\mathsf{M_r} \models \alpha$.*

The soundness claims made in Theorem 3.26 (as well as in Theorem 3.32 to appear later) can easily be demonstrated. In all cases, one shows that the axioms are valid and that the rules of inferences (MP) and RE($\triangle$) preserve validity on the class of generalized MSASs in question.

For $\Lambda \in \{\mathrm{EMN}, \mathrm{EMNT}\}$, let $\mathbb{M}_\Lambda$ be the set of all Λ-maximal consistent sets of $\mathcal{L}(\triangle)$. Recall that the usual canonical relation R^Λ on $\mathbb{M}_\Lambda$ is defined as follows. $(\Gamma, \Gamma') \in \mathrm{R}^\Lambda$ if and only if for all wff $\beta \in \mathcal{L}(\triangle)$,

$$\triangle\beta \in \Gamma \text{ implies } \beta \in \Gamma'. \tag{3.5}$$

For each wff β, condition (3.5) defines a relation on $\mathbb{M}_\Lambda$. Hence, we obtain a countable collection of relations as there is a countable collection of wffs. We use this countable collection of relations to define the required canonical model. Formally, we have the following definition for $\Lambda \in \{\text{EMN}, \text{EMNT}\}$.

Definition 3.27 (Canonical Model)**.** $\mathfrak{M}^\Lambda := (\mathfrak{F}^\Lambda, m^\Lambda)$, where

- $\mathfrak{F}^\Lambda := (\mathbb{M}_\Lambda, \{\mathrm{R}_i^\Lambda\}_{i\in\mathbb{N}})$ is a generalized MSAS such that
  - $\mathbb{M}_\Lambda := \{\Gamma : \Gamma$ is a Λ-maximal consistent set of $\mathcal{L}(\triangle)\}$;
  - for $\Gamma \in \mathbb{M}_\Lambda$, and $i \in \mathbb{N}$,

$$\mathrm{R}_i^\Lambda(\Gamma) := \{\Gamma' \in \mathbb{M}_\Lambda : \triangle\alpha \ \notin \Gamma, \text{ or, } \alpha \in \Gamma'\}, \text{ where } f(i) = \alpha;$$

- $m^\Lambda(p) := \{\Gamma \in \mathbb{M}_\Lambda : p \in \Gamma\}$.

One can easily verify that

$$\mathrm{R}^\Lambda = \bigcap_{i\in\mathbb{N}} \mathrm{R}_i^\Lambda . \tag{3.6}$$

Proposition 3.28 (Truth Lemma)**.** *For any wff $\alpha \in \mathcal{L}(\triangle)$, and $\Gamma \in \mathbb{M}_\Lambda$,*

$$\alpha \in \Gamma \textit{ if and only if } \mathfrak{M}^\Lambda, \Gamma \models \alpha.$$

Proof. Proof is by induction on the complexity of the wff α. We only provide the proof of the case when α is of the form $\triangle\beta$. First, we assume $\mathfrak{M}^\Lambda, \Gamma \models \triangle\beta$, and we show $\triangle\beta \in \Gamma$.

$$\begin{aligned} & \mathfrak{M}^\Lambda, \Gamma \models \triangle\beta \\ \Longrightarrow\ & \text{there exists } i \in \mathbb{N} \text{ such that } \mathfrak{M}^\Lambda, \Gamma' \models \beta \text{ for all } \Gamma' \in \mathrm{R}_i^\Lambda(\Gamma) \\ \Longrightarrow\ & \text{there exists } i \in \mathbb{N} \text{ such that } \beta \in \Gamma' \text{ for all } \Gamma' \in \mathrm{R}_i^\Lambda(\Gamma) \qquad (3.7) \\ & \text{(by induction hypothesis).} \end{aligned}$$

Let $f(i) = \gamma$.

Consider the set

$$\Theta := \begin{cases} \{\gamma, \neg\beta\} & \text{if } \triangle\gamma \in \Gamma \\ \{\neg\beta\} & \text{otherwise.} \end{cases} \tag{3.8}$$

Claim: The set Θ in (3.8) is not Λ-consistent.

If the set Θ is Λ-consistent, then by Lindenbaum's Lemma, we will obtain a Λ-maximal consistent set Γ' containing Θ. But, then $\Gamma' \in \mathrm{R}_i^{\Lambda}(\Gamma)$, and hence by (3.7), $\beta \in \Gamma'$. But, this is not possible as $\neg\beta \in \Theta \subseteq \Gamma'$. Thus we have proved the claim.

Case 1: $\triangle\gamma \notin \Gamma$

Since $\{\neg\beta\}$ is not Λ-consistent, we have $\vdash_\Lambda \beta$, and hence $\vdash_\Lambda \triangle\beta$. This, in turn, implies $\triangle\beta \in \Gamma$.

Case 2: $\triangle\gamma \in \Gamma$

Since $\{\gamma, \neg\beta\}$ is not Λ-consistent, we have $\vdash_\Lambda \gamma \to \beta$, and this gives $\vdash_\Lambda \triangle\gamma \to \triangle\beta$ (using inference rule RM($\triangle$)), and thus we obtain $\triangle\beta \in \Gamma$ as $\triangle\gamma \in \Gamma$.

Now, we prove the converse. That is, we show that if $\triangle\beta \in \Gamma$, then $\mathfrak{M}^\Lambda, \Gamma \models \triangle\beta$. Consider $j \in \mathbb{N}$, where $f(j) = \beta$. It is enough to show that $\mathfrak{M}^\Lambda, \Gamma' \models \beta$ for all Γ' with $\Gamma' \in \mathrm{R}_j^{\Lambda}(\Gamma)$. In fact,

$$
\begin{aligned}
& \Gamma' \in \mathrm{R}_j^{\Lambda}(\Gamma) \\
\Longrightarrow\ & \triangle\, \beta \notin \Gamma, \text{ or, } \beta \in \Gamma' \\
\Longrightarrow\ & \beta \in \Gamma', \ (\because \triangle\beta \in \Gamma) \\
\Longrightarrow\ & \mathfrak{M}^\Lambda, \Gamma' \models \beta \text{ (by induction hypothesis).}
\end{aligned}
$$

This completes the proof. □

Proposition 3.29. *The generalized MSAS $\mathfrak{F}^{\mathrm{EMNT}}$ belongs to the class $\mathsf{M_r}$.*

Proof. Choose an arbitrary $\Gamma \in \mathbb{M}_{\mathrm{EMNT}}$ and $i \in \mathbb{N}$. Let $f(i) = \alpha$. Due to axiom T($\triangle$), either $\triangle\alpha \notin \Gamma$, or $\alpha \in \Gamma$. This gives $\Gamma \in \mathrm{R}_i^{\mathrm{EMNT}}(\Gamma)$. □

Now, we can prove the desired completeness theorem.

Theorem 3.30 (Completeness Theorem)**.** *For $\alpha \in \mathcal{L}(\triangle)$, the following hold.*

- *If $\mathsf{M} \models \alpha$, then $\vdash_{\mathrm{EMN}} \alpha$.*
- *If $\mathsf{M_r} \models \alpha$, then $\vdash_{\mathrm{EMNT}} \alpha$.*

Proof. Proof argument is very standard in modal logic and makes use of Lindenbaum's Lemma (Theorem 3.25) and Truth Lemma (Proposition 3.28). We sketch the proof for the logic EMNT for readers who are not familiar with it. An exactly similar argument works for the logic EMN.

Let $\mathsf{M_r} \models \alpha$ and, if possible, $\nvdash_{\mathrm{EMNT}} \alpha$. Then we get $\{\neg\alpha\}$ as EMNT-consistent, and hence, by Theorem 3.25, we obtain an EMNT-maximal consistent set Γ containing $\neg\alpha$. Then Proposition 3.28 gives us $\mathfrak{M}^{\text{EMNT}}, \Gamma \models \neg\alpha$. But this contradicts $\mathsf{M_r} \models \alpha$ as, by Proposition 3.29, $\mathfrak{F}^{\text{EMNT}} \in \mathsf{M_r}$. This completes the proof. □

As a consequence of Theorems 3.26 and 3.30, we obtain the following.

Theorem 3.31.

- EMN $= \Lambda(\triangle)$.
- EMNT $= \Lambda_{\mathrm{r}}(\triangle)$.

Proof. Note that for each $\alpha \in \mathcal{L}(L)$, there exists a $\beta \in \mathcal{L}(\triangle)$ such that $\beta^\star$ is α. Now,

$$\begin{aligned} & \vdash_{\text{EMNT}} \alpha \\ \iff & \vdash_{\text{EMNT}} \beta \text{ (by Proposition 3.24)} \\ \iff & \mathsf{M_r} \models \beta \text{ (by Theorems 3.26 and 3.30)} \\ \iff & \alpha \in \Lambda_{\mathrm{r}}(\triangle). \end{aligned}$$

Thus, we have obtained EMNT $= \Lambda_{\mathrm{r}}(\triangle)$. We can prove EMN $= \Lambda(\triangle)$ proceeding in the same way. □

It follows from Theorem 3.31 that the logics for weak lower and strong upper approximations based on the generalized MSASs belonging to the classes M and $\mathsf{M_r}$ are the monotonic logic EMN and EMNT, respectively.

3.5.1.2. *Axiomatization relative to the classes* $\mathsf{M_t}$ *and* $\mathsf{M_{rt}}$. It is not difficult to prove the following soundness theorem.

Theorem 3.32 (Soundness Theorem)**.** *For $\alpha \in \mathcal{L}(\triangle)$, the following hold.*

- *If $\vdash_{\text{EMN4}^0} \alpha$, then $\mathsf{M_t} \models \alpha$.*
- *If $\vdash_{\text{EMNT4}} \alpha$, then $\mathsf{M_{rt}} \models \alpha$.*

For $n \in \mathbb{N}$, we will write $\triangle^n\alpha$ for $\underbrace{\triangle \cdots \triangle}_{n-\text{times}} \alpha$. Moreover, $\triangle^0\alpha$ will denote the wff α.

The canonical model given in Definition 3.27 does not work for the modal systems EMN4^0 and EMNT4, as the corresponding canonical models does not belong to the classes $\mathsf{M_t}$ and $\mathsf{M_{rt}}$, respectively. Therefore, we need to make some changes in the definition of the canonical relation. In order to do that, let us recall that the usual canonical relation corresponding to a unary modal operator can also be defined equivalently as follows. $(\Gamma, \Gamma') \in \mathrm{R}^{\Lambda}$ if and only if for all wff $\beta \in \mathcal{L}(\triangle)$,

$$\triangle^n \beta \in \Gamma \text{ implies } \triangle^{n-1} \beta \in \Gamma', \text{ for each } n \in \mathbb{N}. \tag{3.9}$$

Therefore, as in the case of Definition 3.27, we consider a countable collection of relations obtained by using (3.9) for different β. That is, we define the canonical models for $\Lambda \in \{\mathrm{EMN4}^0, \mathrm{EMNT4}\}$ as follows.

Definition 3.33 (Canonical Model)**.** $\mathfrak{M}^{\Lambda} := (\mathfrak{F}^{\Lambda}, m^{\Lambda})$, where

- $\mathfrak{F}^{\Lambda} := (\mathbb{M}_{\Lambda}, \{\mathrm{R}_i^{\Lambda}\}_{i \in \mathbb{N}})$ is a generalized MSAS such that
  - $\mathbb{M}_{\Lambda} := \{\Gamma : \Gamma$ is a Λ-maximal consistent set of $\mathcal{L}(\triangle)\}$;
  - for $\Gamma \in \mathbb{M}_{\Lambda}$, and $i \in \mathbb{N}$,

    $$\mathrm{R}_i^{\Lambda}(\Gamma) := \{\Gamma' : \text{ for each } n \in \mathbb{N}, \ \triangle^n \alpha \notin \Gamma, \text{ or, } \triangle^{n-1} \alpha \in \Gamma'\}, \tag{3.10}$$

    where $f(i) = \alpha$;
- $m^{\Lambda}(p) := \{\Gamma \in \mathbb{M}_{\Lambda} : p \in \Gamma\}$.

As in (3.6), we again obtain $\mathrm{R}^{\Lambda} = \bigcap_{i \in \mathbb{N}} \mathrm{R}_i^{\Lambda}$, where R_i^{Λ} is now given by (3.10). Moreover, as desired, we also have the following.

Proposition 3.34.

1. *The generalized MSAS* $\mathfrak{F}^{\mathrm{EMN4}^0}$ *belongs to the class* $\mathsf{M_t}$*.*
2. *The generalized MSAS* $\mathfrak{F}^{\mathrm{EMNT4}}$ *belongs to the class* $\mathsf{M_{rt}}$*.*

Proof. Let us first show that $\mathfrak{F}^{\mathrm{EMN4}^0}$ and $\mathfrak{F}^{\mathrm{EMNT4}}$ belong to the class $\mathsf{M_t}$. Let Λ be one of the systems $\mathrm{EMN4}^0$, or EMNT4.
Suppose $\Gamma' \in \mathrm{R}_i^{\Lambda}(\Gamma)$, and $\Gamma'' \in \mathrm{R}_i^{\Lambda}(\Gamma')$, and we prove that $\Gamma'' \in \mathrm{R}_i^{\Lambda}(\Gamma)$. Let $f(i) = \alpha$. Let us choose an arbitrary $n \in \mathbb{N}$, and we prove that either $\triangle^n \alpha \notin \Gamma$, or, $\triangle^{n-1} \alpha \in \Gamma''$. So, let us assume $\triangle^{n-1}\alpha \notin \Gamma''$, and we prove $\triangle^n \alpha \notin \Gamma$. Since $\Gamma'' \in \mathrm{R}_i^{\Lambda}(\Gamma')$, $\triangle^{n-1}\alpha \notin \Gamma''$ implies $\triangle^n \alpha \notin \Gamma'$. Again, since $\Gamma' \in \mathrm{R}_i^{\Lambda}(\Gamma)$, $\triangle^n \alpha \notin \Gamma'$ implies $\triangle^{n+1}\alpha \notin \Gamma$. Thus, using axiom $4(\triangle)$, we obtain $\triangle^n \alpha \notin \Gamma$.
Next, we show that $\mathfrak{F}^{\mathrm{EMNT4}} \in \mathsf{M_r}$. We assume $\triangle^{n-1}\alpha \notin \Gamma$. Then, using axiom $\mathrm{T}(\triangle)$, we obtain $\triangle^n \alpha \notin \Gamma$. Thus, we have shown $\Gamma \in \mathrm{R}_i^{\mathrm{EMNT4}}(\Gamma)$. □

Proposition 3.35 (Truth Lemma). *Let* $\Lambda \in \{\text{EMN4}^0, \text{EMNT4}\}$. *For any wff* $\alpha \in \mathcal{L}(\triangle)$ *and a* Λ*-maximal consistent set* Γ*,*

$$\alpha \in \Gamma \textit{ if and only if } \mathfrak{M}^\Lambda, \Gamma \models \alpha.$$

Proof. Proof is by induction on the complexity of the wff α. We only provide the proof of the case when α is of the form $\triangle\beta$.
($\Leftarrow$): First, we assume $\mathfrak{M}^\Lambda, \Gamma \models \triangle\beta$, and we show $\triangle\beta \in \Gamma$. Here,

$$\begin{aligned} & \mathfrak{M}^\Lambda, \Gamma \models \triangle\beta \\ \Longrightarrow & \text{ there exists } i \in \mathbb{N} \text{ such that } \mathfrak{M}^\Lambda, \Gamma' \models \beta \text{ for all } \Gamma' \in \mathrm{R}_i^\Lambda(\Gamma), \\ \Longrightarrow & \text{ there exists } i \in \mathbb{N} \text{ such that } \beta \in \Gamma' \text{ for all } \Gamma' \in \mathrm{R}_i^\Lambda(\Gamma), \qquad (3.11) \\ & \text{(by induction hypothesis).} \end{aligned}$$

Let $f(i) = \gamma$.
Consider the set

$$\Theta := \{\triangle^{n-1}\gamma : \triangle^n\gamma \in \Gamma,\ n \in \mathbb{N}\} \cup \{\neg\beta\}. \qquad (3.12)$$

Claim: The set Θ in (3.12) is not Λ-consistent.
If the set Θ is Λ-consistent, then by Lindenbaum's Lemma, we will obtain a Λ-maximal consistent set Γ' containing Θ. But, then $\Gamma' \in \mathrm{R}_i^\Lambda(\Gamma)$, and hence by (3.11), $\beta \in \Gamma'$. But, this is not possible as $\neg\beta \in \Theta \subseteq \Gamma'$. Thus we have proved the claim.
Case 1: $\{\triangle^{n-1}\gamma : \triangle^n\gamma \in \Gamma,\ n \in \mathbb{N}\} = \emptyset$
In this case, as the set Θ of (3.12) is not Λ-consistent, we obtain $\vdash_\Lambda \beta$, and hence $\vdash_\Lambda \triangle\beta$. This, in turn, implies $\triangle\beta \in \Gamma$.
Case 2: $\{\triangle^{n-1}\gamma : \triangle^n\gamma \in \Gamma,\ n \in \mathbb{N}\} \neq \emptyset$
In this case, again using the fact that the set Θ of (3.12) is not Λ-consistent, we obtain

$$\vdash_\Lambda \triangle^{k_1-1}\gamma \wedge \cdots \wedge \triangle^{k_m-1}\gamma \to \beta, \qquad (3.13)$$

where $\{\triangle^{k_1}\gamma, \ldots, \triangle^{k_m}\gamma\} \subseteq \Gamma$. Let $n = \min\{k_1 - 1, \ldots, k_m - 1\}$.
Case 2(a): $n = 0$.
In this case $\triangle\gamma \in \Gamma$. Let $t = \min\{\{k_1 - 1, \ldots, k_m - 1\} \setminus \{0\}\}$. Recall that axiom $4(\triangle)$ is a theorem of the modal system EMN4^0. Therefore, from (3.13), and axiom $4(\triangle)$, we

obtain

$$\begin{aligned} & \vdash_\Lambda \gamma \wedge \triangle^t\gamma \to \beta \\ \Longrightarrow\ & \vdash_\Lambda \gamma \wedge \triangle\gamma \to \beta\ (\because \vdash_\Lambda \triangle\gamma \to \triangle^t\gamma) \\ \Longrightarrow\ & \vdash_\Lambda \triangle(\gamma \wedge \triangle\gamma) \to \triangle\beta \\ \Longrightarrow\ & \vdash_\Lambda \triangle\gamma \to \triangle\beta\ (\because \vdash_\Lambda \triangle\gamma \to \triangle(\gamma \wedge \triangle\gamma)) \end{aligned}$$

Thus, we get $\triangle\beta \in \Gamma$ as $\triangle\gamma \in \Gamma$.

Case 2(b): $n \neq 0$.

Note that $\triangle^{n+1}\gamma \in \Gamma$. From (3.13), and axiom $4(\triangle)$, we obtain

$$\begin{aligned} & \vdash_\Lambda \triangle^n\gamma \to \beta \\ \Longrightarrow\ & \vdash_\Lambda \triangle^{n+1}\gamma \to \triangle\beta. \end{aligned}$$

Thus, we again obtain $\triangle\beta \in \Gamma$ as $\triangle^{n+1}\gamma \in \Gamma$.

($\Rightarrow$): We assume $\triangle\beta \in \Gamma$, and we prove that $\mathfrak{M}^\Lambda, \Gamma \models \triangle\beta$.

Let $j \in \mathbb{N}$, where $f(j) = \beta$. Consider an arbitrary Γ' with $\Gamma' \in \mathrm{R}_j^\Lambda(\Gamma)$. Then, we have

$$\begin{aligned} \Gamma' \in \mathrm{R}_j^\Lambda(\Gamma) \Longrightarrow\ & \triangle\beta \notin \Gamma,\ \text{or},\ \beta \in \Gamma' \\ \Longrightarrow\ & \beta \in \Gamma',\ (\because \triangle\beta \in \Gamma) \\ \Longrightarrow\ & \mathfrak{M}^\Lambda, \Gamma' \models \beta \text{ (by induction hypothesis).} \end{aligned}$$

Thus, we obtained $\mathfrak{M}^\Lambda, \Gamma \models \triangle\beta$. □

As in the case of Theorem 3.30, Lindenbaum's Lemma (Theorem 3.25) and Truth Lemma (Proposition 3.35) give us the following completeness theorem.

Theorem 3.36 (Completeness Theorem)**.** *For $\alpha \in \mathcal{L}(\triangle)$, the following hold.*

- *If $\mathsf{M_t} \models \alpha$, then $\vdash_{\mathrm{EMN4^0}} \alpha$.*
- *If $\mathsf{M_{rt}} \models \alpha$, then $\vdash_{\mathrm{EMNT4}} \alpha$.*

Theorems 3.32 and 3.36 give us the following.

Theorem 3.37.

- $\mathrm{EMN4^0} = \Lambda_{\mathrm{t}}(\triangle)$.
- $\mathrm{EMNT4} = \Lambda_{\mathrm{rt}}(\triangle)$.

Proof of Theorem 3.37 is very similar to the proof of Theorem 3.31 and we omit the same. From Theorem 3.37, it follows that the logics for weak lower and strong upper approximations based on the MSASs belonging to the classes $\mathsf{M_t}$ and $\mathsf{M_{rt}}$ are respectively the monotonic logic EMN4^0 and EMNT4.

3.5.1.3. *Axiomatization relative to the classes involving symmetric relations.* Let $\models^*$ denote the standard satisfiability relation of the basic modal logic based on Kripke frames for the language $\mathcal{L}(\triangle)$ (or, $\mathcal{L}(\Box)$), treating $\triangle$ (respectively, $\Box$) as the 'necessity' modal operator. Then one can easily verify the following result.

Proposition 3.38. *Consider a model $\mathfrak{M} := (\mathfrak{F}, m)$ based on a Kripke frame $\mathfrak{F} := (W, R)$. Then the following hold for each $w \in W$.*

- *For each $\alpha \in \mathcal{L}(\triangle)$, $\mathfrak{M}, w \models^* \alpha$ if and only if $\mathfrak{M}, w \models \alpha$.*
- *For each $\alpha \in \mathcal{L}(\Box)$, $\mathfrak{M}, w \models^* \alpha$ if and only if $\mathfrak{M}, w \models \alpha$.*

We make use of Proposition 3.38 to get the following.

Proposition 3.39. *For $\alpha \in \mathcal{L}(\triangle)$, the following hold.*

1. *If $\mathsf{M_s} \models \alpha$, then $\vdash_{\mathrm{B}} \alpha$.*
2. *If $\mathsf{M_{rs}} \models \alpha$, then $\vdash_{\mathrm{KTB}} \alpha$.*
3. *If $\mathsf{M_{st}} \models \alpha$, then $\vdash_{\mathrm{KB4}} \alpha$.*
4. *If $\mathsf{M_e} \models \alpha$, then $\vdash_{\mathrm{S5}} \alpha$.*

Proof. We prove Item 1. Similar argument works for rest of the items.
Let $\mathsf{M_s} \models \alpha$, and we prove $\vdash_{\mathrm{B}} \alpha$. We claim that relative to the satisfiability relation $\models^*$ given above, α is valid in the class of all symmetric Kripke frames. If not, then there exists a model $\mathfrak{M} := (\mathfrak{F}, m)$ based on a symmetric Kripke frame $\mathfrak{F} := (W, R)$ and a $w \in W$ such that $\mathfrak{M}, w \models^* \neg\alpha$. But then, from Proposition 3.38, we obtain $\mathfrak{M}, w \models \neg\alpha$. But this contradicts that $\mathsf{M_s} \models \alpha$ as $\mathfrak{F} \in \mathsf{M_s}$. Thus, we have shown that α is valid in the class of all symmetric Kripke frames. This gives $\vdash_{\mathrm{B}} \alpha$, using completeness of the modal system B. □

We also note the following fact.

Proposition 3.40. *The axioms $C(\triangle)$ and $B(\triangle)$ are not valid in the class $\mathsf{M_e}$.*

Proof. Consider the models $\mathfrak{M} := (\mathfrak{F}, m)$ and $\mathfrak{M}' := (\mathfrak{F}', m')$ based on the MSASs $\mathfrak{F} := (\{v, w, u\}, \{R_1, R_2\})$ and $\mathfrak{F}' := (\{v, w, u\}, \{R_1', R_2'\})$ given by Figure 3.5 and 3.6, respectively. Note that $\mathfrak{M}, w \not\models \triangle p \wedge \triangle q \to \triangle(p \wedge q)$, and $\mathfrak{M}', w \not\models p \to \triangle \bigtriangledown p$. □

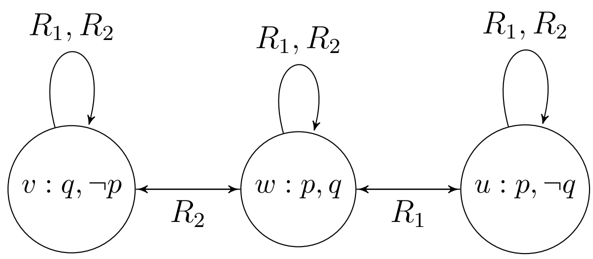


Figure 3.5

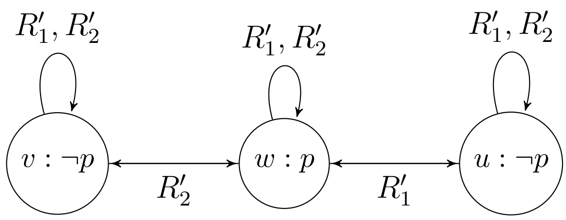


Figure 3.6

Finally, we obtain the following main result of the section.

Theorem 3.41.

1. EMN $\subseteq \Lambda_{\mathrm{s}}(\triangle) \subsetneq$ B.
2. EMNT $\subseteq \Lambda_{\mathrm{rs}}(\triangle) \subsetneq$ KTB.
3. EMN4^0 $\subseteq \Lambda_{\mathrm{st}}(\triangle) \subsetneq$ KB4.
4. EMNT4 $\subseteq \Lambda_{\mathrm{e}}(\triangle) \subsetneq$ S5.

Proof. We demonstrate the proof of Item 4. Rest of the items can be proved in the same way.

Let us first prove EMNT4 $\subseteq \Lambda_{\mathrm{e}}(\triangle)$. Let $\alpha \in$ EMNT4, that is, $\vdash_{\mathrm{EMNT4}} \alpha$. Then there exists a $\beta \in \mathcal{L}(\triangle)$ such that $\beta^\star$ is α. Since $\vdash_{\mathrm{EMNT4}} \alpha$, from Proposition 3.24, we obtain $\vdash_{\mathrm{EMNT4}} \beta$. Then, due to Theorem 3.32, we get $\mathsf{M}_{\mathsf{rt}} \models \beta$, and hence $\mathsf{M}_{\mathsf{e}} \models \beta$. This implies that $\beta^\star$, that is, α belongs to $\Lambda_{\mathrm{e}}(\triangle)$.

Next, we prove $\Lambda_{\mathrm{e}}(\triangle) \subseteq$ S5. Let $\alpha \in \Lambda_{\mathrm{e}}(\triangle)$. Then there exists a $\beta \in \mathcal{L}(\triangle)$ such that $\beta^\star$ is α and $\mathsf{M}_{\mathsf{e}} \models \beta$. Then, from Proposition 3.39, we obtain $\vdash_{\mathrm{S5}} \beta$ in the language $\mathcal{L}(\triangle)$.

Then, due to Proposition 3.24, we have $\vdash_{S5} \beta^\star$ in the language $\mathcal{L}(L)$, and hence $\alpha \in$ S5. To see S5 $\not\subseteq \Lambda_e(\triangle)$, note that $p \to LMp \in$ S5, M being dual of L, but $\alpha \notin \Lambda_e(\triangle)$ because $p \to \triangle \bigtriangledown p$ is not valid in M_e as shown in Proposition 3.40. $\square$

Thus, it follows that the logic for weak lower and strong upper approximations based on MSASs lies between the monotonic logic EMNT4 and normal modal system S5. However, the exact axiomatization of this logic remains a question.

3.5.2. Axiomatization of the fragment $\mathcal{L}(\Box)$

We now move to the fragment $\mathcal{L}(\Box)$ and discuss its axiomatization with respect to different classes of generalized MSASs.

Let $\mathcal{K}$ be the class of all Kripke frames. Consider the mapping $\Psi : \mathsf{M} \to \mathcal{K}$, which maps a generalized MSAS $(W, \{R_i\}_{i\in N})$ to Kripke frame (W, R), where $R := \{(w, u) : u \in \cup_{i\in N} R_i(w)\}$. It is not difficult to obtain the following.

Proposition 3.42. *Let $\mathfrak{F} := (W, \{R_i\}_{i\in N})$ be a generalized MSAS. Then the following hold.*

- *If $\mathfrak{F} \in \mathsf{M}_\mathsf{r}$, then $\Psi(\mathfrak{F})$ is a reflexive Kripke frame.*
- *If $\mathfrak{F} \in \mathsf{M}_\mathsf{s}$, then $\Psi(\mathfrak{F})$ is a symmetric Kripke frame.*

By induction on the complexity of the wff α, one can prove the following. Recall the satisfiability relation $\models^*$ described in Section 3.5.1.3.

Proposition 3.43. *Let us consider a model $\mathfrak{M} := (\mathfrak{F}, m)$ based on a generalized MSAS $\mathfrak{F}$. Then for all wffs $\alpha \in \mathcal{L}(\Box)$, and w,*

$$\mathfrak{M}, w \models \alpha \text{ if and only if } (\Psi(\mathfrak{F}), m), w \models^* \alpha.$$

Now, we have the required results to give the following soundness and completeness theorem.

Theorem 3.44 (Soundness and Completeness)**.** *For $\alpha \in \mathcal{L}(\Box)$, the following hold.*

1. $\mathsf{M} \models \alpha$ *if and only if* $\vdash_{\mathrm{K}} \alpha$.
2. $\mathsf{M}_\mathsf{r} \models \alpha$ *if and only if* $\vdash_{\mathrm{T}} \alpha$.
3. $\mathsf{M}_\mathsf{s} \models \alpha$ *if and only if* $\vdash_{\mathrm{B}} \alpha$.
4. $\mathsf{M}_\mathsf{rs} \models \alpha$ *if and only if* $\vdash_{\mathrm{KTB}} \alpha$.

Proof. We sketch the proof of Item 4. Rest can be proved in the same way.

The direction $\Longleftarrow$ can easily be proved by showing that the axioms of modal system KTB are valid and that the rules of inferences (MP) and RM($\Box$) preserve validity on the class M_{rs}. To prove the direction $\Longrightarrow$, we assume $\mathsf{M}_{\mathsf{rs}} \models \alpha$. We claim that α is valid in the class of all reflexive and symmetric Kripke frames. If not, then there exists a model $\mathfrak{M} := (\mathfrak{F}, m)$ based on a reflexive and symmetric Kripke frame $\mathfrak{F} := (W, R)$ and a $w \in W$ such that $\mathfrak{M}, w \models^* \neg\alpha$. Since $\Psi(\mathfrak{F}) = \mathfrak{F}$, from Proposition 3.43, we obtain $\mathfrak{M}, w \models \neg\alpha$. This contradicts $\mathsf{M}_{\mathsf{rs}} \models \alpha$. Thus, we have proved the claim. Now, as α is valid in the class of all reflexive and symmetric Kripke frames, we obtain $\vdash_{\mathrm{KTB}} \alpha$ from the completeness theorem of modal system KTB. This completes the proof. $\Box$

Following steps of the above proof, we obtain $\vdash_{\mathrm{K4}} \alpha$ from $\mathsf{M}_{\mathsf{t}} \models \alpha$, for all $\alpha \in \mathcal{L}(\Box)$. But, we do not have reverse direction as axiom (4) is not valid in the class M_{t}. The following proposition proves the same.

Proposition 3.45. *The axiom (4) is not valid in the class* M_{t}.

Proof. Consider the model $\mathfrak{M} := (\mathfrak{F}, m)$ based on the generalized MSAS $\mathfrak{F}$ given by Figure 3.7. Note that $\mathfrak{F} \in \mathsf{M}_{\mathsf{t}}$. But, $\mathfrak{M}, w \not\models \Box p \to \Box\Box p$. $\Box$

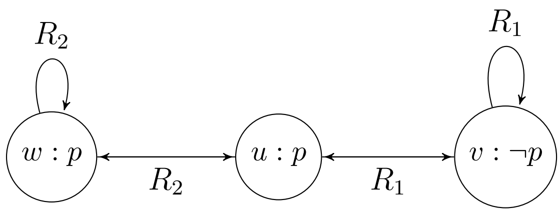


Figure 3.7

Therefore, to obtain a sound and complete modal system for the classes contained in the class M_{t}, we make use of the following theorem.

Proposition 3.46.

1. *Given a finite Kripke frame $\mathcal{F}$, there exists a $\mathfrak{F} \in \mathsf{M}_{\mathsf{t}}$ such that $\Psi(\mathfrak{F}) = \mathcal{F}$.*
2. *Given a finite Kripke frame $\mathcal{F} := (W, R)$ based on a reflexive relation R, there exists a $\mathfrak{F} \in \mathsf{M}_{\mathsf{rt}}$ such that $\Psi(\mathfrak{F}) = \mathcal{F}$.*
3. *Given a finite Kripke frame $\mathcal{F} := (W, R)$ based on a tolerance (i.e. reflexive and symmetric) relation R, there exists a $\mathfrak{F} \in \mathsf{M}_{\mathsf{e}}$ such that $\Psi(\mathfrak{F}) = \mathcal{F}$.*

Proof. Let us first prove Item 1. For each $x, y \in W$ such that $(x, y) \in R$, consider the singleton set $A_{xy} := \{(x, y)\} \subseteq W \times W$. Let $A_1, A_2, ..., A_n$ be an enumeration of all such A_{xy}. Let $N = \{1, 2, \ldots, n\}$. Consider the generalized MSAS $\mathfrak{F} := (W, \{R_i\}_{i \in N})$ where, for each $x \in W$ and $k \in N$,

$$R_k(x) := \{y : (x, y) \in A_k\}. \tag{3.14}$$

Observe that $R = \{(w, u) : u \in \cup_{i \in N} R_i(w)\}$, and hence $\Psi(\mathfrak{F}) = \mathcal{F}$. Also note that $\mathfrak{F} \in \mathsf{M_t}$.

Items 2 and 3 can be proved in the same way. We just need to modify R_k defined in (3.14). For Items 2 and 3, we need to take R_k given by (3.15) and (3.16), respectively, where

$$R_k(x) := \{y : (x, y) \in A_k\} \cup \{x\}, \tag{3.15}$$

$$R_k(x) := \{y : (x, y) \in A_k, \text{or}, (y, x) \in A_k\} \cup \{x\}. \tag{3.16}$$

□

Now, proceeding as in the proof of Theorem 3.44 and using Proposition 3.46 and the *finite model property* of modal systems K, T and KTB, we obtain the following soundness and completeness theorem.

Theorem 3.47 (Soundness and Completeness)**.** *For $\alpha \in \mathcal{L}(\Box)$, the following hold.*

- $\mathsf{M_t} \models \alpha$ *if and only if* $\vdash_{\mathrm{K}} \alpha$.
- $\mathsf{M_{rt}} \models \alpha$ *if and only if* $\vdash_{\mathrm{T}} \alpha$.
- $\mathsf{M_e} \models \alpha$ *if and only if* $\vdash_{\mathrm{KTB}} \alpha$.

By giving argument as in the proof of Theorem 3.31, and using Theorems 3.44 and 3.47, we obtain the following main result of the section.

Theorem 3.48.

- $\mathrm{K} = \Lambda_{\mathrm{t}}(\Box)$.
- $\mathrm{T} = \Lambda_{\mathrm{rt}}(\Box) = \Lambda_{\mathrm{r}}(\Box)$.
- $\mathrm{KTB} = \Lambda_{\mathrm{e}}(\Box) = \Lambda_{\mathrm{rs}}(\Box)$.
- $\mathrm{B} = \Lambda_{\mathrm{s}}(\Box)$.

As far as the class $\mathsf{M_{st}}$ of generalized MSASs is concerned, we have the following result.

Theorem 3.49. $\mathrm{B} \subseteq \Lambda_{\mathrm{st}}(\Box) \subsetneq \mathrm{KB4}$.

Proof is very much in the line of Theorem 3.41, and we omit the same. $\mathrm{KB4} \not\subseteq \Lambda_{\mathrm{st}}(\Box)$ is obtained from Proposition 3.45.

We give a summary of the results obtained in Section 3.5 through Figure 3.8. The dashed arrows and the solid arrows from A to B are used to denote $A \subseteq B$ and $A \subsetneq B$, respectively.

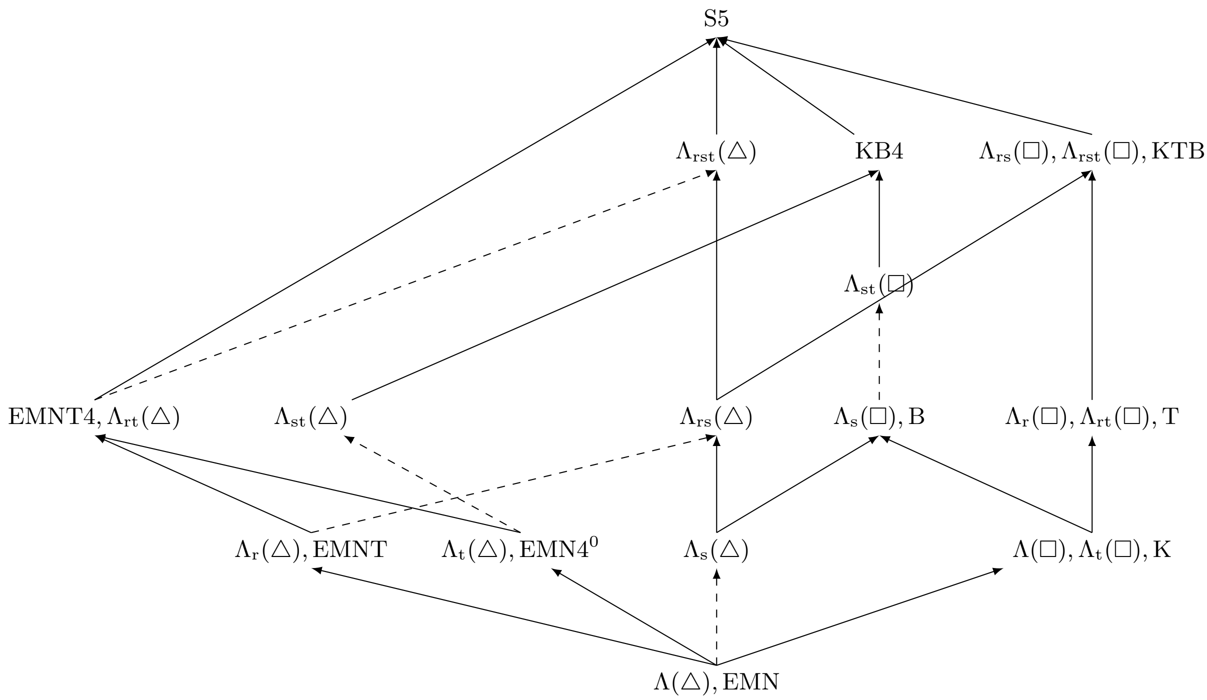


Figure 3.8. A summary of results of Section 3.5

As mentioned at the beginning of Section 3.5, it is evident from Figure 3.8 that none of the logics corresponding to the language $\mathcal{L}(\triangle)$ that appeared in this section contains axiom K. On the contrary, all the logics corresponding to $\mathcal{L}(\Box)$ contains axiom K. This is happening because the logic $\Lambda(\Box)$, which is minimal among all the logics corresponding to the language $\mathcal{L}(\Box)$ that appeared in Figure 3.8, coincides with the normal modal system K. Consequently, it also follows that, as far as the notion of validity is concerned, the proposed semantics of the modal operator $\Box$ can be equivalently captured through the standard 'necessity operator' semantics based on Kripke frame. Such a result is not possible for the operator $\triangle$ as it does not satisfy axiom K under the proposed semantics.

3.6. Coalgebraic perspective

This section will illustrate that our study of generalized MSASs can be put under the framework of coalgebras. We will first show that the category consisting of generalized MSASs as objects and bounded morphisms between generalized MSASs as arrows is isomorphic to a category based on a suitable class of coalgebras. Then we will present an equivalent coalgebraic semantics of the language $\mathcal{L}(\Box, \triangle)$ using the notion of *predicate lifting* [72]. To make the chapter self-contained, we briefly present relevant definitions and concepts related to coalgebra, but we refer to [31] for a very accessible introduction to coalgebra. For a comprehensive survey on the topic, readers may consult [76, 88, 101]. The connections between coalgebra and modal logic are well studied in [50, 72].

Let us recall that a *coalgebra* (cf. e.g. [101]) is a state based system consisting of a set A of states endowed with some kind of transition, formally modelled as some map σ from A to another set TA. Here T is some set functor[1] constituting the type or signature of the coalgebra at stake. The transition map σ provides some kind of structure on A. Thus, more formally, given a set functor T, a T-*coalgebra* is a tuple (A, σ), where A is a set and $\sigma : A \to TA$ is a function.

Note that in a generalized MSAS $(W, \{R_i\}_{i\in N})$, each $i \in N$ represents an accessibility relation on W, namely R_i. Let us consider the function σ taking a state/object w and an $i \in N$ (representing the relation R_i) and returning the set of states/objects accessible from w through R_i. Formally, we have $\sigma : W \times N \to \wp(W)$ such that for each $w \in W$ and $i \in N$,

$$\sigma(w, i) := \{w' \in W : (w, w') \in R_i\}.$$

In order to expose the object space W, we may rewrite σ as $\sigma : W \to \wp(W)^N$. Thus $\sigma(w)(i)$ stands for the set of states/objects accessible to w by R_i. Observe that $\sigma(w)$ is a function from N to $\wp(W)$. Thus, the generalized MSAS $(W, \{R_i\}_{i\in N})$ with cardinality N generates a Ω_N-coalgebra (W, σ), where Ω_N is the set functor that maps

- a set X to $\wp(X)^N$, the set of all functions from N to $\wp(X)$, and

[1]A set functor is an operation T which maps every set X to a set TX, and maps every function $f : X \to Y$ to a function $Tf : TX \to TY$ such that $T(id_X) = id_{TX}$ for the identity function id_X on X and $T(g \circ f) = Tg \circ Tf$ whenever g and f are two composable functions.

- a function $f : X \to Y$ to the function $\Omega_N f : \wp(X)^N \to \wp(Y)^N$, where, for $\phi \in \wp(X)^N$, $\Omega_N f(\phi) : N \to \wp(Y)$ is defined as

$$\Omega_N f(\phi)(i) := \{f(x) : x \in \phi(i)\},\ i \in N.$$

It is trivial to note that the standard properties of binary relations (reflexivity, symmetry, transitivity) in a generalized MSAS $(W, \{R_i\}_{i \in N})$ inherits the following properties in the generated Ω_N-coalgebra (W, σ). For each $i \in N$,

- R_i is reflexive if and only if for each $w \in W$, $w \in \sigma(w)(i)$;
- R_i is symmetric if and only if for each $w, u \in W$, $u \in \sigma(w)(i)$ implies $w \in \sigma(u)(i)$;
- R_i is transitive if and only if for each $u, v, w \in W$, $u \in \sigma(w)(i)$ and $v \in \sigma(u)(i)$ imply $v \in \sigma(w)(i)$.

Accordingly, we have the following definition.

Definition 3.50. A Ω_N-coalgebra $\mathbb{S} := (S, \sigma)$ is said to be

- *reflexive* if $w \in \sigma(w)(i)$;
- *symmetric* if $u \in \sigma(w)(i)$ implies $w \in \sigma(u)(i)$;
- *transitive* if $u \in \sigma(w)(i)$ and $v \in \sigma(u)(i)$ imply $v \in \sigma(w)(i)$.

Class of coalgebras	Defining condition	Class of coalgebras	Defining condition
Ω	Class of all coalgebras	Ω_{rs}	$\Omega_r \cap \Omega_s$
Ω_r	Class of reflexive coalgebras	Ω_{rt}	$\Omega_r \cap \Omega_t$
Ω_t	Class of transitive coalgebras	Ω_{st}	$\Omega_s \cap \Omega_t$
Ω_s	Class of symmetric coalgebras	Ω_e	$\Omega_{rs} \cap \Omega_t$

Table 3.3. Classes of coalgebras

Table 3.3 gives different classes of Ω_N-coalgebras that are of interest to us.

Definition 3.51. A function $f : S_1 \to S_2$ is a *homomorphism* from Ω_N-coalgebra $\mathbb{S}_1$ to Ω_N-coalgebra $\mathbb{S}_2$ if $\sigma_2 \circ f = \Omega_N f \circ \sigma_1$. That is, the diagram in Figure 3.9 commutes.

Let $\mathsf{C}(\Omega)$ be the category with objects from the class Ω (cf. Table 3.3) and coalgebraic homomorphisms between the elements of Ω as arrows. Further, let $\mathsf{C}(\mathsf{M})$ be the category

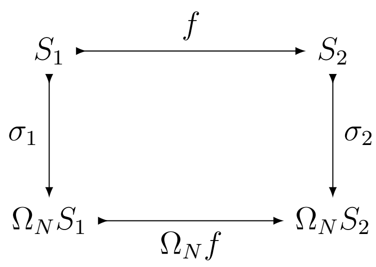


Figure 3.9. Homomorphism

consisting of generalized MSASs as objects and bounded morphisms between generalized MSASs as arrows. Similarly, we define the subcategories corresponding to different classes of generalized MSASs and Ω_N-coalgebras given in Table 3.1 and 3.3. As earlier, indexing with the letters r, s, t, e will be used to denote these categories. For instance, $\mathsf{C}(\mathsf{M_r})$ and $\mathsf{C}(\Omega_\mathsf{r})$ are the categories corresponding to the classes $\mathsf{M_r}$ and Ω_r, respectively.

Definition 3.52. We define functors $\mathcal{H} : \mathsf{C}(\mathsf{M}) \Rightarrow \mathsf{C}(\Omega)$, and $\mathcal{G} : \mathsf{C}(\Omega) \Rightarrow \mathsf{C}(\mathsf{M})$ as follows:

- For a generalized MSAS $\mathfrak{F} := (W, \{R_i\}_{i \in N})$, $\mathcal{H}(\mathfrak{F})$ is the Ω_N-coalgebra (W, σ), where, for $w \in W, i \in N$, $\sigma(w)(i) := \{u \in W : (w, u) \in R_i\}$.
- For a bounded morphism $f : \mathfrak{F} \to \mathfrak{F}'$, $\mathcal{H}(f) = f$.
- For a Ω_N-coalgebra $\mathbb{S} := (W, \sigma)$, $\mathcal{G}(\mathbb{S})$ is the generalized MSAS $(W, \{R_i\}_{i \in N})$, where, $R_i := \{(w, u) : u \in \sigma(w)(i)\}$.
- For a homomorphism $f : \mathbb{S} \to \mathbb{S}'$, $\mathcal{G}(f) = f$.

Note that the functors $\mathcal{H}$ and $\mathcal{G}$ are well defined as (i) a bounded morphism $f : \mathfrak{F} \to \mathfrak{F}'$ from the generalized MSAS $\mathfrak{F}$ to the generalized MSAS $\mathfrak{F}'$ is also a homomorphism from the coalgebra $\mathcal{H}(\mathfrak{F})$ to $\mathcal{H}(\mathfrak{F}')$, and (ii) a homorphism $f : \mathbb{S} \to \mathbb{S}'$ is also a bounded morphism from $\mathcal{G}(\mathbb{S})$ to $\mathcal{G}(\mathbb{S}')$. The following proposition confirms these facts.

Proposition 3.53. *Let $\mathbb{S}_1 = (S_1, \sigma_1)$, and $\mathbb{S}_2 = (S_2, \sigma_2)$ be two Ω_N-coalgebras. Then $f : S_1 \to S_2$ is an homomorphism if and only if the following hold:*

1. *If $v \in \sigma_1(w)(i)$, then $f(v) \in \sigma_2(f(w))(i)$ (the forth condition).*
2. *If $v' \in \sigma_2(f(w))(i)$, then there exists a v with $f(v) = v'$, and $v \in \sigma_1(w)(i)$ (the back condition).*

Proof. Note that for $w \in S_1$, and $i \in N$, $\Omega_N f(\sigma_1(w))(i) := \{f(x) : x \in \sigma_1(w)(i)\}$.
($\Rightarrow$): Let $f : S_1 \to S_2$ be a homomorphism. Then,

$$\begin{aligned} v \in \sigma_1(w)(i) \quad &\implies f(v) \in \Omega_N f(\sigma_1(w))(i) \\ &\implies f(v) \in \sigma_2(f(w))(i) \ (\because \Omega_N f \circ \sigma_1 = \sigma_2 \circ f), \end{aligned}$$

and,

$$\begin{aligned} v' \in \sigma_2(f(w))(i) \quad &\implies v' \in \Omega_N f(\sigma_1(w))(i) \\ &\implies v' = f(v) \text{ for some } v \in \sigma_1(w)(i). \end{aligned}$$

($\Leftarrow$):

$$\begin{aligned} v \in \Omega_N f(\sigma_1(w))(i) \iff & v \in \{f(x) : x \in \sigma_1(w)(i)\} \\ \iff & v = f(u) \text{ for some } u \in \sigma_1(w)(i) \\ \iff & v \in \sigma_2(f(w))(i) \text{ (using given conditions (1), and (2))} \end{aligned}$$

This shows that $\Omega_N f \circ \sigma_1 = \sigma_2 \circ f$, hence f is a homomorphism. □

For two categories $\mathbb{A}$ and $\mathbb{B}$, we write $\mathbb{A} \cong \mathbb{B}$ to denote that $\mathbb{A}$ and $\mathbb{B}$ are isomorphic.

Proposition 3.54.

$$\begin{array}{cccc} \mathsf{C}(\mathsf{M}) \cong \mathsf{C}(\Omega) & \mathsf{C}(\mathsf{M_s}) \cong \mathsf{C}(\Omega_\mathsf{s}) & \mathsf{C}(\mathsf{M_t}) \cong \mathsf{C}(\Omega_\mathsf{t}) & \mathsf{C}(\mathsf{M_r}) \cong \mathsf{C}(\Omega_\mathsf{r}) \\ \mathsf{C}(\mathsf{M_{rs}}) \cong \mathsf{C}(\Omega_\mathsf{rs}) & \mathsf{C}(\mathsf{M_{rt}}) \cong \mathsf{C}(\Omega_\mathsf{rt}) & \mathsf{C}(\mathsf{M_{st}}) \cong \mathsf{C}(\Omega_\mathsf{st}) & \mathsf{C}(\mathsf{M_e}) \cong \mathsf{C}(\Omega_\mathsf{e}). \end{array}$$

Proof. It is not difficult to see that $\mathcal{H} \circ \mathcal{G} = I_{\mathsf{C}(\Omega)}$ and $\mathcal{G} \circ \mathcal{H} = I_{\mathsf{C}(\mathsf{M})}$, where $I_{\mathsf{C}(\Omega)}$ and $I_{\mathsf{C}(\mathsf{M})}$ are identity functors on $\mathsf{C}(\Omega)$ and $\mathsf{C}(\mathsf{M})$, respectively. Thus we obtained $\mathsf{C}(\mathsf{M}) \cong \mathsf{C}(\Omega)$. Similarly, considering the restrictions of $\mathcal{H}$ and $\mathcal{G}$ to respective subcategories, we obtain other isomorphisms. □

Proposition 3.54 gives the identifications of different classes of generalized MSASs with the classes of coalgebras. Once we have this correspondence, we can equivalently view models of the language $\mathcal{L}(\Box, \triangle)$ to be of the form $\mathfrak{M} := (W, \sigma, m)$, where (W, σ) is a Ω_N-coalgebra for some N, and $m : PV \to \wp(W)$. Next, we show that an equivalent coalgebraic semantics of the language $\mathcal{L}(\Box, \triangle)$ can be given using the notion of *predicate lifting* [72].

Recall that a (unary) predicate on a set X is an element of the set $\wp(X)$. What a predicate lifting for a set functor T does is that it takes (lifts) predicates to the realm of T. Formally, a predicate lifting is a set-indexed family of functions $\{\lambda_X\}_{X \in Set}$ of type $\lambda_X : \wp(X) \to \wp(TX)$ such that Figure 3.10 commutes for all functions $f : X \to Y$. Note that for a function $f : X \to Y$, f^{-1} is a function from $\wp(Y)$ to $\wp(X)$ given by $f^{-1}(A) := \{x \in X : f(x) \in A\}$, $A \subseteq Y$.

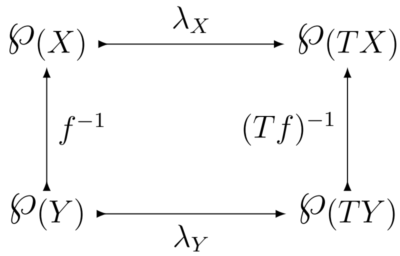


Figure 3.10. Predicate lifting

Observe that a predicate lifting for the functor Ω_N is obtained as a set-indexed family of functions $\{\lambda_X\}_{X \in Set}$ of type $\lambda_X : \wp(X) \to \wp(\wp(X)^N)$. Since the truth set $[\![\alpha]\!]$ of a wff α is a specific example of a predicate, given a predicate lifting $\{\lambda_X\}_{X \in Set}$ for Ω_N, one can define satisfiability of the modal operator $\Box$ (or, $\triangle$) in a model (W, σ, m) based on a Ω_N-coalgebra (W, σ) as follows.

$$\mathfrak{M}, w \models \Box\alpha \iff \sigma(w) \in \lambda_W([\![\alpha]\!]_{\mathfrak{M}}) \tag{3.17}$$

Therefore, we would like to find suitable predicate liftings $\{[\Box_N]_X\}_{X \in Set}$ and $\{[\triangle_N]_X\}_{X \in Set}$ for the functor Ω_N such that the corresponding satisfiability conditions for the operators $\Box$ and $\triangle$ obtained using (3.17) become equivalent to the satisfiability conditions given by Definition 3.4. For this, we require the following for each Ω_N-coalgebra (W, σ):

$$\sigma(w) \in [\Box_N]_W([\![\alpha]\!]_{\mathfrak{M}}) \iff \cup_{i \in N}\sigma(w)(i) \subseteq [\![\alpha]\!]_{\mathfrak{M}} \tag{3.18}$$

$$\sigma(w) \in [\triangle_N]_W([\![\alpha]\!]_{\mathfrak{M}}) \iff \sigma(w)(i) \subseteq [\![\alpha]\!]_{\mathfrak{M}} \text{ for some } i \in N. \tag{3.19}$$

Thus, it follows that the predicate liftings $\{[\Box_N]_X\}_{X \in Set}$, and $\{[\triangle_N]_X\}_{X \in Set}$ for Ω_N defined by

$$[\Box_N]_X(A) := \{f \in \wp(X)^N : \cup_{i \in N} f(i) \subseteq A\}, \text{ and}$$

$$[\triangle_N]_X(A) := \{f \in \wp(X)^N : f(i) \subseteq A \text{ for some } i\},$$

where $A \subseteq X$, serve the purpose. In fact, as a direct consequence of (3.18) and (3.19), we obtain the following proposition giving us the required equivalence.

Proposition 3.55. *Let $\mathfrak{M} := (W, \sigma, m)$ be a model based on a Ω_N-coalgebra (W, σ). Then the following hold.*

$$\mathfrak{M}, w \models \Box\alpha \iff \sigma(w) \in [\Box_N]_W(\llbracket\alpha\rrbracket_{\mathfrak{M}}).$$

$$\mathfrak{M}, w \models \triangle\alpha \iff \sigma(w) \in [\triangle_N]_W(\llbracket\alpha\rrbracket_{\mathfrak{M}}).$$

3.7. Conclusion

In this chapter, we considered a generalized notion of multiple-source approximation system (MSAS), called *generalized MSAS*, that contains countable collections $\{R_i\}_{i\in N}$ of relations over the same domain, and where the accessibility relations R_i are of the same type, and may be any binary relation, or have any of the properties of reflexivity, symmetry, transitivity or some combination thereof. The notions of strong/weak lower and upper approximations based on MSASs are extended readily to define these ideas for the generalized MSASs. A logic that can be used to reason about these strong/weak approximations of concepts is obtained. A few invariance results are presented. The axiomatization problem of the proposed logic with respect to different classes of generalized MSASs based on various types of relations are explored. This study gives some insight into the nature of strong/weak approximations. For instance, from Theorem 3.48, one can conclude that the same set of properties involving set-theoretic operations of complementation, union, and intersection hold true for both the strong lower approximation based on MSASs and the lower approximation based on tolerance approximation spaces. At the end, it is also shown that our study of generalized MSASs can be put under the framework of coalgebras.

CHAPTER 4

MODAL SYSTEMS FOR COVERING SEMANTICS AND BOUNDARY OPERATOR

Covering-based rough sets are well studied in the literature [89–91, 106]. For instance, in [91], seventeen covering systems are arranged in a pattern. Many of these systems have been motivated from the angle of applications, but not all. In this chapter, we aim to establish the connections between covering systems and modal systems. For details on the original motivations of the covering systems, we refer to [53, 60, 78, 81, 94, 114]. It is also mentioned that a covering system is a covering along with the lower and upper approximation operators defined in the power set of the universe in a particular way. We have considered here the following covering systems, $P_1, P_3, C_1, C_2, C_4, C_5$, and C_{Gr}. Modal systems corresponding to the covering systems P_1, C_2, C_4, and C_5 are presented in [61], but the investigation of modal logics for covering systems P_3, C_1, and C_{Gr} remain open. In this chapter, we investigate modal system for these covering systems. The lower and upper approximations of a set A will be the interpretations of formulas $\Box\alpha$ and $\Diamond\alpha$ respectively, in the modal systems corresponding to covering systems where A is the interpretation of α [vide Definition 4.2]. We also study the modal systems for boundary operator based on generalized approximation spaces and covering systems. Our study also leads to covering semantics for contingency logic and provide its connection with rough set theory. Further, an alternative modal system consisting of contingency modal operator is provided corresponding to covering system P_4.

This chapter has been arranged as follows. Sections 4.1, 4.2 deal with the modal logics developed earlier [61] corresponding to covering systems C_2, C_4, C_5, P_1, and present a modal system corresponding to C_1, P_3, and C_{Gr}. Sections 4.3 and 4.4 present covering semantics to modal logics $\mathbb{CLS}4$ and $\mathbb{CLB}$ [21, 22]. Section 4.5 gives a non-standard modal system $\mathbb{CLS}4B$ based on modalities $\triangle$ and $\triangledown$ for the covering system P_4. Thus, the binding chord of this study is the interplay between various covering systems and their corresponding modal logics. Section 4.6 concludes the chapter.

This is to emphasise that we do not ascribe any preference of covering semantics over others; it is only an alternative which may prove to be handy in certain context. From the rough set angle, however, observing links with modal systems seems to be gratifying.

The work presented in this chapter is based on the article [71].

4.1. Modal logics for covering based rough set models

4.1.1. Syntax

We consider the basic modal language $\mathcal{L}(\Box)$ with a unary modal connective $\Box$. That is, the alphabet of the language $\mathcal{L}(\Box)$ consists of a non-empty countable set PV of propositional variables, a propositional constant $\top$, Boolean connectives $\neg$ (negation) and $\vee$ (disjunction), and the modal connective $\Box$. The wffs of $\mathcal{L}(\Box)$ are defined recursively as:

$$\top \mid p \mid \neg\alpha \mid \alpha \wedge \beta \mid \Box\alpha,$$

where $p \in PV$ and α, β are wffs. Apart from the usual derived connectives $\bot, \vee, \rightarrow, \leftrightarrow$, we have the connective $\Diamond$ defined as follows:

$$\Diamond\alpha := \neg\Box\neg\alpha.$$

The set of all wffs of the language $\mathcal{L}(\Box)$ will be denoted by the same symbol $\mathcal{L}(\Box)$.

4.1.2. Semantics

The semantics of the language $\mathcal{L}(\Box)$ is based on the following notion of model. Let us recall the notion of covering and covering systems based on it from Chapter 2.

Definition 4.1 (Covering Model)**.** A *covering model* is defined as a tuple $\mathfrak{M} := (W, \mathcal{C}, m)$, where $(W, \mathcal{C})$ is a covering space and $m : PV \rightarrow \wp(W)$ is a valuation function.

Definition 4.2 (Truth Set)**.** Let $\mathfrak{M} := (W, \mathcal{C}, m)$ be a covering model and $\sigma \in \{C_1, C_2, C_4, C_5, P_1, P_3, C_{\mathrm{Gr}}\}$. The *truth set* of a wff α in a covering model $\mathfrak{M} :=$

$(W, \mathcal{C}, m)$ under σ semantics, denoted as $[\![\alpha]\!]^{\sigma}_{\mathfrak{M}}$, is defined inductively as follows:

$$[\![\top]\!]^{\sigma}_{\mathfrak{M}} = W.$$

$$[\![p]\!]^{\sigma}_{\mathfrak{M}} = m(p), p \in PV.$$

$$[\![\neg\alpha]\!]^{\sigma}_{\mathfrak{M}} = ([\![\alpha]\!]^{\sigma}_{\mathfrak{M}})^c.$$

$$[\![\alpha \wedge \beta]\!]^{\sigma}_{\mathfrak{M}} = [\![\alpha]\!]^{\sigma}_{\mathfrak{M}} \cap [\![\beta]\!]^{\sigma}_{\mathfrak{M}}.$$

$$[\![\Box\alpha]\!]^{\sigma}_{\mathfrak{M}} = \underline{\sigma}([\![\alpha]\!]^{\sigma}_{\mathfrak{M}}) \quad \text{(cf. Section 2.2.2, page 12,13).}$$

It is not difficult to verify that $[\![\Diamond\alpha]\!]^{\sigma}_{\mathfrak{M}} = \overline{\sigma}([\![\alpha]\!]^{\sigma}_{\mathfrak{M}})$, where $\overline{\sigma}$ is the dual of the operator $\underline{\sigma}$, that is, $\overline{\sigma}(A) := (\underline{\sigma}(A^c))^c$ for all $A \subseteq W$.
Following the standard notation of modal logic, we will write $\mathfrak{M}, x \models_{\sigma} \alpha$ to mean $x \in [\![\alpha]\!]^{\sigma}_{\mathfrak{M}}$.
A wff α is said to be *valid* in a covering model $\mathfrak{M} := (W, \mathcal{C}, m)$ under the σ semantics, denoted by $\mathfrak{M} \models_{\sigma} \alpha$, if $[\![\alpha]\!]^{\sigma}_{\mathfrak{M}} = W$. α is said to be *valid* under the σ semantics, denoted as $\models_{\sigma} \alpha$, if $\mathfrak{M} \models_{\sigma} \alpha$ for all covering model $\mathfrak{M}$.

Remark 4.3. Covering models may remind one of the subset space logic (cf. e.g. [1, 13]) where the model consists of a triple (W, ρ, m), ρ being a non-empty collection of subsets of W and m being a valuation from propositional variables to the power set $\wp(W)$ of S. In the present case, ρ is a covering of W. Also the motivation behind subset logic is completely different. The syntax of the logic is apparently different as there are two modalities, one intended to quantify *over* the sets, whereas the other is intended to quantify *in* the sets. However, a comparative study of covering rough set models and subset space models may be an interesting area of investigation.

4.1.3. Hilbert-style modal systems

Let us recall the well-known axioms K($\Box$), T($\Box$), B($\Box$), 4($\Box$), and inference rules MP and Nec($\Box$) of modal logic (cf. Chapter 3). Table 4.1 gives a few standard modal systems that are of interest to us. For a modal system Λ, as earlier, the notation $\vdash_{\Lambda} \alpha$ will be used to mean that α is a theorem of Λ. The following soundness and completeness theorem is proved in [61].

Theorem 4.4. *For a given wff α, the following hold.*

Modal systems	Axioms and Inference Rules	Modal systems	Axioms and Inference Rules
K	Taut, axiom K(□), MP, Nec(□)	B	K+ axiom B(□)
T	K+ axiom T(□)	KTB	T + axiom B
S4	T+ axiom 4(□)	S5	S4+ axiom B(□)

Table 4.1. A few modal systems

- $\models_{C_2} \alpha$ *if and only if* $\vdash_{\mathrm{S4}} \alpha$.
- $\models_{C_5} \alpha$ *if and only if* $\vdash_{\mathrm{S4}} \alpha$.
- $\models_{P_1} \alpha$ *if and only if* $\vdash_{\mathrm{KTB}} \alpha$.
- $\models_{C_4} \alpha$ *if and only if* $\vdash_{\mathrm{KTB}} \alpha$.

Thus, it follows that the modal system S4 captures C_2 and C_5 semantics. Further, the modal system KTB corresponds to P_1 and C_4 semantics. As far as C_1 semantics is concerned, we are not familiar with any work determining a modal system capturing this semantics. In this chapter, this issue will be resolved and we will provide a modal system for C_1 semantics. It is worth mentioning here that, unlike the technique used in [61], the corresponding completeness theorem for C_1 semantics will be proved by constructing a canonical covering model.

4.2. Modal system for C_1 semantics

This section presents a modal system for C_1 semantics and the corresponding soundness and completeness theorems. Let $\mathfrak{M} := (W, \mathcal{C}, m)$ be a covering model, where $\mathcal{C} := \{C_i\}_{i \in I}$. Recall that we write $\mathfrak{M}, x \models_{C_1} \alpha$ to mean $x \in [\![\alpha]\!]^{C_1}_{\mathfrak{M}}$. First note that, unfolding the truth set definition, we obtain the following equivalent satisfiability condition.

Proposition 4.5. *1. $\mathfrak{M}, x \models_{C_1} \Box\alpha$ if and only if there exists a $C_i \in \mathcal{C}$ such that $x \in C_i$ and $\mathfrak{M}, y \models_{C_1} \alpha$ for all $y \in C_i$.*

2. $\mathfrak{M}, x \models_{C_1} \Diamond\alpha$ if and only if for all $C_i \in \mathcal{C}$, either $x \notin C_i$ or there exists a $y \in C_i$ such that $\mathfrak{M}, y \models_{C_1} \alpha$.

Proof. 1. Suppose $\mathfrak{M}, x \models_{C_1} \Box\alpha$. Then $x \in [\![\Box\alpha]\!]^{C_1}_{\mathfrak{M}}$, and thus $x \in \underline{C_1}([\![\alpha]\!]^{C_1}_{\mathfrak{M}})$. Using the definition of operator $\underline{C_1}$, we obtain a $C_i \in \mathcal{C}$ such that $x \in C_i$ and $C_i \subseteq [\![\alpha]\!]^{C_1}_{\mathfrak{M}}$. This guarantees the existence of a $C_i \in \mathcal{C}$ such that $x \in C_i$ and $\mathfrak{M}, y \models_{C_1} \alpha$ for all $y \in C_i$.
For the converse part, let there exist a $C_i \in \mathcal{C}$ with $x \in C_i$ and $\mathfrak{M}, y \models_{C_1} \alpha$ for all $y \in C_i$. That is, we have $y \in [\![\alpha]\!]^{C_1}_{\mathfrak{M}}$ for all $y \in C_i$ and $x \in C_i$. This gives $x \in \underline{C_1}([\![\alpha]\!]^{C_1}_{\mathfrak{M}})$ and hence $x \in [\![\Box\alpha]\!]^{C_1}_{\mathfrak{M}}$. That is, $\mathfrak{M}, x \models_{C_1} \Box\alpha$.

2. Let $\mathfrak{M}, x \models_{C_1} \Diamond\alpha$ and we show that for all $C_i \in \mathcal{C}$ either $x \notin C_i$ or there exists a $y \in C_i$ such that $\mathfrak{M}, y \models_{C_1} \alpha$. Since $\mathfrak{M}, x \models_{C_1} \Diamond\alpha$, we obtain $x \in \overline{C_1}([\![\alpha]\!]^{C_1}_{\mathfrak{M}})$. By the definition of the operator $\overline{C_1}$, either $x \notin C_i$ or $C_i \cap [\![\alpha]\!]^{C_1}_{\mathfrak{M}} \neq \emptyset$. Further, $C_i \cap [\![\alpha]\!]^{C_1}_{\mathfrak{M}} \neq \emptyset$ guarantees the existence of a $y \in C_i$ such that $\mathfrak{M}, y \models_{C_1} \alpha$. Thus, either $x \notin C_i$ or there exists a $y \in C_i$ such that $\mathfrak{M}, y \models_{C_1} \alpha$.
For the other direction, suppose that either $x \notin C_i$ or there exists a $y \in C_i$ such that $\mathfrak{M}, y \models_{C_1} \alpha$. That is, we have either $x \notin C_i$ or $y \in [\![\alpha]\!]^{C_1}_{\mathfrak{M}}$. This implies $x \in \overline{C_1}([\![\alpha]\!]^{C_1}_{\mathfrak{M}})$ and hence $x \in [\![\Diamond\alpha]\!]^{C_1}_{\mathfrak{M}}$. Thus $\mathfrak{M}, x \models_{C_1} \Diamond\alpha$. ☐

Consider the modal system $\mathrm{ML}_{\mathrm{C}_1}$ which consists of axioms (Taut), M(☐), N(☐), T(☐), 4(☐), and inference rules MP and RE(☐) given in Chapter 3. Note that $\mathrm{ML}_{\mathrm{C}_1}$ is the extension of monotonic logic EMN [67] with the axioms T(☐) and 4(☐).

One can easily verify the following soundness theorem by proving the validity of the axioms and soundness of the inference rules.

Theorem 4.6 (Soundness)**.** *For each wff α, if $\vdash_{\mathrm{ML}_{\mathrm{C}_1}} \alpha$, then $\models_{C_1} \alpha$.*

In the remaining part of the section, we provide a proof of the corresponding completeness theorem. Recall the notion of maximal consistent set [7]. The following notion of canonical covering based on maximal consistent sets is introduced which plays a key role in the proof. In this chapter, we use $\mathbb{M}$, instead of $\mathbb{M}_{\mathrm{ML}_{\mathrm{C}_1}}$, to denote the set of all $\mathrm{ML}_{\mathrm{C}_1}$-maximal consistent sets. Further, let $\alpha_1, \alpha_2, \ldots$ be an enumeration of all the wffs of the language $\mathcal{L}(\Box)$.

Definition 4.7 (Canonical Covering Model)**.** The *canonical covering model* is defined as the tuple $\mathfrak{M}^0 = (\mathbb{M}, \mathcal{C}^0, m^0)$, where

- for each $i \in \mathbb{N}$, $C_i^0 := \{\Gamma \in \mathbb{M} : \alpha_i \wedge \Box\alpha_i \in \Gamma\}$;
- $\mathcal{C}^0 := \{C_i^0\}_{i\in\mathbb{N}}$;
- $m^0(p) = \{\Gamma \in \mathbb{M} : p \in \Gamma\}$.

It may be noted that C_i^0 may be an empty set for some i. Further, we also have the following.

Lemma 4.8. $(\mathbb{M}, \mathcal{C}^0)$ *is a covering space.*

Proof. We need to show that $\bigcup_{i\in\mathbb{N}} C_i^0 = \mathbb{M}$. Obviously $\bigcup_{i\in\mathbb{N}} C_i^0 \subseteq \mathbb{M}$ holds. So, let $\Gamma \in \mathbb{M}$. Choose an α_j such that $\vdash_{\mathrm{ML_{C_1}}} \alpha_j$. Then $\alpha_j \wedge \Box\alpha_j \in \Gamma$ and hence $\Gamma \in C_j^0$. This completes the proof. □

Recall the following well-known result.

Lemma 4.9 (Lindenbaum's Lemma)**.** *Let* Γ *be a* $\mathrm{ML_{C_1}}$*-consistent set of wffs. Then there exists a* $\mathrm{ML_{C_1}}$*-maximal consistent set* Γ^+ *containing* Γ*.*

We also have the following counterpart of the standard existence lemma.

Lemma 4.10 (Existence Lemma)**.** *Let* $\Gamma \in \mathbb{M}$*. Then, we have the following.*

1. *If* $\Box\alpha \in \Gamma$*, then there exists a* $C_i^0 \in \mathcal{C}^0$ *such that* $\Gamma \in C_i^0$ *and* $\alpha \in \Gamma'$ *for all* $\Gamma' \in C_i^0$*.*

2. *If* $\Diamond\alpha \in \Gamma$*, then for all* $C_i^0 \in \mathcal{C}^0$*, either* $\Gamma \notin C_i^0$ *or there exists a* $\Gamma' \in C_i^0$ *such that* $\alpha \in \Gamma'$*.*

Proof. (1). Let $\Box\alpha \in \Gamma$ and suppose α is the wff α_j. Since $\Box\alpha_j \in \Gamma$, we get $\alpha_j \in \Gamma$ ($\because \vdash_{\mathrm{ML_{C_1}}} \Box\alpha_j \to \alpha_j$). Now the definition of C_j^0 implies $\Gamma \in C_j^0$. We also have $\alpha \in \Gamma'$ for all $\Gamma' \in C_j^0$ as α is the wff α_j.

(2). Assume $\Diamond\alpha \in \Gamma$ and consider a $C_i^0 \in \mathcal{C}^0$. If $\Gamma \notin C_i^0$, then we are done. So, let $\Gamma \in C_i^0$. By the definition of C_i^0, we obtain $\alpha_i \wedge \Box\alpha_i \in \Gamma$. Let α be the wff α_j. Consider the set $\{\alpha_j, \Box\alpha_i\}$.

Claim: $\{\alpha_j, \Box\alpha_i\}$ is a $\mathrm{ML_{C_1}}$-consistent set.

If possible, suppose $\{\alpha_j, \Box\alpha_i\}$ is not a $\mathrm{ML}_{\mathrm{C}_1}$-consistent set. Then, we must have $\vdash_{\mathrm{ML}_{\mathrm{C}_1}} \neg(\alpha_j \wedge \Box\alpha_i)$. That is, $\vdash_{\mathrm{ML}_{\mathrm{C}_1}} \neg\alpha_j \vee \neg\Box\alpha_i$, and thus,

$$
\begin{aligned}
&\vdash_{\mathrm{ML}_{\mathrm{C}_1}} \alpha_j \to \neg\Box\alpha_i \\
&\Longrightarrow \vdash_{\mathrm{ML}_{\mathrm{C}_1}} \alpha_j \to \Diamond\neg\alpha_i \\
&\Longrightarrow \vdash_{\mathrm{ML}_{\mathrm{C}_1}} \Diamond\alpha_j \to \Diamond\Diamond\neg\alpha_i \\
&\Longrightarrow \vdash_{\mathrm{ML}_{\mathrm{C}_1}} \Diamond\alpha_j \to \Diamond\neg\alpha_i \quad \text{(using axioms 4}(\Box)\text{ and Taut).}
\end{aligned}
$$

Since $\Diamond\alpha_j \in \Gamma$, it follows that $\Diamond\neg\alpha_i \in \Gamma$. This is contrary to the fact that $\Box\alpha_i \in \Gamma$. Hence $\{\alpha_j, \Box\alpha_i\}$ is a $\mathrm{ML}_{\mathrm{C}_1}$-consistent set. This proves the claim. Next, using Lindenbaum's lemma, we obtain a $\mathrm{ML}_{\mathrm{C}_1}$-maximal consistent set Γ' containing $\Box\alpha_i$ and α_j. Also $\Box\alpha_i \in \Gamma'$ implies $\alpha_i \in \Gamma'$, and hence $\Gamma' \in C_i^0$. This completes the proof. □

Lemma 4.11 (Truth Lemma)**.** *For any wff α and $\Gamma \in \mathbb{M}$,*

$$\alpha \in \Gamma \textit{ if and only if } \mathfrak{M}^0, \Gamma \models_{C_1} \alpha.$$

Proof. The proof is by induction on the number of connectives in the wff α. We provide the proof when α is of the form $\Diamond\beta$. Rest of the cases can be proved easily. Assume that $\Diamond\beta \in \Gamma$ and we show that $\mathfrak{M}^0, \Gamma \models_{C_1} \Diamond\beta$. If possible, let $\mathfrak{M}^0, \Gamma \not\models_{C_1} \Diamond\beta$, that is, $\mathfrak{M}^0, \Gamma \models_{C_1} \Box\neg\beta$. Then, there must be a $C_j^0 \in \mathcal{C}^0$ such that $\Gamma \in C_j^0$ and $\mathfrak{M}^0, \Gamma' \models_{C_1} \neg\beta$ for all $\Gamma' \in C_j^0$. By induction hypothesis, it follows that $\neg\beta \in \Gamma'$ for all $\Gamma' \in C_j^0$. Since $\Diamond\beta \in \Gamma$ and $\Gamma \in C_j^0$, using Item 2 of Lemma 4.10, we obtain a $\Gamma'' \in C_j^0$ such that $\beta \in \Gamma''$. But this is not possible as $\neg\beta \in \Gamma''$. Hence $\mathfrak{M}^0, \Gamma \models_{C_1} \Diamond\beta$.
For the converse, let $\mathfrak{M}^0, \Gamma \models_{C_1} \Diamond\beta$ and we prove that $\Diamond\beta \in \Gamma$. If possible, let $\neg\Diamond\beta \in \Gamma$, that is, $\Box\neg\beta \in \Gamma$. Then by Item 1 of Lemma 4.10, there exists a $C_i^0 \in \mathcal{C}^0$ such that $\Gamma \in C_i^0$ and $\neg\beta \in \Gamma'$ for all $\Gamma' \in C_i^0$. Since $\mathfrak{M}^0, \Gamma \models_{C_1} \Diamond\beta$ and $\Gamma \in C_i^0$, we must have a $\Gamma'' \in C_i^0$ such that $\mathfrak{M}, \Gamma'' \models_{C_1} \beta$. Use of induction hypothesis implies $\beta \in \Gamma''$. This is not possible as $\neg\beta \in \Gamma'$ for all $\Gamma' \in C_i^0$. Thus $\Diamond\beta \in \Gamma$. □

Theorem 4.12 (Completeness Theorem)**.** *For any wff α, if $\models_{C_1} \alpha$, then $\vdash_{\mathrm{ML}_{\mathrm{C}_1}} \alpha$.*

Proof. If possible, let $\nvdash_{\mathrm{ML}_{\mathrm{C}_1}} \alpha$. Then $\{\neg\alpha\}$ is a $\mathrm{ML}_{\mathrm{C}_1}$-consistent set and hence there exists a $\mathrm{ML}_{\mathrm{C}_1}$-maximal consistent set Γ containing $\neg\alpha$. By Truth Lemma 4.11, $\mathfrak{M}^0, \Gamma \models_{C_1} \neg\alpha$. This contradicts the given condition that $\models_{C_1} \alpha$. Hence $\vdash_{\mathrm{ML}_{\mathrm{C}_1}} \alpha$. □

We have $\underline{P_3}(A) = \underline{C_1}(A) = \underline{C_{\mathrm{Gr}}}(A)$, and $\overline{P_3}(A) = \overline{C_1}(A) = \overline{C_{\mathrm{Gr}}}(A)$ (cf. Chapter 2, page 13). Thus using these facts, and Theorems 4.6 and 4.12, the following can be obtained.

Corollary 4.13. *Let $\sigma \in \{P_3, C_{\mathrm{Gr}}\}$. For any wff α, $\models_\sigma \alpha$ if and only if $\vdash_{\mathrm{MLC_1}} \alpha$.*

4.3. The boundary operator

Recall that a *Kripke frame* consists of a non-empty set W and a binary relation R on W. Thus, it is mathematically the same as *generalized approximation space* studied in rough set literature. In this chapter, we will use both the names interchangeably. The lower and upper approximation operator based on generalized approximation space will be denoted by L_R and U_R, respectively. That is, for $A \subseteq W$,

$$L_R(A) := \underline{A}_R;$$

$$U_R(A) := \overline{A}_R.$$

Given a Kripke frame (W, R), consider the operator $D_R : \wp(W) \to \wp(W)$ defined as follows. Let $A \subseteq W$.

$$D_R(A) := \{x \in W : R(x) \cap A = \emptyset\} \cup \{x \in W : R(x) \subseteq A\}.$$

The dual of the operator D_R, denoted as B_R, is obtained as follows.

$$B_R(A) := \{x \in W : R(x) \cap A \neq \emptyset \ \& \ R(x) \not\subseteq A\} = Bd_R(A).$$

Note 4.14. In the rest of the chapter, we will omit the subscript R in the notations L_R, U_R, B_R and D_R to make the writing simple.

Thus, D maps a set A to its *decidable region*, that is, the union of positive and negative region of A. On the other hand, B maps a set to its boundary region. Accordingly, the operator B will be called the *boundary operator*. It is worth to mentioning that operator D is the same as the knowledge operator considered in [14].

It is not difficult to see that operators D and B can be defined through operators L and U as shown below.

Proposition 4.15. *Let (W, R) be a generalized approximation space. Then, we have the following.*

$$B(A) = U(A) \setminus L(A) \text{ and}$$
$$D(A) = L(A) \cup L(A^c).$$

On the other hand, in general, it is not possible to define operators L, U using operators D, B. But, they are definable if the relation is reflexive as shown in the following proposition.

Proposition 4.16. *Let (W, R) be a generalized approximation space, where R is a reflexive relation. Then, we have the following.*

$$L(A) = A \cap D(A) \text{ and}$$
$$U(A) = A \cup B(A).$$

Proof. Let $x \in L(A)$. Then, $R(x) \subseteq A$, and thus $x \in D(A)$. As R is reflexive, we also obtain $x \in A$. Hence $x \in A \cap D(A)$.
Next, assume that $x \in A \cap D(A)$. Then, from $x \in D(A)$, $R(x) \subseteq A$ or $R(x) \subseteq A^c$. Since R is reflexive and $x \in A$, we must have $R(x) \subseteq A$ and hence $x \in L(A)$. Thus, it is shown that $L(A) = A \cap D(A)$.
$U(A) = A \cup B(A)$ follows directly from $L(A) = A \cap D(A)$ using the fact that U and B are duals of the operators L and D, respectively. □

One can find extensive studies on the lower and upper approximation operators in rough set literature (cf. e.g. [3, 4, 111]). On the contrary, the boundary operator lacks such a detailed study. In this chapter, we will present a study of the boundary operator with the modal logic perspective. But before moving to this, we would like to mention a few points that also highlight the significance of the boundary-operator approach in rough set theory.

Rough sets are defined in several ways even when the base is taken as (W, R), where R is an equivalence relation (cf. e.g. [5]). Of these, two are $(\underline{A}_R, \overline{A}_R)$ and $(\underline{A}_R, \overline{A}_R{}^c)$. It is clear that the second definition focuses on the positive-negative regions of the applicability of a concept. It is observed in rough set studies that different algebras emerge with respect to different definitions of rough sets and appropriately defined operations. For example, while the first definition gives rise to pre-rough/rough algebra or 3-valued Łukasiewicz algebra [4], the second gives semi simple Nelson algebra [68].

Again when the approximation space has no singleton elementary sets, rough sets defined by the collection of pairs in reverse order $(\overline{A}_R, \underline{A}_R)$ with appropriate operation give rise to Post algebras of order three [5]. Besides, a rough set is also defined as a pair (D_1, D_2), where D_1 and D_2 are definable sets and $D_1 \subseteq D_2$ [12, 62]. In this case, the algebra turns out to be a Kleene algebra [49].

Considering the study of rough set from this angle, it becomes interesting and important to visualize rough set as the pair $(B(A), A)$ and to investigate the algebra that comes up (this aspect, however, has not been dealt with in this chapter). Also the obtained structures not necessarily remain the same when the base set constitutes of relations other than equivalence or constitutes of a covering.

There is another point to make. In practical applications it is sometimes required to make the boundary region as thin as possible subject to certain constraints. Not all boundary elements are regarded as having equal status, some elements are dropped depending upon the rough membership value of the elements [11, 110]. But there is no study of minimal formal properties that are to be retained during the thinning procedure. The present study may open up investigations in this direction.

We first shed some light on the boundary operator based on generalized approximation space (Kripke frame). Then, this study will be extended to covering based models.

At this point, it is pertinent mentioning the work on contingency logics [22] as it is very relevant to the study of boundary operators.

In the papers [21, 22], the authors have presented a bunch of contingency logics corresponding to various classes of frames related with standard modal logic systems viz. $\mathrm{K}, \mathrm{D}, \mathrm{T}, \mathrm{S4}, \mathrm{B}$. The contingency logic corresponding to the class of the frame of the system K has been named as $\mathbb{CL}$. In all these logics, a new modality $\triangle$ and its dual $\triangledown$ have been taken instead of the standard ones that is $\Box$ (necessity) and $\Diamond$ (possibility). The formula $\triangle\alpha$ is read as 'α is non-contingent', and $\triangledown\alpha$ is 'α is contingent.' A whole lot of axiom systems have been introduced in terms of the new modalities. The motivation of the authors behind introducing the new modalities and the axiom systems is to capture the notion of contingency and non-contingency of a proposition from the doxastic as well as epistemic contexts. We, however, are interested in these operators from the angle of rough set theory.

4.3.1. Modal systems for boundary operator based on generalized approximation space

In our study of boundary operators, we will consider the basic modal language $\mathcal{L}(\triangle)$ with unary modal connective $\triangle$. Moreover, the connective $\triangledown$ represents the dual of $\triangle$, that is, $\triangledown\alpha := \neg \triangle \neg\alpha$. Recall the axioms Taut, 4($\triangle$) and inference rules MP, Nec($\triangle$) and RE($\triangle$) of modal logic (cf. Chapter 3). Further, consider the following axioms based on the language $\mathcal{L}(\triangle)$.

$$\triangle (\gamma \to \alpha) \wedge \triangle(\neg\gamma \to \alpha) \to \triangle\alpha, \qquad (Con^0)$$

$$\triangle \alpha \to \triangle(\alpha \to \beta) \vee \triangle(\neg\alpha \to \gamma), \qquad (Dis^0)$$

$$\triangle \alpha \leftrightarrow \triangle\neg\alpha, \qquad (\leftrightarrow)$$

$$\triangle \alpha \wedge \triangle(\alpha \to \beta) \wedge \alpha \to \triangle\beta, \qquad (T^0)$$

$$\alpha \to \triangle(\triangle\alpha \wedge \triangle(\alpha \to \beta) \wedge \neg \triangle \beta \to \gamma), \qquad (B^0)$$

$$\alpha \to \triangle(\triangle\alpha \to \alpha), \qquad (B')$$

$$\triangle \alpha \to \triangle(\triangle\alpha \vee \beta), \qquad (4_{\triangle})$$

$$\neg \triangle \alpha \to \triangle(\neg \triangle \alpha \vee \beta), \qquad (5^0)$$

$$\neg \triangle \alpha \to \triangle\neg \triangle \alpha, \qquad (5)$$

Table 4.2 gives a few modal systems based on the language $\mathcal{L}(\triangle)$. In order to keep the presentation uniform, we present the semantics using truth set as it is done in Section 4.1.2.

Let us recall that a model based on a Kripke frame (generalized approximation space) $\mathfrak{F} := (W, R)$ is defined as a tuple $\mathfrak{M} := (W, R, m)$, where $m : PV \to \wp(W)$. Given a model $\mathfrak{M} := (W, R, m)$, the operator D is used to define the truth set. That is, the truth set of a wff α in $\mathfrak{M}$ under D semantics, denoted as $[\![\alpha]\!]^D_{\mathfrak{M}}$, is defined inductively as Definition 4.2, except the semantic clause

$$[\![\triangle\alpha]\!]^D_{\mathfrak{M}} := D([\![\alpha]\!]^D_{\mathfrak{M}}).$$

Modal Systems	Axioms and Inference Rules	Modal Systems	Axioms and Inference Rules
$\mathbb{CL}$	Taut, Con^0, Dis^0, $\leftrightarrow$, MP, Nec($\triangle$), RE($\triangle$)	$\mathbb{CLT}$	$\mathbb{CL}$ + axiom T^0
$\mathbb{CL}4$	$\mathbb{CL}$ + axiom 4_Δ	$\mathbb{CL}5$	$\mathbb{CL}$ + axiom 5^0
$\mathbb{CL}45$	$\mathbb{CL}4$ + axiom 5^0	$\mathbb{CLB}$	$\mathbb{CL}$ + axiom B^0
$\mathbb{CLTB}$	$\mathbb{CLT}$ + axiom B'	$\mathbb{CLB}5$	$\mathbb{CLB}$ + axiom 5^0
$\mathbb{CLS}4$	$\mathbb{CLT}$ + axiom $4(\triangle)$	$\mathbb{CLS}5$	$\mathbb{CLT}$ + axiom 5
$\mathbb{CLB}4$	$\mathbb{CLB}$ + axiom 4_Δ		

Table 4.2. A few modal systems

The truth set clause for operator $\bigtriangledown$ is obtained as

$$[\![\bigtriangledown\alpha]\!]^D_{\mathfrak{M}} := B([\![\alpha]\!]^D_{\mathfrak{M}}).$$

An equivalent satisfiability condition for operator $\triangle$ under D semantics is given by the following proposition.

Proposition 4.17. *$\mathfrak{M}, x \models_D \triangle\alpha$ if and only if, for all y_1, y_2 with xRy_1 and xRy_2, $\mathfrak{M}, y_1 \models_D \alpha$ if and only if $\mathfrak{M}, y_2 \models_D \alpha$.*

Table 4.3 lists a few classes of Kripke frames.

A wff α is said to be valid in a class $\mathfrak{C}$ of Kripke frames if $\mathfrak{M} \models_D \alpha$ for all models $\mathfrak{M}$ based on Kripke frames from $\mathfrak{C}$. Table 4.4 summarizes the known soundness and completeness results for various classes of frames.

Notation	Frame Property	Notation	Frame Property
$\mathcal{K}$	–	$\mathcal{D}$	seriality
$\mathcal{T}$	reflexivity	$\mathcal{B}$	symmetry
4	transitivity	5	euclidicity
45	transitivity, euclidicity	$\mathcal{KD}45$	seriality, transitivity, euclidicity
$\mathcal{S}4$	reflexivity, transitivity	$\mathcal{S}5$	reflexivity, euclidicity
$\mathcal{D}4$	seriality, transitivity	$\mathcal{D}5$	seriality, euclidicity
$\mathcal{B}4$	symmetry, transitivity	$\mathcal{B}5$	symmetry, euclidicity
$\mathcal{TB}$	reflexivity, symmetry		

Table 4.3. Classes of Kripke frames

Modal Systems	Frame Classes	References for Completeness Result
$\mathbb{CL}$	$\mathcal{K}, \mathcal{D}$	[22, 30, 117]
$\mathbb{CLT}$	$\mathcal{T}$	[22, 63]
$\mathbb{CL}4$	$4, \mathcal{D}4$	[22, 47, 117]
$\mathbb{CL}5$	$5, \mathcal{D}5$	[22, 117]
$\mathbb{CL}45$	$45, \mathcal{KD}45$	[22, 117]
$\mathbb{CLB}$	$\mathcal{B}$	[22]
$\mathbb{CLTB}$	$\mathcal{TB}$	[21]
$\mathbb{CLB}5$	$\mathcal{B}5, \mathcal{B}4$	[21]
$\mathbb{CLS}4$	$\mathcal{S}4$	[22, 63]
$\mathbb{CLS}5$	$\mathcal{S}5$	[22, 63]

Table 4.4. Modal systems for various classes of frames

4.4. Modal systems for boundary operator based on covering space

In this section, the idea of boundary operator based on the Kripke frame will be extended to define the boundary operators for covering space. Then, we will investigate the modal systems for these boundary operators.

Let $(W, \mathcal{C})$ be a covering space. Taking a cue from the notion of boundary operator defined on the Kripke frame, we consider the following notions of boundary operators and their duals. Let us recall the definition of functions $N_{\mathcal{C}} : W \to \wp(W)$ and $F_{\mathcal{C}} : W \to \wp(W)$ given in Chapter 2 (cf. page 12). Let $A \subseteq W$. Three pairs of covering based operators are defined as below.

$$\underline{C_2^*}(A) = \{x \in W : N_{\mathcal{C}}(x) \subseteq A\} \cup \{x \in W \mid N_{\mathcal{C}}(x) \subseteq A^c\};$$
$$\overline{C_2^*}(A) = \{x \in W : N_{\mathcal{C}}(x) \cap A \neq \emptyset\} \cap \{x \in W \mid N_{\mathcal{C}}(x) \cap A^c \neq \emptyset\};$$

$$\underline{C_5^*}(A) = \{x \in W : y \in A \text{ for all } y \text{ with } x \in N_{\mathcal{C}}(y)\} \cup \{x \in W : y \in A^c \text{ for all } y \text{ with } x \in N_{\mathcal{C}}(y)\};$$
$$\overline{C_5^*}(A) = \{x \in W : \text{ there exists a } y \text{ such that } x \in N_{\mathcal{C}}(y) \text{ and } y \in A\} \cup \{x \in W : \text{ there exists a } y \text{ such that } x \in N_{\mathcal{C}}(y) \text{ and } y \in A^c\};$$

$$\underline{P_1^*}(A) = \{x \in W : F_{\mathcal{C}}(x) \subseteq A\} \cup \{x \in W \mid F_{\mathcal{C}}(x) \subseteq A^c\};$$
$$\overline{P_1^*}(A) = \{x \in W : F_{\mathcal{C}}(x) \cap A \neq \emptyset\} \cap \{x \in W \mid F_{\mathcal{C}}(x) \cap A^c \neq \emptyset\}.$$

One can verify that operators $\underline{C_2^*}, \underline{C_5^*}$ and $\underline{P_1^*}$ are duals of $\overline{C_2^*}, \overline{C_5^*}$ and $\overline{P_1^*}$, respectively. Further, operators $\underline{C_2^*}$, $\overline{C_2^*}$, $\underline{C_5^*}$, $\overline{C_5^*}$, $\underline{P_1^*}$, $\overline{P_1^*}$ are inter-definable with operators $\underline{C_2}$, $\overline{C_2}$, $\underline{C_5}$, $\overline{C_5}$, $\underline{P_1}$, $\overline{P_1}$, respectively, as shown by the following proposition.

Proposition 4.18. *Let $(W, \mathcal{C})$ be a covering space. Then, for all $A \subseteq W$ and $\sigma \in \{C_2, C_5, P_1\}$, we have the following.*

1. $\underline{\sigma^*}(A) = \underline{\sigma}(A) \cup \underline{\sigma}(A^c)$.
2. $\underline{\sigma}(A) = A \cap \underline{\sigma^*}(A)$.
3. $\overline{\sigma^*}(A) = \overline{\sigma}(A) \cap \overline{\sigma}(A^c)$.

4. $\overline{\sigma}(A) = A \cup \overline{\sigma^*}(A)$.

Proof. Let us prove the result for C_2. One can similarly prove it for C_5 and P_1.
Item 1 follows directly from the definition of $\underline{C_2}^*(A)$. Let us prove the Item 2. So assume that $x \in \underline{C_2}(A)$. Then $x \in \underline{C_2}^*(A)$ by Item 1 and $N_{\mathcal{C}}(x) \subseteq A$. Since $x \in N_{\mathcal{C}}(x)$, we get $x \in A$. Thus $x \in A \cap \underline{C_2}^*(A)$.
For the converse, let $x \in A \cap \underline{C_2}^*(A)$. From $x \in \underline{C_2}^*(A)$ and Item 1, $x \in \underline{C_2}(A)$ or $x \in \underline{C_2}(A^c)$. Since $x \in A$ and $x \in N_{\mathcal{C}}(x)$, x must be in $\underline{C_2}(A)$. This completes the proof.
Items 3 and 4 follows from the fact that $\underline{C_2}$ and $\underline{C_2}^*$ are the dual of $\overline{C_2}$ and $\overline{C_2^*}$, respectively. □

The σ semantics for $\sigma \in \{C_2^*, C_5^*, P_1^*\}$ are defined in the natural way using the clause

$$[\![\triangle\alpha]\!]^{\sigma}_{\mathfrak{M}} := \underline{\sigma}([\![\alpha]\!]^{\sigma}_{\mathfrak{M}}).$$

The truth set for $\triangledown\alpha$ is obtained as

$$[\![\triangledown\alpha]\!]^{\sigma}_{\mathfrak{M}} := \overline{\sigma}([\![\alpha]\!]^{\sigma}_{\mathfrak{M}}).$$

Next, we obtain the modal systems for σ semantics, where $\sigma \in \{C_2^*, C_5^*, P_1^*\}$.

4.4.1. Modal systems for σ semantics for $\sigma \in \{C_2^*, C_5^*, P_1^*\}$

Recall again that $\mathfrak{M}, x \models_\sigma \alpha$ denotes $x \in [\![\alpha]\!]^{\sigma}_{\mathfrak{M}}$ for $\sigma \in \{C_2^*, C_5^*, P_1^*\}$. The following proposition gives the equivalent satisfiability condition for the operators $\triangle$ and $\triangledown$ under σ semantics.

Proposition 4.19.

- $\mathfrak{M}, x \models_{C_2^*} \triangle\alpha$ *if and only if for all* $y \in N_{\mathcal{C}}(x), \mathfrak{M}, y \models_{C_2^*} \alpha$ *or, for all* $y \in N_{\mathcal{C}}(x)$, $\mathfrak{M}, y \models_{C_2^*} \neg\alpha$.
- $\mathfrak{M}, x \models_{C_2^*} \triangledown\alpha$ *if and only if there exist* $y_1, y_2 \in N_{\mathcal{C}}(x)$ *such that* $\mathfrak{M}, y_1 \models_{C_2^*} \alpha$ *and* $\mathfrak{M}, y_2 \models_{C_2^*} \neg\alpha$.
- $\mathfrak{M}, x \models_{C_5^*} \triangle\alpha$ *if and only if* $\mathfrak{M}, y \models_{C_5^*} \alpha$ *for all* y *with* $x \in N_{\mathcal{C}}(y)$, *or* $\mathfrak{M}, y \models_{C_5^*} \neg\alpha$ *for all* y *with* $x \in N_{\mathcal{C}}(y)$.
- $\mathfrak{M}, x \models_{C_5^*} \triangledown\alpha$ *if and only if there exist* $y_1, y_2 \in W$ *such that* $x \in N_{\mathcal{C}}(y_1) \cap N_{\mathcal{C}}(y_2)$, *and* $\mathfrak{M}, y_1 \models_{C_5^*} \alpha$ *and* $\mathfrak{M}, y_2 \models_{C_5^*} \neg\alpha$.

- $\mathfrak{M}, x \models_{P_1^*} \triangle\alpha$ *if and only if for all* $C_i \in \mathcal{C}$ *with* $x \in C_i$ *and for all* $y \in C_i$, $\mathfrak{M}, y \models_{P_1^*} \alpha$, *or for all* $C_i \in \mathcal{C}$ *with* $x \in C_i$ *and for all* $y \in C_i$, $\mathfrak{M}, y \models_{P_1^*} \neg\alpha$.
- $\mathfrak{M}, x \models_{P_1^*} \triangledown\alpha$ *if and only if there exist* $C_i, C_j \in \mathcal{C}$, $y_1 \in C_i$ *and* $y_2 \in C_j$ *such that* $x \in C_i \cap C_j$, $\mathfrak{M}, y_1 \models_{P_1^*} \alpha$ *and* $\mathfrak{M}, y_2 \models_{P_1^*} \neg\alpha$.

It will be proved that the modal system $\mathbb{CLS4}$ corresponds to C_2^* and C_5^* semantics. Further, $\mathbb{CLTB}$ is the system for P_1^* semantics. We begin with the following soundness theorems.

Theorem 4.20 (Soundness Theorem for C_2^* Semantics)**.** *For each wff* α, *if* $\vdash_{\mathbb{CLS4}} \alpha$, *then* $\models_{C_2^*} \alpha$.

Proof. It will be shown that all the axioms of the modal system $\mathbb{CLS4}$ are valid under C_2^* semantics.

Axiom Con^0: Consider a model $\mathfrak{M}$ and an object x such that $\mathfrak{M}, x \models_{C_2^*} \triangle(\gamma \to \alpha) \wedge \triangle(\neg\gamma \to \alpha)$ and we prove $\mathfrak{M}, x \models_{C_2^*} \triangle\alpha$. That is, we show that for all $y \in N_{\mathcal{C}}(x)$, $\mathfrak{M}, y \models_{C_2^*} \alpha$ or for all $y \in N_{\mathcal{C}}(x)$, $\mathfrak{M}, y \models_{C_2^*} \neg\alpha$. If for all $y \in N_{\mathcal{C}}(x)$, $\mathfrak{M}, y \models_{C_2^*} \neg\alpha$, then we are done. So, suppose there exists a $y_1 \in N_{\mathcal{C}}(x)$ such that $\mathfrak{M}, y_1 \models_{C_2^*} \alpha$. Let us take an arbitrary $y_2 \in N_{\mathcal{C}}(x)$ and we prove that $\mathfrak{M}, y_2 \models_{C_2^*} \alpha$. Since $\mathfrak{M}, x \models_{C_2^*} \triangle(\gamma \to \alpha)$ and $\mathfrak{M}, y_1 \models_{C_2^*} \alpha$, we get $\mathfrak{M}, y_1 \models_{C_2^*} \gamma \to \alpha$ and $\mathfrak{M}, y_2 \models_{C_2^*} \gamma \to \alpha$. Similarly, from $\mathfrak{M}, x \models_{C_2^*} \triangle(\neg\gamma \to \alpha)$ and $\mathfrak{M}, y_1 \models_{C_2^*} \alpha$, we obtain $\mathfrak{M}, y_1 \models_{C_2^*} \neg\gamma \to \alpha$ and $\mathfrak{M}, y_2 \models_{C_2^*} \neg\gamma \to \alpha$. Now from $\mathfrak{M}, y_2 \models_{C_2^*} \gamma \to \alpha$ and $\mathfrak{M}, y_2 \models_{C_2^*} \neg\gamma \to \alpha$, we conclude $\mathfrak{M}, y_2 \models_{C_2^*} \alpha$. Thus $\mathfrak{M}, x \models_{C_2^*} \triangle\alpha$. Hence the axiom Con^0 is valid under C_2^* semantics.

Axiom Dis^0: Let $\mathfrak{M}, x \models_{C_2^*} \neg\triangle(\alpha \to \beta) \wedge \neg\triangle(\neg\alpha \to \gamma)$ and we show that $\mathfrak{M}, x \models_{C_2^*} \neg\triangle\alpha$. Since $\mathfrak{M}, x \models_{C_2^*} \neg \triangle (\alpha \to \beta)$, there exist $y_1, y_2 \in N_{\mathcal{C}}(x)$ such that $\mathfrak{M}, y_1 \models_{C_2^*} (\alpha \to \beta)$ and $\mathfrak{M}, y_2 \models_{C_2^*} \neg(\alpha \to \beta)$. Further, $\mathfrak{M}, y_2 \models_{C_2^*} \neg(\alpha \to \beta)$ implies $\mathfrak{M}, y_2 \models_{C_2^*} \alpha$ and $\mathfrak{M}, y_2 \models_{C_2^*} \neg\beta$. From $\mathfrak{M}, x \models_{C_2^*} \neg \triangle (\neg\alpha \to \gamma)$, we obtain $y_3, y_4 \in N_{\mathcal{C}}(x)$ such that $\mathfrak{M}, y_3 \models_{C_2^*} (\neg\alpha \to \gamma)$ and $\mathfrak{M}, y_4 \models_{C_2^*} \neg(\neg\alpha \to \gamma)$. Thus it follows that $\mathfrak{M}, y_4 \models_{C_2^*} \neg\alpha$ and $\mathfrak{M}, y_4 \models_{C_2^*} \neg\gamma$. We have $\mathfrak{M}, y_2 \models_{C_2^*} \alpha, \mathfrak{M}, y_4 \models_{C_2^*} \neg\alpha$ and $y_2, y_4 \in N_{\mathcal{C}}(x)$ and therefore $\mathfrak{M}, x \models_{C_2^*} \neg \triangle \alpha$, as desired.

Axiom $\leftrightarrow$: Follows directly from the definition of satisfiability condition of $\triangle\alpha$.

Axiom T^0: Assume $\mathfrak{M}, x \models_{C_2^*} \triangle\alpha \wedge \triangle(\alpha \to \beta) \wedge \alpha$ and we show $\mathfrak{M}, x \models_{C_2^*} \triangle\beta$. If for all $y \in N_{\mathcal{C}}(x)$, $\mathfrak{M}, y \models_{C_2^*} \neg\beta$, then we are done. If not, then there exists a $y_1 \in N_{\mathcal{C}}(x)$ such that $\mathfrak{M}, y_1 \models_{C_2^*} \beta$. Consider an arbitrary $y_2 \in N_{\mathcal{C}}(x)$ and we prove that $\mathfrak{M}, y_2 \models_{C_2^*} \beta$. Since $\mathfrak{M}, x \models_{C_2^*} \triangle\alpha \wedge \alpha$ and $x \in N_{\mathcal{C}}(x)$, we get $\mathfrak{M}, y_1 \models_{C_2^*} \alpha$ and $\mathfrak{M}, y_2 \models_{C_2^*} \alpha$. Further, $\mathfrak{M}, x \models_{C_2^*} \triangle(\alpha \to \beta)$ and $\mathfrak{M}, y_1 \models_{C_2^*} \beta$ implies $\mathfrak{M}, y_1 \models_{C_2^*} \alpha \to \beta$ and $\mathfrak{M}, y_2 \models_{C_2^*} \alpha \to \beta$. We have $\mathfrak{M}, y_2 \models_{C_2^*} \alpha \to \beta$ and $\mathfrak{M}, y_2 \models_{C_2^*} \alpha$, and therefore $\mathfrak{M}, y_2 \models_{C_2^*} \beta$. Hence $\mathfrak{M}, x \models_{C_2^*} \triangle\beta$.

Axiom $4(\triangle)$: Let $\mathfrak{M}, x \models_{C_2^*} \triangle\alpha$ and we show that $\mathfrak{M}, x \models_{C_2^*} \triangle\triangle\alpha$. We prove it by contradiction. So, let $\mathfrak{M}, x \not\models_{C_2^*} \triangle\triangle\alpha$. Then there exist $y_1, y_2 \in N_{\mathcal{C}}(x)$ such that $\mathfrak{M}, y_1 \models_{C_2^*} \triangle\alpha$ and $\mathfrak{M}, y_2 \models_{C_2^*} \neg\triangle\alpha$. From $\mathfrak{M}, y_2 \models_{C_2^*} \neg\triangle\alpha$, there exist $y_3, y_4 \in N_{\mathcal{C}}(y_2)$ such that $\mathfrak{M}, y_3 \models_{C_2^*} \alpha$ and $\mathfrak{M}, y_4 \models_{C_2^*} \neg\alpha$. We claim that $y_3, y_4 \in N_{\mathcal{C}}(x)$. Consider a $C_i \in \mathcal{C}$ such that $x \in C_i$ and we need to prove that $y_3 \in C_i$. Since $y_2 \in N_{\mathcal{C}}(x)$, we derive that $y_2 \in C_i$. Since $y_3 \in N_{\mathcal{C}}(y_2)$, we conclude $y_3 \in C_i$. Similarly, it can be shown that $y_4 \in N_{\mathcal{C}}(x)$. We have $y_3, y_4 \in N_{\mathcal{C}}(x)$ such that $\mathfrak{M}, y_3 \models_{C_2^*} \alpha$ and $\mathfrak{M}, y_4 \models_{C_2^*} \neg\alpha$. Thus, it follows that $\mathfrak{M}, x \models_{C_2^*} \neg\triangle\alpha$, which is contrary to our assumption $\mathfrak{M}, x \models_{C_2^*} \triangle\alpha$. Therefore $\mathfrak{M}, x \models_{C_2^*} \triangle\triangle\alpha$. □

Theorem 4.21 (Soundness Theorem for C_5^* Semantics)**.** *For each wff α, if $\vdash_{\mathbb{CLS}4} \alpha$, then $\models_{C_5^*} \alpha$.*

Proof. We prove that all the axioms of the modal system $\mathbb{CLS}4$ are valid under C_5^* semantics.

Axiom Con^0: Consider a model $\mathfrak{M}$ and an object x such that $\mathfrak{M}, x \models_{C_5^*} \triangle(\gamma \to \alpha) \wedge \triangle(\neg\gamma \to \alpha)$ and we prove that $\mathfrak{M}, x \models_{C_5^*} \triangle\alpha$. If for all y with $x \in N_{\mathcal{C}}(y)$, $\mathfrak{M}, y \models_{C_5^*} \neg\alpha$, then we are done. If not, then there exists a y_1 with $x \in N_{\mathcal{C}}(y_1)$ such that $\mathfrak{M}, y_1 \models_{C_5^*} \alpha$. Let us take an arbitrary y_2 with $x \in N_{\mathcal{C}}(y_2)$ and we show that $\mathfrak{M}, y_2 \models_{C_5^*} \alpha$. Since $\mathfrak{M}, x \models_{C_5^*} \triangle(\gamma \to \alpha)$ and $\mathfrak{M}, y_1 \models_{C_5^*} \alpha$, we must have $\mathfrak{M}, y_1 \models_{C_5^*} \gamma \to \alpha$ and $\mathfrak{M}, y_2 \models_{C_5^*} \gamma \to \alpha$. Further $\mathfrak{M}, x \models_{C_5^*} \triangle(\neg\gamma \to \alpha)$ and $\mathfrak{M}, y_1 \models_{C_5^*} \alpha$ implies $\mathfrak{M}, y_1 \models_{C_5^*} \neg\gamma \to \alpha$ and $\mathfrak{M}, y_2 \models_{C_5^*} \neg\gamma \to \alpha$. Now using $\mathfrak{M}, y_2 \models_{C_5^*} \gamma \to \alpha$ and $\mathfrak{M}, y_2 \models_{C_5^*} \neg\gamma \to \alpha$, we conclude that $\mathfrak{M}, y_2 \models_{C_5^*} \alpha$. Thus axiom Con^0 is valid under C_5^* semantics.

Axiom $\leftrightarrow$: The proof for axiom $(\triangle \leftrightarrow)$ is trivial, we omit it here.

Axiom Dis^0: Let $\mathfrak{M}, x \models_{C_5^*} \neg\triangle(\alpha \to \beta) \wedge \neg\triangle(\neg\alpha \to \gamma)$ and we show that $\mathfrak{M}, x \models_{C_5^*} \neg\triangle\alpha$. Since $\mathfrak{M}, x \models_{C_5^*} \neg\triangle(\alpha \to \beta)$, there exist $y_1, y_2 \in W$ such that $x \in N_{\mathcal{C}}(y_1) \cap N_{\mathcal{C}}(y_2)$, and $\mathfrak{M}, y_1 \models_{C_5^*} (\alpha \to \beta)$ and $\mathfrak{M}, y_2 \models_{C_5^*} \neg(\alpha \to \beta)$. Further, $\mathfrak{M}, y_2 \models_{C_5^*} \neg(\alpha \to \beta)$ implies

$\mathfrak{M}, y_2 \models_{C_5^*} \alpha$ and $\mathfrak{M}, y_2 \models_{C_5^*} \neg\beta$. Using $\mathfrak{M}, x \models_{C_5^*} \neg \triangle (\neg\alpha \to \gamma)$, we obtain $y_3, y_4 \in W$ such that $x \in N_{\mathcal{C}}(y_3) \cap N_{\mathcal{C}}(y_4)$, $\mathfrak{M}, y_3 \models_{C_5^*} (\neg\alpha \to \gamma)$ and $\mathfrak{M}, y_4 \models_{C_5^*} \neg(\neg\alpha \to \gamma)$. By $\mathfrak{M}, y_4 \models_{C_5^*} \neg(\neg\alpha \to \gamma)$, it follows $\mathfrak{M}, y_4 \models_{C_5^*} \neg\alpha$ and $\mathfrak{M}, y_4 \models_{C_5^*} \neg\gamma$. We have $\mathfrak{M}, y_2 \models_{C_5^*} \alpha, \mathfrak{M}, y_4 \models_{C_5^*} \neg\alpha$ and $x \in N_{\mathcal{C}}(y_2) \cap N_{\mathcal{C}}(y_4)$. Thus $\mathfrak{M}, x \models_{C_5^*} \neg \triangle \alpha$.

Axiom T^0: Assume that $\mathfrak{M}, x \models_{C_5^*} \triangle\alpha \wedge \triangle(\alpha \to \beta) \wedge \alpha$ and we need to show $\mathfrak{M}, x \models_{C_5^*} \triangle\beta$. If for every y with $x \in N_{\mathcal{C}}(y)$, $\mathfrak{M}, y \models_{C_5^*} \neg\beta$, then we are done. If not, there exists a y_1 with $x \in N_{\mathcal{C}}(y_1)$ and $\mathfrak{M}, y_1 \models_{C_5^*} \beta$. Let us take a y_2 with $x \in N_{\mathcal{C}}(y_2)$ and we show that $\mathfrak{M}, y_2 \models_{C_5^*} \beta$. As $\mathfrak{M}, x \models_{C_5^*} \triangle\alpha \wedge \alpha$ and $x \in N_{\mathcal{C}}(x)$, we must have $\mathfrak{M}, y_1 \models_{C_5^*} \alpha$ and $\mathfrak{M}, y_2 \models_{C_5^*} \alpha$. Since $\mathfrak{M}, x \models_{C_5^*} \triangle(\alpha \to \beta)$ and $\mathfrak{M}, y_1 \models_{C_5^*} \beta$, it follows that $\mathfrak{M}, y_1 \models_{C_5^*} \alpha \to \beta$ and $\mathfrak{M}, y_2 \models_{C_5^*} \alpha \to \beta$. By $\mathfrak{M}, y_2 \models_{C_5^*} \alpha \to \beta$ and $\mathfrak{M}, y_2 \models_{C_5^*} \alpha$, $\mathfrak{M}, y_2 \models_{C_5^*} \beta$ holds and hence $\mathfrak{M}, x \models_{C_5^*} \triangle\beta$.

Axiom $4(\triangle)$: Assume $\mathfrak{M}, x \models_{C_5^*} \triangle\alpha$. Towards proving a contradiction, let $\mathfrak{M}, x \not\models_{C_5^*} \triangle \triangle \alpha$. Then there exist $y_1, y_2 \in W$ such that $x \in N_{\mathcal{C}}(y_1) \cap N_{\mathcal{C}}(y_2)$, $\mathfrak{M}, y_1 \models_{C5^*} \triangle\alpha$ and $\mathfrak{M}, y_2 \models_{C_5^*} \neg \triangle \alpha$. From $\mathfrak{M}, y_2 \models_{C_5^*} \neg \triangle \alpha$, there exist $y_3, y_4 \in W$ such that $y_2 \in N_{\mathcal{C}}(y_3) \cap N_{\mathcal{C}}(y_4)$, $\mathfrak{M}, y_3 \models_{C_5^*} \alpha$ and $\mathfrak{M}, y_4 \models_{C_5^*} \neg\alpha$. Now, we show that $x \in N_{\mathcal{C}}(y_3) \cap N_{\mathcal{C}}(y_4)$. So, consider a $C_i \in \mathcal{C}$ such that $y_3 \in C_i$. Then $y_2 \in C_i$ as $y_2 \in N_{\mathcal{C}}(y_3)$. Since $x \in N_{\mathcal{C}}(y_2)$, we get $x \in C_i$. Similarly, it can be shown that $x \in N_{\mathcal{C}}(y_4)$. We have $x \in N_{\mathcal{C}}(y_3) \cap N_{\mathcal{C}}(y_4)$, $\mathfrak{M}, y_3 \models_{C_5^*} \alpha$ and $\mathfrak{M}, y_4 \models_{C_5^*} \neg\alpha$. Thus, it follows that $\mathfrak{M}, x \models_{C_5^*} \neg \triangle \alpha$, which is not possible as $\mathfrak{M}, x \models_{C_5^*} \triangle\alpha$. Hence $\mathfrak{M}, x \models_{C_5^*} \triangle \triangle \alpha$.

□

Theorem 4.22 (Soundness Theorem for P_1^* Semantics)**.** *For each wff α, if $\vdash_{\mathbb{CLTB}} \alpha$, then $\models_{P_1^*} \alpha$.*

Proof. Axiom Con^0: Let us assume $\mathfrak{M}, x \models_{P_1^*} \triangle(\gamma \to \alpha) \wedge \triangle(\neg\gamma \to \alpha)$ and we prove that $\mathfrak{M}, x \models_{P_1^*} \triangle\alpha$. If for each $C_i \in \mathcal{C}$ with $x \in C_i$ and for all $y \in C_i$, $\mathfrak{M}, y \models_{P_1^*} \neg\alpha$, then we are done. If not, then there exist a $C_1 \in \mathcal{C}$ having $x \in C_1$ and a $y_1 \in C_1$ such that $\mathfrak{M}, y_1 \models_{P_1^*} \alpha$. Consider an arbitrary $C_2 \in \mathcal{C}$ with $x \in C_2$ and a $y_2 \in C_2$, and we show that $\mathfrak{M}, y_2 \models_{P_1^*} \alpha$. From $\mathfrak{M}, x \models_{P_1^*} \triangle(\gamma \to \alpha)$ and $\mathfrak{M}, y_1 \models_{P_1^*} \alpha$, it follows that $\mathfrak{M}, y_1 \models_{P_1^*} \gamma \to \alpha$ and $\mathfrak{M}, y_2 \models_{P_1^*} \gamma \to \alpha$. Again from $\mathfrak{M}, x \models_{P_1^*} \triangle(\neg\gamma \to \alpha)$ and $\mathfrak{M}, y_1 \models_{P_1^*} \alpha$, we obtain $\mathfrak{M}, y_1 \models_{P_1^*} \neg\gamma \to \alpha$ and $\mathfrak{M}, y_2 \models_{P_1^*} \neg\gamma \to \alpha$. We have $\mathfrak{M}, y_2 \models_{P_1^*} \gamma \to \alpha$ and $\mathfrak{M}, y_2 \models_{P_1^*} \neg\gamma \to \alpha$, and thus $\mathfrak{M}, y_2 \models_{P_1^*} \alpha$. Thus axiom Con^0 is valid with respect to P_1^* semantics.

Axiom Dis^0: Let $\mathfrak{M}, x \models_{P_1^*} \neg\triangle(\alpha \to \beta) \wedge \neg\triangle(\neg\alpha \to \gamma)$ and we show that $\mathfrak{M}, x \models_{P_1^*} \neg\triangle\alpha$. Since $\mathfrak{M}, x \models_{P_1^*} \neg \triangle (\alpha \to \beta)$, there exist $C_1, C_2 \in \mathcal{C}$, $y_1, y_2 \in W$ such that $x, y_1 \in C_1$, $\mathfrak{M}, y_1 \models_{P_1^*} \alpha \to \beta$ and $x, y_2 \in C_2$, $\mathfrak{M}, y_2 \models_{P_1^*} \neg(\alpha \to \beta)$. Further, $\mathfrak{M}, y_2 \models_{P_1^*} \neg(\alpha \to \beta)$ implies $\mathfrak{M}, y_2 \models_{P_1^*} \alpha$ and $\mathfrak{M}, y_2 \models_{C_2^*} \neg\beta$. From $\mathfrak{M}, x \models_{P_1^*} \neg \triangle (\neg\alpha \to \gamma)$, there exist $C_3, C_4 \in \mathcal{C}, y_3, y_4 \in W$ such that $x, y_3 \in C_3$, $\mathfrak{M}, y_3 \models_{P_1^*} (\neg\alpha \to \gamma)$ and $x, y_4 \in C_4$, $\mathfrak{M}, y_4 \models_{P_1^*} \neg(\neg\alpha \to \gamma)$. Thus it follows that $\mathfrak{M}, y_4 \models_{P_1^*} \neg\alpha$ and $\mathfrak{M}, y_4 \models_{P_1^*} \neg\gamma$. We have $\mathfrak{M}, y_2 \models_{P_1^*} \alpha, \mathfrak{M}, y_4 \models_{P_1^*} \neg\alpha$ and $y_2 \in C_2, y_4 \in C_4$, and therefore $\mathfrak{M}, x \models_{P_1^*} \neg \triangle \alpha$.

Axiom $\leftrightarrow$: The proof of validity of the axiom is very obvious.

Axiom B': We assume that $\mathfrak{M}, x \not\models_{P_1^*} \triangle(\triangle\alpha \to \alpha)$ and we prove that $\mathfrak{M}, x \not\models_{P_1^*} \alpha$. From $\mathfrak{M}, x \not\models_{P_1^*} \triangle(\triangle\alpha \to \alpha)$, there exist $C_1, C_2 \in \mathcal{C}$, $y_1, y_2 \in W$ such that $x, y_1 \in C_1$, $\mathfrak{M}, y_1 \models_{P_1^*} \triangle\alpha \to \alpha$ and $x, y_2 \in C_2$, $\mathfrak{M}, y_2 \models_{P_1^*} \neg(\triangle\alpha \to \alpha)$. Since $\mathfrak{M}, y_2 \models_{P_1^*} \neg(\triangle\alpha \to \alpha)$, we get $\mathfrak{M}, y_2 \models_{P_1^*} \triangle\alpha$ and $\mathfrak{M}, y_2 \models_{P_1^*} \neg\alpha$. From $\mathfrak{M}, y_2 \models_{P_1^*} \triangle\alpha$ and $\mathfrak{M}, y_2 \models_{P_1^*} \neg\alpha$, it follows that $\mathfrak{M}, x \models_{P_1^*} \neg\alpha$.

Axiom T^0: Consider $\mathfrak{M}, x \models_{P_1^*} \triangle(\alpha \to \beta) \wedge \triangle\alpha \wedge \alpha$ and we prove that $\mathfrak{M}, x \models_{P_1^*} \triangle\beta$. If for each $C_i \in \mathcal{C}$ with $x \in C_i$ and for all $y \in C_i$, $\mathfrak{M}, y \models_{P_1^*} \neg\beta$, then we are done. If not, then there exist a $C_1 \in \mathcal{C}$ having $x \in C_1$ and a $y_1 \in C_1$ such that $\mathfrak{M}, y_1 \models_{P_1^*} \beta$. Consider a $y_2 \in C_1$ and we show that $\mathfrak{M}, y_2 \models_{P_1^*} \beta$. From $\mathfrak{M}, x \models_{P_1^*} \triangle(\alpha \to \beta) \wedge \triangle\alpha \wedge \alpha$, it follows $\mathfrak{M}, x \models_{P_1^*} \triangle(\alpha \to \beta)$, $\mathfrak{M}, x \models_{P_1^*} \triangle\alpha$ and $\mathfrak{M}, x \models_{P_1^*} \alpha$. Thus we obtain $\mathfrak{M}, y_1 \models_{P_1^*} \alpha$, $\mathfrak{M}, y_2 \models_{P_1^*} \alpha$, $\mathfrak{M}, y_1 \models_{P_1^*} \alpha \to \beta$ and $\mathfrak{M}, y_2 \models_{P_1^*} \alpha \to \beta$. Therefore, $\mathfrak{M}, y_2 \models_{P_1^*} \beta$. Hence $\mathfrak{M}, x \models_{P_1^*} \triangle\beta$. □

Now we move to obtain the completeness results. First note the following notion of lifting of a frame [61].

Definition 4.23. Let $\mathfrak{F} := (W, R)$ be a Kripke frame.

- The C_2 *lifting* of $\mathfrak{F}$ is defined as the structure $\mathfrak{F}^{C_2} = (W, \mathcal{C}_R^{C_2})$, where

$$\mathcal{C}_R^{C_2} := \{R(x) : x \in W\}.$$

- The C_5 *lifting* of $\mathfrak{F}$ is defined as the structure $\mathfrak{F}^{C_5} = (W, \mathcal{C}_R^{C_5})$, where

$$\mathcal{C}_R^{C_5} = \{R^{-1}(x) : x \in W\}.$$

  $R^{-1}(x)$ is being used to denote the set $\{y \in W : yRx\}$.

- The P_1 *lifting* of $\mathfrak{F}$ is defined as the structure $\mathfrak{F}^{P_1} = (W, \mathcal{C}_R^{P_1})$, where

$$\mathcal{C}_R^{P_1} = \{\{x, y\} \subseteq W : xRy\}.$$

Recall the function $N_{\mathcal{C}}$ and $F_{\mathcal{C}}$ given in Chapter 2 (cf. page 12). Then we have the following.

Proposition 4.24 ([61]). *1. Let $\mathfrak{F} := (W, R) \in \mathcal{S}4$. Then the following hold.*

- *$\mathcal{C}_R^{C_2}$ is a covering of W.*
- *$N_{\mathcal{C}_R^{C_2}}(x) = R(x)$ for each $x \in W$.*

2. Let $\mathfrak{F} := (W, R) \in \mathcal{S}4$. Then the following hold.

- *$\mathcal{C}_R^{C_5}$ is a covering of W.*
- *$N_{\mathcal{C}_R^{C_5}}(x) = R^{-1}(x)$ for each $x \in W$.*

3. Let $\mathfrak{F} := (W, R) \in \mathcal{TB}$. Then the following hold.

- *$\mathcal{C}_R^{P_1}$ is a covering of W.*
- *$F_{\mathcal{C}_R^{P_1}}(x) = R(x)$ for each $x \in W$.*

Lemma 4.25. *Let $\mathfrak{F} := (W, R) \in \mathcal{S}4$ and $\mathfrak{M} := (W, R, m)$ be a model based on $\mathfrak{F}$. Consider the covering model $\mathfrak{M}^{C_2} = (W, \mathcal{C}_R^{C_2}, m)$. Then, for any wff α and $x \in W$, we have the following.*

$$\mathfrak{M}, x \models_D \alpha \textit{ if and only if } \mathfrak{M}^{C_2}, x \models_{C_2^*} \alpha.$$

Proof. Proof is by induction on the number of connectives in α. We only provide a proof for the case when α is of the form $\triangle\beta$. We will use the symbol $\iff$ to mean 'if and only if.'

$$\begin{aligned}
&\mathfrak{M}^{C_2}, x \models_{C_2^*} \triangle\beta \\
\iff &\text{(for all } y \in N_{\mathcal{C}_R^{C_2}}(x),\ \mathfrak{M}^{C_2}, y \models_{C_2^*} \beta) \text{ or} \\
&\qquad\qquad\text{(for all } y \in N_{\mathcal{C}_R^{C_2}}(x),\ \mathfrak{M}^{C_2}, y \models_{C_2^*} \neg\beta) \\
\iff &\text{(for all } y \in R(x),\ \mathfrak{M}, y \models_D \beta) \text{ or (for all } y \in R(x),\ \mathfrak{M}, y \models_D \neg\beta) \\
&\text{(using induction hypothesis and Proposition 4.24)} \\
\iff &\mathfrak{M}, x \models_D \triangle\beta.
\end{aligned}$$

□

Lemma 4.26. *Let* $\mathfrak{F} := (W, R) \in \mathcal{S}4$ *and* $\mathfrak{M} := (W, R, m)$ *be a model based on* $\mathfrak{F}$. *Consider the covering model* $\mathfrak{M}^{C_5} = (W, \mathcal{C}_R^{C_5}, m)$. *Then, for any wff* α *and* $x \in W$, *we have the following.*

$$\mathfrak{M}, x \models_D \alpha \textit{ if and only if } \mathfrak{M}^{C_5}, x \models_{C_5^*} \alpha.$$

Proof. Proof is by induction on the number of connectives in α. We only provide a proof for the case when α is of the form $\triangle\beta$.

$$\begin{aligned}
&\mathfrak{M}^{C_5}, x \models_{C_5^*} \triangle\beta \\
\Longleftrightarrow\ &(\text{for all } y \text{ with } x \in N_{\mathcal{C}_R^{C_5}}(y), \mathfrak{M}^{C_5}, y \models_{C_5^*} \beta) \text{ or} \\
&\qquad\qquad (\text{for all } y \text{ with } x \in N_{\mathcal{C}_R^{C_5}}(y), \mathfrak{M}^{C_5}, y \models_{C_5^*} \neg\beta) \\
\Longleftrightarrow\ &(\text{for all } y \text{ with } x \in R^{-1}(y), \mathfrak{M}, y \models_D \beta) \text{ or} \\
&\qquad\qquad (\text{for all } y \text{ with } x \in R^{-1}(y), \mathfrak{M}, y \models_D \neg\beta) \\
&\quad (\text{by induction case and Proposition 4.24}) \\
\Longleftrightarrow\ &\mathfrak{M}, x \models_D \triangle\beta.
\end{aligned}$$

□

Lemma 4.27. *Let* $\mathfrak{F} := (W, R) \in \mathcal{TB}$ *and* $\mathfrak{M} := (W, R, m)$ *be a model based on* $\mathfrak{F}$. *Consider the covering model* $\mathfrak{M}^{P_1} = (W, \mathcal{C}_R^{P_1}, m)$. *Then, for any wff* α *and* $x \in W$, *we have the following.*

$$\mathfrak{M}, x \models_D \alpha \textit{ if and only if } \mathfrak{M}^{P_1}, x \models_{P_1^*} \alpha.$$

Proof. Proof is by induction on the number of connectives in α. We only provide a proof for the case when α is of the form $\triangle\beta$. We have

$$\begin{aligned}
&\mathfrak{M}^{P_1}, x \models_{P_1^*} \triangle\beta \\
\Longleftrightarrow\ &(\text{for all } C_i \in \mathcal{C}_R^{P_1} \text{ with } x \in C_i \text{ and for all } y \in C_i, \mathfrak{M}, y \models_{P_1^*} \beta) \text{ or} \\
&\quad (\text{for all } C_i \in \mathcal{C}_R^{P_1} \text{ with } x \in C_i \text{ and for all } y \in C_i, \mathfrak{M}, y \models_{P_1^*} \neg\beta) \\
\Longleftrightarrow\ &(\text{for all } C_i \in \mathcal{C}_R^{P_1} \text{ with } x \in C_i \text{ and for all } y \in C_i, \mathfrak{M}, y \models_D \beta) \text{ or}
\end{aligned}$$

(for all $C_i \in \mathcal{C}_R^{P_1}$ with $x \in C_i$ and for all $y \in C_i$, $\mathfrak{M}, y \models_D \neg\beta$)

(by induction case)

$\Longleftrightarrow \mathfrak{M}, x \models_D \triangle\beta$. (Using Proposition 4.24)

□

Now, we are in a position to prove the completeness theorem.

Theorem 4.28 (Completeness Theorem)**.** *For each wff α, we have the following.*

1. *If $\models_{C_2^*} \alpha$, then $\vdash_{\mathbb{CLS}4} \alpha$.*
2. *If $\models_{C_5^*} \alpha$, then $\vdash_{\mathbb{CLS}4} \alpha$.*
3. *If $\models_{P_1^*} \alpha$, then $\vdash_{\mathbb{CLTB}} \alpha$.*

Proof. We will provide the proof of Item 1 only. Rest can be done in the same way. If possible, let $\not\vdash_{\mathbb{CLS}4} \alpha$. Then using the completeness theorem of the modal system $\mathbb{CLS}4$ with respect to the class $\mathcal{S}4$ of frames (cf. Table 4.4), we obtain a model $\mathfrak{M} := (W, R, m)$ based on a frame $\mathfrak{F} \in \mathcal{S}4$ and a $x \in W$ such that $\mathfrak{M}, x \not\models_D \alpha$. Thus, by Lemma 4.25, $\mathfrak{M}^{C_2}, x \not\models_{C_2^*} \alpha$. This contradicts that $\models_{C_2^*} \alpha$. Hence $\vdash_{\mathbb{CLS}4} \alpha$. □

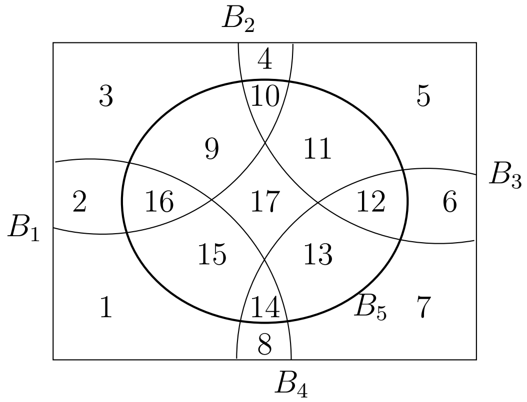
4.5. P_4 semantics revisited

The covering system P_4 [91] is defined in terms of the basic granules $P_x^{\mathcal{C}}$ where $\mathcal{C} = \{C_i \subseteq W : i \in I\}$ is the covering of W and for each $x \in W$, $P_x^{\mathcal{C}}$ is defined by

$$P_x^{\mathcal{C}} := \{y \in W : \text{ for all } C_i \in \mathcal{C}, x \in C_i \text{ if and only if } y \in C_i\}.$$

One can see that $\{P_x^{\mathcal{C}} \mid x \in W\}$ is the partition of W generated by the covering $\mathcal{C}$.

Example 4.29. The partition generated by the covering $\{B_1, B_2, B_3, B_4, B_5\}$ of W in the picture constitutes of the regions 1-17.



The lower and upper approximations of a set $A \subseteq W$ are defined by

$$\underline{P_4}(A) = \cup\{P_x^{\mathcal{C}} : P_x^{\mathcal{C}} \subseteq A\},$$
$$\overline{P_4}(A) = \cup\{P_x^{\mathcal{C}} : P_x^{\mathcal{C}} \cap A \neq \emptyset\}. \text{ (cf. [8,74,77,91])}$$

So, these are exactly the same as Pawlakian approximations. The corresponding modal system being S5, the modal operators $\Box$ and $\Diamond$ are interpreted as the lower and upper approximation respectively. It follows that

$$\vdash_{S5} \alpha \text{ if and only if } \models_{P_4} \alpha. \tag{4.1}$$

The objective of this section is to obtain an alternative modal system for the covering system P4 in terms of modalities $\triangle$ and $\triangledown$.

Let us recall the language $\mathcal{L}(\Box)$ which has the primitive modal operator $\Box$ and its dual $\Diamond$. In the language $\mathcal{L}(\Box)$, we consider $\triangle$ and $\triangledown$ as defined connectives given as follows:

$$\left.\begin{aligned} \triangle\, \alpha &:= \Box\alpha \vee \Box\neg\alpha; \\ \triangledown\, \alpha &:= \neg \triangle \neg\alpha. \end{aligned}\right\} \tag{4.2}$$

Similarly, in the language $\mathcal{L}(\triangle)$, which has primitive modal operator $\triangle$ and its dual $\triangledown$, we define the connectives $\Box$ and $\Diamond$ as follows:

$$\left.\begin{aligned} \Box\alpha &:= \alpha \wedge \triangle\alpha; \\ \Diamond\alpha &:= \neg\Box\neg\alpha. \end{aligned}\right\} \tag{4.3}$$

Thus, using (4.2) and (4.3), a wff of $\mathcal{L}(\Box)$ can be treated as a wff of $\mathcal{L}(\triangle)$ and conversely. Because of the above interdefinability, any wff of one language may be considered as a wff of the other.

Recalling the definition of $\models_D$ (cf. Section 4.3.1) and using the symbol $\models$ to denote the standard satisfiability relation based on Kripke frame, we obtain Proposition 4.30.

Proposition 4.30. *Let* $\mathfrak{M} := (W, R, m)$ *be a model based on a Kripke frame* $\mathfrak{M} := (W, R, m)$ *with reflexive relation* R*. Then, we have the following for all wffs* α*:*

$$\mathfrak{M}, x \models \alpha \textit{ if and only if } \mathfrak{M}, x \models_D \alpha.$$

Proof. First note that in $\mathfrak{M}, x \models \alpha$, α is meant to be a wff of $\mathcal{L}(\Box)$ and in $\mathfrak{M}, x \models_D \alpha$, we treat α as a wff of $\mathcal{L}(\triangle)$. The result is proved by induction on the number of connectives $\neg, \vee, \Box, \triangle$ in the wff α. We provide the proof of the cases when α is of the form $\Box\beta$ and $\triangle\beta$. So, assume that the result holds for β and we prove it for $\Box\beta$ and $\triangle\beta$. We begin with $\Box\beta$ case. Let $\mathfrak{M}, x \models \Box\beta$ and we prove that $\mathfrak{M}, x \models_D \Box\beta$. Due to (4.3), it is enough to prove that $\mathfrak{M}, x \models_D \beta \wedge \triangle\beta$. From $\mathfrak{M}, x \models \Box\beta$, it follows that for all $y \in R(x)$, $\mathfrak{M}, y \models \beta$. Using induction hypothesis, we conclude that for all $y \in R(x)$, $\mathfrak{M}, y \models_D \beta$. Thus $\mathfrak{M}, x \models_D \beta$ as R is reflexive relation. From Proposition 4.17 and the fact that for all $y \in R(x)$, $\mathfrak{M}, y \models_D \beta$, we obtain $\mathfrak{M}, x \models_D \triangle\beta$. We have $\mathfrak{M}, x \models_D \beta \wedge \triangle\beta$ and hence $\mathfrak{M}, x \models_D \Box\beta$.
For the other part, let $\mathfrak{M}, x \models_D \Box\beta$. Then by (4.3), $\mathfrak{M}, x \models_D \beta \wedge \triangle\beta$. Using reflexivity of R and induction hypothesis, it follows that $\mathfrak{M}, x \models \Box\beta$.
Next, Assume $\mathfrak{M}, x \models \triangle\beta$ and towards a contradiction, let $\mathfrak{M}, x \not\models_D \triangle\beta$. Then there exist $y_1, y_2 \in R(x)$ such that $\mathfrak{M}, y_1 \models_D \beta$ and $\mathfrak{M}, y_2 \models_D \neg\beta$ (cf. Proposition 4.17). From induction hypothesis, we obtain $\mathfrak{M}, y_1 \models \beta$ and $\mathfrak{M}, y_2 \models \neg\beta$. This is not possible as $\mathfrak{M}, x \models \triangle\beta$, that is, $\mathfrak{M}, x \models \Box\beta \vee \Box\neg\beta$. Thus $\mathfrak{M}, x \models_D \triangle\beta$.
The converse follows easily by induction hypothesis due to (4.2). □

As a consequence of Proposition 4.30, it follows that α is valid in the class $\mathcal{S}4$ of frames with respect to the standard modal logic semantics if and only if α is valid in the class $\mathcal{S}4$ of frames with respect to the semantics presented in Section 4.3.1. Therefore, using the soundness and completeness theorems for the modal system $\mathbb{CLS}4$ with respect to the class $\mathcal{S}4$ of frames [22] and the well-known soundness and completeness theorems for the modal system S4, we obtain the following result.

Proposition 4.31. *For every wff* α*, we have*

$$\vdash_{\mathbb{CLS}4} \alpha \textit{ if and only if } \vdash_{\mathrm{S4}} \alpha.$$

Let us use $\mathbb{CLS}4B$ to denote the modal system $\mathbb{CLS}4$ + axiom B(□). We know that modal system S5 consists of modal system S4 along with axiom B(□). So, modal system $\mathbb{CLS}4B$ is syntactically equivalent to modal system S5. That is, for all wffs α,

$$\vdash_{\mathrm{S5}} \alpha \text{ if and only if } \vdash_{\mathbb{CLS}4B} \alpha.$$

Hence, from (4.1), we obtain the following for each wff α:

$$\vdash_{\mathbb{CLS}4B} \alpha \text{ if and only if } \models_{P_4} \alpha.$$

Thus, it is shown that axioms of $\mathbb{CLS}4B$ gives a axiomatization of modal system S5 and P_4 gives a covering semantics to this system.

4.6. Conclusion

In [91], seventeen covering based rough set systems are presented with the purpose of investigating their modal logic aspects. These systems can be divided in two categories depending on whether the lower and upper approximation operators are dual or not. The systems $C_1, C_2, C_3, C_4, C_5, C_{\mathrm{Gr}}, P_1, P_2, P_3$ and P_4 belong to the first category. The non-dual group consists of the systems $C_*, C_-, C_\#, C_@, C_+, C_\%$ and C_t. In [61], modal systems for the covering systems C_2, C_4, C_5 and P_1 were presented, but the question of modal system for $C_1, C_3, C_{\mathrm{Gr}}, P_2$ and P_3 remained unanswered. Note that the axiom K is not valid under the semantics corresponding to these later systems and hence the corresponding modal systems, if exist, are not normal. In this chapter, a modal system $\mathrm{ML}_{\mathrm{C}_1}$ for covering systems C_1, C_{Gr} and P_3 is presented. $\mathrm{ML}_{\mathrm{C}_1}$ turns out to be the monotonic modal system EMN with two more axioms T(□) and 4(□). Rough set systems P_2 and C_3 are left for future work.

The chapter also presents a formal study of the boundary operators. The σ semantics, where $\sigma \in \{C_2, C_5, P_1\}$, is used to obtain the σ^* semantics for the language where the modal operator corresponds to the boundary operator. It is shown that the modal system $\mathbb{CLS}4$ corresponds to C_2^* and C_5^* semantics, and $\mathbb{CLTB}$ is the system for P_1^* semantics. The question of axiom systems for the boundary operators relative to the remaining covering systems is still open. It is also worth to mention that our study leads us to the study of a few contingency logics via covering semantics based on systems C_2, C_5, and P_1, and finds its connection with rough sets. It will be interesting to investigate the connections between the other contingency logics and covering semantics. At the end, an alternative modal axiom system for P_4 is presented in terms of contingency modality.

As far as non-dual systems are concerned, the modal logics for the covering systems $C_*, C_-, C_\#, C_@, C_+, C_\%$ are still pending, but the issue is resolved for C_t in [19]. The lower and upper approximations in C_t are defined as below [48].

$$\underline{C_t}(A) := \bigcup\{E \in \mathbb{D} : E \subseteq A\}$$
$$\overline{C_t}(A) := \bigcap\{E \in \mathbb{D} : A \subseteq E\},$$

where $\mathbb{D} := \{E \subseteq W : E = \bigcup_{x \in B} N_{\mathcal{C}}(x) \text{ for some } B \subseteq W\}$. It can be observed that the above operators are not dual. A modal system $\mathrm{ML}_{\mathrm{C_t}}$ is obtained and corresponding soundness and completeness theorems are proved. As expected, the language of $\mathrm{ML}_{\mathrm{C_t}}$ contains two primitive unary modal operators $\Box$ and $\Diamond$ and it consists of the following axioms and inference rules.

- $\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)$
- $\neg(\Diamond\alpha \to \Diamond\beta) \to \Diamond(\neg(\alpha \to \beta))$
- $\Box\alpha \to \alpha$
- $\alpha \to \Diamond\alpha$
- $\Box\alpha \to \Box\Box\alpha$
- $\Diamond\Diamond\alpha \to \alpha$
- $\Diamond\Box\alpha \to \Box\alpha$
- $\Diamond\alpha \to \Box\Diamond\alpha$
- From α and $\alpha \to \beta$, infer β
- From α, infer $\Box\alpha$
- From $\neg\alpha$, infer $\neg\Diamond\alpha$.

It is also pertinent to note that for non-normal modal systems in [67], mostly neighborhood semantics are studied. In this chapter, a new semantics based on covering of a set is presented for one such modal system namely $\mathrm{ML}_{\mathrm{C_1}}$. Relationship between covering semantics and neighborhood semantics may be investigated in future. However, this study has partly been initiated in chapter 15 of [70].

CHAPTER 5

A MODAL LOGIC FOR GENERALIZED ROUGH SET MODEL BASED ON SUBSET APPROXIMATION STRUCTURE

In this dissertation, we have already seen a few generalizations of Pawlak's approximation space. In literature, one can find several other generalizations and the notions of approximations based on these generalizations. In [75], a generalized approximation space is defined as a tuple $AS := (W, I, v)$, where I is a function from W to $\wp(W)$, and v is a function from $\wp(W) \times \wp(W) \to [0,1]$ measuring the degree of inclusion of sets. I and v are called *uncertainty function* and *inclusion function*, respectively. The lower and upper approximations are defined in AS as follows.

$$AS_*(X) := \{x \in W : v(I(x), X) = 1\},$$
$$AS^*(X) := \{x \in W : v(I(x), X) > 0\}.$$

Given a Pawlak's approximation space (W, R) with finite W, it can be viewed as an instance of the generalized approximation space given above by taking $I(x) = R(x)$ and $v(X, Y) = \frac{|X \cap Y|}{|X|}$ for any $X, Y \subseteq W$, $X \neq \emptyset$. In this case, $AS_*(X)$ and $AS^*(X)$ coincides with the standard lower and upper approximations $\underline{X}_R$ and $\overline{X}_R$, respectively.

In [93], the generalized approximation space discussed above is further generalized by considering a system $\mathfrak{R} := (W, I, v, P)$, where I, v are defined as above and $P : I(W) \to \{0, 1\}$, known as *structurality function*. Given a $R \subseteq W \times W$, the following lower and upper approximations are proposed.

$$L_R(\mathfrak{R}, X) := \{x \in W : P(I(x)) = 1 \ \& \text{ for all } y \text{ such that } (x,y) \in R \text{ implies } v(I(y), X) = 1\},$$
$$U_R(\mathfrak{R}, X) := \{x \in W : P(I(x)) = 1 \ \& \text{ for all } y \text{ such that } (x,y) \in R \text{ implies } v(I(y), X) > 0\}.$$

The rough set model based on neighborhood is also well studied in the literature [107, 113]. In this approach, each element x of the domain W is associated with a subset $n(x) \subseteq W$, called a *neighborhood of* x. A neighborhood system $NS(x)$ of x is a non-empty family of neighborhoods of x. A *neighborhood system* of W, denoted by $NS(W)$, is the collection of $NS(x)$ for all $x \in W$. The tuple $(W, NS(W))$ is known as *Frechet (V)Space*. Various lower and upper approximations are proposed based on the neighborhood system, and we refer to [107, 113] for details.

The notions of approximation operators discussed so far, including the one based on neighborhood system, depend on all the objects of the underlined domain. However, it may be preferable to approximate concepts relative to some subsets of the domain instead of the whole domain in some cases. In this chapter, we study approximations based on this approach. Our study will be based on the following generalization of approximation space.

Definition 5.1. A *subset approximation structure* (in brief, SAS) is defined as a tuple $\mathfrak{F} := (W, \rho, R)$, where W is a non-empty set of objects, ρ is a non-empty collection of non-empty subsets of W, and $R \subseteq W \times W$.

In addition to a relation R, a SAS consists of a collection of subsets of W, called the *sets of interest*. For each $x \in W$, let us use ρ_x to denote the collection $\{U \in \rho : x \in U\}$. The elements of ρ_x are called *neighborhood* of x. These sets are used in determining the relative approximations of subsets of the domain. In the next section, we propose and study approximations based on SAS. As mentioned earlier, various types of relations are worth studying in the context of approximations in rough set theory (c.f. [43,56,66,69,93]). Therefore, we explore the sub-classes of SASs listed in Table 5.1. It is worth mentioning that given a SAS (W, ρ, R), (W, ρ) can be viewed as a Frechet (V)Space, where we take ρ_x to be the neighborhood system $NS(x)$ of x.

To the best of our knowledge, we have not seen any proposals of logics describing rough sets (including normal modal systems) that can capture the approximations of sets proposed in this chapter. This is due to the fact that the proposed approximations are defined relative to elements of ρ. In Section 5.2, we introduce a modal logic for SAS that can be used for this purpose. The satisfiability of a wff is evaluated at an ordered pair whose first component corresponds to an object from the domain of discourse, and the

Class of SAS	Defining condition	Class of SAS	Defining condition
A	Class of all SAS	$\mathsf{A_{rs}}$	$\mathsf{A_r} \cap \mathsf{A_s}$
$\mathsf{A_r}$	R is reflexive	$\mathsf{A_{rt}}$	$\mathsf{A_r} \cap \mathsf{A_t}$
$\mathsf{A_t}$	R is transitive	$\mathsf{A_{st}}$	$\mathsf{A_s} \cap \mathsf{A_t}$
$\mathsf{A_s}$	R is symmetric	$\mathsf{A_e}$	$\mathsf{A_{rs}} \cap \mathsf{A_t}$

Table 5.1. Classes of SASs

second component corresponds to a set from ρ. Section 5.3 is intended to discuss the interpretation of proposed logic wffs within the framework of rough set theory. Section 5.4 provides sound and complete modal systems for different classes of SASs. Section 5.5 is devoted to the proof of the corresponding completeness results. Section 5.6 presents a comparison of the proposed semantics with the well-known multi-modal logic semantics. This study also leads us to the fact that the problem of the decidability of the proposed logics is equivalent to that of some known multi-modal logics. This result is useful in obtaining the decidability results for the proposed logic. Section 5.7 discusses the decidability problems. Section 5.8 presents some invariance results related to the presented logic. We also return to the issue of the expressibility power of the logic and provide a few classes of SASs that can be defined through wffs of the logic. Section 5.9 concludes the chapter.

5.1. Notion of approximations based on SASs

We begin with the following notion of approximations defined relative to a set from ρ.

Definition 5.2. Consider a SAS $\mathfrak{F} := (W, \rho, R)$ and $S \in \rho$. We define the lower and upper approximations of a set $Z \subseteq W$ relative to S, denoted by $[R]_S(Z)$ and $\langle R\rangle_S(Z)$ respectively, as follows:

$$[R]_S(Z) := \{z \in S : R(z) \cap S \subseteq Z\}, \text{ and}$$

$$\langle R\rangle_S(Z) := \{z \in S : R(z) \cap S \cap Z \neq \emptyset\}.$$

For a given $S \in \rho$, consider the approximation space $(S, R|_S)$ where $R|_S$ is restriction of the relation R on the set S, that is, $(x, y) \in R|_S$ if and only if $x, y \in S$ and $(x, y) \in R$. Let $Z \subseteq W$. Then one can verify that

$$[R]_S(Z) = \underline{(Z \cap S)}_{R|_S}, \text{ and}$$
$$\langle R \rangle_S(Z) = \overline{(Z \cap S)}_{R|_S}.$$

We list some properties of aforementioned approximations in the following proposition. For a better understanding, modal logic notations are used to label the properties.

Proposition 5.3. **(Dual):** $[R]_S(X^c) = (\langle R \rangle_S(X))^c \cap S$.

(K): $[R]_S(X^c \cup Y) \subseteq ([R]_S(X))^c \cup [R]_S(Y)$.

(T): *If* $\mathfrak{F} \in \mathsf{A_r}$, *then we have* $[R]_S(X) \subseteq X$.

(B): *If* $\mathfrak{F} \in \mathsf{A_s}$, *then we have* $X \cap S \subseteq [R]_S(\langle R \rangle_S(X))$.

(4): *If* $\mathfrak{F} \in \mathsf{A_t}$, *then we have* $[R]_S(X) \subseteq [R]_S([R]_S(X))$.

The proof of the above proposition is very much in the line of the proof of similar properties of the standard rough set approximation operators [74], and we omit the same here.

Given a set Z, in a study under the framework of SASs, the following sets may be of interest.

- The set consisting of objects that belongs to the positive region of Z relative to all the sets of interest.
- The set consisting of objects that belongs to the positive region of Z relative to some sets of interest.

Accordingly, we define the following concepts of approximations. Suppose $\mathfrak{F} := (W, \rho, R)$ be a SAS and $X \subseteq W$.

Definition 5.4. The *necessity lower approximation* $L^n_{\mathfrak{F}}(X)$, *possibility lower approximation* $L^p_{\mathfrak{F}}(X)$, *necessity upper approximation* $U^n_{\mathfrak{F}}(X)$, and *possibility upper approximation* $U^p_{\mathfrak{F}}(X)$ are defined as follows.

$$L^n_{\mathfrak{F}}(X) := \bigcap_{S \in \rho} [R]_S(X); \qquad L^p_{\mathfrak{F}}(X) := \bigcup_{S \in \rho} [R]_S(X);$$
$$U^n_{\mathfrak{F}}(X) := \bigcap_{S \in \rho} \langle R \rangle_S(X); \qquad U^p_{\mathfrak{F}}(X) := \bigcup_{S \in \rho} \langle R \rangle_S(X).$$

Thus, $L^n_{\mathfrak{F}}(X)$ ($L^p_{\mathfrak{F}}(X)$) consists of elements that are in the lower approximation of the concept X relative to all sets (respectively, some set) from ρ. Similarly, $U^n_{\mathfrak{F}}(X)$ ($U^p_{\mathfrak{F}}(X)$) consists of elements that are in the upper approximation of the concept X relative to all sets (respectively, some set) from ρ. Observe that $L^n_{\mathfrak{F}}(X) \subseteq L^p_{\mathfrak{F}}(X)$ and $U^n_{\mathfrak{F}}(X) \subseteq U^p_{\mathfrak{F}}(X)$ for all X, and $\underline{X}_R \subseteq L^p_{\mathfrak{F}}(X)$ and $\overline{X}_R \subseteq U^p_{\mathfrak{F}}(X)$ provided $W \in \rho$. Further, one can identify the generalized approximation space (W, R) with the SAS $\mathfrak{F} := (W, \rho, R)$, where $\rho := \{W\}$. Moreover, in this case, we get

$$L^n_{\mathfrak{F}}(X) := \underline{X}_R = L^p_{\mathfrak{F}}(X) \text{ and } U^n_{\mathfrak{F}}(X) := \overline{X}_R = U^p_{\mathfrak{F}}(X).$$

Note 5.5. In order to make the notation simple, we will avoid using $\mathfrak{F}$ as the subscript in the above definitions. For instance, we will write $L^n_{\mathfrak{F}}(X)$ simply as $L^n(X)$.

We list below a few properties of the approximations defined above. As earlier, modal logic notations are used to label the properties.

Proposition 5.6. *Let $\mathfrak{F} \in \mathsf{A}$ be a SAS. Then the following hold.*

(Dual): *(a) $L^n(X^c) \subseteq (U^p(X))^c$.*

(b) $L^p(X^c) \subseteq (U^n(X))^c$.

(c) $\bigcap_{S\in\rho} S \cap (U^p(X))^c = \bigcap_{S\in\rho} S \cap L^n(X^c) \subseteq L^n(X^c)$.

(d) $\bigcap_{S\in\rho} S \cap (U^n(X))^c \subseteq L^p(X^c)$.

(K): *$L^n(X^c \cup Y) \subseteq (L^n(X))^c \cup L^n(Y)$.*

(T): *If $\mathfrak{F} \in \mathsf{A_r}$, then the following is true.*

(a) $L^n(X) \subseteq X$.

(b) $X \subseteq U^n(X)$ if and only if $X \subseteq \bigcap_{S\in\rho} S$.

(c) $L^p(X) \subseteq X$.

(d) $X \subseteq U^p(X)$ if and only if $X \subseteq \bigcup_{S\in\rho} S$.

(B): *Let $\mathfrak{F} \in \mathsf{A_s}$. Then, $X \subseteq L^n(U^p(X))$ if and only if $X \subseteq \bigcap_{S\in\rho} S$.*

(4): *Let $\mathfrak{F} \in \mathsf{A_t}$. Then $L^n(X) \subseteq L^n(L^n(X))$ if $R(W) \subseteq \bigcap_{S\in\rho} S$, where $R(W) = \{y \in W : (x, y) \in R$ for some $x \in W\}$.*

Proof. **(Dual)**(a) Let $x \in L^n(X^c)$ and we prove that $x \in (U^p(X))^c$. If possible, let $x \in U^p(X)$. Then there exists an $S_1 \in \rho$ such that $x \in \langle R\rangle_{S_1}(X)$, therefore, $R(x) \cap S_1 \cap X \neq \emptyset$. From $x \in L^n(X^c)$, we get $x \in [R]_S(X^c)$ for all $S \in \rho$. Thus $x \in [R]_{S_1}(X^c)$ and

$R(x) \cap S_1 \subseteq X^c$. We have $R(x) \cap S_1 \subseteq X^c$ and $R(x) \cap S_1 \cap X \neq \emptyset$, which is not possible. Hence $L^n(X^c) \subseteq (U^p(X))^c$.

(b) Let $x \in L^p(X^c)$ and we show that $x \in (U^n(X))^c$. If possible, let $x \in U^n(X)$. Then $x \in \langle R \rangle_S(X)$ for all $S \in \rho$. From $x \in L^p(X^c)$, we obtain an $S_1 \in \rho$ such that $x \in [R]_{S_1}(X^c)$. That is, $R(x) \cap S_1 \subseteq X^c$. Also, we have $x \in \langle R \rangle_{S_1}(X)$. This gives $R(x) \cap S_1 \cap X \neq \emptyset$. This is a contradiction. Hence $L^p(X^c) \subseteq (U^n(X))^c$.

(c) Let us first show that $\bigcap_{S \in \rho} S \cap (U^p(X))^c \subseteq \bigcap_{S \in \rho} S \cap L^n(X^c)$. So, let us consider $x \in \bigcap_{S \in \rho} S \cap (U^p(X))^c$ and we prove that $x \in \bigcap_{S \in \rho} S \cap (L^n(X^c))$. Let us take an $S_1 \in \rho$. Since $x \in \bigcap_{S \in \rho} S$, we get $x \in S_1$. We need to show that $x \in [R]_{S_1}(X^c)$. From $x \in (U^p(X))^c$, we obtain for all $S \in \rho$, $R(x) \cap S \subseteq X^c$. Thus $R(x) \cap S_1 \subseteq X^c$ and $x \in [R]_{S_1}(X^c)$.

For the converse, let $x \in \bigcap_{S \in \rho} S \cap L^n(X^c)$ and we prove that $x \in \bigcap_{S \in \rho} S \cap (U^p(X))^c$. If possible, let $x \in U^p(X)$. Then there exist an $S_1 \in \rho$ such that $x \in \langle R \rangle_{S_1}(X)$. From $x \in L^n(X^c)$, we obtain $x \in [R]_{S_1}(X^c)$. Thus $R(x) \cap S_1 \subseteq X^c$. Since $x \in \langle R \rangle_{S_1}(X)$, we get a $y \in R(x) \cap S_1 \cap X$. This is not possible. Hence $\bigcap_{S \in \rho} S \cap L^n(X^c) \subseteq \bigcap_{S \in \rho} S \cap (U^p(X))^c$.

(d) Let $x \in \bigcap_{S \in \rho} S \cap (U^n(X))^c$ and we show that $x \in L^p(X^c)$. From $x \in (U^n(X))^c$, we get $x \notin U^n(X)$. Then there exists an $S_1 \in \rho$ such that $x \notin \langle R \rangle_{S_1}(X)$. This gives $R(x) \cap S_1 \cap X = \emptyset$, that is, $R(x) \cap S_1 \subseteq X^c$. Since $x \in S_1$ and $R(x) \cap S_1 \subseteq X^c$, $x \in [R]_{S_1}(X^c)$. Thus $x \in L^p(X^c)$.

(K). Assume that $x \in L^n(X^c \cup Y)$ and we prove that $x \in (L^n(X))^c \cup L^n(Y)$. If $x \in (L^n(X))^c$, then we are done. So, suppose $x \notin (L^n(X))^c$ and we show that $x \in L^n(Y)$. Let us take $S_1 \in \rho$. Since $x \in L^n(X^c \cup Y)$, $x \in [R]_S(X^c \cup Y)$ for all $S \in \rho$. Thus $x \in S_1$ and $x \in [R]_{S_1}(X^c \cup Y)$. Since $x \in L^n(X)$, we get $x \in [R]_{S_1}(X)$. Let us take a $y \in R(x) \cap S_1$ and we show that $y \in Y$. From $x \in [R]_{S_1}(X)$ and $x \in [R]_{S_1}(X^c \cup Y)$, we obtain $y \in Y$. Thus $x \in L^n(Y)$.

(T) Let $\mathfrak{F} \in \mathsf{A_r}$ and we show the following.

(a) Assume that $x \in L^n(X)$ and we show that $x \in X$. From $x \in L^n(X)$, we get $x \in [R]_S(X)$ for all $S \in \rho$. Since ρ is a non-empty collection, there exists an $S_1 \in \rho$ and $x \in S_1$. As $\mathfrak{F} \in \mathsf{A_r}$, $x \in R(x)$. We have $R(x) \cap S_1 \subseteq X$ and $x \in R(x) \cap S_1$. Hence $x \in X$.

(b) Let $X \subseteq U^n(X)$ and we show that $X \subseteq \bigcap_{S \in \rho} S$. Assume that $x \in X$ and consider an S_1 from ρ. We obtain $x \in U^n(X)$. This gives $x \in \langle R \rangle_S(X)$ for all $S \in \rho$. Hence $x \in S_1$.

For the other direction, suppose that $X \subseteq \bigcap_{S\in\rho} S$ and we show that $X \subseteq U^n(X)$. So, let $x \in X$ and let us take $S_1 \in \rho$. We have $x \in S_1$ and $x \in R(x) \cap X$. Thus $x \in \langle R\rangle_{S_1}(X)$.

(c) Suppose that $x \in L^p(X)$ and we show that $x \in X$. From $x \in L^p(X)$, there exist an $S_1 \in \rho$ such that $x \in [R]_{S_1}(X)$. Since $x \in R(x) \cap S_1$ and $x \in [R]_{S_1}(X)$, we obtain $x \in X$.

(d) Let $X \subseteq U^p(X)$ and we show that $X \subseteq \bigcup_{S\in\rho} S$. So, let us take a $x \in X$. Then $x \in U^p(X)$ and there exists an $S_1 \in \rho$ such that $x \in \langle R\rangle_{S_1}(X)$. That is $x \in S_1$ and $x \in \bigcup_{S\in\rho} S$.

For the converse, let $X \subseteq \bigcup_{S\in\rho} S$ and we show that $X \subseteq U^p(X)$. So, let $x \in X$. From $x \in \bigcup_{S\in\rho} S$, there exists an $S_1 \in \rho$ such that $x \in S_1$. We have $x \in S_1$ and $x \in R(x) \cap X$. Thus $x \in \langle R\rangle_{S_1}(X)$ and $x \in U^p(X)$.

(B) Let $\mathfrak{F} \in \mathsf{A_s}$. Assume that $X \subseteq L^n(U^p(X))$ and we prove that $X \subseteq \bigcap_{S\in\rho} S$. So, let $x \in X$. Then $x \in L^n(U^p(X))$. We obtain for all $S \in \rho$, $x \in [R]_S(U^p(X))$ and hence $x \in S$ for all $S \in \rho$.

For the converse, let $X \subseteq \bigcap_{S\in\rho} S$ and we show that $X \subseteq L^n(U^p(X))$. So, let $x \in X$ and let us take an $S_1 \in \rho$. Since $X \subseteq \bigcap_{S\in\rho} S$, we get $x \in S_1$. We prove that $x \in [R]_{S_1}(U^p(X))$. Let us consider $y \in R(x) \cap S_1$. Since R is symmetric, we obtain $x \in R(y)$. We have $x \in R(y) \cap S_1 \cap X$ and $y \in S_1$. Thus $y \in U^p(X)$.

(4) Let $\mathfrak{F} \in \mathsf{A_t}$ and $R(W) \subseteq \bigcap_{S\in\rho} S$. We prove that $L^n(X) \subseteq L^n(L^n(X))$. So, suppose that $x \in L^n(X)$. Consider an $S_1 \in \rho$. We have $x \in S_1$ as $x \in L^n(X)$. We show that $x \in [R]_{S_1}(L^n(X))$. Let us take $y \in R(x) \cap S_1$. Again consider $S_2 \in \rho$ and we show that $y \in [R]_{S_2}(X)$. Since $R(W) \subseteq \bigcap_{S\in\rho} S$, we get $y \in S_2$. Let us take $z \in R(y) \cap S_2$. As R is transitive, we get $z \in R(x)$. We have $z \in R(x) \cap S_2$ and $x \in L^n(X)$. Thus $z \in X$ and $L^n(X) \subseteq L^n(L^n(X))$. □

Note that although we have $L^n(X^c) \subseteq (U^p(X))^c$, but we do not have the reverse inclusion $(U^p(X))^c \subseteq L^n(X^c)$. This shows that the operator L^n is not the dual of the operator U^p, however we have the property Dual(c). Further, while we have $L^n(X) \subseteq X$, we do not have $X \subseteq U^p(X)$ as L^n is not the dual of the operator U^p. Similarly, we have the property $X \subseteq L^n(U^p(X))$ and $L^n(X) \subseteq L^n(L^n(X))$ under some additional restrictions as shown in (B) and (4). We have similar observations for the operator L^p.

Consider the following example.

Example 5.7. Let us consider a SAS (W, ρ, R), where $W := \{1, 2, \ldots, 10\}$, $\rho := \{S, T\}$, $S := \{1, 2, 3, 4, 7, 8\}$, $T := \{1, 2, 3, 4, 6, 8\}$, and $W/R := \{\{1, 5\}, \{2, 6\}, \{4\}, \{3, 7, 8\}, \{9, 10\}\}$. Let us consider the set $X := \{1, 2, 3, 4\}$. One can easily verify the following:

- $\underline{X}_R = \{4\}$; $\overline{X}_R = W \setminus \{9, 10\}$;
- $[R]_S(X) = \{1, 2, 4\}$; $\langle R \rangle_S(X) = \{1, 2, 3, 4, 7, 8\}$;
- $L^p(X) = \{1, 2, 4\}$; $U^p(X) = \{1, 2, 3, 4, 6, 7, 8\}$;
- $L^n(X) = \{1, 4\}$; $U^n(X) = \{1, 2, 3, 4, 8\}$.

Observe that relative to the information provided by the approximation space (W, R), 1 and 2 are undecidable elements of the set X, but when we consider SAS (W, ρ, R), both the elements move to the lower approximation of X relative to some set of interest (in fact, relative to the set S). Also, note that although 2 belongs to the lower approximation of X relative to some set of interest, it is not the case relative to all the sets of interest. On the other hand, 1 belongs to the lower approximation of X relative to all the sets of interest.

5.2. A modal logic for subset approximation structures

In this section, we propose a modal logic with semantics based on SASs. Let us begin with the syntax of the logic.

5.2.1. Syntax

We consider the modal logic language $\mathcal{L}(\Box_1, \Box)$ with two unary modal connectives $\Box_1, \Box$. That is, wffs of $\mathcal{L}(\Box_1, \Box)$ are defined recursively as follows.

$$\top \mid p \mid \neg\alpha \mid \alpha \wedge \beta \mid \Box_1\alpha \mid \Box\alpha,$$

where p is a propositional variable and α, β are wffs. As earlier, we use PV to denote the set of all propositional variables.

Along with the usual derived connectives $\bot, \vee, \rightarrow, \leftrightarrow$, we have the derived modal connectives $\Diamond_1$ and $\Diamond$ defined below.

$$\Diamond_1\alpha := \neg\Box_1\neg\alpha, \text{ and } \Diamond\alpha := \neg\Box\neg\alpha.$$

Consider the fragment $\mathcal{L}(\Box_1)$ of $\mathcal{L}(\Box_1, \Box)$ consisting of wffs that does not involve the modal operator $\Box$. Similarly, $\mathcal{L}(\Box)$ denotes the fragment of $\mathcal{L}(\Box_1, \Box)$ not having the wffs involving the modal operator $\Box_1$.

5.2.2. Semantics

As expected, the semantics is directly based on SASs. Therefore, we take into account the following notion of model.

Definition 5.8. A tuple $\mathfrak{M} := (\mathfrak{F}, m)$ is called a model of $\mathcal{L}(\Box_1, \Box)$, where

- $\mathfrak{F} := (W, \rho, R)$ is a SAS,
- $m : PV \to \wp(W)$ is a *valuation function.*

The set $\{(x, U) \in W \times \rho : x \in U\}$ is denoted by $E(\mathfrak{F})$. Also recall that for $x \in W$, $\rho_x := \{U \in \rho : x \in U\}$. The *satisfiability* of a wff α in a model $\mathfrak{M} := (\mathfrak{F}, m)$ at $(x, U) \in E(\mathfrak{F})$, denoted as $\mathfrak{M}, x, U \models \alpha$, is defined inductively as follows.

$$
\begin{array}{lcl}
\mathfrak{M}, x, U \models \top & & \text{always.} \\
\mathfrak{M}, x, U \models p & \Longleftrightarrow & x \in m(p),\ \text{for } p \in PV. \\
\mathfrak{M}, x, U \models \neg\alpha & \Longleftrightarrow & \mathfrak{M}, x, U \not\models \alpha. \\
\mathfrak{M}, x, U \models \alpha \wedge \beta & \Longleftrightarrow & \mathfrak{M}, x, U \models \alpha \text{ and } \mathfrak{M}, x, U \models \beta. \\
\mathfrak{M}, x, U \models \Box_1\alpha & \Longleftrightarrow & \text{for every } V \in \rho_x,\ \mathfrak{M}, x, V \models \alpha. \\
\mathfrak{M}, x, U \models \Box\alpha & \Longleftrightarrow & \text{for every } y \in U \text{ with } (x, y) \in R,\ \mathfrak{M}, y, U \models \alpha.
\end{array}
$$

Note that the satisfiability of a propositional variable p at $(x, U) \in E(\mathfrak{F})$ depends on x, but not on U. Also, the modal operator $\Box_1$ captures the quantification over elements of ρ. We refer to [13] for more details for such modal operator.

The satisfiability condition for the derived connectives are obtained as follows.

$$
\begin{array}{lcl}
\mathfrak{M}, x, U \models \Diamond\alpha & \Longleftrightarrow & \text{there is a } y \in U \text{ with } (x, y) \in R \text{ such that } \mathfrak{M}, y, U \models \alpha. \\
\mathfrak{M}, x, U \models \Diamond_1\alpha & \Longleftrightarrow & \text{there is a } V \in \rho_x \text{ such that } \mathfrak{M}, x, V \models \alpha.
\end{array}
$$

In the line of the standard definition of truth set of a wff in a modal logic, we consider the following sets. For any wff α, model $\mathfrak{M}$ and $U \in \rho$, let

$$[\![\alpha]\!]_{\mathfrak{M},U} := \{x \in W : \mathfrak{M}, x, U \models \alpha\};$$

$$[\![\alpha]\!]^*_{\mathfrak{M}} := \{x \in W : \mathfrak{M}, x, U \models \alpha \text{ for some } (x, U) \in E(\mathfrak{F})\};$$

$$[\![\alpha]\!]_{\mathfrak{M}} := \{(x, U) \in E(\mathfrak{F}) : \mathfrak{M}, x, U \models \alpha\}.$$

In the classical rough set theory, a concept is represented/given by a subset of the domain of the underlined information system. Accordingly, a wff of modal logic proposed for classical rough set theory represents a concept given by the truth set of the wff. Similarly, a wff α of the above proposed language also represents a concept (relative to the set $U \in \rho$) given by $[\![\alpha]\!]_{\mathfrak{M},U}$. Let us see the following proposition.

Proposition 5.9. *Let $\mathfrak{M} = (\mathfrak{F}, m)$, where $\mathfrak{F} = (W, \rho, R)$, be a model, $\alpha \in \mathcal{L}(\Box_1)$, and $\mathfrak{M}, x, U_1 \models \alpha$ for some $(x, U_1) \in E(\mathfrak{F})$. Then for all $(x, U) \in E(\mathfrak{F})$, $\mathfrak{M}, x, U \models \alpha$.*

Proof. The proof follows by induction on the complexity of the wff α. First let us see the case when α is a propositional variable. From $\mathfrak{M}, x, U_1 \models p$, we obtain $x \in m(p)$. We need to show that $\mathfrak{M}, x, U \models p$ for all $(x, U) \in E(\mathfrak{F})$. So, let us take a $U \in \rho$ with $x \in U$. Then $\mathfrak{M}, x, U \models p$ as $x \in m(p)$. Boolean cases can be proved easily.
Next, consider the case when α is of the form $\Box_1\beta$. Consider a $U \in \rho$ with $x \in U$ and we prove that $\mathfrak{M}, x, U \models \Box_1\beta$. So, let us take $U_2 \in \rho$ with $x \in U_2$. Then $\mathfrak{M}, x, U_2 \models \beta$ as we have $\mathfrak{M}, x, U_1 \models \Box_1\beta$. Thus $\mathfrak{M}, x, U \models \Box_1\beta$. $\Box$

Here, it is pertinent to note that for $\alpha \in \mathcal{L}(\Box_1)$,

$$[\![\alpha]\!]^*_{\mathfrak{M}} := \{x \in W : \mathfrak{M}, x, U \models \alpha \text{ for all } (x, U) \in E(\mathfrak{F})\} \cap (\bigcup_{U \in \rho} U), \text{ and} \tag{5.1}$$

$$(\bigcap_{U \in \rho} U) \cap [\![\alpha]\!]^*_{\mathfrak{M}} := [\![\alpha]\!]_{\mathfrak{M},U} \text{ for all } U \in \rho. \tag{5.2}$$

Hence, we may say that the wffs from $\mathcal{L}(\Box_1)$ represent concepts that do not depend on the individual elements of ρ but on the whole ρ.

We conclude the section with the definitions of the standard concepts of the validity and satisfaction of wffs. We say that a wff α is *valid* in a model $\mathfrak{M} := (\mathfrak{F}, m)$, denoted as $\mathfrak{M} \models \alpha$, if $[\![\alpha]\!]_{\mathfrak{M}} = E(\mathfrak{F})$. A wff α is *valid* in a SAS $\mathfrak{F}$, notation: $\mathfrak{F} \models \alpha$, if $\mathfrak{M} \models \alpha$ for all models $\mathfrak{M}$ based on $\mathfrak{F}$.

Class of Models	Defining condition
Θ	Class of all models having R as a binary relation
Θ_{r}	Class of all elements of Θ having R as a reflexive relation.
Θ_{t}	Class of all elements of Θ having R as a transitive relation.
Θ_{s}	Class of all elements of Θ having R as a symmetric relation.
Θ_{rs}	$\Theta_{\mathsf{r}} \cap \Theta_{\mathsf{s}}$
Θ_{rt}	$\Theta_{\mathsf{r}} \cap \Theta_{\mathsf{t}}$
Θ_{st}	$\Theta_{\mathsf{s}} \cap \Theta_{\mathsf{t}}$
Θ_{e}	$\Theta_{\mathsf{st}} \cap \Theta_{\mathsf{r}}$

Table 5.2. Classes of Models

A wff α is *valid* in a given class $\mathfrak{G}$ of SASs, denoted as $\mathfrak{G} \models \alpha$, if α is valid in every SAS $\mathfrak{F}$ in $\mathfrak{G}$. For a SAS $\mathfrak{F}$, and a set Δ of wffs, we will write $\mathfrak{F} \models \Delta$ if $\mathfrak{F} \models \alpha$ for all $\alpha \in \Delta$. Similarly, for a class $\mathfrak{G}$ of SAS, we write $\mathfrak{G} \models \Delta$ if $\mathfrak{G} \models \alpha$ for all $\alpha \in \Delta$.
A wff α is said to be *satisfiable* in a model $\mathfrak{M}$ if $[\![\alpha]\!]_{\mathfrak{M}} \neq \emptyset$. α is said to be *satisfiable* in a given class $\mathfrak{G}$ of SAS if α is satisfiable in a model $\mathfrak{M} := (\mathfrak{F}, m)$ based on a SAS $\mathfrak{F} \in \mathfrak{G}$.

5.3. Rough set interpretation

We begin with the proposition given below.

Proposition 5.10. *For a model $\mathfrak{M} := (\mathfrak{F}, m)$, where $\mathfrak{F} := (W, \rho, R)$, $\alpha \in \mathcal{L}(\Box_1, \Box)$ and $\beta \in \mathcal{L}(\Box_1)$, the following holds.*

1. $[\![\Box\alpha]\!]_{\mathfrak{M},U} = [R]_U([\![\alpha]\!]_{\mathfrak{M},U})$, $[\![\Diamond\alpha]\!]_{\mathfrak{M},U} = \langle R\rangle_U([\![\alpha]\!]_{\mathfrak{M},U})$;
2. $[\![\Box\beta]\!]_{\mathfrak{M},U} = [R]_U([\![\beta]\!]^*_{\mathfrak{M}})$, $[\![\Diamond\beta]\!]_{\mathfrak{M},U} = \langle R\rangle_U([\![\beta]\!]^*_{\mathfrak{M}})$.

Proof. 1. First we prove that $[\![\Box\alpha]\!]_{\mathfrak{M},U} = [R]_U([\![\alpha]\!]_{\mathfrak{M},U})$. So, let $x \in [\![\Box\alpha]\!]_{\mathfrak{M},U}$ and we show that $x \in [R]_U([\![\alpha]\!]_{\mathfrak{M},U})$. Consider a $y \in U$ such that $(x, y) \in R$. As $x \in [\![\Box\alpha]\!]_{\mathfrak{M},U}$, $\mathfrak{M}, x, U \models \Box\alpha$. Thus $\mathfrak{M}, y, U \models \alpha$, and $y \in [\![\alpha]\!]_{\mathfrak{M},U}$. Hence $x \in [R]_U([\![\alpha]\!]_{\mathfrak{M},U})$.
For the other part, let $x \in [R]_U([\![\alpha]\!]_{\mathfrak{M},U})$ and we prove that $x \in [\![\Box\alpha]\!]_{\mathfrak{M},U}$. Let us take a $y \in U$ such that $(x, y) \in R$. Then $y \in [\![\alpha]\!]_{\mathfrak{M},U}$ due to $x \in [R]_U([\![\alpha]\!]_{\mathfrak{M},U})$. We get $\mathfrak{M}, y, U \models \alpha$ and thus $\mathfrak{M}, x, U \models \Box\alpha$, that is, $x \in [\![\Box\alpha]\!]_{\mathfrak{M},U}$.

Next, we show that $[\![\Diamond\alpha]\!]_{\mathfrak{M},U} = \langle R\rangle_U([\![\alpha]\!]_{\mathfrak{M},U})$. So, assume that $x \in [\![\Diamond\alpha]\!]_{\mathfrak{M},U}$ and we prove that $x \in \langle R\rangle_U([\![\alpha]\!]_{\mathfrak{M},U})$. From $x \in [\![\Diamond\alpha]\!]_{\mathfrak{M},U}$, there exists a $y \in U$ such that $(x, y) \in R$ and $\mathfrak{M}, y, U \models \alpha$. We have $(x, y) \in R$, $y \in U$ and $y \in [\![\alpha]\!]_{\mathfrak{M},U}$. Thus $x \in \langle R\rangle_U([\![\alpha]\!]_{\mathfrak{M},U})$.
For the other part, let $x \in \langle R\rangle_U([\![\alpha]\!]_{\mathfrak{M},U})$ and we prove that $x \in [\![\Diamond\alpha]\!]_{\mathfrak{M},U}$. From $x \in \langle R\rangle_U([\![\alpha]\!]_{\mathfrak{M},U})$, there exists a $y \in U$ such that $(x, y) \in R$ and $y \in [\![\alpha]\!]_{\mathfrak{M},U}$. That is, $\mathfrak{M}, y, U \models \alpha$ and hence $\mathfrak{M}, x, U \models \Diamond\alpha$, that is, $x \in [\![\Diamond\alpha]\!]_{\mathfrak{M},U}$.

2. We need to show that $[\![\Box\beta]\!]_{\mathfrak{M},U} = [R]_U([\![\beta]\!]^*_{\mathfrak{M}})$. So, let $x \in [\![\Box\beta]\!]_{\mathfrak{M},U}$ and we prove that $x \in [R]_U([\![\beta]\!]^*_{\mathfrak{M}})$. Let us take a $y \in U$ with $(x, y) \in R$. Then $\mathfrak{M}, y, U \models \beta$ as $\mathfrak{M}, x, U \models \Box\beta$. Thus, we obtain $y \in [\![\beta]\!]^*_{\mathfrak{M}}$ and $x \in [R]_U([\![\beta]\!]^*_{\mathfrak{M}})$.
For the other part, let us assume that $x \in [R]_U([\![\beta]\!]^*_{\mathfrak{M}})$ and we need to prove that $x \in [\![\Box\beta]\!]_{\mathfrak{M},U}$. Consider a $y \in U$ with $(x, y) \in R$ and we show that $\mathfrak{M}, y, U \models \beta$. From $x \in [R]_U([\![\beta]\!]^*_{\mathfrak{M}})$, we get $y \in [\![\beta]\!]^*_{\mathfrak{M}}$. Thus there exists a $U_1 \in \rho$ such that $y \in U_1$ and $\mathfrak{M}, y, U_1 \models \beta$. Using Proposition 5.9, we obtain $\mathfrak{M}, y, U \models \beta$ and hence $x \in [\![\Box\beta]\!]_{\mathfrak{M},U}$.
Next, we prove that $[\![\Diamond\beta]\!]_{\mathfrak{M},U} = \langle R\rangle_U([\![\beta]\!]^*_{\mathfrak{M}})$. First, let $x \in [\![\Diamond\beta]\!]_{\mathfrak{M},U}$ and we show that $x \in \langle R\rangle_U([\![\beta]\!]^*_{\mathfrak{M}})$. From $x \in [\![\Diamond\beta]\!]_{\mathfrak{M},U}$, there exists a $y \in U$ with $(x, y) \in R$ and $\mathfrak{M}, y, U \models \beta$. Thus $y \in [\![\beta]\!]^*_{\mathfrak{M}}$ and $x \in \langle R\rangle_U([\![\beta]\!]^*_{\mathfrak{M}})$.
For the other direction, suppose that $x \in \langle R\rangle_U([\![\beta]\!]^*_{\mathfrak{M}})$ and we prove that $x \in [\![\Diamond\beta]\!]_{\mathfrak{M},U}$. Since $x \in \langle R\rangle_U([\![\beta]\!]^*_{\mathfrak{M}})$, we get a $y \in U$ such that $(x, y) \in R$ and $y \in [\![\beta]\!]^*_{\mathfrak{M}}$. Thus there exists a $U_1 \in \rho$ with $y \in U_1$ and $\mathfrak{M}, y, U_1 \models \beta$. Then by Proposition 5.9, $\mathfrak{M}, y, U \models \beta$ and thus $\mathfrak{M}, x, U \models \Diamond\beta$. $\Box$

It is evident from Proposition 5.10 that the operators $\Box$ and $\Diamond$ capture the lower and upper approximation, respectively, with respect to relation R relative to the set at which wffs are evaluated. Therefore, Proposition 5.3 translates into the following proposition.

Proposition 5.11. **(K):** $\Theta \models \Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)$.

(T): $\Theta_{\mathsf{r}} \models \Box\alpha \to \alpha$.

(B): $\Theta_{\mathsf{s}} \models \alpha \to \Box\Diamond\alpha$.

(4): $\Theta_{\mathsf{t}} \models \Box\alpha \to \Box\Box\alpha$.

The proof of the above proposition is very standard, and we omit it here.
As far as the necessity and possibility approximations are concerned, we do not have a result similar to Proposition 5.10 for these approximations, although we have some partial

results (cf. Proposition 5.14). But before going to Proposition 5.14, let us first take a note of the following.

Proposition 5.12.

$$[\![\Box_1\alpha]\!]_{\mathfrak{M},U} \cap \bigcap_{V\in\rho} V = \bigcap_{V\in\rho} [\![\alpha]\!]_{\mathfrak{M},V}; \quad [\![\Diamond_1\alpha]\!]_{\mathfrak{M},U} = \bigcup_{V\in\rho} [\![\alpha]\!]_{\mathfrak{M},V} \cap U.$$

Further, we now define the following connectives.

Definition 5.13.

$$\blacktriangle^n\alpha := \Box_1\Box\alpha, \qquad \blacktriangle^p\alpha := \Diamond_1\Box\alpha,$$

$$\blacktriangledown^p\alpha := \neg\blacktriangle^n\neg\alpha, \qquad \blacktriangledown^n\alpha := \neg\blacktriangle^p\neg\alpha.$$

The following proposition addresses how the connectives mentioned above capture possibility and necessity approximation operators.

Proposition 5.14. *Consider a model $\mathfrak{M} = (\mathfrak{F}, m)$ and wffs $\alpha \in \mathcal{L}(\Box_1, \Box)$ and $\beta \in \mathcal{L}(\Box_1)$. Then, the following hold.*

1. $[\![\blacktriangle^n\alpha]\!]_{\mathfrak{M},U} \cap \bigcap_{V\in\rho} V = \bigcap_{V\in\rho} [R]_V([\![\alpha]\!]_{\mathfrak{M},V})$, $[\![\blacktriangledown^p\alpha]\!]_{\mathfrak{M},U} = \bigcup_{V\in\rho} \langle R\rangle_V([\![\alpha]\!]_{\mathfrak{M},V}) \cap U$;
2. $[\![\blacktriangle^p\alpha]\!]_{\mathfrak{M},U} = \bigcup_{V\in\rho} [R]_V([\![\alpha]\!]_{\mathfrak{M},V}) \cap U$, $[\![\blacktriangledown^n\alpha]\!]_{\mathfrak{M},U} \cap \bigcap_{V\in\rho} V = \bigcap_{V\in\rho} \langle R\rangle_V([\![\alpha]\!]_{\mathfrak{M},V})$;
3. $[\![\blacktriangle^n\beta]\!]_{\mathfrak{M},U} \cap \bigcap_{V\in\rho} V = L^n([\![\beta]\!]^*_{\mathfrak{M}})$, $[\![\blacktriangledown^p\beta]\!]_{\mathfrak{M},U} = U^p([\![\beta]\!]^*_{\mathfrak{M}}) \cap U$;
4. $[\![\blacktriangle^p\beta]\!]_{\mathfrak{M},U} = L^p([\![\beta]\!]^*_{\mathfrak{M}}) \cap U$, $[\![\blacktriangledown^n\beta]\!]_{\mathfrak{M},U} \cap \bigcap_{V\in\rho} V = U^n([\![\beta]\!]^*_{\mathfrak{M}})$.

Proof. Items 3 and 4 follow directly from items 1 and 2, respectively. We will prove item 1. The proof of Item 2 follows in similar steps.

$$\begin{aligned}[\![\blacktriangle^n\alpha]\!]_{\mathfrak{M},U} \cap \bigcap_{V\in\rho} V &= [\![\Box_1\Box\alpha]\!]_{\mathfrak{M},U} \cap \bigcap_{V\in\rho} V \\ &= \bigcap_{V\in\rho} [\![\Box\alpha]\!]_{\mathfrak{M},V} \quad \text{(Using Proposition 5.12)} \\ &= \bigcap_{V\in\rho} [R]_V([\![\alpha]\!]_{\mathfrak{M},V}) \quad \text{(Using Proposition 5.10).}\end{aligned}$$

$$
\begin{aligned}
[\![\blacktriangledown^p \alpha]\!]_{\mathfrak{M},U} &= [\![\Diamond_1 \Diamond \alpha]\!]_{\mathfrak{M},U} \\
&= \bigcup_{V \in \rho} [\![\Diamond \alpha]\!]_{\mathfrak{M},V} \cap U \quad \text{(Using Proposition 5.12)} \\
&= \bigcup_{V \in \rho} \langle R \rangle_V ([\![\alpha]\!]_{\mathfrak{M},V}) \cap U \quad \text{(Using Proposition 5.10).}
\end{aligned}
$$

□

Thus, it follows that an element x of $\bigcap_{V \in \rho} V$ belongs to the necessity lower approximation of a concept represented by a $\beta \in \mathcal{L}(\Box_1)$ if and only if $x \in [\![\blacktriangle^n \beta]\!]_{\mathfrak{M},U}$ for some $U \in \rho$. Similarly, an element x belongs to the possibility lower approximation of a concept represented by a $\beta \in \mathcal{L}(\Box_1)$ if and only if $x \in [\![\blacktriangle^p \beta]\!]_{\mathfrak{M},U}$ for some $U \in \rho$. With these interpretations in mind, we list some wffs illustrating the fact that the proposed semantics enable us to formally reason about SAS and the defined approximations based upon it.

- Interpretation of the wff $\blacktriangle^n p \to p$ is as follows: if an element x of $\bigcap_{V \in \rho} V$ belongs to the necessity lower approximation of a set X (represented by p), then x is also an element of the set X. Observe that $\blacktriangle^n p \to p$ is valid in the class $\mathsf{A_r}$ of SASs.
- Interpretation of the wff $p \to \blacktriangledown^p p$ is as follows: if an element x of $\bigcup_{V \in \rho} V$ belongs to a set X then x also belongs to the possibility upper approximation of X. It may be noted that $p \to \blacktriangledown^p p$ is valid in the class $\mathsf{A_r}$ of SASs.
- Interpretation of the wff $\blacktriangle^n (p \wedge q) \to (\blacktriangle^n p \wedge \blacktriangle^n q)$ is as follows: if an element x of $\bigcap_{V \in \rho} V$ belongs to the necessity lower approximation of the set $X \cap Y$, then x belongs to the necessity lower approximations of both the sets X and Y. Observe that $\blacktriangle^n (p \wedge q) \to (\blacktriangle^n p \wedge \blacktriangle^n q)$ is valid in the class A of all SASs.

We pause our discussion on expressibility power here and move to the axiomatization issue in the next section. In Section 5.8.2, we shall further illustrate the expressibility power of the proposed syntax and semantics.

5.4. Axiomatization

This section presents Hilbert-style proof systems for modal logic given in Section 5.2. Recall the axioms (Taut), K(□), K(□$_1$), T(□$_1$), T(□$_1$), B(□), B(□$_1$), 4(□), 4(□$_1$), and the rules of inferences MP, Nec(□), and Nec(□$_1$) (cf. Chapter 3). Also note the following

Modal Systems	Axioms and inferene rules	Classes of Models
J	Taut, K($\Box$), K($\Box_1$), T($\Box_1$), S($\Box_1$), NS($\Box_1$), B($\Box_1$), 4($\Box_1$), MP, Nec($\Box$), Nec($\Box_1$)	Θ
J(T)	J+T($\Box$)	Θ_{r}
J(B)	J+B($\Box$)	Θ_{s}
J(4)	J+4($\Box$)	Θ_{t}
J(TB)	J+T($\Box$) + B($\Box$)	Θ_{rs}
J(T4)	J+T($\Box$) + 4($\Box$)	Θ_{rt}
J(B4)	J+B($\Box$) + 4($\Box$)	Θ_{st}
J(E)	J +T($\Box$) + 4($\Box$)+ B($\Box$)	Θ_{e}

Table 5.3. Defined modal systems with their corresponding classes of models

axioms.

$$p \to \Box_1 p. \tag{S($\Box_1$)}$$

$$\neg p \to \Box_1 \neg p. \tag{NS($\Box_1$)}$$

The axioms S($\Box_1$) and NS($\Box_1$) correspond to the fact that the satisfiability of propositional variables at a point $(x, U) \in E(\mathfrak{F})$ depends on x but not on U.

A few modal systems are listed in Table 5.3. The entries occurring in the right end column of the table denote the model classes for which we expect to have completeness results.

The following soundness theorem can be easily established.

Theorem 5.15 (Soundness)**.** *Let (Λ, Φ) be a pair consisting of a modal system Λ and a class Φ of models from the same row of Table 5.3. Then, for all wff α from $\mathcal{L}(\Box_1, \Box)$, $\vdash_\Lambda \alpha$ implies $\Phi \models \alpha$.*

The corresponding completeness theorem will be presented in the next section.

5.5. Completeness

We will use the well-known step by step method to prove the completeness theorem. Let us first recall a few standard definitions and results. Let us take a modal system Λ from Table 5.3, and denote the set consisting of all Λ-maximal consistent sets by $\mathbb{M}_\Lambda$.

Lemma 5.16 (Lindenbaum's Lemma)**.** *Given a Λ-consistent set Γ of wffs, there exists a Λ-maximal consistent set Γ^+ containing Γ.*

As usual, the canonical relations for modal operators $\Box_1$ and $\Box$ on $\mathbb{M}_\Lambda$ is defined as follows.

$(\Gamma, \Delta) \in \mathrm{R}^\Lambda_{\Box_1}$ if $\Box_1\alpha \in \Gamma$ implies $\alpha \in \Delta$.

$(\Gamma, \Delta) \in \mathrm{R}^\Lambda_{\Box}$ if $\Box\alpha \in \Gamma$ implies $\alpha \in \Delta$.

Let us note a few results that can be concluded easily by applying standard modal logic arguments.

Proposition 5.17. *Consider a modal system Λ from Table 5.3. Then we have the following.*

1. *For each modal system Λ, $\mathrm{R}^\Lambda_{\Box_1}$ is an equivalence relation.*
2. *(i) If Λ includes $\mathrm{T}(\Box)$, then $\mathrm{R}^\Lambda_{\Box}$ is reflexive.*
   (ii) If Λ includes $4(\Box)$, then $\mathrm{R}^\Lambda_{\Box}$ is transitive.
   (iii) If Λ includes $\mathrm{B}(\Box)$, then $\mathrm{R}^\Lambda_{\Box}$ is symmetric.

Lemma 5.18 (Existence Lemma)**.** *Let Γ be in $\mathbb{M}_\Lambda$. Then the following holds.*

1. *If $\Diamond_1\alpha \in \Gamma$, then there exists a $\Delta \in \mathbb{M}_\Lambda$ such that $(\Gamma, \Delta) \in \mathrm{R}^\Lambda_{\Box_1}$ and $\alpha \in \Delta$.*
2. *If $\Diamond\alpha \in \Gamma$, then there exists a $\Delta \in \mathbb{M}_\Lambda$ such that $(\Gamma, \Delta) \in \mathrm{R}^\Lambda_{\Box}$ and $\alpha \in \Delta$.*

We are now in a position to give the following definition of network.

Definition 5.19 (Network)**.** A tuple $\mathcal{N} := (W, \{i(a) : a \in P\}, R, \mu)$ is called a Λ-*network*, where

- W and P are non-empty sets;
- $i : P \to \wp(W)$;
- $R \subseteq W \times W$;
- $\mu : \chi \to \mathbb{M}_\Lambda$ where $\chi = \{(x, a) \in W \times P \mid x \in i(a)\}$.

Definition 5.20 (Coherent Network)**.** Assume that $\mathcal{N} := (W, \{i(a) : a \in P\}, R, \mu)$ is a Λ-network. Then it is called a *coherent* Λ-network if the following conditions hold.

(C1) If $x, y \in i(a)$ with $(x, y) \in R$, then $(\mu(x, a), \mu(y, a)) \in \mathrm{R}^{\Lambda}_{\Box}$.

(C2)
- If $T(\Box)$ axiom is in Λ, then relation R is reflexive;
- If $4(\Box)$ axiom is in Λ, then relation R is transitive;
- If $B(\Box)$ axiom is in Λ, then relation R is symmetric;
- If $\Lambda \in \{\mathrm{J}, \mathrm{J(B)}, \mathrm{J(4)}\}$, then relation R is irreflexive.

(C3) If $x \in i(a) \cap i(b)$, then $(\mu(x, a), \mu(x, b)) \in \mathrm{R}^{\Lambda}_{\Box_1}$.

Definition 5.21 (Saturated network)**.** A Λ-network $\mathcal{N}$ is known as *saturated* if the following properties are true.

(S1) If $\Diamond\alpha \in \mu(x, a)$, then there must be a $y \in i(a)$ satisfying $(x, y) \in R$ and $\alpha \in \mu(y, a)$.

(S2) If $\Diamond_1\alpha \in \mu(x, a)$, then there must exist a $b \in P$ such that $\alpha \in \mu(x, b)$.

Definition 5.22 (Perfect network)**.** A Λ-network $\mathcal{N}$ is said to be perfect network if it is both coherent and saturated.

Definition 5.23 (Defects)**.** For a Λ-network $\mathcal{N} := (W, \{i(a) : a \in P\}, R, \mu)$, let us consider the following.

(D1) The tuple $\langle x, a, \Diamond\alpha\rangle$ forms a $\Diamond$-*defect* of $\mathcal{N}$ if $\Diamond\alpha \in \mu(x, a)$, but there is no $y \in i(a)$ satisfying $(x, y) \in R$ and $\alpha \in \mu(y, a)$.

(D2) The tuple $\langle x, a, \Diamond_1\alpha\rangle$ forms a $\Diamond_1$-*defect* of $\mathcal{N}$ if $\Diamond_1\alpha \in \mu(x, a)$, but there is no $b \in P$ such that $\alpha \in \mu(x, b)$.

Definition 5.24 (Induced Model)**.** Suppose $\mathcal{N} := (W, \{i(a) : a \in P\}, R, \mu)$ is a Λ-network. Then $\mathcal{N}$ gives rise to a model $\mathfrak{M}^{\mathcal{N}} := (\mathfrak{F}^{\mathcal{N}}, m^{\mathcal{N}})$, where

- $\mathfrak{F}^{\mathcal{N}} := (W, \{i(a) : a \in P\}, R)$;
- $m^{\mathcal{N}}(p) := \{x \in W : p \in \mu(x, a) \text{ for some } a \in P\}$, $p \in PV$.

Proposition 5.25. *If $\mathcal{N}$ is reflexive, symmetric, or transitive network, then $\mathfrak{M}^{\mathcal{N}}$ is also reflexive, symmetric, or transitive model, respectively.*

Theorem 5.26 (Truth Lemma)**.** *Consider a perfect Λ-network $\mathcal{N}$. Then for every wff α in $\mathcal{L}(\Box_1, \Box)$,*

$$\mathfrak{M}^{\mathcal{N}}, x, i(a) \models \alpha \textit{ if and only if } \alpha \in \mu(x, a).$$

Proof. The proof is by induction on the number of connectives in α. First, let us discuss the proof for propositional variables. So, let $\mathfrak{M}^{\mathcal{N}}, x, i(a) \models p$ and we show that $p \in \mu(x, a)$. From $\mathfrak{M}^{\mathcal{N}}, x, i(a) \models p$, we get $x \in m^{\mathcal{N}}(p)$. By the definition of $m^{\mathcal{N}}(p)$, we obtain $p \in \mu(x, b)$ for some $b \in P$. Thus we have two cases.
Case(i). When b is same as a. Then we have $p \in \mu(x, a)$.
Case(ii). When b is not same as a. Then from $p \in \mu(x, b)$ and $\mathrm{S}(\Box_1)$ axiom, we get $\Box_1 p \in \mu(x, b)$. Since $x \in i(b) \cap i(a)$, we obtain $(\mu(x, b), \mu(x, a)) \in \mathrm{R}^{\Lambda}_{\Box_1}$, because of (C3). From $\Box_1 p \in \mu(x, b)$ and $(\mu(x, b), \mu(x, a)) \in \mathrm{R}^{\Lambda}_{\Box_1}$, we obtain $p \in \mu(x, a)$.
For the other direction, suppose that $p \in \mu(x, a)$ and we show that $\mathfrak{M}^{\mathcal{N}}, x, i(a) \models p$. Since $p \in \mu(x, a)$, by the definition of $m^{\mathcal{N}}$, we get $x \in m^{\mathcal{N}}(p)$ and hence $\mathfrak{M}^{\mathcal{N}}, x, i(a) \models p$.
Next, let $\mathfrak{M}^{\mathcal{N}}, x, i(a) \models \Diamond\beta$. Then there exists a $y \in i(a)$ with $(x, y) \in R$, and $\mathfrak{M}^{\mathcal{N}}, y, i(a) \models \beta$. By induction hypothesis, we obtain $\beta \in \mu(y, a)$. Since $x, y \in i(a)$ with $(x, y) \in R$, using (C1), we get $(\mu(x, a), \mu(y, a)) \in \mathrm{R}^{\Lambda}_{\Box}$. Thus $\Diamond\beta \in \mu(x, a)$ as $\beta \in \mu(y, a)$.
For the other part, consider $\Diamond\beta \in \mu(x, a)$ and we have to show that $\mathfrak{M}^{\mathcal{N}}, x, i(a) \models \Diamond\beta$. From $\Diamond\beta \in \mu(x, a)$, there exists a $y \in i(a)$ such that $(x, y) \in R$ and $\beta \in \mu(y, a)$. By induction hypothesis, we get $\mathfrak{M}^{\mathcal{N}}, y, i(a) \models \beta$. We have $(x, y) \in R$ and $\mathfrak{M}^{\mathcal{N}}, y, i(a) \models \beta$. Therefore, we derive $\mathfrak{M}^{\mathcal{N}}, x, i(a) \models \Diamond\beta$.
Let us discuss the case for $\Diamond_1\beta$. So, let $\mathfrak{M}^{\mathcal{N}}, x, i(a) \models \Diamond_1\beta$ and we show that $\Diamond_1\beta \in \mu(x, a)$. From $\mathfrak{M}^{\mathcal{N}}, x, i(a) \models \Diamond_1\beta$, we get a $c \in P$ such that $x \in i(c)$, and $\mathfrak{M}^{\mathcal{N}}, x, i(c) \models \beta$. By induction hypothesis, we conclude $\beta \in \mu(x, c)$. Since $x \in i(a) \cap i(c)$, using (C3), we get $(\mu(x, a), \mu(x, c)) \in \mathrm{R}^{\Lambda}_{\Box_1}$. Thus, we obtain $\Diamond_1\beta \in \mu(x, a)$ due to $\beta \in \mu(x, c)$.
Further, let $\Diamond_1\beta \in \mu(x, a)$ and we prove that $\mathfrak{M}^{\mathcal{N}}, x, i(a) \models \Diamond_1\beta$. From $\Diamond_1\beta \in \mu(x, a)$, we get a $c \in P$ satisfying $x \in i(c)$ and $\beta \in \mu(x, c)$. By induction hypothesis, we conclude $\mathfrak{M}^{\mathcal{N}}, x, i(c) \models \beta$, thus $\mathfrak{M}^{\mathcal{N}}, x, i(a) \models \Diamond_1\beta$. This completes the proof. □

Definition 5.27. Consider two Λ-networks $\mathcal{N} := (W, \{i(a) : a \in P\}, R, \mu)$ and $\mathcal{N}' := (W', \{i'(a) : a \in P'\}, R', \mu')$. We say $\mathcal{N}'$ *extends* $\mathcal{N}$, notation $\mathcal{N}' \rhd \mathcal{N}$, if we have the following.

- $W \subseteq W'$;
- $P \subseteq P'$;
- $i(a) = i'(a)$ for all $a \in P$;
- $R = R' \cap (W \times W)$;

- $\mu(x,a) = \mu'(x,a)$ for all $(x,a) \in \chi'$.

Next, we discuss the Repair lemma for (D1) and (D2) defects.

Theorem 5.28 (Repair Lemma in case of $\Diamond$-Defect)**.** *Let $\mathcal{N} := (W, \{i(a) : a \in P\}, R, \mu)$ be a coherent Λ-network having $\Diamond$-defect $\langle x_0, a_0, \Diamond\alpha\rangle$, where W and P contains finite number of elements. Then there is a finite coherent Λ-network $\mathcal{N}'$ extending $\mathcal{N}$ such that $\langle x_0, a_0, \Diamond\alpha\rangle$ is not a $\Diamond$-defect of $\mathcal{N}'$.*

Proof. Given $\Diamond\alpha \in \mu(x_0, a_0)$, by Existence lemma 5.18, there exists a $\Delta_0 \in \mathbb{M}_\Lambda$ such that

$$(\mu(x_0,a_0), \Delta_0) \in \mathrm{R}^{\Lambda}_{\Box} \text{ and } \alpha \in \Delta_0. \tag{5.3}$$

Let y_0 be a point that does not occur in W. Consider a network $\mathcal{N}' := (W', \{i'(a) : a \in P'\}, R', \mu')$, where

$$W' := W \cup \{y_0\}$$

$$P' := P$$

$$i'(r) := \begin{cases} i(r) \cup \{y_0\}, & \text{if } r = a_0 \\ i(r), & \text{otherwise} \end{cases}$$

$$\mu'(x,a) := \begin{cases} \Delta_0, & \text{if } x = y_0, a = a_0 \\ \mu(x,a), & \text{otherwise} \end{cases}$$

$$R' := \begin{cases} R \cup \{(x_0,y_0)\} & \text{if } \Lambda = \mathrm{J} \\ R \cup \{(x_0,y_0),(y_0,y_0)\} & \text{if } \Lambda = \mathrm{J(T)} \\ R \cup \{(x_0,y_0),(y_0,x_0)\} & \text{if } \Lambda = \mathrm{J(B)} \\ R \cup \{(x_0,y_0)\} \cup \{(x,y_0) \mid (x,x_0)\} & \text{if } \Lambda = \mathrm{J(4)} \\ R \cup \{(x_0,y_0),(y_0,x_0),(y_0,y_0)\} & \text{if } \Lambda = \mathrm{J(TB)} \\ R \cup \{(x,y_0) \mid (x,x_0)\} \cup \{(y_0,y_0)\} & \text{if } \Lambda = \mathrm{J(T4)} \\ R \cup \{(x,y_0),(y_0,x) \mid (x,x_0)\} \cup \{(y_0,y_0)\} & \text{if } \Lambda = \mathrm{J(E)}. \end{cases}$$

We show that $\mathcal{N}'$ is the required network.

(C1): Let us take $x, y \in i'(b)$ such that $(x,y) \in R'$ and we show that $(\mu'(x,b), \mu'(y,b)) \in$

$R^{\Lambda}_{\Box}$. If $b \neq a_0$, then already we have the desired result. So, let us take $b = a_0$. If $x = x_0$ and $y = y_0$, then by (5.3) and definition of μ', we derive the result. If $x = y = y_0$, then Λ must be in $\{J(T), J(TB), J(T4), J(E)\}$, and hence $(\mu'(y_0, a_0) = \Delta_0, \mu'(y_0, a_0)) \in R^{\Lambda}_{\Box}$. If $x \neq x_0, y = y_0$, then we must have $(x, x_0) \in R$ and $\Lambda \in \{J(4), J(T4), J(E)\}$. By the coherency of $\mathcal{N}$, we have

$$(\mu(x, a_0), \mu(x_0, a_0)) \in R^{\Lambda}_{\Box}. \tag{5.4}$$

From (5.3), (5.4), transitivity of $R^{\Lambda}_{\Box}$ and using definition of μ', we obtain

$$(\mu'(x, a_0) = \mu(x, a_0), \mu'(y_0, a_0) = \Delta_0) \in R^{\Lambda}_{\Box}$$

as required. Due to the definition of R', (C2) follows.

(C3): Let us take $a, b \in P$ such that x belongs to $i'(a) \cap i'(b)$ and we show that $(\mu'(x, a), \mu'(x, b)) \in R^{\Lambda}_{\Box_1}$. By the construction of i', x cannot be y_0. Since $\mathcal{N}$ is a coherent network, we have the desired result.

Thus $\mathcal{N}'$ is the required network. □

Theorem 5.29 (Repair Lemma for $\Diamond_1$-Defect)**.** *Let $\mathcal{N} := (W, \{i(a) : a \in P\}, R, \mu)$ be a coherent Λ-network having $\Diamond_1$-defect $\langle x_0, a_0, \Diamond_1\alpha\rangle$, where W and P contains finite number of elements. Then there is a finite coherent Λ-network $\mathcal{N}'$ extending $\mathcal{N}$ such that $\langle x_0, a_0, \Diamond_1\alpha\rangle$ is not a $\Diamond_1$-defect of $\mathcal{N}'$.*

Proof. As we have $\Diamond_1\alpha \in \mu(x_0, a_0)$, by Existence lemma 5.18, there exists a $\Delta_0 \in \mathbb{M}_\Lambda$ such that

$$(\mu(x_0, a_0), \Delta_0) \in R^{\Lambda}_{\Box_1} \text{ and } \alpha \in \Delta_0. \tag{5.5}$$

Let b_0 be a point not in P. Consider a network $\mathcal{N}' := (W', \{i'(a) : a \in P'\}, R', \mu')$, where

$$W' := W$$

$$P' := P \cup \{b_0\}$$

$$R' := R$$

$$i'(r) := \begin{cases} x_0, & \text{if } r = b_0 \\ i(r), & \text{otherwise} \end{cases}$$

$$\mu'(x, a) := \begin{cases} \mu(x, a), & \text{if } a \neq b_0 \\ \Delta_0, & \text{if } x = x_0 \quad \text{and } a = b_0. \end{cases}$$

We need to prove that $\mathcal{N}'$ is our required network.
(C1): Let us take $x, y \in i'(a)$ such that $(x, y) \in R'$ and we show that $(\mu'(x, a), \mu'(y, a)) \in \mathrm{R}^{\Lambda}_{\Box}$. If $a \neq b_0$, then obviously we have the required result. So, let $a = b_0$, thus from the definition of i', x, y must be x_0 and $\Lambda \in \{\mathrm{J(T), J(TB), J(T4), J(E)}\}$. If $\Lambda \in \{\mathrm{J, J(B), J(4)}\}$, then $(x_0, x_0) \in R'$ is not possible as R is irreflexive relation. Using reflexivity of $\mathrm{R}^{\Lambda}_{\Box}$, we obtain $(\mu'(x_0, b_0) = \Delta_0, \mu'(x_0, b_0) = \Delta_0) \in \mathrm{R}^{\Lambda}_{\Box}$.
(C2): R' obviously satisfies the required properties.
(C3): Suppose there are $a, b \in P$ such that $x \in i'(a) \cap i'(b)$ and we show that $(\mu'(x, a), \mu'(x, b)) \in \mathrm{R}^{\Lambda}_{\Box_1}$. If $x \neq x_0$ and $a, b \neq b_0$, then obviously we have the required property. So, let us consider the case when $x = x_0$ and $b = b_0$. Since $x_0 \in i(a) \cap i(a_0)$, we have

$$(\mu(x_0, a), \mu(x_0, a_0)) \in \mathrm{R}^{\Lambda}_{\Box_1} . \tag{5.6}$$

From (5.6), (5.5) and using transitivity of $\mathrm{R}^{\Lambda}_{\Box_1}$, we obtain

$$(\mu(x_0, a) = \mu'(x_0, a), \Delta_0 = \mu'(x_0, b_0)) \in \mathrm{R}^{\Lambda}_{\Box_1}$$

as required. Hence $\mathcal{N}'$ is the required network. □

Theorem 5.30 (Completeness Theorem)**.** *Let us consider a pair* (Λ, Φ)*, where* Λ *and* Φ *denote a modal system and a class of models, respectively, occurring in same row in Table 5.3. Then, for all wff* α *from* $\mathcal{L}(\Box_1, \Box)$*,* $\Phi \models \alpha$ *implies* $\vdash_\Lambda \alpha$*.*

Proof. Let $\nvdash_\Lambda \alpha$ hold, if possible. Then, we obtain $\{\neg\alpha\}$ as a Λ-consistent set and thus there exists a Λ-maximal consistent set $\Gamma \in \mathbb{M}_\Lambda$ containing $\neg\alpha$. Consider the network

$$\mathcal{N}_0 = (W_0, \{i_0(a_0) \mid a_0 \in P_0\}, R_0, \mu_0),$$

where

- $W_0 = \{x_0\}$;
- $P_0 = \{a_0\}$;
- $i_0 : P_0 \to \wp(W_0)$ such that $i_0(a_0) = x_0$;
- $R_0 = \begin{cases} \emptyset & \text{if } \Lambda \in \{\mathrm{J}, \mathrm{J(B)}, \mathrm{J(4)}, \mathrm{J(B4)}\}; \\ (x_0, x_0) & \text{if } \Lambda \in \{\mathrm{J(T)}, \mathrm{J(TB)}, \mathrm{J(T4)}, \mathrm{J(E)}\}; \end{cases}$
- $\mu_0 : \chi_0 \to \mathbb{M}_\Lambda$ where $\chi_0 = \{(x_0, a_0)\}$ such that $\mu_0(x_0, a_0) = \Gamma$.

It is not difficult to verify that $\mathcal{N}_0$ is a finite coherent Λ-network for Γ. In addition, by means of repeated applications of Repair Lemma 5.28 and 5.29, we obtain a perfect Λ-network $\mathcal{N}$ extending $\mathcal{N}_0$. The proof is standard and, for details, we refer to [7]. It should also be noted that, due to Proposition 5.25, the induced model $\mathfrak{M}^{\mathcal{N}} := (\mathfrak{F}^{\mathcal{N}}, m^{\mathcal{N}})$ belongs to the class Φ. Since $\neg\alpha \in \mu_0(x_0, a_0)$, by Truth Lemma 5.26, we get $\mathfrak{M}^{\mathcal{N}}, x_0, i(a_0) \models \neg\alpha$, which is a contradiction. Hence $\vdash_\Lambda \alpha$. □

5.6. A comparison with multi-modal logic

Let us recall that the language $\mathcal{L}(\Box_1, \Box)$ is the standard modal language consisting of two unary modal operators $\Box$ and $\Box_1$. Thus, following the usual modal logic approach, models for this language consist of a set bearing two binary relations. Therefore, we have the below mentioned notion of a model.

Definition 5.31. A tuple $\mathcal{F} := (\mathbb{S}, R, T)$, where R, T are binary relations on a non-empty set $\mathbb{S}$, is defined as an *auxiliary frame*. An *auxiliary model* for $\mathcal{L}(\Box_1, \Box)$ is a tuple $\mathcal{M} := (\mathcal{F}, m)$, where $\mathcal{F}$ is an auxiliary frame and $m : PV \to \wp(\mathbb{S})$ is a valuation function.

Definition 5.32. The satisfiability of a wff α in an auxiliary model $\mathcal{M} := (\mathcal{F}, m)$, denoted as $\mathcal{M}, s \mid\models \alpha$, is defined inductively as follows:

$$\begin{array}{lll} \mathcal{M}, s \models \top & & \text{always.} \\ \mathcal{M}, s \mid\models p & \Longleftrightarrow & s \in m(p), \text{ for } p \in PV. \\ \mathcal{M}, s \mid\models \neg\alpha & \Longleftrightarrow & \mathcal{M}, s \not\mid\models \alpha. \\ \mathcal{M}, s \mid\models \alpha \wedge \beta & \Longleftrightarrow & \mathcal{M}, s \mid\models \alpha \text{ and } \mathcal{M}, s \mid\models \beta. \\ \mathcal{M}, s \mid\models \Box_1\alpha & \Longleftrightarrow & \text{for all } r \in \mathbb{S} \text{ with } (s, r) \in R, \mathcal{M}, r \mid\models \alpha. \\ \mathcal{M}, s \mid\models \Box\alpha & \Longleftrightarrow & \text{for all } s \in \mathbb{S} \text{ with } (s, r) \in T, \mathcal{M}, r \mid\models \alpha. \end{array}$$

The notions of validity and satisfiability in a class $\mathfrak{G}$ of auxiliary models are defined in the usual way:

- $\alpha \in \mathcal{L}(\Box_1, \Box)$ is *satisfiable* in $\mathfrak{G}$ if there exists a model $\mathcal{M} := (\mathcal{F}, m)$ in $\mathfrak{G}$, and an element $s \in \mathbb{S}$, where $\mathbb{S}$ is the domain of $\mathcal{F}$, such that $\mathcal{M}, s \mid\models \alpha$;
- α is *valid* in $\mathfrak{G}$, denoted as $\mathfrak{G} \mid\models \alpha$, if for all models $\mathcal{M} := (\mathcal{F}, m)$ in $\mathfrak{G}$, and all elements $s \in \mathbb{S}$, $\mathcal{M}, s \mid\models \alpha$.

Since we aim to connect the above standard semantics with the semantics proposed in Section 5.2, we need to impose some conditions on the auxiliary models. For instance, keeping in view the axioms S($\Box_1$) and NS($\Box_1$), we consider the following property of an auxiliary model $\mathcal{M} := (\mathcal{F}, m)$.

(Q): If $(s, r) \in R$, then $s \in m(p)$ if and only if $r \in m(p)$ for all $p \in PV$.

A few classes of auxiliary models that we require are listed in Table 5.4.

It is relevant to note that every model $\mathfrak{M} := (\mathfrak{F}, m)$ based on a SAS $\mathfrak{F} := (W, \rho, S)$ gives an auxiliary model $\mathcal{M}_{\mathfrak{M}} := (\mathcal{F}_{\mathfrak{F}}, m')$, $\mathcal{F}_{\mathfrak{F}} := (E(\mathfrak{F}), R, T)$, where

- $((x, U), (y, V)) \in R$ if and only if $x = y$,
- $((x, U), (y, V)) \in T$ if and only if $(x, y) \in S$ and $U = V$,
- $m'(p) = \{(x, U) \in E(\mathfrak{F}) : x \in m(p)\}$, $p \in PV$.

It is not difficult to show that $\mathcal{M}_{\mathfrak{M}} \in \Upsilon$. Moreover, if (Φ, Ψ) is a tuple consisting of a class Φ of models and a class Ψ of auxiliary models from the same row of Table 5.5 and $\mathfrak{M} \in \Phi$, then $\mathcal{M}_{\mathfrak{M}} \in \Psi$. Further, we have the following.

Class of Models	Defining condition
Υ	Class of all auxiliary models satisfying property (Q) and the relation R is an equivalence relation
Υ_{r}	Class of all elements of Υ with T as a reflexive relation
Υ_{s}	Class of all elements of Υ with T as a symmetric relation
Υ_{t}	Class of all elements of Υ with T as a transitive relation
Υ_{rs}	$\Upsilon_{\mathsf{r}} \cap \Upsilon_{\mathsf{s}}$
Υ_{st}	$\Upsilon_{\mathsf{s}} \cap \Upsilon_{\mathsf{t}}$
Υ_{rt}	$\Upsilon_{\mathsf{r}} \cap \Upsilon_{\mathsf{t}}$
Υ_{e}	$\Upsilon_{\mathsf{rt}} \cap \Upsilon_{\mathsf{s}}$

Table 5.4. A few classes of auxiliary models

Proposition 5.33. *For any* $(x, U) \in E(\mathfrak{F})$ *and wff* $\alpha \in \mathcal{L}(\Box_1, \Box)$*, we have*

$$\mathfrak{M}, x, U \models \alpha \iff \mathcal{M}_{\mathfrak{M}}, (x, U) \mid\models \alpha.$$

Proof. The proof is by induction on the number of connectives in α. The result holds for propositional variable in a obvious way. Boolean cases can also be proved easily. So, we provide the proof for modal cases $\Diamond_1$ and $\Diamond$. First we discuss $\Diamond_1$ case. So, let $\mathfrak{M}, x, U \models \Diamond_1\beta$ and we prove that $\mathcal{M}_{\mathfrak{M}}, (x, U) \models \Diamond_1\beta$. From $\mathfrak{M}, x, U \models \Diamond_1\beta$, we obtain a $V \in \rho$ such that $x \in V$ and $\mathfrak{M}, x, V \models \beta$. Using induction hypothesis, we get $\mathcal{M}_{\mathfrak{M}}, (x, V) \mid\models \beta$. By the definition of R, we conclude $((x, U), (x, V)) \in R$. From $\mathcal{M}_{\mathfrak{M}}, (x, V) \mid\models \beta$ and $((x, U), (x, V)) \in R$, we obtain $\mathcal{M}_{\mathfrak{M}}, (x, U) \mid\models \Diamond_1\beta$.

For the converse, suppose that $\mathcal{M}_{\mathfrak{M}}, (x, U) \mid\models \Diamond_1\beta$ and we have to prove that $\mathfrak{M}, x, U \models \Diamond_1\beta$. Since $\mathcal{M}_{\mathfrak{M}}, (x, U) \mid\models \Diamond_1\beta$, there exists $(x, V) \in E(\mathfrak{F})$ such that $((x, U), (x, V)) \in R$ and $\mathcal{M}_{\mathfrak{M}}, (x, V) \mid\models \beta$. Due to induction hypothesis, $\mathfrak{M}, x, V \models \beta$. Since $((x, U), (x, V)) \in R$, we get $\mathfrak{M}, x, U \models \Diamond_1\beta$.

Next, let us discuss the case for $\Diamond$ modality. So, assume that $\mathfrak{M}, x, U \models \Diamond\beta$ and we need to prove that $\mathcal{M}_{\mathfrak{M}}, (x, U) \mid\models \Diamond\beta$. Since $\mathfrak{M}, x, U \models \Diamond\beta$, there must be a $y \in U$

such that $(x, y) \in S$ and $\mathfrak{M}, y, U \models \beta$. By the use of induction case, we deduce that $\mathcal{M}_{\mathfrak{M}}, (y, U) \mid\models \beta$. From the definition of T, we get $((x, U), (y, U)) \in T$ as $(x, y) \in S$. Since $((x, U), (y, U)) \in T$ and $\mathcal{M}_{\mathfrak{M}}, (y, U) \mid\models \beta$, we obtain $\mathcal{M}_{\mathfrak{M}}, (x, U) \mid\models \Diamond\beta$.
For the converse, suppose that $\mathcal{M}_{\mathfrak{M}}, (x, U) \mid\models \Diamond\beta$ and we prove that $\mathfrak{M}, x, U \models \Diamond\beta$. Since $\mathcal{M}_{\mathfrak{M}}, (x, U) \mid\models \Diamond\beta$, there exists $(y, U) \in E(\mathfrak{F})$ such that $((x, U), (y, U)) \in T$ and $\mathcal{M}_{\mathfrak{M}}, (y, U) \mid\models \beta$. Again, due to induction hypothesis, $\mathfrak{M}, y, U \models \beta$ holds. From $((x, U), (y, U)) \in T$, we obtain $(x, y) \in S$. We have $(x, y) \in S$ and $\mathfrak{M}, y, U \models \beta$, and hence $\mathfrak{M}, x, U \models \Diamond\beta$. $\square$

Logic	Class of Model	Class of Auxiliary Model
J	Θ	Υ
J(T)	Θ_{r}	Υ_{r}
J(B)	Θ_{s}	Υ_{s}
J(4)	Θ_{t}	Υ_{t}
J(TB)	Θ_{rs}	Υ_{rs}
J(T4)	Θ_{rt}	Υ_{rt}
J(E)	Θ_{e}	Υ_{e}

Table 5.5. Soundness and completeness theorems relative to various classes of models and auxiliary models for the fragment $\mathcal{L}(\Box_1, \Box)$

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Let (Φ, Ψ) be a tuple consisting of a class Φ of models and a class Ψ of auxiliary models occurring in the same row of Table 5.5. As a consequence of Proposition 5.33, it follows that if a wff α is satisfiable in the class Φ of models, then α is also satisfiable in the class Ψ of auxiliary models.
At this juncture, it is natural to ask if we can find a translation of auxiliary models to models (cf. Definition 5.8) preserving satisfiability. Such a translation together with Proposition 5.33, will lead us to the fact that the class Φ of models and the class Ψ of auxiliary models have the same set of valid wffs. Unfortunately, we are not able to obtain such a translation. But the good news is that we are still able to prove the following result.

Theorem 5.34. *Consider a modal system Λ, a class of models Φ, and a class of auxiliary models Ψ from the same row of Table 5.5. Then we obtain the following for every wff α in $\mathcal{L}(\Box_1, \Box)$.*

$$\Phi \models \alpha \iff \vdash_\Lambda \alpha \iff \Psi \mid\models \alpha.$$

We have already proved $\Phi \models \alpha \iff \vdash_\Lambda \alpha$ (cf. Theorems 5.15 and 5.30). The part $\vdash_\Lambda \alpha \iff \Psi \mid\models \alpha$ can be shown following the standard modal logic technique, and we exclude its proof here. As a consequence of Theorem 5.34, it follows that properties concerning the validity of wffs with respect to the standard multi-modal logic semantics given by Definition 5.32 relative to the class of auxiliary models give the corresponding properties concerning the validity of wffs with respect to the class Φ of models given in Table 5.3. As an application of this result, we obtain the decidability results for various classes of models by proving the same for various classes of auxiliary models. The following section presents the decidability results for various classes of auxiliary models.

5.7. Decidability

This section aims to prove the following decidability result.

Theorem 5.35. *Let Φ be a class of models from the list of classes of models given in Table 5.5. Then, for a given wff $\alpha \in \mathcal{L}(\Box_1, \Box)$, we can determine whether α is satifiable in the class Φ.*

Due to Theorem 5.34, we obtain Theorem 5.35 directly from the following result.

Theorem 5.36. *Let Ψ be a class of auxiliary models from the list of classes of auxiliary models given in Table 5.5. Then, for a given wff $\alpha \in \mathcal{L}(\Box_1, \Box)$, we can determine whether α is satifiable in the class Ψ.*

Theorem 5.36 can be proved following the standard filtration technique. In the rest of the section, we provide its proof sketch. Consider a finite subset Σ of $\mathcal{L}(\Box_1, \Box)$ which is closed under sub-formulas. Let $\mathcal{M} := (\mathcal{F}, m)$, $\mathcal{F} := (\mathbb{S}, R, T)$, be an auxiliary model. An equivalence relation $\equiv_\Sigma$ on $\mathbb{S}$ is defined as follows.

$$s \equiv_\Sigma s' \text{ if and only if for all } \beta \in \Sigma, \mathcal{M}, s \mid\models \beta \text{ if and only if } \mathcal{M}, s' \mid\models \beta.$$

Definition 5.37 (Filtration model)**.** Consider an auxiliary model $\mathcal{M} := (\mathcal{F}, m)$ and subset Σ as aforementioned.

- Let us define an auxiliary filtration model $\mathcal{M}^f := (\mathcal{F}^f, m^f)$, $\mathcal{F}^f := (\mathbb{S}^f, R^f, T^f)$, where
  - $\mathbb{S}^f := \{[s] : s \in \mathbb{S}\}, [s]$ represents the equivalence class of s with respect to the equivalence relation $\equiv_\Sigma$;
  - $([s], [s']) \in R^f$ if and only if there exist $s_1 \in [s]$ and $s_2 \in [s']$ such that $(s_1, s_2) \in R$;
  - $([s], [s']) \in T^f$ if and only if there exists $s_1 \in [s]$ and $s_2 \in [s']$ such that $(s_1, s_2) \in T$;
  - $m^f(p) := \{[s] \in \mathbb{S}^f : s \in m(p)\}$, $p \in PV$.
- For a class of auxiliary models Ψ, we define the auxiliary model $\mathcal{M}^\Psi := (\mathcal{F}^\Psi, m^f)$, $\mathcal{F}^\Psi := (\mathbb{S}^f, R^{f^*}, T^\Psi)$, where
  - R^{f^*} is the transitive closure of R^f.
  - $T^\Psi := \begin{cases} T^f & \text{if } \Psi \in \{\Upsilon, \Upsilon_{\mathsf{r}}, \Upsilon_{\mathsf{s}}, \Upsilon_{\mathsf{rs}}\} \\ T^{f^*} & \text{if } \Psi \in \{\Upsilon_{\mathsf{t}}, \Upsilon_{\mathsf{rt}}, \Upsilon_{\mathsf{st}}, \Upsilon_{\mathsf{e}}\}, \end{cases}$

    where T^{f^*} is the transitive closure of T^f.

Proposition 5.38. *If $\mathcal{M} \in \Psi$, then $\mathcal{M}^\Psi \in \Psi$.*

Proposition 5.39. *The domain $\mathbb{S}^f$ of the auxiliary model $\mathcal{M}^\Psi$ can have at most $2^{|\Sigma|}$ elements.*

Proof. Define the map $\Xi : \mathbb{S}^f \to 2^\Sigma$ such that

$$\Xi([s]) = \{\beta \in \Sigma : \mathcal{M}, s \mid\models \beta\}.$$

Since Ξ is injective, $|\mathbb{S}^f|$ is less than or equal to $2^{|\Sigma|}$. □

Proposition 5.40 (Filtration Theorem)**.** *For each wff β in Σ, and every element $s \in \mathbb{S}$,*

$$\mathcal{M}, s \mid\models \beta \iff \mathcal{M}^\Psi, [s] \mid\models \beta.$$

Using Proposition 5.38, 5.39 and 5.40, we finally obtain the following result.

Proposition 5.41 (Finite Model Property)**.** *Consider a wff α and a set Σ consisting of all sub-wffs of α. If α is satisfiable in the class Ψ, then it is satisfiable in a finite auxiliary model having at most $2^{|\Sigma|}$ elements belonging to the class Ψ.*

5.8. Invariance and definability

We now provide some results on invariance and definability related to the presented logic. These results also give us an insight into the expressibility power of the logic.

5.8.1. Invariance

Recall that for $\mathfrak{F} := (W, \rho, R)$, $E(\mathfrak{F})$ denotes the set $\{(x, U) \in W \times \rho : x \in U\}$. The following definition suggests when two states in distinct models are considered indistinguishable by means of the language $\mathcal{L}(\Box_1, \Box)$.

Definition 5.42. Let $\mathfrak{M} = (\mathfrak{F}, m)$ and $\mathfrak{M}' = (\mathfrak{F}', m')$ be two models based on SASs $\mathfrak{F} = (W, \rho, R)$ and $\mathfrak{F}' = (W', \rho', R')$. Let $(x, U) \in E(\mathfrak{F})$ and $(x', U') \in E(\mathfrak{F}')$. The set $\{\alpha \in \mathcal{L}(\Box_1, \Box) : \mathfrak{M}, x, U \models \alpha\}$ is called theory of (x, U). Two states (x, U) and (x', U') are known as equivalent, denoted by $\mathfrak{M}, (x, U) \leftrightsquigarrow \mathfrak{M}', (x', U')$, whenever their theories are identical.

We define the following notion of bisimulation, and we will show that under it, the proposed semantics is invariant.

Definition 5.43 (Bisimulation)**.** Consider two models $\mathfrak{M} = (\mathfrak{F}, m)$ and $\mathfrak{M}' = (\mathfrak{F}', m')$ based on SASs $\mathfrak{F} = (W, \rho, R)$ and $\mathfrak{F}' = (W', \rho', R')$, respectively. A non-empty relation $Z \subseteq E(\mathfrak{F}) \times E(\mathfrak{F}')$ is called a *bisimulation* between $\mathfrak{M}$ and $\mathfrak{M}'$ if the following conditions hold.

1. If $((x, U), (x', U')) \in Z$, then for each $p \in PV$, $x \in m(p)$ if and only if $x' \in m'(p)$.
2. (B1) If $((x, U), (x', U')) \in Z$ and $(x', V') \in E(\mathfrak{F}')$, then there exists a $(x, V) \in E(\mathfrak{F})$ such that $((x, V), (x', V')) \in Z$.

   (F1) If $((x, U), (x', U')) \in Z$ and $(x, V) \in E(\mathfrak{F})$, then there exists a $(x', V') \in E(\mathfrak{F}')$ such that $((x, V), (x', V')) \in Z$.
3. (B2) If $((x, U), (x', U')) \in Z$ and there is a $y' \in U'$ such that $(x', y') \in R'$, then there exists a $y \in U$ with $(x, y) \in R$ and $((y, U), (y', U')) \in Z$.

   (F2) If $((x, U), (x', U')) \in Z$ and there is a $y \in U$ such that $(x, y) \in R$, then there exists a $y' \in U'$ with $(x', y') \in R'$ and $((y, U), (y', U')) \in Z$.

When Z is a bisimulation linking two states $(x, U) \in E(\mathfrak{F})$ and $(x', U') \in E(\mathfrak{F}')$, we say that (x, U) and (x', U') are bisimilar, and we write

$$Z : \mathfrak{M}, (x, U) \underline{\longleftrightarrow} \mathfrak{M}', (x', U').$$

If there is a bisimulation between $\mathfrak{M}$ and $\mathfrak{M}'$, then we write $Z : \mathfrak{M} \underline{\longleftrightarrow} \mathfrak{M}'$.

The following theorem shows that we have the invariance result under bisimulation.

Theorem 5.44 (Invariance Theorem)**.** *Consider models $\mathfrak{M} := (\mathfrak{F}, m)$ and $\mathfrak{M}' := (\mathfrak{F}', m')$ based on SASs $\mathfrak{F} = (W, \rho, R)$ and $\mathfrak{F}' = (W', \rho', R')$, respectively. Then, for each $(x, U) \in E(\mathfrak{F})$ and $(x', U') \in E(\mathfrak{F}')$, $Z : \mathfrak{M}, (x, U) \underline{\longleftrightarrow} \mathfrak{M}', (x', U')$ implies $\mathfrak{M}, (x, U) \leftrightsquigarrow \mathfrak{M}', (x', U')$.*

Proof. Let $Z : \mathfrak{M}, (x, U) \underline{\longleftrightarrow} \mathfrak{M}', (x', U')$ and we prove that $\mathfrak{M}, (x, U) \leftrightsquigarrow \mathfrak{M}', (x', U')$. The proof follows by induction. The case for propositional variable follows from the definition of bisimulation. Boolean cases are also obvious. We discuss the proof of modal cases. So, first suppose that $\mathfrak{M}, x, U \models \Box_1 \beta$ and we prove that $\mathfrak{M}', x', U' \models \Box_1 \beta$. Let us take a $V' \in \rho'$ such that $x' \in V'$. From 2(B1) condition, there is a $V \in \rho$ with $x \in V$ and $((x, V), (x', V')) \in Z$. Since $\mathfrak{M}, x, U \models \Box_1 \beta$, we get $\mathfrak{M}, x, V \models \beta$. Using induction hypothesis, it follows that $\mathfrak{M}', x', V' \models \beta$. Thus $\mathfrak{M}', x', U' \models \Box_1 \beta$. The other direction can be proved using 2(F1) condition.
Next, assume that $\mathfrak{M}, x, U \models \Box \beta$ and we prove that $\mathfrak{M}', x', U' \models \Box \beta$. Consider a $y' \in R(x') \cap U'$, then by 3(B2) condition, there exists a $y \in R(x) \cap U$ and $((y, U), (y', U')) \in Z$. Since $\mathfrak{M}, x, U \models \Box \beta$, we get $\mathfrak{M}, y, U \models \beta$. By the use of induction case, we derive $\mathfrak{M}', y', U' \models \beta$, thus $\mathfrak{M}', x', U' \models \Box \beta$. Similary, other direction can be proved using 3(F2) condition. □

The converse of above theorem is not true in general. Like the standard modal logic [7], it is possible to prove a restricted converse. Let us first note the following properties of a SAS $\mathfrak{F} = (W, \rho, R)$.

- **(I1):** For each $x \in W$, the set $\rho_x = \{V \in \rho : x \in V\}$ is finite.
- **(I2):** For each $U \in \rho$ and for each $x \in U$, the set $\{y \in U \mid (x, y) \in R\}$ contains finite number of elements.

Theorem 5.45. *Consider models $\mathfrak{M} = (\mathfrak{F}, m)$ and $\mathfrak{M}' = (\mathfrak{F}', m')$ depending on SASs $\mathfrak{F} = (W, \rho, R)$ and $\mathfrak{F}' = (W', \rho', R')$, respectively, where $\mathfrak{F}$ and $\mathfrak{F}'$ satisfy the properties*

(I1) and (I2). Let $(x,U) \in E(\mathfrak{F})$ and $(x',U') \in E(\mathfrak{F}')$. Then $\mathfrak{M},(x,U) \leftrightsquigarrow \mathfrak{M}',(x',U')$ implies $Z : \mathfrak{M},(x,U)\underline{\longleftrightarrow}\mathfrak{M}',(x',U')$ for some Z.

Proof. We will show that $\leftrightsquigarrow$ is itself a bisimulation. The condition (1) of Definition 5.43 follows trivially. For 2(B1), let us take a $V' \in \rho'$ with $x' \in V'$. If possible, assume that there does not exist any $V \in \rho$ with $x \in V$ and $\mathfrak{M},(x,V) \leftrightsquigarrow \mathfrak{M}',(x',V')$. Consider the set $\rho_x = \{V \in \rho \mid x \in V\}$. By the given conditions, ρ_x is non-empty as $U \in \rho_x$ and finite. Let $\rho_x = \{V_1, V_2, \cdots, V_n\}$. By assumption, for each $V_i \in \rho_x$, there must be a wff α_i having $\mathfrak{M}',x',V' \models \alpha_i$ and $\mathfrak{M},x,V_i \not\models \alpha_i$. Thus we have $\mathfrak{M}',x',V' \models \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_n$ and hence $\mathfrak{M}',x',U' \models \Diamond_1(\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_n)$. But $\mathfrak{M},x,U \not\models \Diamond_1(\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_n)$. This cannot be true due to the assumption $\mathfrak{M},x,U \leftrightsquigarrow \mathfrak{M}',x',U'$. Thus 2(B1) holds. The 2(F1) condition can be proved following similar steps.
Next, let us take a $y' \in U'$ with $(x',y') \in R'$. If possible, assume that there does not exist any $y \in U$ with $(x,y) \in R$ and $(y,U) \leftrightsquigarrow (y',U')$. Let $S = \{y \in U \mid (x,y) \in R\}$. Note that S must be non-empty as $\mathfrak{M}',x',U' \models \Diamond\top$ and $\mathfrak{M},(x,U) \leftrightsquigarrow \mathfrak{M}',(x',U')$. By the given condition, S must be finite. Let $S = \{y_1, y_2, \ldots, y_m\}$. By assumption, for each $y_i \in S$, there exists a wff β_i such that $\mathfrak{M}',y',U' \models \beta_i$ but $\mathfrak{M},y_i,U \not\models \beta_i$. We have $\mathfrak{M}',y',U' \models \beta_1 \wedge \beta_2 \cdots \wedge \beta_m$ and $(x',y') \in R'$. Thus we obtain $\mathfrak{M}',x',U' \models \Diamond(\beta_1 \wedge \beta_2 \cdots \wedge \beta_m)$. But $\mathfrak{M},x,U \not\models \Diamond(\beta_1 \wedge \beta_2 \cdots \wedge \beta_m)$ which contradicts our assumption $\mathfrak{M},(x,U) \leftrightsquigarrow \mathfrak{M}',(x',U')$. Therefore 3(B2) condition holds. Similarly, we can show 3(F2) condition. Thus $\leftrightsquigarrow$ is a bisimulation between $\mathfrak{M}$ and $\mathfrak{M}'$. □

5.8.2. Definability

We are now in a position to present some results emphasizing the expressive power of modal logic given in Section 5.2. Let us first see the following definition of definability.

Definition 5.46. We say that a wff α defines a class $\mathfrak{G}$ of SASs if for every SAS $\mathfrak{F}$, $\mathfrak{F}$ belongs to $\mathfrak{G}$ if and only if $\mathfrak{F} \models \alpha$. In a similar way, a set Δ of wffs defines a class $\mathfrak{G}$ of SASs if for each SAS $\mathfrak{F}$, $\mathfrak{F}$ belongs to $\mathfrak{G}$ if and only if $\mathfrak{F} \models \Delta$. A class of SASs is definable whenever we have a set of wffs defining it.

The following theorem presents a few classes of SASs that can be defined through the wffs of the proposed logic. It demonstrates the expressibility power of the proposed syntax and semantics. Let $\mathfrak{F} := (W, \rho, R)$ be a SAS. Retrieve that $\rho_x := \{U \in \rho : x \in U\}$.

Proposition 5.47. *Let $\mathfrak{F} := (W, \rho, R)$ be a SAS. Then the following hold.*

1. $\mathfrak{F} \models \Diamond_1\Box p \wedge \Diamond_1\Box q \to \Diamond_1\Box(p \wedge q)$ *if and only if for all $x \in W$ and $U, V \in \rho_x$, there exists a $Z \in \rho_x$ such that $R(x) \cap Z \subseteq R(x) \cap U \cap V$.*
2. $\mathfrak{F} \models \Diamond_1\Diamond p \to \Diamond p$ *if and only if for every $x, y \in W$ with $y \in R(x)$, if $U, V \in \rho_x$ and $V \in \rho_y$, then $U \in \rho_y$.*
3. $\mathfrak{F} \models \Box p \to \Diamond p$ *if and only if for all $x \in W$ and for all $U \in \rho_x$, $U \in \rho_y$ for some $y \in R(x)$.*
4. $\mathfrak{F} \models p \to \Diamond_1\Box p$ *if and only if for all $x \in W$ and for all $U \in \rho_x$, there is a $V \in \rho_x$ satisfying $R(x) \cap V \subseteq \{x\}$.*

Proof. (1). First, assume that $\mathfrak{F} \models \Diamond_1\Box p \wedge \Diamond_1\Box q \to \Diamond_1\Box(p \wedge q)$ and we show that for every $x \in W$ and $U, V \in \rho_x$, there exists a $Z \in \rho_x$ such that $R(x) \cap Z \subseteq R(x) \cap U \cap V$. If $R(x) \cap U \subseteq R(x) \cap U \cap V$ or $R(x) \cap V \subseteq R(x) \cap U \cap V$, then we are done. So, let $R(x) \cap U \not\subseteq R(x) \cap U \cap V$ and $R(x) \cap V \not\subseteq R(x) \cap U \cap V$. Let us consider a model $\mathfrak{M} := (\mathfrak{F}, m)$ where $m(p) = R(x) \cap U$ and $m(q) = R(x) \cap V$. Note that sets $R(x) \cap U$ and $R(x) \cap V$ are non-empty. Since $m(p) = R(x) \cap U$ and $x \in U$, we get $\mathfrak{M}, x, U \models \Box p$. Similarly, we have $\mathfrak{M}, x, V \models \Box q$. Thus, we obtain $\mathfrak{M}, x, U \models \Diamond_1\Box p \wedge \Diamond_1\Box q$. This, in turn, gives $\mathfrak{M}, x, U \models \Diamond_1\Box(p \wedge q)$ as $\mathfrak{F} \models \Diamond_1\Box p \wedge \Diamond_1\Box q \to \Diamond_1\Box(p \wedge q)$. Thus, we get a $Z \in \rho_x$ having $\mathfrak{M}, x, Z \models \Box(p \wedge q)$. This gives $R(x) \cap Z \subseteq m(p) \cap m(q) = R(x) \cap U \cap V$ as required.

For the other direction, suppose that for each $x \in W$ and $U, V \in \rho_x$, there exists a $Z \in \rho_x$ such that $R(x) \cap Z \subseteq R(x) \cap U \cap V$. We prove that $\mathfrak{F} \models \Diamond_1\Box p \wedge \Diamond_1\Box q \to \Diamond_1\Box(p \wedge q)$. Let us take a model $\mathfrak{M}$ such that $\mathfrak{M}, x, U \models \Diamond_1\Box p \wedge \Diamond_1\Box q$. We show that $\mathfrak{M}, x, U \models \Diamond_1\Box(p \wedge q)$. From $\mathfrak{M}, x, U \models \Diamond_1\Box p \wedge \Diamond_1\Box q$, we obtain $V_1, V_2 \in \rho_x$ such that

$$\mathfrak{M}, x, V_1 \models \Box p \text{ and } \mathfrak{M}, x, V_2 \models \Box q. \tag{5.7}$$

Since $V_1, V_2 \in \rho_x$, by our assumption, we obtain a $Z \in \rho_x$ such that $R(x) \cap Z \subseteq R(x) \cap V_1 \cap V_2$. We prove that $\mathfrak{M}, x, Z \models \Box(p \wedge q)$. So, let us take $y \in R(x) \cap Z$. We have $y \in V_1 \cap V_2$ as $R(x) \cap Z \subseteq R(x) \cap V_1 \cap V_2$. From (5.7), we get $\mathfrak{M}, y, V_1 \models p$ and $\mathfrak{M}, y, V_2 \models q$, therefore $\mathfrak{M}, y, Z \models p \wedge q$. Thus, we have shown $\mathfrak{M}, x, Z \models \Box(p \wedge q)$.

(2) First, let $\mathfrak{F} \models \Diamond_1\Diamond p \to \Diamond p$ and we show that for all $x, y \in W$ with $y \in R(x)$, if $U, V \in \rho_x$ and $V \in \rho_y$, then $U \in \rho_y$. If possible, assume that there exist $x, y \in W$ with

$y \in R(x)$ and $U, V \in \rho_x, V \in \rho_y$ but $U \notin \rho_y$. Let us take a model $\mathfrak{M} = (\mathfrak{F}, m)$ such that $m(p) = \{y\}$. Then $\mathfrak{M}, x, U \models \Diamond_1 \Diamond p$ but $\mathfrak{M}, x, U \not\models \Diamond p$. Hence $\mathfrak{M}, x, U \not\models \Diamond_1 \Diamond p \rightarrow \Diamond p$.
For the other direction, assume that for all $x, y \in W$ with $y \in R(x)$, if $U, V \in \rho_x$ and $V \in \rho_y$, then $U \in \rho_y$. We will prove that $\mathfrak{F} \models \Diamond_1 \Diamond p \rightarrow \Diamond p$. So, let us consider a model $\mathfrak{M}$ and (x, U) such that $\mathfrak{M}, x, U \models \Diamond_1 \Diamond p$ and we show that $\mathfrak{M}, x, U \models \Diamond p$. From $\mathfrak{M}, x, U \models \Diamond_1 \Diamond p$, there exists a $V_1 \in \rho$ such that $x \in V_1$ and $\mathfrak{M}, x, V_1 \models \Diamond p$. Then, we obtain a $y \in R(x) \cap V_1$ such that $\mathfrak{M}, y, V_1 \models p$. Since $U, V_1 \in \rho_x, V_1 \in \rho_y$ and $y \in R(x)$, by assumption we obtain $U \in \rho_y$. Thus $\mathfrak{M}, y, U \models p$ and $\mathfrak{M}, x, U \models \Diamond p$.

(3) First, let $\mathfrak{F} \models \Box p \rightarrow \Diamond p$ and we show that for all $U \in \rho_x$, $U \in \rho_y$ for some $y \in R(x)$. If possible, suppose that there exist a $x \in W$ and $U \in \rho_x$ such that $R(x) \cap U = \emptyset$. Let us consider a model $\mathfrak{M} = (\mathfrak{F}, m)$ such that $m(p) = U$. Then we have $\mathfrak{M}, x, U \models \Box p$ but $\mathfrak{M}, x, U \not\models \Diamond p$. This is not possible as $\mathfrak{F} \models \Box p \rightarrow \Diamond p$.
For the converse part, assume that for all $U \in \rho_x$, $U \in \rho_y$ for some $y \in R(x)$. We prove that $\mathfrak{F} \models \Box p \rightarrow \Diamond p$. Let us take a model $\mathfrak{M}$ and (x, U) such that $\mathfrak{M}, x, U \models \Box p$ and we show that $\mathfrak{M}, x, U \models \Diamond p$. From the assumption, there exists a $y \in R(x)$ such that $U \in \rho_y$. Since $\mathfrak{M}, x, U \models \Box p$, we get $\mathfrak{M}, y, U \models p$. Thus $\mathfrak{M}, x, U \models \Diamond p$.

(4) First, let $\mathfrak{F} \models p \rightarrow \Diamond_1 \Box p$ and we show that for all $x \in W$ and for all $U \in \rho_x$, there exists a $V \in \rho_x$ such that $R(x) \cap V \subseteq \{x\}$. If possible, suppose that there exist $x \in W$, $U \in \rho_x$, and for all $V \in \rho$ with $V \in \rho_x, R(x) \cap V \neq \{x\}$. Consider a model $\mathfrak{M} = (\mathfrak{F}, m)$ such that $m(p) = \{x\}$. Then $\mathfrak{M}, x, U \models p$. Further, we prove that $\mathfrak{M}, x, U \not\models \Diamond_1 \Box p$, that is, $\mathfrak{M}, x, U \models \Box_1 \Diamond \neg p$. Let us take a $V_1 \in \rho_x$. By the assumption, we obtain a $y \in R(x) \cap V_1$ and $y \neq x$. Also, $\mathfrak{M}, y, V_1 \not\models p$. Thus $\mathfrak{M}, x, V_1 \models \Diamond \neg p$. We have $\mathfrak{M}, x, U \models p$ and $\mathfrak{M}, x, U \not\models \Diamond_1 \Box p$. This is not possible as $\mathfrak{F} \models p \rightarrow \Diamond_1 \Box p$. Thus for all $x \in W$ and for every $U \in \rho_x$, there is a $V \in \rho_x$ satisfying $R(x) \cap V \subseteq \{x\}$.
For the other part, suppose that for all $x \in W$ and for all $U \in \rho_x$, there exists a $V \in \rho_x$ such that $R(x) \cap V \subseteq \{x\}$. We prove that $\mathfrak{F} \models p \rightarrow \Diamond_1 \Box p$. Let us take a model $\mathfrak{M}$ and (x, U) such that $\mathfrak{M}, x, U \models p$ and we prove that $\mathfrak{M}, x, U \models \Diamond_1 \Box p$. From the assumption, there exist a $V_1 \in \rho_x$ such that $R(x) \cap V_1 \subseteq \{x\}$. In both the cases, $\mathfrak{M}, x, V_1 \models \Box p$. Hence $\mathfrak{M}, x, U \models \Diamond_1 \Box p$. □

Now, we provide a few classes of SASs which are not definable.

Proposition 5.48. *The class* $\mathsf{A_r}$ *of SASs is not definable.*

Proof. Let $\mathfrak{F}_1 := (W_1, \rho_1, R_1)$ and $\mathfrak{F}_2 := (W_2, \rho_2, R_2)$ be two SASs, where

$$W_1 = \{x, y\}, \rho_1 = \{\{x\}\}, R_1 = \{(x, x)\}, \text{ and}$$
$$W_2 = \{x\}, \rho_2 = \{\{x\}\}, R_2 = \{(x, x)\}.$$

Towards a contradiction, consider a subset Δ of $\mathcal{L}(\Box_1, \Box)$ defining the class $\mathsf{A_r}$. We have $\mathfrak{F}_1 \notin \mathsf{A_r}$ and $\mathfrak{F}_2 \in \mathsf{A_r}$, thus $\mathfrak{F}_1 \not\models \Delta$ and $\mathfrak{F}_2 \models \Delta$. Since $\mathfrak{F}_1 \not\models \Delta$, we get a model $\mathfrak{M}_1 := (\mathfrak{F}_1, m_1)$ depending on $\mathfrak{F}_1$ and $\alpha \in \Delta$ such that $\mathfrak{M}_1, x, \{x\} \not\models \alpha$. Let us take a model $\mathfrak{M}_2 := (\mathfrak{F}_2, m_2)$, where $m_2(p) = m_1(p) \cap W_2$ for all $p \in PV$. Note that the relation $Z := \{(x, \{x\}), (x, \{x\})\}$ is a bisimulation between $\mathfrak{M}_1$ and $\mathfrak{M}_2$. Since $\mathfrak{M}_1, x, \{x\} \not\models \alpha$, using Theorem 5.44, we conclude $\mathfrak{M}_2, x, \{x\} \not\models \alpha$. But this is a contradiction to the fact $\mathfrak{F}_2 \models \Delta$. Hence, the class $\mathsf{A_r}$ is not definable. □

Proposition 5.49. *The class $\mathsf{A_t}$ of SASs is not definable.*

Proof. Let $\mathfrak{F}_1 := (W_1, \rho_1, R_1)$ and $\mathfrak{F}_2 := (W_2, \rho_2, R_2)$ be SASs, where

$$W_1 = \{x, y, z\}, \rho_1 = \{\{x\}, \{y\}, \{z\}\}, R_1 = \{(x, y), (y, z), (z, x)\}, \text{ and}$$
$$W_2 = \{x, y, z\}, \rho_2 = \{\{x\}, \{y\}, \{z\}\}, R_2 = \{(x, y), (y, z)\}.$$

As $\mathfrak{F}_1 \in \mathsf{A_t}$ and $\mathfrak{F}_2 \notin \mathsf{A_t}$, we obtain $\mathfrak{F}_1 \models \Delta$ and $\mathfrak{F}_2 \not\models \Delta$. From $\mathfrak{F}_2 \not\models \Delta$, we derive that there is a model $\mathfrak{M}_2 := (\mathfrak{F}_2, m_2)$ based on $\mathfrak{F}_2$, $t \in \{x, y, z\}$ and $\alpha \in \Delta$ such that $\mathfrak{M}_2, t, \{t\} \not\models \alpha$. Let us take a model $\mathfrak{M}_1 := (\mathfrak{F}_1, m_2)$. Note that the relation $Z := \{((x, \{x\}), (x, \{x\})), ((y, \{y\}), (y, \{y\})), ((z, \{z\}), (z, \{z\}))\}$ is a bisimulation between $\mathfrak{M}_1$ and $\mathfrak{M}_2$. Therefore, using $\mathfrak{M}_2, t, \{t\} \not\models \alpha$ and Theorem 5.44, we derive $\mathfrak{M}_1, t, \{t\} \not\models \alpha$. But this cannot be true as $\mathfrak{F}_1 \models \Delta$. Hence, the class $\mathsf{A_t}$ is not definable. □

Proposition 5.50. *The class $\mathsf{A_s}$ of SASs is not definable.*

Proof. Let $\mathfrak{F}_1 := (W_1, \rho_1, R_1)$ and $\mathfrak{F}_2 := (W_2, \rho_2, R_2)$ be two SASs, where

$$W_1 = \{x, y\}, \rho_1 = \{\{x\}, \{y\}\}, R_1 = \{(x, y), (y, x)\}, \text{ and}$$
$$W_2 = \{x, y\}, \rho_2 = \{\{x\}, \{y\}\}, R_2 = \{(x, y)\}.$$

To the contrary, assume that there is a set Δ consisting of wffs defining $\mathsf{A_s}$. Since $\mathfrak{F}_1 \in \mathsf{A_s}$ and $\mathfrak{F}_2 \notin \mathsf{A_s}$, we get $\mathfrak{F}_1 \models \Delta$ and $\mathfrak{F}_2 \not\models \Delta$. But this leads to a contradiction as the relation $Z := \{((x, \{x\}), (x, \{x\})), ((y, \{y\}), (y, \{y\}))\}$ is a bisimulation between $\mathfrak{F}_1$ and $\mathfrak{F}_2$. □

We end this section with an interesting observation. We have seen in Section 5.5 that the axiom $\Box\alpha \to \Box\Box\alpha$ leads us to the completeness theorem for the class $\mathsf{A_t}$. Moreover, it is not difficult to show that the class Υ_{t} of auxiliary models can be defined by the wff $\Box\alpha \rightarrow \Box\Box\alpha$. Therefore, we were expecting that the wff $\Box\alpha \rightarrow \Box\Box\alpha$ will be able to define the class A_{t} of SASs. But, as shown above, it is not the case. We have similar observation for the classes A_{r} and A_{s}.

5.9. Conclusion

We introduced and studied a generalized rough set model based on subset approximation structure (SAS). This work is just a beginning toward the studies in the rough set model based on SASs, and the model needs further investigation. For instance, other rough set concepts such as definability of sets, membership functions deserve to be explored. Similarly, it is crucial to obtain the characterization results for the proposed approximation operators.

In the second part of the chapter, we have provided a modal logic with semantics based on SASs. The logic is shown to be decidable. Moreover, sound and complete modal systems for different classes of SASs are obtained. Regarding the expressibility power of the logic, it is observed that the logic can describe many concepts related to approximations defined on SASs, but it also needs some improvements.

CHAPTER 6

A MODAL LOGIC TO STUDY KNOWLEDGE AND APPROXIMATION OPERATORS

In Chapters 3 and 4, the unary modal operator is interpreted as the approximation operator from rough set theory. We also know that, in epistemic logic, the unary modal operator is used to study the knowledge and belief of agents [20, 99, 100]. In this research direction, usually, the possible-worlds semantics is used where the objects of the domain are interpreted as possible states. These possible states carry atomic information represented by propositional variables. Besides the true state of affairs, an agent may consider many other states to be possible due to his/her partial knowledge. This possibility of the states is captured through binary relations between the states. A natural generalization would be the case where each state carries information about a set of objects regarding a set of attributes. In other words, each state is assigned an information system. Thus each state gives a collection of approximation operators indexed with subsets of the attribute set. In this chapter, we will have a study based on such a model where knowledge operator as well as approximation operators comes into the picture. Therefore, it is interesting to reason about the statements like 'agent knows/safely believes that the object under consideration belongs to the upper approximation of the set' or 'the object is in the lower approximation of the set in some states which are considered to be at least as good as the current state'. We propose a modal logic that can express these types of statements. Our study will be based on the notion of possible-worlds information system proposed in Section 6.1. It consists of a set of states where each state is assigned a DIS. It is shown in Section 6.1 that the possible-worlds information systems can also represent the situations captured by various types of information systems viz. incomplete, non-deterministic, and probabilistic information systems. In Section 6.2, we propose a modal logic for possible-worlds information system. Section 6.3 gives the rough set interpretations of the proposed logic with illustrations of the kind of statements one can express through the language.

In Section 6.4, we present modal systems for different classes of models. Section 6.5 provides the corresponding completeness theorems. Again the step by step technique [7] of modal logic is used for the completeness proofs. Section 6.6 presents a comparison of the proposed semantics with the well-known multi-modal logic semantics. In Section 6.7, decidability results related to the proposed logics are discussed. We conclude the chapter in Section 6.8.

The work presented in this chapter is based on the article [42].

6.1. Possible-worlds information system

We propose the following notion of possible-worlds information system and we show that the situations captured by various types of information systems discussed in the Chapter 2 can also be represented by it.

Definition 6.1. A *possible-worlds information system* (PWIS) is defined as a tuple $\mathfrak{F} := (\mathbb{S}, R, \{\mathcal{K}_s\}_{s \in \mathbb{S}})$, where

- $\mathbb{S}$ is a non-empty set of states,
- $R \subseteq \mathbb{S} \times \mathbb{S}$,
- for each $s \in \mathbb{S}$, $\mathcal{K}_s := (W_s, \mathcal{A}, \bigcup_{a \in \mathcal{A}} \mathcal{V}_a, f_s)$ is a DIS with the fixed attribute and attribute-value sets $\mathcal{A}$ and $\bigcup_{a \in \mathcal{A}} \mathcal{V}_a$, respectively.

The above definition of PWIS is based on the intuitive idea that the agent, due to his/her partial knowledge, considers several information systems possible besides the actual information system. So, at the state s, the agent considers all the DISs $\mathcal{K}_t$, where $t \in R(s)$, to be possible, but he/she is not certain about the DIS, which gives the correct information. One can impose different conditions on R depending on the requirement. For instance, we may take R to be a *plausibility order* [99], that is a reflexive and transitive relation, and in that case, $(s, t) \in R$ signifies that the DIS $\mathcal{K}_t$ is at least as plausible as the DIS $\mathcal{K}_s$. Similarly, to reason about the agent's knowledge, one needs to take R to be an equivalence relation [20].

6.1.1. Representations of NISs, IISs, PISs through PWISs

The situations captured by NISs, IISs, PISs can also be represented through PWISs in a natural way. For instance, consider the following example.

	Disease	
	Presence	Absence
Patient 1	$\frac{3}{4}$	$\frac{1}{4}$
Patient 2	$\frac{1}{3}$	$\frac{2}{3}$

Table 6.1. PIS $\mathcal{S}$

	Disease
Patient 1	Presence
Patient 2	Presence

Table 6.2. $\mathcal{K}_{s_1}$

	Disease
Patient 1	Absence
Patient 2	Absence

Table 6.3. $\mathcal{K}_{s_2}$

	Disease
Patient 1	Presence
Patient 2	Absence

Table 6.4. $\mathcal{K}_{s_3}$

	Disease
Patient 1	Absence
Patient 2	Presence

Table 6.5. $\mathcal{K}_{s_4}$

Example 6.2. Let us consider a PIS $\mathcal{S}$ given by Table 6.1 that gives information about two patients regarding the presence/absence of a disease. According to this PIS, the probabilities of Patient 1 and Patient 2 suffering from the disease are $\frac{3}{4}$ and $\frac{1}{3}$, respectively. Note that we must have either of the following four situations.

- s_1: Both the Patients 1 and 2 are suffering from the disease.
- s_2: None of the Patients 1 and 2 are suffering from the disease.
- s_3: Patient 1 is suffering from the disease, but Patient 2 is not.
- s_4: Patient 2 is suffering from the disease, but Patient 1 is not.

The above four situations can be captured by the four DISs given by Tables 6.2-6.5. Observe that, based on the information given by PIS $\mathcal{S}$ of Table 6.1, possibility of having situation s_3 is more compared to the other three situations. In fact, we have the following probability measure for having the four situations.

- Probability of having situation $s_1 = \frac{3}{4} \times \frac{1}{3} = \frac{3}{12}$.
- Probability of having situation $s_2 = \frac{1}{4} \times \frac{2}{3} = \frac{2}{12}$.
- Probability of having situation $s_3 = \frac{3}{4} \times \frac{2}{3} = \frac{6}{12}$.
- Probability of having situation $s_4 = \frac{1}{4} \times \frac{1}{3} = \frac{1}{12}$.

Therefore, the above mentioned four possible situations and their comparative chances to occur can be captured through the PWIS

$$\mathfrak{F} := (\{s_1, \ldots, s_4\}, R^*, \{\mathcal{K}_{s_1}, \mathcal{K}_{s_2}, \mathcal{K}_{s_3}, \mathcal{K}_{s_4}\}),$$

where the plausibility relation R^* is the transitive closure of the relation R given by Figure 6.1.

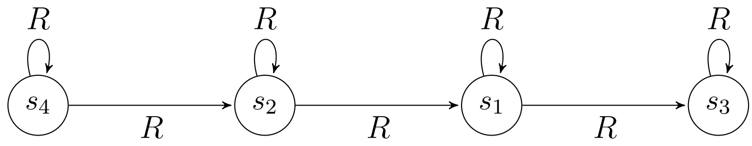

Figure 6.1

The idea of above example can be extended for any PIS with finite domain as follows. Let $\mathcal{S} := (W, \mathcal{A}, \bigcup_{a\in\mathcal{A}} \mathcal{V}_a, F)$ be a PIS with finite W, and consider the set $\mathcal{I}$ of all DISs with domain W, attribute set $\mathcal{A}$ and attribute-value set $\mathcal{V}_a$ for each $a \in \mathcal{A}$. Let $\mathbb{S}$ be a set with $|\mathbb{S}| = |\mathcal{I}|$, and g be a bijection from $\mathbb{S}$ to $\mathcal{I}$.
Recall that $F(x, a, \{v\})$ gives the probability of the object x to take the attribute-values v for the attribute a. We use this information to define a function $P_{\mathcal{S}} : \mathcal{I} \to [0, 1]$ as follows.

$$P_{\mathcal{S}}(\mathcal{K}) = \prod_{w\in W} \prod_{a\in\mathcal{A}} F(w, a, f(w, a)), \text{ where } \mathcal{K} := (W, \mathcal{A}, \bigcup_{a\in\mathcal{A}} \mathcal{V}_a, f).$$

Note that $P_{\mathcal{S}}$ gives the probability to have DIS $\mathcal{K}$ based on the information provided by the PIS $\mathfrak{F}$. We make use of this function to obtain a PWIS $\mathfrak{F}_{\mathcal{S}} := (\mathbb{S}, R, \{\mathcal{K}_s\}_{s\in\mathbb{S}})$, where

- $(s, t) \in R$ if and only if $P_{\mathcal{S}}(g(s)) \leq P_{\mathcal{S}}(g(t))$.
- $\mathcal{K}_s := g(s)$.

One can easily verify that R is a plausibility order, that is, reflexive and transitive relation. From the definition of the relation R, it is evident that if $(s, t) \in R$, then the DIS $\mathcal{K}_t$ is at least as plausible as the DIS $\mathcal{K}_s$.

Let us now move to see how PWISs can also be used to capture the situations represented by NISs. Let us consider a finite domain NIS $\mathcal{S} := (W, \mathcal{A}, \bigcup_{a\in\mathcal{A}} \mathcal{V}_a, f)$ under the assumption that $f(x, a) = V$ represents a situation where we do not know what attribute-value the object x takes for the attribute a, but we know that it is one of the members of

V. Further, we also assume that $f(x,a) \neq \emptyset$ for each x and a. This condition is natural as we know that each object takes attribute-value from the set $\mathcal{V}_a$ for each attribute a. Under these assumptions, let us define the function $F(x,a,\{v\})$, which gives the probability of the object x to take the attribute-values v for the attribute a, as follows.

$$F(x,a,\{v\}) := \begin{cases} \frac{1}{|f(x,a)|}, & \text{if } v \in f(x,a) \\ 0, & \text{otherwise.} \end{cases} \tag{6.1}$$

We can now define PWIS $\mathfrak{F}_{\mathcal{S}}$ as in the case of PIS, using F given by (6.1).
Similarly, for an IIS $\mathcal{S}$, we define the PWIS $\mathfrak{F}_{\mathcal{S}}$ using the function

$$F(x,a,\{v\}) = \begin{cases} 1, & \text{if } f(x,a) = v \\ \frac{1}{|\mathcal{V}_a|}, & \text{if } f(x,a) = * \\ 0, & \text{otherwise.} \end{cases}$$

Here, to define $F(x,a,\{v\})$, the probability of the object x to take the attribute-values v for the attribute a, we have considered the fact that $f(x,a) = *$ denotes the absence of information about x regarding the attribute a. Moreover, in that case, each of the attribute-value $v \in \mathcal{V}_a$ has an equal probability $\frac{1}{|\mathcal{V}_a|}$ to be assigned to the object x for the attribute a.

We have shown above how we can use PWISs with plausibility order R on the states to capture PISs, NISs, and IISs. In fact, depending upon the requirement, one can impose different conditions on the component R of the PWIS $(\mathbb{S}, R, \{\mathcal{K}_s\}_{s\in\mathbb{S}})$. In this chapter, we want to talk about 'knowledge' and 'plausibility', and hence will consider the following classes of PWISs.

- $\mathsf{F_e}$: the class of all PWIS $\mathfrak{F} := (\mathbb{S}, R, \{\mathcal{K}_s\}_{s\in\mathbb{S}})$ where R is an equivalence relation.
- $\mathsf{F_p}$: the class of all PWIS $\mathfrak{F} := (\mathbb{S}, R, \{\mathcal{K}_s\}_{s\in\mathbb{S}})$ where R is a plausibility order (i.e. reflexive and transitive relation).

We are also interested in one another class of PWISs, denoted by $\mathsf{F_c}$, that consists of PWIS $\mathfrak{F}$ where the domains of each of the constituent DISs of $\mathfrak{F}$ is the same set. Elements of $\mathsf{F_c}$ will be called a *constant domain PWIS*. So a constant domain PWIS represents a situation where each state carries information about the same set of objects.

6.2. A modal logic for possible-worlds information system

This section presents a modal logic that can be used to reason about statements (on PWISs) involving basic statements like 'the object is in the lower/ upper approximation of the set in some/all states which are considered to be *at least as good as the current state'*, 'agent *knows/safely believes* that the object belongs to the lower/upper approximation of the set'.

6.2.1. Syntax

The alphabet of the language $\mathcal{L}$ contains (i) a non-empty finite set $\mathcal{A}$ of attribute constants, (ii) for each $a \in \mathcal{A}$, a non-empty finite set $\mathcal{V}_a$ of attribute-value constants, (iii) a non-empty countable set PV of propositional variables, and (iv) the propositional constant $\top$. The tuples (a, v), where $a \in \mathcal{A}$ and $v \in \mathcal{V}_a$, are called *descriptors* [74]. The set of all descriptors will be denoted by $\mathcal{D}$. The propositional variables, descriptors and propositional constant $\top$ form the set of atomic well-formed formulae. Using atomic well-formed formulae, the standard Boolean logical connectives $\neg$ (negation) and $\wedge$ (conjunction), the modal connectives $\Box, \Box_C$, where $C \subseteq \mathcal{A}$, the wffs of $\mathcal{L}$ are then defined recursively as:

$$\top \mid p \mid (a, v) \mid \neg\alpha \mid \alpha \wedge \beta \mid \Box\alpha \mid \Box_C\alpha,$$

where $p \in PV$, $(a, v) \in \mathcal{D}$, and α, β are wffs. The intuitive readings of components $(a, v), \Box\alpha$ and $\Box_C\alpha$ are as follows.

(a, v) : Object takes the value v for the attribute a.

$\Box\alpha$: Agent knows (safely believes) α.

$\Box_C\alpha$: Object belongs to the lower approximation (with respect to attribute set C) of the set represented by α.

Apart from the usual derived connectives $\bot, \vee, \rightarrow, \leftrightarrow$, we have the connectives $\Diamond$ and $\Diamond_C$ defined as follows:

$$\Diamond\alpha := \neg\Box\neg\alpha,$$

$$\Diamond_C\alpha := \neg\Box_C\neg\alpha.$$

We will make use of the same symbol $\mathcal{L}$ to denote the set of all wffs of the language $\mathcal{L}$. Moreover, we will use $\mathcal{L}(\Box, \Box_\emptyset)$ to denote the set of all wffs which does not involve modal operators $\Box_C$, for any non-empty $C \subseteq \mathcal{A}$.

Classes of Models	Defining conditions
Ω	Class of all models
Ω_{c}	Class of models based on constant domain PWISs
Ω_{e}	Class of all models based on PWISs from $\mathsf{F_e}$
Ω_{p}	Class of all models based on PWISs from $\mathsf{F_p}$
Ω_{ce}	$\Omega_{\mathsf{c}} \cap \Omega_{\mathsf{e}}$
Ω_{cp}	$\Omega_{\mathsf{c}} \cap \Omega_{\mathsf{p}}$

Table 6.6. Classes of Models

6.2.2. Semantics

The semantics of $\mathcal{L}$ will be based on PWISs. Thus, we have the following definition of a model.

Definition 6.3. A *model* of $\mathcal{L}$ is a tuple $\mathfrak{M} := (\mathfrak{F}, m)$, where

$$\mathfrak{F} := (\mathbb{S}, R, \{\mathcal{K}_s\}_{s\in\mathbb{S}}),$$

is a PWIS with the constituent DISs $\mathcal{K}_s := (W_s, \mathcal{A}, \bigcup_{a\in\mathcal{A}} \mathcal{V}_a, f_s)$, and $m : PV \to 2^{D(\mathfrak{F})}$ is a *valuation function*, $D(\mathfrak{F})$ being the set $\bigcup_{s\in\mathbb{S}}(\{s\} \times W_s)$.

Table 6.6 lists a few classes of models that are of interest to us in this chapter.

The symbol $\iff$ used below means 'if and only if'. This symbol will be used for such a purpose throughout the chapter. Moreover, we will use the symbol $\implies$ to mean 'this implies'.

Definition 6.4. The *satisfiability* of a wff α in a model $\mathfrak{M} := (\mathfrak{F}, m)$ at $(s, w) \in D(\mathfrak{F})$, denoted as $\mathfrak{M}, s, w \models \alpha$, is defined inductively:

$$
\begin{array}{lcl}
\mathfrak{M}, s, w \models \top & & \text{always.} \\
\mathfrak{M}, s, w \models p & \iff & (s, w) \in m(p), \text{ for } p \in PV. \\
\mathfrak{M}, s, w \models (a, v) & \iff & f_s(w, a) = v, \text{ for } (a, v) \in \mathcal{D}. \\
\mathfrak{M}, s, w \models \neg\alpha & \iff & \mathfrak{M}, s, w \not\models \alpha. \\
\mathfrak{M}, s, w \models \alpha \wedge \beta & \iff & \mathfrak{M}, s, w \models \alpha \text{ and } \mathfrak{M}, s, w \models \beta. \\
\mathfrak{M}, s, w \models \Box\alpha & \iff & \text{for all } r \in \mathbb{S} \text{ with } (s, r) \in R \text{ and } w \in W_r, \\
 & & \mathfrak{M}, r, w \models \alpha. \\
\mathfrak{M}, s, w \models \Box_C\alpha & \iff & \text{for all } u \in W_s \text{ with } (w, u) \in \text{Ind}_C^{\mathcal{K}_s}, \\
 & & \mathfrak{M}, s, u \models \alpha.
\end{array}
$$

It is pertinent to note here that, since $\text{Ind}_\emptyset^{\mathcal{K}_s} = W_s \times W_s$, the modal operator $\Box_\emptyset$ behaves like the global modal operator [7] on the domain W_s of the constituent DIS corresponding to the state s where the modal operator is evaluated.

Conditions of satisfiability of the derived connectives $\Diamond$ and $\Diamond_C$ are then obtained as follows:

$$
\begin{array}{lcl}
\mathfrak{M}, s, w \models \Diamond\alpha & \iff & \mathfrak{M}, r, w \models \alpha \text{ for some } r \text{ with } (s, r) \in R. \\
\mathfrak{M}, s, w \models \Diamond_C\alpha & \iff & \mathfrak{M}, s, u \models \alpha \text{ for some } u \text{ with } (w, u) \in \text{Ind}_C^{\mathcal{K}_s}.
\end{array}
$$

For any wff α and a model $\mathfrak{M}$, let

$$
\begin{aligned}
&[\![\alpha]\!]_{\mathfrak{M},s} := \{w \in W_s : \mathfrak{M}, s, w \models \alpha\}, \text{ and} \\
&[\![\alpha]\!]_{\mathfrak{M}} := \{(s, w) \in D(\mathfrak{F}) : \mathfrak{M}, s, w \models \alpha\}.
\end{aligned}
$$

Thus, at each state s, α represents a set of objects given by $[\![\alpha]\!]_{\mathfrak{M},s}$. A wff α is said to be *valid* in $\mathfrak{M} := (\mathfrak{F}, m)$, notation: $\mathfrak{M} \models \alpha$, if $[\![\alpha]\!]_{\mathfrak{M}} = D(\mathfrak{F})$. A wff α is *valid in a class* $\mathfrak{G}$ of models, notation: $\mathfrak{G} \models \alpha$, if α is valid in every model $\mathfrak{M} \in \mathfrak{G}$.
A wff α is *valid in a PWIS* $\mathfrak{F}$ if $\mathfrak{M} \models \alpha$ for all models $\mathfrak{M} := (\mathfrak{F}, V)$ based on $\mathfrak{F}$.
A wff α is said to be *satisfiable* in a model $\mathfrak{M}$ if $[\![\alpha]\!]_{\mathfrak{M}} \neq \emptyset$. α is satisfiable in a given class $\mathfrak{G}$ of models, if it is satisfiable in some model $\mathfrak{M} \in \mathfrak{G}$.

Observe that the proposed semantics is directly based on PWISs and is 2-dimensional, having the dimensions for states and objects of the domains. We would like to add here that the combination of modal operators proposed above is very much in the line of the one considered in [39] but distinct from known proposals of combinations of modal logics, e.g. [16, 17, 23, 25, 26, 51, 52, 102]. A detailed comparative study on this issue is done in Section 3 of [39], and we refer readers to it for the same.

6.3. Rough set interpretation

Obviously the operators $\neg$ and $\wedge$ correspond to the set-theoretic operation of complementation and intersection. More formally, we have the following in a model $\mathfrak{M} := (\mathfrak{F}, m)$ based on a PWIS $\mathfrak{F} := (\mathbb{S}, R, \{\mathcal{K}_s\}_{s\in\mathbb{S}})$ with the constituent DISs $\mathcal{K}_s := (W_s, \mathcal{A}, \bigcup_{a\in\mathcal{A}} \mathcal{V}_a, f_s)$.

$$\llbracket \neg\alpha \rrbracket_{\mathfrak{M},s} = W_s \setminus \llbracket \alpha \rrbracket_{\mathfrak{M},s};$$

$$\llbracket \alpha \wedge \beta \rrbracket_{\mathfrak{M},s} = \llbracket \alpha \rrbracket_{\mathfrak{M},s} \cap \llbracket \beta \rrbracket_{\mathfrak{M},s}.$$

The following proposition establishes that the operators $\Box_B$ and its dual $\Diamond_B$ capture the lower and upper approximations with respect to the indiscernibility relation relative to B, respectively.

Proposition 6.5. *Let $\mathfrak{F} := (\mathbb{S}, R, \{\mathcal{K}_s\}_{s\in\mathbb{S}})$ be a PWIS with the constituent DISs $\mathcal{K}_s := (W_s, \mathcal{A}, \bigcup_{a\in\mathcal{A}} \mathcal{V}_a, f_s)$. Consider a model $\mathfrak{M} := (\mathfrak{F}, m)$ based on the PWIS $\mathfrak{F}$. Then the following hold.*

1. $\llbracket \Box_B\alpha \rrbracket_{\mathfrak{M},s} = \underline{\llbracket \alpha \rrbracket_{\mathfrak{M},s}}_{\mathrm{Ind}_B^{\mathcal{K}_s}}$;
2. $\llbracket \Diamond_B\alpha \rrbracket_{\mathfrak{M},s} = \overline{\llbracket \alpha \rrbracket_{\mathfrak{M},s}}_{\mathrm{Ind}_B^{\mathcal{K}_s}}$.

We omit the proof as it is very standard.

We define the following connectives for each non-empty subset B of $\mathcal{A}$.

Definition 6.6.

$$\triangle_B\alpha := \Box\Box_B\alpha,\ \blacksquare_B\alpha := \Diamond\Box_B\alpha;$$

$$\triangledown_B\alpha := \Diamond\Diamond_B\alpha,\ \blacklozenge_B\alpha := \Box\Diamond_B\alpha.$$

It is evident from Definition 6.4 that $\Box$ ($\Diamond$) corresponds to universal (existential) quantifier over states and $\Box_B$ ($\Diamond_B$) captures the lower (upper) approximation with respect to attribute set B. Thus, we have the following proposition.

Proposition 6.7. *Let $\mathfrak{F} := (\mathbb{S}, R, \{\mathcal{K}_s\}_{s\in\mathbb{S}})$ be a PWIS with the constituent DISs $\mathcal{K}_s := (W_s, \mathcal{A}, \bigcup_{a\in\mathcal{A}} \mathcal{V}_a, f_s)$. Consider a model $\mathfrak{M} := (\mathfrak{F}, m)$ based on the PWIS $\mathfrak{F}$. Then the following hold for all $s \in \mathbb{S}$ and $w \in W_s$.*

1. $w \in [\![\triangle_B\alpha]\!]_{\mathfrak{M},s} \iff$ *for all $r \in R(s)$ with $w \in W_r$, $w \in \underline{[\![\alpha]\!]_{\mathfrak{M},r}}_{\mathrm{Ind}_B^{\mathcal{K}_r}}$.*
2. $w \in [\![\blacksquare_B\alpha]\!]_{\mathfrak{M},s} \iff$ *there exists a $r \in R(s)$ with $w \in \underline{[\![\alpha]\!]_{\mathfrak{M},r}}_{\mathrm{Ind}_B^{\mathcal{K}_r}}$.*
3. $w \in [\![\triangledown_B\alpha]\!]_{\mathfrak{M},s} \iff$ *there exists a $r \in R(s)$ with $w \in \overline{[\![\alpha]\!]_{\mathfrak{M},r}}_{\mathrm{Ind}_B^{\mathcal{K}_r}}$.*
4. $w \in [\![\blacklozenge_B\alpha]\!]_{\mathfrak{M},s} \iff$ *for all $r \in R(s)$ with $w \in W_r$, $w \in \overline{[\![\alpha]\!]_{\mathfrak{M},r}}_{\mathrm{Ind}_B^{\mathcal{K}_r}}$.*

Proof. We provide the proof of Items 1 and 2. Rest can be done in the same way.

(1):

$$\begin{aligned} w \in [\![\triangle_B\alpha]\!]_{\mathfrak{M},s} &\iff \mathfrak{M}, s, w \models \Box\Box_B\alpha \\ &\iff \text{for all } r \in R(s) \text{ with } w \in W_r,\ \mathfrak{M}, r, w \models \Box_B\alpha \\ &\iff \text{for all } r \in R(s) \text{ with } w \in W_r,\ w \in \underline{[\![\alpha]\!]_{\mathfrak{M},r}}_{\mathrm{Ind}_B^{\mathcal{K}_r}} \\ &\qquad\qquad \text{(using Proposition 6.5).} \end{aligned}$$

(2):

$$\begin{aligned} w \in [\![\blacksquare_B\alpha]\!]_{\mathfrak{M},s} &\iff \mathfrak{M}, s, w \models \Diamond\Box_B\alpha \\ &\iff \text{there exists a } r \in R(s) \text{ with } \mathfrak{M}, r, w \models \Box_B\alpha \\ &\iff \text{there exists a } r \in R(s) \text{ with } w \in \underline{[\![\alpha]\!]_{\mathfrak{M},r}}_{\mathrm{Ind}_B^{\mathcal{K}_r}}. \end{aligned}$$

$\Box$

Depending on the conditions imposed on the underlying relation R, taking a cue from epistemic logic, different interpretations can be given for the operators defined in Definition 6.6. For instance, in the class of models based on equivalence relation R, one can interpret $\triangle_B\alpha$ ($\blacklozenge_B\alpha$) as 'the agent *knows* that the object belongs to the lower (upper, respectively) approximation of the set of objects represented by α.' Similarly, in the class of models based on plausibility order R, one can interpret $\triangle_B\alpha$ ($\blacklozenge_B\alpha$) as 'the agent *safely believes* [99] that the object belongs to the lower (upper, respectively) approximation of the set of objects represented by α.' Keeping in view these interpretations, let us consider a few wffs to illustrate how the proposed semantics can be used to reason about PWISs and approximations of sets based on it.

- The wff $p \to \blacklozenge_B p$ says that if an object is an instance of the set (represented by) p, then the agent knows (or, safely believes) that the object belongs to the upper approximation (with respect to attribute set B) of the set p. Note that $p \to \blacklozenge_B p$ is not valid in the class Ω_{p} of models.
- The wff $\lozenge_B p \to \blacklozenge_B p$ says that if an object belongs to the upper approximation of the set p, then the agent knows (or, safely believes) that the object belongs to the upper approximation of the set p. This wff is not valid in the class Ω_{e} and hence Ω_{p} of models.
- The wff $\Box((a, v) \to \Box_B p)$ represents that the agent knows (or, safely believes) that if an object takes the value v for the attribute a, then the object belongs to the lower approximation of the set p.

Example 6.8. Let us consider a model $\mathfrak{M} := (\mathfrak{F}, m)$, where $\mathfrak{F}$ is the PWIS $\mathfrak{F} := (\{s_1, \ldots, s_4\}, R^*, \{\mathcal{K}_{s_1}, \mathcal{K}_{s_2}, \mathcal{K}_{s_3}, \mathcal{K}_{s_4}\})$, the plausibility relation R^* being the transitive closure of the relation R given by Figure 6.1 (cf. Example 6.2). Let m be such that $m(p) := \{\text{Patient } 1\}$, where p is a propositional variable. Suppose a denotes the attribute 'Disease' and v_1, v_2 denote the attribute-value 'Presence' and 'Absence', respectively. Note that

$$\mathfrak{M}, s_3, \text{Patient } 1 \models \Box\Box_{\{a\}} p, \text{but}$$
$$\mathfrak{M}, s_4, \text{Patient } 1 \not\models \Box\Box_{\{a\}} p.$$

Thus, we can conclude that at the state s_3, the agent safely believes that the Patient 1 belongs to the lower approximation (with respect to attribute set $\{a\}$) of the set $\{\text{Patient } 1\}$. On the other hand, it is not the case at the state s_4.

It is important to note that the proposed language $\mathcal{L}$ can express notions related to dependencies in data and data reduction. The following proposition shows how the language $\mathcal{L}$ can be used to express these notions.

Proposition 6.9. *Let* $\mathfrak{F} := (\mathbb{S}, R, \{\mathcal{K}_s\}_{s \in \mathbb{S}})$ *be a PWIS with the constituent DISs* $\mathcal{K}_s := (W_s, \mathcal{A}, \bigcup_{a \in \mathcal{A}} \mathcal{V}_a, f_s)$*, and* $P, Q \subseteq \mathcal{A}$.

1. *$b \in P$ is dispensable in P in the constituent deterministic information system $\mathcal{K}_s$ of $\mathfrak{F}$ if and only if*

$$[\![\Box_P p \leftrightarrow \Box_{P\setminus\{b\}} p]\!]_{\mathfrak{M},s} = W_s,$$

for every model $\mathfrak{M}$ based on the PWIS $\mathfrak{F}$. Here p is a propositional variable.

2. *For each $b \in P$, let us choose a $p_b \in PV$ such that p_{b_1} and p_{b_2} are distinct for distinct $b_1, b_2 \in P$ (here we have assumed that $|PV| \geq |P|$, where $|X|$ denotes the cardinality of the set X). Then $P \subseteq \mathcal{A}$ is dependent in $\mathcal{K}_s$ if and only if*

$$[\![\bigvee_{b\in P} \Box_\emptyset(\Box_P p_b \leftrightarrow \Box_{P\setminus\{b\}} p_b)]\!]_{\mathfrak{M},s} = W_s,$$

for every model $\mathfrak{M}$ based on $\mathfrak{F}$, $\{p_b : b \in P\}$ being a set of distinct propositional variables.

3. *For each $b \in Q$, let us choose a $p_b \in PV$ such that p_{b_1} and p_{b_2} are distinct for distinct $b_1, b_2 \in Q$. Then, $Q \subseteq P$ is a reduct of P in $\mathcal{K}_s$ if and only if*

$$[\![\bigwedge_{b\in Q} \Diamond_\emptyset \neg(\Box_Q p_b \leftrightarrow \Box_{Q\setminus\{b\}} p_b)]\!]_{\mathfrak{M}',s} \neq \emptyset \tag{6.2}$$

for some model $\mathfrak{M}'$ based on $\mathfrak{F}$ and

$$[\![\Box_Q p \leftrightarrow \Box_P p]\!]_{\mathfrak{M},s} = W_s, \tag{6.3}$$

for every model $\mathfrak{M}$ based on $\mathfrak{F}$.

Proof. 1. Let us first assume that $b \in P$ be dispensable in P in $\mathcal{K}_s$, and we show that $[\![\Box_P p \leftrightarrow \Box_{P\setminus\{b\}} p]\!]_{\mathfrak{M},s} = W_s$ for every model $\mathfrak{M}$ based on the PWIS $\mathfrak{F}$. Since $b \in P$ is dispensable in P in $\mathcal{K}_s$, we obtain

$$\text{Ind}_P^{\mathcal{K}_s} = \text{Ind}_{P\setminus\{b\}}^{\mathcal{K}_s}. \tag{6.4}$$

Let $\mathfrak{M}$ be any model based on $\mathfrak{F}$. From (6.4), we obtain $\mathfrak{M}, s, w \models \Box_P p \leftrightarrow \Box_{P\setminus\{b\}} p$ for all $w \in W_s$. This gives $[\![\Box_P p \leftrightarrow \Box_{P\setminus\{b\}} p]\!]_{\mathfrak{M},s} = W_s$.

To prove the converse, let us assume that $[\![\Box_P p \leftrightarrow \Box_{P\setminus\{b\}} p]\!]_{\mathfrak{M},s} = W_s$ for every model $\mathfrak{M}$ based on the PWIS $\mathfrak{F}$. We need to prove $\text{Ind}_P^{\mathcal{K}_s} = \text{Ind}_{P\setminus\{b\}}^{\mathcal{K}_s}$. Obviously $\text{Ind}_P^{\mathcal{K}_s} \subseteq \text{Ind}_{P\setminus\{b\}}^{\mathcal{K}_s}$, and therefore we just need to show $\text{Ind}_{P\setminus\{b\}}^{\mathcal{K}_s} \subseteq \text{Ind}_P^{\mathcal{K}_s}$. Towards a contradiction, let $\text{Ind}_{P\setminus\{b\}}^{\mathcal{K}_s} \not\subseteq \text{Ind}_P^{\mathcal{K}_s}$. Then there exists $(w, w') \in \text{Ind}_{P\setminus\{b\}}^{\mathcal{K}_s}$, but $(w, w') \notin \text{Ind}_P^{\mathcal{K}_s}$. Let us consider a model $\mathfrak{M} := (\mathfrak{F}, m)$ where m is such that $m(p) = \{(s, u) : u \in \text{Ind}_P^{\mathcal{K}_s}(w)\}$.

Then $\mathfrak{M}, s, w \models \Box_P p$, but $\mathfrak{M}, s, w \not\models \Box_{P\setminus\{b\}} p$. This gives $[\![\Box_P p \leftrightarrow \Box_{P\setminus\{b\}} p]\!]_{\mathfrak{M},s} \neq W_s$, a contradiction. Thus we have shown $\mathrm{Ind}_P^{\mathcal{K}_s} = \mathrm{Ind}_{P\setminus\{b\}}^{\mathcal{K}_s}$.

2. Let P be dependent in $\mathcal{K}_s$ and we prove

$$[\![\bigvee_{b\in P} \Box_\emptyset(\Box_P p_b \leftrightarrow \Box_{P\setminus\{b\}} p_b)]\!]_{\mathfrak{M},s} = W_s \tag{6.5}$$

for every model $\mathfrak{M}$ based on $\mathfrak{F}$. Let $\mathfrak{M}$ be an arbitrary model based on $\mathfrak{F}$, $w \in W_s$ and we prove

$$\mathfrak{M}, s, w \models \bigvee_{b\in P} \Box_\emptyset(\Box_P p_b \leftrightarrow \Box_{P\setminus\{b\}} p_b). \tag{6.6}$$

Since P is dependent in $\mathcal{K}_s$, there exists $b \in P$ such that b is dispensable in P in $\mathcal{K}_s$. Therefore, from Item 1, we obtain $[\![(\Box_P p_b \leftrightarrow \Box_{P\setminus\{b\}} p_b)]\!]_{\mathfrak{M},s} = W_s$. This gives (6.6).
Let us now prove the converse. We know $\mathrm{Ind}_P^{\mathcal{K}_s} \subseteq \mathrm{Ind}_{P\setminus\{b\}}^{\mathcal{K}_s}$ for all $b \in P$. If possible, let P be not dependent in $\mathcal{K}_s$. Then, we have $\mathrm{Ind}_P^{\mathcal{K}_s} \subsetneq \mathrm{Ind}_{P\setminus\{b\}}^{\mathcal{K}_s}$ for all $b \in P$. Hence, for all $b \in P$ we obtain $w_b, w_b' \in W_s$ such that $(w_b, w_b') \in \mathrm{Ind}_{P\setminus\{b\}}^{\mathcal{K}_s}$ and $(w_b, w_b') \notin \mathrm{Ind}_P^{\mathcal{K}_s}$.
Let us consider a model $\mathfrak{M} = (\mathfrak{F}, m)$, where m be such that

$$m(p_b) = \{(s, w') : w' \in \mathrm{Ind}_P^{\mathcal{K}_s}(w_b)\}.$$

Then, for all $b \in P$, we obtain $\mathfrak{M}, s, w_b \not\models \Box_{P\setminus\{b\}} p_b$ and $\mathfrak{M}, s, w_b \models \Box_P p_b$. This gives $\mathfrak{M}, s, w_b \not\models \Box_{P\setminus\{b\}} p_b \leftrightarrow \Box_p p_b$ for all $b \in P$. This, in turn, gives $\mathfrak{M}, s, w \not\models \Box_\emptyset(\Box_{P\setminus\{b\}} p_b \leftrightarrow \Box_p p_b)$ for all $b \in P$, w being an arbitrary element from W_s. Hence, we have $\mathfrak{M}, s, w \not\models \bigvee_{b\in P} \Box_\emptyset(\Box_{P\setminus\{b\}} p_b \leftrightarrow \Box_p p_b)$, that is, $[\![\bigvee_{b\in P} \Box_\emptyset(\Box_P p_b \leftrightarrow \Box_{P\setminus\{b\}} p_b)]\!]_{\mathfrak{M},s} \neq W_s$. But this is a contradiction and hence P must be dependent in $\mathcal{K}_s$.

3. Let $Q \subseteq P$ be a reduct of P and we show that

$$[\![\bigwedge_{b\in Q} \Diamond_\emptyset \neg(\Box_Q p_b \leftrightarrow \Box_{Q\setminus\{b\}} p_b)]\!]_{\mathfrak{M}',s} \neq \emptyset$$

for some model $\mathfrak{M}'$ based on $\mathfrak{F}$ and $[\![\Box_Q p \leftrightarrow \Box_P p]\!]_{\mathfrak{M},s} = W_s$ for every model $\mathfrak{M}$ based on $\mathfrak{F}$. Since Q is a reduct of P in $\mathcal{K}_s$, it follows from the definition of reduct that Q is independent in $\mathcal{K}_s$ and $\mathrm{Ind}_P^{\mathcal{K}_s} = \mathrm{Ind}_Q^{\mathcal{K}_s}$. Also note that

$$\mathrm{Ind}_P^{\mathcal{K}_s} = \mathrm{Ind}_Q^{\mathcal{K}_s} \iff [\![\Box_Q p \leftrightarrow \Box_P p]\!]_{\mathfrak{M},s} = W_s \quad \text{for every model } \mathfrak{M} \text{ based on } \mathfrak{F}. \tag{6.7}$$

Since Q is independent in $\mathcal{K}_s$, using Item 2, we get

$$[\![\neg \bigvee_{b \in Q} \Box_\emptyset(\Box_Q p_b \leftrightarrow \Box_{Q\setminus\{b\}} p_b)]\!]_{\mathfrak{M}',s} \neq \emptyset \text{ for some model } \mathfrak{M}' \text{ based on } \mathfrak{F}.$$

This gives

$$[\![\bigwedge_{b \in Q} \Diamond_\emptyset \neg(\Box_Q p_b \leftrightarrow \Box_{Q\setminus\{b\}} p_b)]\!]_{\mathfrak{M}',s} \neq \emptyset \tag{6.8}$$

for some model $\mathfrak{M}'$ based on $\mathfrak{F}$. Thus, from (6.7) and (6.8), we obtain the result.

Next, let

$$[\![\bigwedge_{b \in Q} \Diamond_\emptyset \neg(\Box_Q p_b \leftrightarrow \Box_{Q\setminus\{b\}} p_b)]\!]_{\mathfrak{M}',s} \neq \emptyset \text{ for some model } \mathfrak{M}' \text{ based on } \mathfrak{F} \tag{6.9}$$

$$\text{and } [\![\Box_Q p \leftrightarrow \Box_P p]\!]_{\mathfrak{M},s} = W_s \text{ for every model } \mathfrak{M} \text{ based on } \mathfrak{F}. \tag{6.10}$$

We show that $Q \subseteq P$ is a reduct of P. From (6.10) and (6.7), it follows that $\mathrm{Ind}_P^{\mathcal{K}_s} = \mathrm{Ind}_Q^{\mathcal{K}_s}$. Further from (6.9) and Item 2, we obtain Q as independent in $\mathcal{K}_s$. Thus, we have shown that Q is a reduct of P in $\mathcal{K}_s$.

□

The above notions related to dependencies in data and data reduction can also be expressed in the language $\mathcal{L}$ without using propositional variables as shown by Proposition 6.10. We will use B-*basic* wffs [74] defined as follows. Let $\emptyset \neq B := \{b_1, \ldots, b_n\} \subseteq \mathcal{A}$. Let $\mathcal{D}_B$ be the set of all B-*basic* wffs, i.e., wffs of the form $(b_1, v_1) \wedge \cdots \wedge (b_n, v_n)$, $v_i \in \mathcal{V}_{b_i}, i = 1, \ldots, n$. In the case when $B = \emptyset$, we take $\mathcal{D}_B := \{\top\}$.

Proposition 6.10. *Let* $\mathfrak{F} := (\mathbb{S}, R, \{\mathcal{K}_s\}_{s \in \mathbb{S}})$ *be a PWIS with the constituent DISs* $\mathcal{K}_s := (W_s, \mathcal{A}, \bigcup_{a \in \mathcal{A}} \mathcal{V}_a, f_s)$*, and* $P, Q \subseteq \mathcal{A}$*. Let* $\mathfrak{M}$ *be a model based on the PWIS* $\mathfrak{F}$*. Then,*

1. $b \in P$ *is dispensable in* P *in the constituent deterministic information system* $\mathcal{K}_s$ *of* $\mathfrak{F}$ *if and only if*

$$[\![\bigwedge_{\beta \in \mathcal{D}_{P\setminus\{b\}}} \bigwedge_{v_b \in \mathcal{V}_b} (\beta \wedge (b, v_b) \rightarrow \Box_\emptyset(\beta \rightarrow (b, v_b)))]\!]_{\mathfrak{M},s} = W_s.$$

2. $P \subseteq \mathcal{A}$ *is dependent in* $\mathcal{K}_s$ *if and only if*

$$[\![\bigvee_{b \in P} \Box_\emptyset(\bigwedge_{\beta \in \mathcal{D}_{P\setminus\{b\}}} \bigwedge_{v_b \in \mathcal{V}_b} (\beta \wedge (b, v_b) \rightarrow \Box_\emptyset(\beta \rightarrow (b, v_b))))]\!]_{\mathfrak{M},s} = W_s.$$

3. $Q \subseteq P$ is a reduct of P in $\mathcal{K}_s$ if and only if

$$[\![\bigvee_{b\in P}\Box_\emptyset(\bigwedge_{\beta\in\mathcal{D}_{P\setminus\{b\}}}\bigwedge_{v_b\in\mathcal{V}_b}(\beta\wedge(b,v_b)\to\Box_\emptyset(\beta\to(b,v_b))))]\!]_{\mathfrak{M},s}\neq W_s,$$

$$\textit{and } [\![\bigwedge_{\beta\in\mathcal{D}_P}\bigwedge_{\gamma\in\mathcal{D}_Q}(\beta\wedge\gamma\to\Box_\emptyset(\beta\leftrightarrow\gamma))]\!]_{\mathfrak{M},s}=W_s.$$

Proof. 1. Let us first assume that $b \in P$ is dispensable in P in $\mathcal{K}_s$, and we show that

$$[\![\bigwedge_{\beta\in\mathcal{D}_{P\setminus\{b\}}}\bigwedge_{v_b\in\mathcal{V}_b}(\beta\wedge(b,v_b)\to\Box_\emptyset(\beta\to(b,v_b)))]\!]_{\mathfrak{M},s}=W_s.$$

Consider an object $w \in W_s$ such that

$$\mathfrak{M},s,w\models\beta\wedge(b,v_b),\ \text{where } \beta\in\mathcal{D}_{P\setminus\{b\}} \text{ and } v_b\in\mathcal{V}_b. \tag{6.11}$$

We show that $\mathfrak{M},s,w \models \Box_\emptyset(\beta \to (b,v_b))$. Let us take an object $w' \in W_s$ such that $\mathfrak{M},s,w' \models \beta$ and we prove that $\mathfrak{M},s,w' \models (b,v_b)$. From $\mathfrak{M},s,w \models \beta$ and $\mathfrak{M},s,w' \models \beta$, we obtain $(w,w') \in \text{Ind}^{\mathcal{K}_s}_{P\setminus\{b\}}$. Since b is dispensable in P, we obtain $\text{Ind}^{\mathcal{K}_s}_P = \text{Ind}^{\mathcal{K}_s}_{P\setminus\{b\}}$ and hence $(w,w') \in \text{Ind}^{\mathcal{K}_s}_P$. From (6.11), we get $\mathfrak{M},s,w' \models (b,v_b)$.
Let us now prove the converse. Let b be not dispensable in P in $\mathcal{K}_s$ and we show that

$$[\![\bigwedge_{\beta\in\mathcal{D}_{P\setminus\{b\}}}\bigwedge_{v_b\in\mathcal{V}_b}(\beta\wedge(b,v_b)\to\Box_\emptyset(\beta\to(b,v_b)))]\!]_{\mathfrak{M},s}\neq W_s.$$

Since b is not dispensable in P in $\mathcal{K}_s$, there exist $w \in W_s$ such that $\text{Ind}^{\mathcal{K}_s}_P(w) \neq \text{Ind}^{\mathcal{K}_s}_{P\setminus\{b\}}(w)$. That is, there exist $w' \in W_s$ such that

$$f_s(w,a)=f_s(w',a) \text{ for all } a\in P\setminus\{b\} \text{ and } f_s(w,b)\neq f_s(w',b). \tag{6.12}$$

Let $\beta = \bigwedge_{a\in P\setminus\{b\}}(a,v_a)$, where $v_a = f_s(w,a)$ for $a \in P \setminus \{b\}$. It is clear that $\beta \in \mathcal{D}_{P\setminus\{b\}}$. Let $f_s(w,b) = v_b$ and $f_s(w',b) = v'_b$. Then $\mathfrak{M},s,w \models \beta \wedge (b,v_b)$, and from (6.12), $\mathfrak{M},s,w' \models \beta$. But $\mathfrak{M},s,w' \not\models (b,v_b)$. Hence,

$$w\notin[\![\bigwedge_{\beta\in\mathcal{D}_{P\setminus\{b\}}}\bigwedge_{v_b\in\mathcal{V}_b}(\beta\wedge(b,v_b)\to\Box_\emptyset(\beta\to(b,v_b)))]\!]_{\mathfrak{M},s}.$$

This gives, $[\![\bigwedge_{\beta\in\mathcal{D}_{P\setminus\{b\}}}\bigwedge_{v_b\in\mathcal{V}_b}(\beta\wedge(b,v_b)\to\Box_\emptyset(\beta\to(b,v_b)))]\!]_{\mathfrak{M},s}\neq W_s$.

2. Let P be dependent in $\mathcal{K}_s$ and we prove

$$[\![\bigvee_{b\in P}\Box_\emptyset(\bigwedge_{\beta\in\mathcal{D}_{P\setminus\{b\}}}\bigwedge_{v_b\in\mathcal{V}_b}(\beta\wedge(b,v_b)\to\Box_\emptyset(\beta\to(b,v_b))))]\!]_{\mathfrak{M},s}=W_s. \tag{6.13}$$

Since P is dependent in $\mathcal{K}_s$, there exist $b \in P$ such that b is dispensable in P. Using Item 1 of Proposition 6.10, we obtain

$$[\![\bigwedge_{\beta \in \mathcal{D}_{P\setminus\{b\}}} \bigwedge_{v_b \in \mathcal{V}_b} (\beta \wedge (b, v_b) \rightarrow \Box_\emptyset(\beta \rightarrow (b, v_b)))]\!]_{\mathfrak{M},s} = W_s.$$

$$\Longrightarrow [\![\Box_\emptyset\big(\bigwedge_{\beta \in \mathcal{D}_{P\setminus\{b\}}} \bigwedge_{v_b \in \mathcal{V}_b} (\beta \wedge (b, v_b) \rightarrow \Box_\emptyset(\beta \rightarrow (b, v_b)))\big)]\!]_{\mathfrak{M},s} = W_s. \tag{6.14}$$

This gives (6.13).

For the converse part, let us assume that

$$[\![\bigvee_{b \in P} \Box_\emptyset\big(\bigwedge_{\beta \in \mathcal{D}_{P\setminus\{b\}}} \bigwedge_{v_b \in \mathcal{V}_b} (\beta \wedge (b, v_b) \rightarrow \Box_\emptyset(\beta \rightarrow (b, v_b)))\big)]\!]_{\mathfrak{M},s} = W_s \tag{6.15}$$

and we show that P is dependent in $\mathcal{K}_s$. Towards a contradiction, let P be not dependent in $\mathcal{K}_s$. Further, consider an object $w \in W_s$. Then from (6.15), we obtain

$$\mathfrak{M}, s, w \models \bigvee_{b \in P} \big(\Box_\emptyset \bigwedge_{\beta \in \mathcal{D}_{P\setminus\{b\}}} \bigwedge_{v_b \in \mathcal{V}_b} (\beta \wedge (b, v_b) \rightarrow \Box_\emptyset(\beta \rightarrow (b, v_b)))\big)$$

and hence there exist $c \in P$ such that

$$\mathfrak{M}, s, w \models \Box_\emptyset\big(\bigwedge_{\beta \in \mathcal{D}_{P\setminus\{c\}}} \bigwedge_{v_c \in \mathcal{V}_c} (\beta \wedge (c, v_c) \rightarrow \Box_\emptyset(\beta \rightarrow (c, v_c)))\big). \tag{6.16}$$

Since P is assumed to be independent, we obtain each $b \in P$ to be indispensable in $\mathcal{K}_s$. Thus, using Item 1 of Proposition 6.10 for $c \in P$, we obtain an object $w_c \in W_s$ such that

$$\mathfrak{M}, s, w_c \not\models \bigwedge_{\beta \in \mathcal{D}_{P\setminus\{c\}}} \bigwedge_{v_c \in \mathcal{V}_c} (\beta \wedge (c, v_c) \rightarrow \Box_\emptyset(\beta \rightarrow (c, v_c))).$$

Hence, for any $w \in W_s$, we get

$$\mathfrak{M}, s, w \not\models \Box_\emptyset\big(\bigwedge_{\beta \in \mathcal{D}_{P\setminus\{c\}}} \bigwedge_{v_c \in \mathcal{V}_c} (\beta \wedge (c, v_c) \rightarrow \Box_\emptyset(\beta \rightarrow (c, v_c)))\big). \tag{6.17}$$

But (6.17) and (6.16) cannot be true together. Hence P is dependent in $\mathcal{K}_s$.

3. First, let us assume that $Q \subseteq P$ be a reduct of P and we show that

$$[\![\bigvee_{b \in P} \Box_\emptyset\big(\bigwedge_{\beta \in \mathcal{D}_{P\setminus\{b\}}} \bigwedge_{v_b \in \mathcal{V}_b} (\beta \wedge (b, v_b) \rightarrow \Box_\emptyset(\beta \rightarrow (b, v_b)))\big)]\!]_{\mathfrak{M},s} \neq W_s, \tag{6.18}$$

$$\text{and } [\![\bigwedge_{\beta \in \mathcal{D}_P} \bigwedge_{\gamma \in \mathcal{D}_Q} (\beta \wedge \gamma \rightarrow \Box_\emptyset(\beta \leftrightarrow \gamma))]\!]_{\mathfrak{M},s} = W_s. \tag{6.19}$$

Since Q is a reduct of P, Q is independent in $\mathcal{K}_s$ and $\mathrm{Ind}_P^{\mathcal{K}_s} = \mathrm{Ind}_Q^{\mathcal{K}_s}$. Using the independence of Q in $\mathcal{K}_s$ and Item 2 of Proposition 6.10, we obtain (6.18). Further, to obtain (6.19), let us prove the following.

$$\mathrm{Ind}_P^{\mathcal{K}_s} = \mathrm{Ind}_Q^{\mathcal{K}_s} \iff [\![\bigwedge_{\beta \in \mathcal{D}_P} \bigwedge_{\gamma \in \mathcal{D}_Q} (\beta \wedge \gamma \to \Box_\emptyset(\beta \leftrightarrow \gamma))]\!]_{\mathfrak{M},s} = W_s. \tag{6.20}$$

First, let us assume that $\mathrm{Ind}_P^{\mathcal{K}_s} = \mathrm{Ind}_Q^{\mathcal{K}_s}$ and we prove

$$[\![\bigwedge_{\beta \in \mathcal{D}_P} \bigwedge_{\gamma \in \mathcal{D}_Q} (\beta \wedge \gamma \to \Box_\emptyset(\beta \leftrightarrow \gamma))]\!]_{\mathfrak{M},s} = W_s.$$

So, consider an object $w \in W_s$ such that

$$\mathfrak{M}, s, w \models \beta \wedge \gamma \text{ where } \beta \in \mathcal{D}_P \text{ and } \gamma \in \mathcal{D}_Q. \tag{6.21}$$

We show that $\mathfrak{M}, s, w \models \Box_\emptyset(\beta \leftrightarrow \gamma)$. Let $w' \in W_s$ and we prove that

$$\mathfrak{M}, s, w' \models (\beta \leftrightarrow \gamma). \tag{6.22}$$

Using (6.21) and $\mathrm{Ind}_P^{\mathcal{K}_s} = \mathrm{Ind}_Q^{\mathcal{K}_s}$, we obtain (6.22).

Next, let us assume $[\![\bigwedge_{\beta \in \mathcal{D}_P} \bigwedge_{\gamma \in \mathcal{D}_Q} (\beta \wedge \gamma \to \Box_\emptyset(\beta \leftrightarrow \gamma))]\!]_{\mathfrak{M},s} = W_s$ and we prove $\mathrm{Ind}_P^{\mathcal{K}_s} = \mathrm{Ind}_Q^{\mathcal{K}_s}$. Towards a contradiction, let $\mathrm{Ind}_P^{\mathcal{K}_s} \neq \mathrm{Ind}_Q^{\mathcal{K}_s}$. Then, there exist $w, w' \in W_s$ such that $(w, w') \in \mathrm{Ind}_P^{\mathcal{K}_s}$ but $(w, w') \notin \mathrm{Ind}_Q^{\mathcal{K}_s}$. From $(w, w') \notin \mathrm{Ind}_Q^{\mathcal{K}_s}$, there exists a $b \in Q$ such that $f_s(w, b) \neq f_s(w', b)$. Let $\beta = \bigwedge_{a \in P}(a, v_a), \gamma = \bigwedge_{c \in Q}(c, v_c)$, where $v_a = f_s(w, a) = f_s(w', a)$ and $v_c = f_c(w, c)$. It is obvious that $\beta \in \mathcal{D}_p$ and $\gamma \in \mathcal{D}_Q$. Then, we get $\mathfrak{M}, s, w \models \beta \wedge \gamma$ and hence by assumption, we obtain $\mathfrak{M}, s, w \models \Box_\emptyset(\beta \leftrightarrow \gamma)$. This gives $\mathfrak{M}, s, w' \models (\beta \leftrightarrow \gamma)$. Since $\mathfrak{M}, s, w' \models \beta$, we get $\mathfrak{M}, s, w' \models \gamma$. This is not possible as $f_s(w, b) \neq f_s(w', b)$. Hence $\mathrm{Ind}_P^{\mathcal{K}_s} = \mathrm{Ind}_Q^{\mathcal{K}_s}$. This completes the proof of (6.20).

We have $\mathrm{Ind}_P^{\mathcal{K}_s} = \mathrm{Ind}_Q^{\mathcal{K}_s}$. Thus, using (6.20), we obtain (6.19).

For the converse, let us assume (6.18) and (6.19). We show that $Q \subseteq P$ is a reduct of P in $\mathcal{K}_s$. Using (6.18) and Item 2 of Proposition 6.10, we obtain that Q is independent in $\mathcal{K}_s$. From (6.19) and (6.20), we get $\mathrm{Ind}_P^{\mathcal{K}_s} = \mathrm{Ind}_Q^{\mathcal{K}_s}$. Hence $Q \subseteq P$ is a reduct of P in $\mathcal{K}_s$. □

6.4. Axiomatization

We first note that for each non-empty $C \subseteq \mathcal{A}$, $\Box_C$ is definable in the language $\mathcal{L}(\Box, \Box_\emptyset)$, as shown by the following proposition. Recall that the modal operator $\Box_\emptyset$

behaves like the global modal operator on the domain W_s of the constituent DIS corresponding to the state s where the modal operator is evaluated.

Proposition 6.11. *For any wff α and attribute set B, we have*

$$\Omega \models \Box_B\alpha \leftrightarrow \bigwedge_{\beta\in\mathcal{D}_B}(\beta \to \Box_\emptyset(\beta \to \alpha)).$$

Proof. Obviously we have the result when $B = \emptyset$. So, let $B \neq \emptyset$. Let us first show that $\Omega \models \Box_B\alpha \to \bigwedge_{\beta\in\mathcal{D}_B}(\beta \to \Box_\emptyset(\beta \to \alpha))$. Let $\mathfrak{M}$ be an arbitrary model such that

$$\mathfrak{M}, s, w \models \Box_B\alpha, \tag{6.23}$$

$$\mathfrak{M}, s, w \models \beta, \text{where } \beta \in \mathcal{D}_B, \tag{6.24}$$

and we show that $\mathfrak{M}, s, w \models \Box_\emptyset(\beta \to \alpha)$. Let us take an arbitrary $w' \in W_s$ such that

$$\mathfrak{M}, s, w' \models \beta, \tag{6.25}$$

and we prove that $\mathfrak{M}, s, w' \models \alpha$. From (6.24) and (6.25), it follows that $f_s(w, a) = f_s(w', a)$ for all $a \in B$ and hence $w' \in \text{Ind}_B^{\mathcal{K}_s}(w)$. Therefore, using (6.23), we obtain $\mathfrak{M}, s, w' \models \alpha$. Next, we show that $\Omega \models \bigwedge_{\beta\in\mathcal{D}_B}(\beta \to \Box_\emptyset(\beta \to \alpha)) \to \Box_B\alpha$. So, let $\mathfrak{M}$ be an arbitrary model such that

$$\mathfrak{M}, s, w \models \bigwedge_{\beta\in\mathcal{D}_B}(\beta \to \Box_\emptyset(\beta \to \alpha)), \tag{6.26}$$

and we show that $\mathfrak{M}, s, w \models \Box_B\alpha$. Let $w' \in \text{Ind}_B^{\mathcal{K}_s}(w)$ and we show that $\mathfrak{M}, s, w' \models \alpha$. For each $a \in B$, let $v_a \in \mathcal{V}_a$ be such that $f_s(w, a) = v_a = f_s(w', a)$. Let β be the wff $\bigwedge_{a\in B}(a, v_a)$. Note that $\beta \in \mathcal{D}_B$, $\mathfrak{M}, s, w \models \beta$ and $\mathfrak{M}, s, w' \models \beta$. Thus, from (6.26), we get $\mathfrak{M}, s, w' \models \alpha$. $\Box$

Therefore, due to Proposition 6.11, we will restrict ourselves to the language $\mathcal{L}(\Box, \Box_\emptyset)$, and provide the axiomatization and decidability results for this language. In the rest of the section, we will work with the language $\mathcal{L}(\Box, \Box_\emptyset)$.

Modal Systems	Axioms and inference rules	Classes of Models
I	Taut, Des(1), Des(2), K($\Box$), K($\Box_\emptyset$), T($\Box_\emptyset$), B($\Box_\emptyset$), 4($\Box_\emptyset$), MP, Nec($\Box$), Nec($\Box_\emptyset$)	Ω
I(C)	I+C	Ω_{c} [1]
I(P)	I+T($\Box$) + 4($\Box$)	Ω_{p}
I(E)	I +T($\Box$) + 4($\Box$)+ B($\Box$)	Ω_{e}
I(CE)	I +T($\Box$) + 4($\Box$)+ B($\Box$)+ C	Ω_{ce}
I(CP)	I+T($\Box$) + 4($\Box$)+ C	Ω_{cp} [1]

Table 6.7. Proposed modal systems & corresponding classes of models (cf. Table 6.6 on page 123)

6.4.1. Modal Systems and soundness

Recall the axioms Taut, K($\Box$), K($\Box_\emptyset$), T($\Box_\emptyset$), B($\Box_\emptyset$), 4($\Box_\emptyset$) and the rules of inferences MP, Nec($\Box$), and Nec($\Box_\emptyset$) of modal logic (cf. Chapter 3). Also, note the following axioms.

$$(a,u) \to \neg(a,v), \text{ for } u,v \in \mathcal{V}_a, \text{ and } u \neq v. \tag{Des(1)}$$

$$\bigvee_{v \in \mathcal{V}_a} (a,v), \text{ where } a \in \mathcal{A}. \tag{Des(2)}$$

$$\Diamond_\emptyset \Diamond \alpha \to \Diamond \Diamond_\emptyset \alpha. \text{ (Axiom for constant domain)} \tag{C}$$

Des(1) and Des(2) are axioms for descriptors. These axioms correspond to the fact that each object takes precisely one attribute-value for each attribute. Axiom C is required to impose the condition of constant domain. Table 6.7 provides a few modal systems based on these axioms and inference rules. The column on the right-hand side in the table gives model classes for which we expect to have completeness.

It is not difficult to obtain the following soundness theorem.

Theorem 6.12 (Soundness)**.** *Let* (Λ, Φ) *be a pair consisting of a modal system* Λ *and a class* Φ *of models from the same row of Table 6.7. Then, for each wff* $\alpha \in \mathcal{L}(\Box, \Box_\emptyset)$*,* $\vdash_\Lambda \alpha$ *implies* $\Phi \models \alpha$*.*

---

[1] The corresponding completeness theorem is not yet proved, but we expect the result.

Proof. Proof of this theorem is very standard and we omit the details. We only show that the axiom C is valid in the class Ω_{c} of models. Let $\mathfrak{M} := (\mathfrak{F}, m)$ be a model from Ω_{c}, where $\mathfrak{F} := (\mathbb{S}, R, \{\mathcal{K}_s\}_{s\in\mathbb{S}})$ and $\mathcal{K}_s := (W, \mathcal{A}, \bigcup_{a\in\mathcal{A}} \mathcal{V}_a, f_s)$. Let us assume that $\mathfrak{M}, s, w \models \Diamond_\emptyset \Diamond\alpha$ and we show that $\mathfrak{M}, s, w \models \Diamond\Diamond_\emptyset\alpha$. Since $\mathfrak{M}, s, w \models \Diamond_\emptyset\Diamond\alpha$, there exists a $w' \in W$ such that $\mathfrak{M}, s, w' \models \Diamond\alpha$. This gives a $r \in \mathbb{S}$ such that $(s, r) \in R$ and $\mathfrak{M}, r, w' \models \alpha$. Thus, we obtain $\mathfrak{M}, r, w \models \Diamond_\emptyset\alpha$ and hence $\mathfrak{M}, s, w \models \Diamond\Diamond_\emptyset\alpha$ as $(s, r) \in R$. This completes the proof. $\square$

We present the completeness theorem in the next section.

6.5. Completeness

Let Λ be one of the modal systems listed in Table 6.7. We denote the set of all Λ-maximal consistent sets by $\mathbb{M}_\Lambda$. As in normal modal logic, we have the following.

Lemma 6.13 (Lindenbaum's Lemma)**.** *Let Γ be a Λ-consistent set of wffs. Then there exists a Λ-maximal consistent set Γ^+ containing Γ.*

Let us now recall the notion of canonical relations on $\mathbb{M}_\Lambda$ corresponding to the modal operators $\Box, \Box_\emptyset$ defined as follows.

$$(\Gamma, \Delta) \in \mathrm{R}^\Lambda_\Box \text{ if } \Box\alpha \in \Gamma \text{ implies } \alpha \in \Delta; \tag{6.27}$$

$$(\Gamma, \Delta) \in \mathrm{R}^\Lambda_{\Box_\emptyset} \text{ if } \Box_\emptyset\alpha \in \Gamma \text{ implies } \alpha \in \Delta. \tag{6.28}$$

Giving the standard modal logic arguments, we obtain the following.

Proposition 6.14. *Let Λ be one of the modal systems listed in Table 6.7. Then the following hold.*

1. *$\mathrm{R}^\Lambda_{\Box_\emptyset}$ is an equivalence relation.*
2. *(a) If Λ contains $\mathrm{T}(\Box)$, then $\mathrm{R}^\Lambda_\Box$ is reflexive.*
   (b) If Λ contains $4(\Box)$, then $\mathrm{R}^\Lambda_\Box$ is transitive.
   (c) If Λ contains $\mathrm{B}(\Box)$, then $\mathrm{R}^\Lambda_\Box$ is symmetric.

Lemma 6.15 (Existence Lemma)**.** *For any $\Gamma \in \mathbb{M}_\Lambda$, the following hold.*

- *If $\Diamond\alpha \in \Gamma$, then there exists a $\Delta \in \mathbb{M}_\Lambda$ such that $(\Gamma, \Delta) \in \mathrm{R}^\Lambda_\Box$ and $\alpha \in \Delta$.*
- *If $\Diamond_\emptyset\alpha \in \Gamma$, then there exists a $\Delta \in \mathbb{M}_\Lambda$ such that $(\Gamma, \Delta) \in \mathrm{R}^\Lambda_{\Box_\emptyset}$ and $\alpha \in \Delta$.*

Lemma 6.16 (Existence Lemma)**.** *Let* Λ *be one of the modal systems listed in Table 6.7 that contains axiom C. Let* $\Gamma, \Gamma', \Delta \in \mathbb{M}_\Lambda$ *be such that* $(\Delta, \Gamma) \in \mathrm{R}^\Lambda_{\Box_\emptyset}$ *and* $(\Gamma, \Gamma') \in \mathrm{R}^\Lambda_{\Box}$*. Then there exists a* $\Delta' \in \mathbb{M}_\Lambda$ *such that* $(\Delta, \Delta') \in \mathrm{R}^\Lambda_{\Box}$ *and* $(\Delta', \Gamma') \in \mathrm{R}^\Lambda_{\Box_\emptyset}$*.*

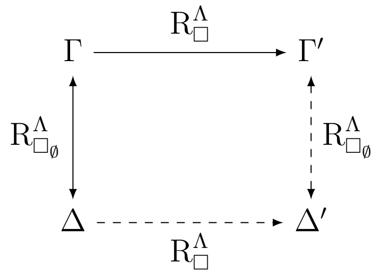

Figure 6.2. Lemma 6.16

Proof. Consider the set

$$\theta = \{\alpha : \Box\alpha \in \Delta\} \cup \{\Diamond_\emptyset\beta : \beta \in \Gamma'\}.$$

Due to Lemma 6.13, it is enough to show that θ is Λ-consistent. If possible, let θ be not Λ-consistent. Then there exist $\Box\alpha_1, \Box\alpha_2, \ldots, \Box\alpha_n \in \Delta$ and $\beta_1, \beta_2, \ldots, \beta_m \in \Gamma'$ such that

$$\vdash_\Lambda \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_n \to \neg(\Diamond_\emptyset\beta_1 \wedge \Diamond_\emptyset\beta_2 \wedge \cdots \wedge \Diamond_\emptyset\beta_m)$$

$$\Longrightarrow \vdash_\Lambda \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_n \to \neg\Diamond_\emptyset(\beta_1 \wedge \beta_2 \wedge \cdots \wedge \beta_m)$$

$$\Longrightarrow \vdash_\Lambda \Box(\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_n) \to \Box\neg\Diamond_\emptyset(\beta_1 \wedge \beta_2 \wedge \cdots \wedge \beta_m)$$

$$\Longrightarrow \vdash_\Lambda (\Box\alpha_1 \wedge \Box\alpha_2 \wedge \cdots \wedge \Box\alpha_n) \to \Box\neg\Diamond_\emptyset(\beta_1 \wedge \beta_2 \wedge \cdots \wedge \beta_m)$$

$$\Longrightarrow \Box\neg\Diamond_\emptyset(\beta_1 \wedge \beta_2 \wedge \cdots \wedge \beta_m) \in \Delta \ (\because \Box\alpha_1 \wedge \Box\alpha_2 \wedge \cdots \wedge \Box\alpha_n \in \Delta)$$

$$\Longrightarrow \neg\Diamond\Diamond_\emptyset(\beta_1 \wedge \beta_2 \wedge \cdots \wedge \beta_m) \in \Delta$$

$$\Longrightarrow \neg\Diamond_\emptyset\Diamond(\beta_1 \wedge \beta_2 \wedge \cdots \wedge \beta_m) \in \Delta \text{ (using axiom C)}$$

$$\Longrightarrow \Box_\emptyset\Box\neg(\beta_1 \wedge \beta_2 \wedge \cdots \wedge \beta_m) \in \Delta$$

$$\Longrightarrow \Box\neg(\beta_1 \wedge \beta_2 \wedge \cdots \wedge \beta_m) \in \Gamma \ (\because (\Delta, \Gamma) \in \mathrm{R}^\Lambda_{\Box_\emptyset})$$

$$\Longrightarrow \neg(\beta_1 \wedge \beta_2 \wedge \cdots \wedge \beta_m) \in \Gamma' \ (\because (\Gamma, \Gamma') \in \mathrm{R}^\Lambda_{\Box}).$$

But this is not possible as $\beta_1 \wedge \beta_2 \wedge \cdots \wedge \beta_m \in \Gamma'$. This completes the proof. □

We again use step by step technique [7] to prove the completeness theorem. Let us begin with a few standard definitions.

Definition 6.17 (Network)**.** A Λ-*network* is defined as a tuple

$$\mathcal{N} := (\mathbb{S}, R, \{W_s\}_{s\in\mathbb{S}}, \mu),$$

where

- $\mathbb{S}$ is a non-empty set of states;
- $R \subseteq \mathbb{S} \times \mathbb{S}$;
- For each $s \in \mathbb{S}$, W_s is a non-empty set of objects;
- $\mu : \bigcup_{s\in\mathbb{S}}(\{s\} \times W_s) \to \mathbb{M}_\Lambda$.

A network $\mathcal{N}$ is said to be a *constant domain network* if $W_s = W_{s'}$ for all $s, s' \in \mathbb{S}$. Thus, a constant domain network will be simply written as $\mathcal{N} := (\mathbb{S},\ R,\ W,\ \mu)$, where $\mu : \mathbb{S} \times W \to \mathbb{M}_\Lambda$. A network $\mathcal{N}$ is said to be *finite* if $\mathbb{S}$ and W_s for each $s \in \mathbb{S}$ are finite. Further, a network will be called reflexive, symmetric, or transitive according to the relation R is reflexive, symmetric, or transitive, respectively.

Definition 6.18 (Coherent Network)**.** A Λ-network $\mathcal{N} := (\mathbb{S}, R, \{W_s\}_{s\in\mathbb{S}}, \mu)$ is said to be Λ-*coherent* if it satisfies the following:

(C1): If $(s,t) \in R$, then $(\mu(s,w), \mu(t,w)) \in \mathrm{R}^\Lambda_\Box$ for all $w \in W_s \cap W_t$.

(C2): If $w, w' \in W_s$, then $(\mu(s,w), \mu(s,w')) \in \mathrm{R}^\Lambda_{\Box_\emptyset}$.

(C3):
- If Λ contains axiom $T(\Box)$, then R is reflexive;
- If Λ contains axiom $4(\Box)$, then R is transitive;
- If Λ contains axiom $B(\Box)$, then R is symmetric;
- If $\Lambda \in \{\mathrm{I}, \mathrm{I(C)}\}$, then R is irreflexive.

A Λ-network for $\Gamma \in \mathbb{M}_\Lambda$ is a Λ-network such that $\mu(s,w) = \Gamma$ for some (s,w).

It may look a little odd to have the condition of irreflexivity of R for $\Lambda \in \{\mathrm{I}, \mathrm{I(C)}\}$ under the condition (C3). In fact, it is required for our methods to work to give completeness theorem. Its requirement will become more clear in Remark 6.27.

We need to impose a few more properties on networks to make it strong enough to give us the Truth Lemma (i.e. Theorem 6.22). Thus, we have the following definition.

Definition 6.19 (Saturated and Perfect Network)**.** A Λ-network $\mathcal{N}$ is said to be *saturated* if it satisfies the following:

- If $\Diamond\alpha \in \mu(s,w)$, then there exists a $s' \in \mathbb{S}$ such that $(s,s') \in R$ and $\alpha \in \mu(s',w)$.

- If $\Diamond_\emptyset \alpha \in \mu(s,w)$, then there exists a $u \in W_s$ such that $\alpha \in \mu(s,u)$.

$\mathcal{N}$ is said to be Λ-*perfect* if it is both Λ-coherent and saturated.

Definition 6.20 (Defects)**.** Let $\mathcal{N} := (\mathbb{S}, R, \{W_s\}_{s\in\mathbb{S}}, \mu)$ be a Λ-network. Then we say that:

- the tuple $\langle s, w, \Diamond\alpha\rangle$ constitutes a $\Diamond$-*defect* of $\mathcal{N}$ if $\Diamond\alpha \in \mu(s,w)$, but there is no s' with $(s,s') \in R$ and $\alpha \in \mu(s',w)$.
- the tuple $\langle s, w, \Diamond_\emptyset\alpha\rangle$ constitutes a $\Diamond_\emptyset$-*defect* of $\mathcal{N}$ if $\Diamond_\emptyset\alpha \in \mu(s,w)$, but there is no w' with $\alpha \in \mu(s,w')$.

A Λ-network induces a model as follows.

Definition 6.21 (Induced Model)**.** Let $\mathcal{N} := (\mathbb{S}, R, \{W_s\}_{s\in\mathbb{S}}, \mu)$ be a Λ-network. Then $\mathcal{N}$ induces a model $\mathfrak{M}^{\mathcal{N}} := (\mathfrak{F}^{\mathcal{N}}, m^{\mathcal{N}})$, where

- $\mathfrak{F}^{\mathcal{N}} := (\mathbb{S}, R, \{\mathcal{K}_s^{\mathcal{N}}\}_{s\in\mathbb{S}})$;
- $\mathcal{K}_s^{\mathcal{N}} := (W_s, \mathcal{A}, \bigcup_{a\in\mathcal{A}} \mathcal{V}_a, f_s^{\mathcal{N}})$;
- $f_s^{\mathcal{N}} : W_s \times \mathcal{A} \to \bigcup_{a\in\mathcal{A}} \mathcal{V}_a$ such that $f_s^{\mathcal{N}}(w,a) = v$ if and only if $(a,v) \in \mu(s,w)$;
- $m^{\mathcal{N}}(p) := \{(s,w) : p \in \mu(s,w)\}$.

Note that Axiom Des guarantees that for each $s \in \mathbb{S}$, $f_s^{\mathcal{N}}$ is a function from $W_s \times \mathcal{A}$ to $\bigcup_{a\in\mathcal{A}} \mathcal{V}_a$, and hence $\mathcal{K}_s^{\mathcal{N}}$ is obtained as a DIS.

A Λ-perfect network is good enough to give us the following Truth Lemma for the induced model.

Theorem 6.22 (Truth Lemma)**.** *Let $\mathcal{N}$ be a Λ-perfect network. Then, for all wff $\alpha \in \mathcal{L}(\Box, \Box_\emptyset)$,*

$$\mathfrak{M}^{\mathcal{N}}, s, w \models \alpha \textit{ if and only if } \alpha \in \mu(s,w).$$

Proof. We use induction on the number of connectives in the wff α. When α is a propositional variable or descriptor, the result holds due to the definition of $m^{\mathcal{N}}$ and $f_s^{\mathcal{N}}$ respectively. The Boolean cases are straight forward. So, let us consider the case when α is of the form $\Diamond\beta$. Let us assume that $\mathfrak{M}^{\mathcal{N}}, s, w \models \Diamond\beta$ and we show that $\Diamond\beta \in \mu(s,w)$. Since $\mathfrak{M}^{\mathcal{N}}, s, w \models \Diamond\beta$, there exist $r \in \mathbb{S}$ such that $(s,r) \in R$, $w \in W_r$, and $\mathfrak{M}^{\mathcal{N}}, r, w \models \beta$. Using induction hypothesis, we obtain $\beta \in \mu(r,w)$. Since $(s,r) \in R$, $w \in W_s \cap W_r$ and $\mathcal{N}$

is a perfect network, we get $(\mu(s,w), \mu(r,w)) \in \mathrm{R}^{\Lambda}_{\Box}$. We have $(\mu(s,w), \mu(r,w)) \in \mathrm{R}^{\Lambda}_{\Box}$ and $\beta \in \mu(r,w)$. Thus, we get $\Diamond\beta \in \mu(s,w)$.

For the converse, let us assume that $\Diamond\beta \in \mu(s,w)$ and we show that $\mathfrak{M}^{\mathcal{N}}, s, w \models \Diamond\beta$. Due to $\Diamond\beta \in \mu(s,w)$, there exists a $r \in \mathbb{S}$ such that $(s,r) \in R$ and $\beta \in \mu(r,w)$ ($\because \mathcal{N}$ is a perfect network). Again using induction hypothesis, we get $\mathfrak{M}^{\mathcal{N}}, r, w \models \beta$. We have $(s,r) \in R$, $w \in W_s \cap W_r$ and $\mathfrak{M}^{\mathcal{N}}, r, w \models \beta$. This gives $\mathfrak{M}^{\mathcal{N}}, s, w \models \Diamond\beta$.

The case when α is of the form $\Diamond_{\emptyset}\beta$ can be proved in the same way. $\Box$

Definition 6.23. Let $\mathcal{N} := (\mathbb{S}, R, \{W_s\}_{s\in\mathbb{S}}, \mu)$ and $\mathcal{N}' := (\mathbb{S}', R', \{W'_s\}_{s\in\mathbb{S}'}, \mu')$ be two Λ-networks. We say that $\mathcal{N}'$ *extends* $\mathcal{N}$, denoted as $\mathcal{N}' \rhd \mathcal{N}$, if the following hold.

- $\mathbb{S} \subseteq \mathbb{S}'$;
- $W_s \subseteq W'_s$ for all $s \in \mathbb{S}$;
- $R = R' \cap (\mathbb{S} \times \mathbb{S})$;
- $\mu(s,w) = \mu'(s,w)$ for all $s \in \mathbb{S}$ and $w \in W_s$.

Let α be an arbitrary Λ-consistent wff. Then, by Lemma 6.13, there exists a $\Gamma \in \mathbb{M}_{\Lambda}$ containing α. Let us consider a Λ-network $\mathcal{N} := (\mathbb{S}, R, \{W_s\}_{s\in\mathbb{S}}, \mu)$, where $\mathbb{S} := \{s_0\}$, $W_{s_0} := \{w_0\}$, $\mu(s_0, w_0) := \Gamma$ and

$$R := \begin{cases} \emptyset, & \text{if } \Lambda \in \{\mathrm{I}, \mathrm{I(C)}\} \\ (s_0, s_0), & \text{if } \Lambda \in \{\mathrm{I(E)}, \mathrm{I(P)}, \mathrm{I(CE)}, \mathrm{I(CP)}\}. \end{cases}$$

Trivially $\mathcal{N}$ is a Λ-coherent network for Γ. If we are guaranteed to obtain a Λ-perfect network $\mathcal{N}' \rhd \mathcal{N}$, then due to Truth Lemma 6.22, we will obtain a completeness result for the logic Λ. Moreover, this completeness result will be with respect to the class of models of which $\mathfrak{M}_{\mathcal{N}'}$ is an instance. Thus, it becomes clear that our task to obtain completeness theorem has reduced to find a way to obtain $\mathcal{N}'$ as described above. This is done by removing defects present in a Λ-coherent network without affecting coherency, making subsequent use of results known as *Repair Lemmas*. We present these results for the modal systems with and without axioms C separately in Section 6.5.1 and Section 6.5.2, respectively.

6.5.1. Completeness for modal systems without axiom C

Let Λ be one of the modal systems listed in Table 6.7 that does not contain axiom C. The following proposition is obvious.

Proposition 6.24. *Let Φ be a class of models such that Λ and Φ lie in the same row of Table 6.7. Then we have $\mathfrak{M}^{\mathcal{N}} \in \Phi$ for every Λ-coherent network $\mathcal{N}$.*

Proof. We provide the proof for the pair $(\mathrm{I(E)}, \Omega_{\mathsf{e}})$. Rest cases can be proved similarly. So, let $\mathcal{N}$ be a I(E)-coherent network. We prove that $\mathfrak{M}^{\mathcal{N}} := (\mathbb{S}, R, \{\mathcal{K}_s^{\mathcal{N}}\}_{s\in\mathbb{S}}, m^{\mathcal{N}})$, where $\mathcal{K}_s^{\mathcal{N}} := (W_s, \mathcal{A}, \bigcup_{a\in\mathcal{A}} \mathcal{V}_a, f_s^{\mathcal{N}})$ is in Ω_{e}. Due to the coherency of $\mathcal{N}$, we obtain R as an equivalence relation. Hence $\mathfrak{M}^{\mathcal{N}} \in \Omega_{\mathsf{e}}$. □

Theorem 6.25 (Repair Lemma for $\Diamond$-Defect)**.** *Let $\langle s_0, w_0, \Diamond\alpha\rangle$ constitute a* $\Diamond$-defect *of a Λ-coherent network $\mathcal{N} := (\mathbb{S}, R, \{W_s\}_{s\in\mathbb{S}}, \mu)$, where $\mathbb{S}$ and W_s for each $s \in \mathbb{S}$, are finite sets. Then there exists a finite Λ-coherent network $\mathcal{N}' \rhd \mathcal{N}$ such that $\langle s_0, w_0, \Diamond\alpha\rangle$ no longer constitutes a $\Diamond$-defect of $\mathcal{N}'$.*

Proof. Since $\Diamond\alpha \in \mu(s_0, w_0)$, by Existence Lemma 6.15, there exists a $\Delta_0 \in \mathbb{M}_\Lambda$ such that

$$(\mu(s_0, w_0), \Delta_0) \in \mathrm{R}_{\Box}^{\Lambda}, \text{ and } \alpha \in \Delta_0. \tag{6.29}$$

Let s' be a new symbol, not used in $\mathbb{S}$. Consider the network $\mathcal{N}' := (\mathbb{S}', R', \{W_s'\}_{s\in\mathbb{S}'}, \mu')$, where

$$\mathbb{S}' := \mathbb{S} \cup \{s'\}$$

$$W_s' := \begin{cases} W_s, & \text{if } s \neq s' \\ \{w_0\}, & \text{if } s = s', \end{cases}$$

$$R' := \begin{cases} R \cup \{(s_0, s')\}, & \text{if } \Lambda = \mathrm{I} \\ R \cup \{(s', s')\} \cup \{(s, s') : (s, s_0) \in R\}, & \text{if } \Lambda = \mathrm{I(P)} \\ R \cup \{(s', s')\} \cup \{(s, s'), (s', s) : (s, s_0) \in R\}, & \text{if } \Lambda = \mathrm{I(E)} \end{cases}$$

$$\mu'(s, w) := \begin{cases} \mu(s, w), & \text{if } s \neq s' \\ \Delta_0, & \text{if } s = s'. \end{cases}$$

We claim that $\mathcal{N}'$ is the required network. In fact, obviously $\mathcal{N}' \rhd \mathcal{N}$ and $\langle s_0, w_0, \Diamond\alpha\rangle$ no longer constitutes a $\Diamond$-defect of $\mathcal{N}'$.

(C1): Let $(s, t) \in R'$, $w \in W_s' \cap W_t'$ and we show that $(\mu'(s, w), \mu'(t, w)) \in \mathrm{R}_{\Box}^{\Lambda}$. If $s, t \in \mathbb{S}$ or $s = t = s'$, then obviously we have the result. So, let us assume that $t = s'$, and $s \neq s'$. Then, using the definition of R', we obtain $w = w_0$. If $s = s_0$, then we obtain

$(\mu'(s,w_0),\mu'(t,w_0)) \in \mathrm{R}^{\Lambda}_{\Box}$ directly from (6.29). If $s \neq s_0$, then $\Lambda \neq \mathrm{I}$, and in that case $\mathrm{R}^{\Lambda}_{\Box}$ is obtained as a transitive relation and $(s,s_0) \in R$. Therefore, we get

$$(\mu(s,w_0),\mu(s_0,w_0)) \in \mathrm{R}^{\Lambda}_{\Box}. \tag{6.30}$$

Now, using (6.29), (6.30), transitivity of $\mathrm{R}^{\Lambda}_{\Box}$ and definition of μ', we obtain

$$(\mu'(s,w_0),\mu'(s',w_0)) \in \mathrm{R}^{\Lambda}_{\Box}.$$

One can similarly obtain $(\mu'(s,w),\mu'(t,w)) \in \mathrm{R}^{\Lambda}_{\Box}$ when $s = s'$, and $t \neq s'$.

(C2) follows from the fact that $\mathrm{R}^{\Lambda}_{\Box_\emptyset}$ is reflexive. Moreover, (C3) follows directly from the definition of R'. □

Theorem 6.26 (Repair Lemma for $\Diamond_\emptyset$-Defect)**.** *Let $\langle s_0, w_0, \Diamond_\emptyset\alpha\rangle$ constitute a $\Diamond_\emptyset$-defect of a Λ-coherent network $\mathcal{N} := (\mathbb{S}, R, \{W_s\}_{s\in\mathbb{S}}, \mu)$, where $\mathbb{S}$ and W_s for each $s \in \mathbb{S}$, are finite sets. Then there exists a finite Λ-coherent network $\mathcal{N}' \rhd \mathcal{N}$ such that $\langle s_0, w_0, \Diamond_\emptyset\alpha\rangle$ no longer constitutes a $\Diamond_\emptyset$-defect of $\mathcal{N}'$.*

Proof. Since $\Diamond_\emptyset\alpha \in \mu(s_0,w_0)$, by Existence Lemma 6.15, there exists a Δ_0 such that $(\mu(s_0,w_0),\Delta_0) \in \mathrm{R}^{\Lambda}_{\Box_\emptyset}$ and $\alpha \in \Delta_0$. Let w' be a new symbol, not used in $\bigcup_{s\in\mathbb{S}} W_s$. Consider the network

$$\mathcal{N}' := (\mathbb{S}', R', \{W'_s\}_{s\in\mathbb{S}'}, \mu'),$$

where,

$$\mathbb{S}' := \mathbb{S}$$

$$W'_s := \begin{cases} W_s, & \text{if } s \neq s_0 \\ W_s \cup \{w'\}, & \text{if } s = s_0, \end{cases}$$

$$R' := R$$

$$\mu'(s,w) := \begin{cases} \mu(s,w), & \text{if } w \neq w' \\ \Delta_0, & \text{otherwise.} \end{cases}$$

$\mathcal{N}'$ is our required network. In fact, obviously $\mathcal{N}' \rhd \mathcal{N}$ and $\langle s_0, w_0, \Diamond_\emptyset\alpha\rangle$ no longer constitutes an $\Diamond_\emptyset$-defect of $\mathcal{N}'$. Thus, it remains to show that $\mathcal{N}'$ is a Λ-coherent network. (C1): Let $(s,t) \in R'$, $w \in W'_s \cap W'_t$ and we show that $(\mu'(s,w),\mu'(t,w)) \in \mathrm{R}^{\Lambda}_{\Box}$. Let us first consider the case when $w \neq w'$. Then by construction of $\mathcal{N}'$, we obtain $w \in W_s \cap W_t$, $\mu'(s,w) = \mu(s,w)$ and $\mu'(t,w) = \mu(t,w)$. Since $\mathcal{N}$ is a coherent network, we get $(\mu(s,w) = \mu'(s,w), \mu(t,w) = \mu'(t,w)) \in \mathrm{R}^{\Lambda}_{\Box}$. Next, let us consider the case when $w = w'$. In that case we must have $s = t = s_0$ and hence $(s_0, s_0) \in R$. This shows that $\Lambda \in \{\mathrm{I(P)}, \mathrm{I(E)}\}$ as R is irreflexive for $\Lambda = \mathrm{I}$. Hence $\mathrm{R}^{\Lambda}_{\Box}$ is obtained as a reflexive relation. Thus we obtain $(\mu'(s,w) = \Delta_0, \Delta_0 = \mu'(t,w)) \in \mathrm{R}^{\Lambda}_{\Box}$.

(C2): Let $s \in \mathbb{S}$, $w, u \in W_s$ and we show that $(\mu'(s,w), \mu'(s,u)) \in \mathrm{R}^{\Lambda}_{\Box_\emptyset}$. Obviously, we have the result when both w and u are different from w', or $w = w' = u$. So, without loss of generality, let $w = w'$, $u \neq w'$ and $s = s_0$. Then, we obtain

$$(\mu'(s,w) = \mu'(s_0,w') = \Delta_0, \mu(s_0,w_0)) \in \mathrm{R}^{\Lambda}_{\Box_\emptyset} \text{ and } (\mu(s_0,w_0), \mu(s_0,u) = \mu'(s,u)) \in \mathrm{R}^{\Lambda}_{\Box_\emptyset}.$$

Thus, we obtain $(\mu'(s,w), \mu'(s,u)) \in \mathrm{R}^{\Lambda}_{\Box_\emptyset}$ using the transitivity of $\mathrm{R}^{\Lambda}_{\Box_\emptyset}$.

Obviously $\mathcal{N}'$ satisfies the condition (C3) and this completes the proof. □

Remark 6.27. Note that the proof of (C1) condition in the above proof will not work for the modal system I if we do not keep the requirement of irreflexivity in the definition of Λ-coherent network under (C3). In order to see it, observe that, in the absence of irreflexivity, we may have $(s_0, s_0) \in R$, but $(\Delta_0, \Delta_0) \notin \mathrm{R}^{\Lambda}_{\Box}$ and this will break down the proof arguments for the condition (C1).

Theorem 6.28 (Completeness Theorem)**.** *Let (Λ, Φ) be a pair consisting of a modal system Λ that does not contain axiom C and a class Φ of models from the same row of Table 6.7. Then, for each wff $\alpha \in \mathcal{L}(\Box, \Box_\emptyset)$, $\Phi \models \alpha$ implies $\vdash_\Lambda \alpha$.*

Proof. If possible, let $\nvdash_\Lambda \alpha$. Then, in that case, $\{\neg\alpha\}$ is obtained as Λ-consistent, and hence there exists a $\Gamma \in \mathbb{M}_\Lambda$ containing $\neg\alpha$.

Consider $\mathcal{N} := (\mathbb{S}, R, \{W_s\}_{s\in\mathbb{S}}, \mu)$, where $\mathbb{S} := \{s_0\}$, $W_{s_0} := \{w_0\}$, $\mu(s_0, w_0) := \Gamma$ and

$$R := \begin{cases} \emptyset, & \text{if } \Lambda = \mathrm{I} \\ (s_0, s_0), & \text{if } \Lambda \in \{\mathrm{I(E)}, \mathrm{I(P)}\}. \end{cases}$$

Trivially $\mathcal{N}$ is finite, Λ-coherent network for Γ. Moreover, by repeated applications of Repair Lemma 6.25 and 6.26, we obtain a Λ-perfect network $\mathcal{N}'$ such that $\mathcal{N}' \rhd \mathcal{N}$. The proof is very standard, and we refer to [7] for details. Also note that, due to Proposition 6.24, we obtain $\mathfrak{M}_{\mathcal{N}'} \in \Phi$. Now, since $\neg\alpha \in \Gamma = \mu'(s_0, w_0)$, by Truth Lemma 6.22, we obtain $\mathfrak{M}_{\mathcal{N}'}, s_0, w_0 \models \neg\alpha$. But this contradicts that $\Phi \models \alpha$. Hence, we must have $\vdash_\Lambda \alpha$. This completes the proof. $\square$

6.5.2. Completeness for the modal system I(CE)

In this section we will provide the repair lemmas for the modal system I(CE). We begin with the $\Diamond$-defect.

Theorem 6.29 (Repair Lemma for $\Diamond$-Defect)**.** *Let* $\langle s_0, w_0, \Diamond\alpha\rangle$ *constitute a* $\Diamond$-defect *of a finite,* I(CE)*-coherent and constant domain network* $\mathcal{N} := (\mathbb{S}, R, W, \mu)$*. Then there exists a finite,* I(CE)*-coherent and constant domain network* $\mathcal{N}' \rhd \mathcal{N}$ *such that* $\langle s_0, w_0, \Diamond\alpha\rangle$ *no longer constitutes a* $\Diamond$*-defect of* $\mathcal{N}'$*.*

Proof. Since $\Diamond\alpha \in \mu(s_0, w_0)$, by Existence Lemma 6.15, there exists a $\Delta_{w_0} \in \mathbb{M}_{\mathrm{I(CE)}}$ such that

$$(\mu(s_0, w_0), \Delta_{w_0}) \in \mathrm{R}_{\Box}^{\mathrm{I(CE)}}, \text{ and } \alpha \in \Delta_{w_0}. \tag{6.31}$$

Further, for each $w(\neq w_0) \in W$, Lemma 6.16 guarantees the existence of a Δ_w satisfying the conditions described in Figure 6.3. Let s' be a new symbol, not used in $\mathbb{S}$. Consider

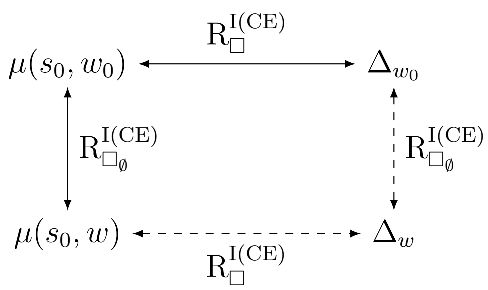

Figure 6.3

the network $\mathcal{N}' := (\mathbb{S}', R', W, \mu')$, where

$$\mathbb{S}' := \mathbb{S} \cup \{s'\}$$

$$R' := R \cup \{(s', s')\} \cup \{(s, s'), (s', s) : (s, s_0) \in R\}$$

$$\mu'(s, w) := \begin{cases} \mu(s, w), & \text{if } s \neq s' \\ \Delta_w, & \text{if } s = s'. \end{cases}$$

We claim that $\mathcal{N}'$ is the required network. In fact, obviously $\mathcal{N}' \rhd \mathcal{N}$ and $\langle s_0, w_0, \Diamond\alpha\rangle$ no longer constitutes an $\Diamond$-defect of $\mathcal{N}'$. Thus, it remains to show that $\mathcal{N}'$ is a I(CE)-coherent network.

(C1): Let $(s,t) \in R'$ and $w \in W$. We need to show that $(\mu'(s,w), \mu'(t,w)) \in \mathrm{R}_{\Box}^{\mathrm{I(CE)}}$. If $s,t \in \mathbb{S}$, or $s = t = s'$, then we have the result. So, without loss of generality, we assume that $s = s'$, and $t \neq s'$. If $t = s_0$, then we get the result using the property of $\mu'(s', w) = \Delta_w$ described in Figure 6.3. If $t \neq s_0$, then we obtain $(s_0, t) \in R$ by the construction of R'. This gives $(\mu'(s_0,w), \mu'(t,w)) \in \mathrm{R}_{\Box}^{\mathrm{I(CE)}}$. We also have $(\mu'(s',w), \mu'(s_0,w)) \in \mathrm{R}_{\Box}^{\mathrm{I(CE)}}$ (cf. Figure 6.3). Therefore, using transitivity of $\mathrm{R}_{\Box}^{\mathrm{I(CE)}}$, we obtain $(\mu'(s',w), \mu'(t,w)) \in \mathrm{R}_{\Box}^{\mathrm{I(CE)}}$.

(C2): Let $s \in \mathbb{S}'$, $w, u \in W$, and we show that $(\mu'(s,w), \mu'(s,u)) \in \mathrm{R}_{\Box_\emptyset}^{\mathrm{I(CE)}}$. If $s \neq s'$, then coherency of the network $\mathcal{N}$ gives the required result. So, let us assume that $s = s'$. Since $\mathrm{R}_{\Box_\emptyset}^{\mathrm{I(CE)}}$ is an equivalence relation, using Figure 6.3, we get

$$(\mu'(s',w) = \Delta_w, \Delta_{w_0}) \in \mathrm{R}_{\Box_\emptyset}^{\mathrm{I(CE)}} \text{ and } (\Delta_{w_0}, \Delta_u = \mu'(s',u)) \in \mathrm{R}_{\Box_\emptyset}^{\mathrm{I(CE)}},$$

and hence $(\mu'(s',w), \mu'(s',u)) \in \mathrm{R}_{\Box_\emptyset}^{\mathrm{I(CE)}}$.

Obviously $\mathcal{N}'$ satisfies the condition (C3) and this completes the proof. □

Theorem 6.30 (Repair Lemma for $\Diamond_\emptyset$-Defect)**.** *Let $\langle s_0, w_0, \Diamond_\emptyset\alpha\rangle$ constitute a $\Diamond_\emptyset$-defect of a finite,* I(CE)*-coherent and constant domain network $\mathcal{N} := (\mathbb{S}, R, \{W_s\}_{s\in\mathbb{S}}, \mu)$. Then there exists a finite,* I(CE)*-coherent and constant domain network $\mathcal{N}' \rhd \mathcal{N}$ such that $\langle s_0, w_0, \Diamond_\emptyset\alpha\rangle$ no longer constitutes a $\Diamond_\emptyset$-defect of $\mathcal{N}'$.*

Proof. Since $\Diamond_\emptyset\alpha \in \mu(s_0, w_0)$, by Existence Lemma 6.15, there exists a Δ_{s_0} such that

$$(\mu(s_0, w_0), \Delta_{s_0}) \in \mathrm{R}_{\Box_\emptyset}^{\mathrm{I(CE)}} \text{ and } \alpha \in \Delta_{s_0}.$$

Further, for each $s(\neq s_0)$ with $(s_0, s) \in R$, Lemma 6.16 guarantees the existence of a Δ_s satisfying the conditions described in Figure 6.4. Let w' be a new symbol, not used in W. Consider the network

$$\mathcal{N}' := (\mathbb{S}', R', W', \mu'),$$

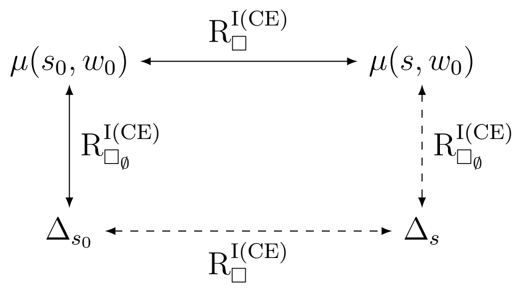

Figure 6.4

where, $\mathbb{S}' := \mathbb{S}$, $W' := W \cup \{w'\}$, $R' := R$ and

$$\mu'(s,w) := \begin{cases} \mu(s,w), & \text{if } w \neq w' \\ \Delta_s, & \text{if } w = w' \text{ and } (s_0,s) \in R \\ \mu(s,w_0), & \text{if } w = w' \text{ and } (s_0,s) \notin R. \end{cases}$$

We claim that $\mathcal{N}'$ is the required network. In fact, obviously $\mathcal{N}' \rhd \mathcal{N}$ and $\langle s_0, w_0, \Diamond_\emptyset \alpha \rangle$ no longer constitutes a $\Diamond_\emptyset$-defect of $\mathcal{N}'$. Thus, it remains to show that $\mathcal{N}'$ is an I(CE)-coherent network.

(C1): Let $(s,t) \in R'$ and $w \in W'$ and we need to show that $(\mu'(s,w), \mu'(t,w)) \in \mathrm{R}_\Box^{\mathrm{I(CE)}}$. We obviously have the result when $w \neq w'$. So, let us consider the case when $w = w'$. Note that, since R is an equivalence relation, it cannot be the case that (i) $(s_0,s) \in R$, but $(s_0,t) \notin R$, or (ii) $(s_0,t) \in R$, but $(s_0,s) \notin R$. Thus, we need to consider the following two cases:

Case (A): Let $(s_0,s), (s_0,t) \in R$. Then, since $\mathrm{R}_\Box^{\mathrm{I(CE)}}$ is an equivalence relation, we obtain

$$(\mu'(s,w) = \Delta_s, \Delta_{s_0}) \in \mathrm{R}_\Box^{\mathrm{I(CE)}} \text{ and } (\Delta_{s_0}, \Delta_t = \mu'(t,w)) \in \mathrm{R}_\Box^{\mathrm{I(CE)}},$$

and hence $(\mu'(s,w), \mu'(t,w)) \in \mathrm{R}_\Box^{\mathrm{I(CE)}}$.

Case (B): Let $(s_0,s), (s_0,t) \notin R$. Then,

$$(\mu'(s,w) = \mu(s,w_0), \mu'(t,w_0) = \mu(t,w)) \in \mathrm{R}_\Box^{\mathrm{I(CE)}}.$$

(C2): Let $s \in \mathbb{S}'$, $w, u \in W'$ and we show that $(\mu'(s,w), \mu'(s,u)) \in \mathrm{R}_{\Box_\emptyset}^{\mathrm{I(CE)}}$. Obviously, we have the result when both w and u are different from w', or $w = w' = u$. So, without loss of generality, let $w = w'$ and $u \neq w'$.

If $(s_0, s) \in R$, then we obtain

$$(\mu'(s,w) = \Delta_s, \mu(s,w_0)) \in \mathrm{R}_{\Box_\emptyset}^{\mathrm{I(CE)}} \text{ and } (\mu(s,w_0), \mu(s,u) = \mu'(s,u)) \in \mathrm{R}_{\Box_\emptyset}^{\mathrm{I(CE)}}.$$

This gives $(\mu'(s,w), \mu'(s,u))\, \mathrm{R}_{\Box_\emptyset}^{\mathrm{I(CE)}}$ as $\mathrm{R}_{\Box_\emptyset}^{\mathrm{I(CE)}}$ is an equivalence relation.
Similarly, if $(s_0, s) \notin R$, then we get

$$(\mu'(s,w) = \mu(s,w_0), \mu(s,u) = \mu'(s,u)) \in \mathrm{R}_{\Box_\emptyset}^{\mathrm{I(CE)}}.$$

Obviously $\mathcal{N}'$ satisfies the condition (C3) and this completes the proof. □

Once we have Theorems 6.29 and 6.30, proceeding as in the proof of Theorem 6.28, we obtain the following completeness theorem.

Theorem 6.31 (Completeness Theorem). *For each wff $\alpha \in \mathcal{L}(\Box, \Box_\emptyset)$, $\Omega_{\mathsf{ce}} \models \alpha$ implies $\vdash_{\mathrm{I(CE)}} \alpha$.*

It is pertinent to note here that we are not able to obtain a repair lemma for $\Diamond_\emptyset$-defect for modal systems I(C) and I(CP) and, as a result, we still do not have the completeness theorem for the modal systems I(C) and I(CP) with respect to the classes Ω_{c} and Ω_{cp}, respectively.

6.6. A comparison with the standard multi-modal logic semantics

Recall that the semantics proposed in Section 6.2.2 is 2-dimensional, having the dimensions for states and objects of the domains. In this section, we shall prove that, as far as the notion of the validity of wffs of $\mathcal{L}(\Box, \Box_\emptyset)$ is concerned, it can be equivalently captured through the standard multi-modal logic semantics. This result is important from the view point of modal logic. For instance, it will lead us to a few decidability results in Section 6.7.

Consider the following notion of auxiliary frame.

Definition 6.32. An *auxiliary frame* is defined as a tuple $\mathcal{F} := (\mathbb{S}, R, R_\emptyset)$, where R is a binary relation on $\mathbb{S}$, and $R_\emptyset$ is an equivalence relation on $\mathbb{S}$.
An *auxiliary model* $\mathcal{M}$ for $\mathcal{L}(\Box, \Box_\emptyset)$ consists of an auxiliary frame $\mathcal{F} := (\mathbb{S}, R, R_\emptyset)$ together with the valuation function $m : PV \to 2^{\mathbb{S}}$ and $f : \mathcal{D} \to 2^{\mathbb{S}}$, where

- $f(a,v) \cap f(a,u) = \emptyset$ for distinct $u, v \in \mathcal{V}_a$;

- $\mathbb{S} = \bigcup_{v \in \mathcal{V}_a} f(a, v)$ for all $a \in \mathcal{A}$.

Here, to make the writing simple, we have preferred to write $f(a, v)$ instead of $f((a, v))$.

Let us consider the following standard multi-modal logic semantics for $\mathcal{L}(\Box, \Box_\emptyset)$ based on auxiliary frames.

Definition 6.33. The *satisfiability* of a wff $\alpha \in \mathcal{L}(\Box, \Box_\emptyset)$ in an auxiliary model $\mathcal{M} := (\mathcal{F}, m, f)$, denoted as $\mathcal{M}, s \models \alpha$, is defined inductively:

$$
\begin{array}{lcl}
\mathcal{M}, s \models \top & & \text{always.} \\
\mathcal{M}, s \models p & \iff & s \in m(p), \text{ for } p \in PV. \\
\mathcal{M}, s \models (a, v) & \iff & s \in f(a, v), \text{ for } (a, v) \in \mathcal{D}. \\
\mathcal{M}, s \models \neg\alpha & \iff & \mathcal{M}, s \not\models \alpha. \\
\mathcal{M}, s \models \alpha \wedge \beta & \iff & \mathcal{M}, s \models \alpha \text{ and } \mathcal{M}, s \models \beta. \\
\mathcal{M}, s \models \Box\alpha & \iff & \text{for all } r \in \mathbb{S} \text{ with } (s, r) \in R, \ \mathcal{M}, r \models \alpha. \\
\mathcal{M}, s \models \Box_\emptyset\alpha & \iff & \text{for all } r \in \mathbb{S} \text{ with } (s, r) \in R_\emptyset, \ \mathcal{M}, r \models \alpha.
\end{array}
$$

The notions of validity and satisfiability in a class $\mathfrak{G}$ of auxiliary models are defined in the usual way:

- $\alpha \in \mathcal{L}(\Box, \Box_\emptyset)$ is *satisfiable* in $\mathfrak{G}$ if there exists a model $\mathcal{M} := (\mathcal{F}, m, f)$ in $\mathfrak{G}$, and an element $w \in \mathbb{S}$, where $\mathbb{S}$ is the domain of $\mathcal{F}$, such that $\mathcal{M}, w \models \alpha$;
- α is *valid* in $\mathfrak{G}$, denoted as $\mathfrak{G} \models \alpha$, if for all models $\mathcal{M} := (\mathcal{F}, m, f)$ in $\mathfrak{G}$, and all elements $w \in \mathbb{S}$, $\mathcal{M}, w \models \alpha$.

Let us consider the property P1 for an auxiliary model $\mathcal{M}$ that is defined as follows:

P1.: If there exist $s, t, r \in \mathbb{S}$ with $(s, t) \in R_\emptyset$ and $(t, r) \in R$, then there exists a $w \in \mathbb{S}$ such that $(s, w) \in R$ and $(w, r) \in R_\emptyset$.

Table 6.8 lists a few classes of auxiliary models that are of interest to us.

Let us recall the notion of canonical model corresponding to a modal system Λ.

Definition 6.34 (Canonical Auxiliary Model)**.** Let Λ be a modal system listed in Table 6.7. Let $\mathcal{M}^\Lambda := (\mathcal{F}^\Lambda, m^\Lambda, g^\Lambda)$, $\mathcal{F}^\Lambda := (\mathbb{M}_\Lambda, \mathrm{R}^\Lambda_\Box, \mathrm{R}^\Lambda_{\Box_\emptyset})$, be the canonical auxiliary model

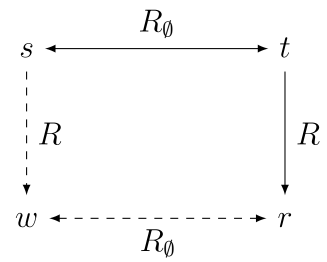

Figure 6.5. Property P1 for an auxiliary model

Class of Models	Defining condition
Υ	Class of all auxiliary models
Υ_{c}	Class of all auxiliary models satisfying P1
Υ_{e}	Class of all auxiliary models with equivalence relation R
Υ_{p}	Class of all auxiliary models with reflexive and transitive relation R
Υ_{ce}	$\Upsilon_{\mathsf{c}} \cap \Upsilon_{\mathsf{e}}$
Υ_{cp}	$\Upsilon_{\mathsf{c}} \cap \Upsilon_{\mathsf{p}}$

Table 6.8. Classes of auxiliary models

corresponding to the modal system Λ, where

$$\mathrm{R}^{\Lambda}_{\Box} \text{ and } \mathrm{R}^{\Lambda}_{\Box_\emptyset} \text{ are given by (6.27) and (6.28);}$$

$$m^{\Lambda}(p) := \{\Gamma \in \mathbb{M}_{\Lambda} : p \in \Gamma\} \text{ for } p \in PV;$$

$$g^{\Lambda}(a, v) := \{\Gamma \in \mathbb{M}_{\Lambda} : (a, v) \in \Gamma\} \text{ for } (a, v) \in \mathcal{D}.$$

Theorem 6.35. *Let* (Λ, Ψ) *be a tuple consisting of a modal system* Λ *and a class* Ψ *of auxiliary models from the same row of Table 6.9. Then* $\mathcal{M}^{\Lambda} \in \Psi$.

Theorem 6.36 (Truth Lemma)**.** *Let* Λ *be a modal system listed in Table 6.9. Then for all wff* $\alpha \in \mathcal{L}(\Box, \Box_\emptyset)$,

$$\mathcal{M}^{\Lambda}, \Gamma \mid\models \alpha \text{ if and only if } \alpha \in \Gamma.$$

[2]The corresponding completeness theorem is not yet proved, but we expect the result.

Modal System	Class of Model	Class of Auxiliary Model
I	Ω	Υ
I(C)	$\Omega_{c}{}^{2}$	Υ_{c}
I(E)	Ω_{e}	Υ_{r}
I(P)	Ω_{p}	Υ_{p}
I(CE)	Ω_{ce}	Υ_{ce}
I(CP)	$\Omega_{cp}{}^{2}$	Υ_{cp}

Table 6.9. Soundness and completeness theorems relative to different classes of models and auxiliary models

Table 6.9 gives the soundness and completeness theorems for different modal systems relative to different classes of auxiliary models. That is, we have the following theorem.

Theorem 6.37. *Let* (Λ, Φ, Ψ) *be a tuple consisting of a modal system* Λ*, a class* Φ *of models and a class* Ψ *of auxiliary models from the same row of Table 6.9. Then, for each wff* $\alpha \in \mathcal{L}(\Box, \Box_{\emptyset})$*, we have the following.*

1. *For* $\Lambda \in \{\mathrm{I}, \mathrm{I(E)}, \mathrm{I(P)}, \mathrm{I(CE)}\}$*,*

$$\Phi \models \alpha \iff \vdash_{\Lambda} \alpha \iff \Psi \mid\models \alpha.$$

2. *For* $\Lambda \in \{\mathrm{I(C)}, \mathrm{I(CP)}\}$*,* $\vdash_{\Lambda} \alpha \iff \Psi \mid\models \alpha.$

Proof. We provide the proof when Λ is the modal system I(CE). The proof of other systems follow in a similar way. The part $\Phi \models \alpha \iff \vdash_{\Lambda} \alpha$ is proved in Section 6.4.1 and Section 6.5 (cf. Theorem 6.12 and Theorem 6.31). We will discuss the proof of the part $\vdash_{\Lambda} \alpha \iff \Psi \mid\models \alpha$. Soundness part $\vdash_{\Lambda} \alpha \implies \Psi \mid\models \alpha$ is straightforward and we omit its proof. Let us assume that $\Psi \mid\models \alpha$ and we show that $\vdash_{\Lambda} \alpha$. Towards a contradiction, let $\nvdash_{\Lambda} \alpha$. Then, we obtain $\{\neg\alpha\}$ as a Λ-consistent set. By Lindenbaum's Lemma, there exists a Λ-maximal consistent set Γ containing $\{\neg\alpha\}$. From Truth Lemma 6.36, we get $\mathcal{M}^{\Lambda}, \Gamma \mid\models \neg\alpha$. Since $\mathcal{M}^{\Lambda} \in \Psi$ due to Theorem 6.35 and $\Psi \mid\models \alpha$, we obtain $\mathcal{M}^{\Lambda}, \Gamma \mid\models \alpha$. But $\mathcal{M}^{\Lambda}, \Gamma \mid\models \alpha$ and $\mathcal{M}^{\Lambda}, \Gamma \mid\models \neg\alpha$ cannot be true together. Hence, we must have $\vdash_{\Lambda} \alpha$. $\Box$

As a consequence of Theorem 6.37, it follows that properties concerning the validity of wffs with respect to the standard multi-modal logic semantics given by Definition 6.33

relative to the class $\Psi \in \{\Upsilon, \Upsilon_{\mathsf{p}}, \Upsilon_{\mathsf{e}}, \Upsilon_{\mathsf{ce}}\}$ of auxiliary models gives the corresponding properties concerning the validity of wffs with respect to the class Φ of models that lies in the same row as Ψ in Table 6.9. For instance, in Section 6.7, we obtain the decidability of the validity problem for the class Υ_{e}, and this, in turn, gives the decidability of the validity problem for the classes Ω_{e}. We end this section with the remark that, although we have the correspondence as mentioned above between different *classes* of models and auxiliary models related to the notion of validity (satisfiability) of wffs (cf. Theorem 6.37), we still do not know any validity (satisfiability) preserving correspondence between the models and auxiliary models. More precisely, we are not able to find a function F from the class of models to the class of auxiliary models (or, from the class of auxiliary models to the class of models) such that a wff α is satisfiable in a model $\mathfrak{M}$ (auxiliary model $\mathcal{M}$) if and only if α is satisfiable in the auxiliary model $F(\mathfrak{N})$ (model $F(\mathcal{M})$).

6.7. Decidability

We would like to prove here the decidability of the satisfiability (validity) problem of the semantics of Section 6.2.2 with respect to various classes of models listed in Table 6.6. It is pertinent to note that we are not able to obtain the finite model property for the semantics of Section 6.2.2 with respect to these classes of models. Hence we are not able to prove the above decidability result using the finite model property of the semantics presented in Section 6.2.2. However, the good news is that we can obtain the finite model property for the semantics of Section 6.6 with respect to the classes $\Upsilon, \Upsilon_{\mathsf{e}}, \Upsilon_{\mathsf{p}}, \Upsilon_{\mathsf{c}}, \Upsilon_{\mathsf{cp}}$ of auxiliary models (cf. Theorems 6.45 and 6.47) and, as a consequence of this, we will obtain the following result.

Theorem 6.38. *Let $\Psi \in \{\Upsilon, \Upsilon_{\mathsf{e}}, \Upsilon_{\mathsf{p}}, \Upsilon_{\mathsf{c}}, \Upsilon_{\mathsf{cp}}\}$ be a class of auxiliary models. Then, we can decide for a given wff $\alpha \in \mathcal{L}(\Box, \Box_{\emptyset})$, whether α is satisfiable in the class Ψ.*

Then, using Theorem 6.37 and Theorem 6.38, we immediately obtain the following decidability result.

Theorem 6.39. *Let $\Phi \in \{\Omega, \Omega_{\mathsf{e}}, \Omega_{\mathsf{p}}\}$ be a class of models. Then, we can decide for a given wff $\alpha \in \mathcal{L}(\Box, \Box_{\emptyset})$, whether α is satisfiable in the class Φ.*

We want to add here that we do not have Theorem 6.39 for the classes $\Omega_{\mathsf{c}}, \Omega_{\mathsf{cp}}$ as we cannot prove the completeness theorem for these classes with respect to the modal systems I(C), I(CP), respectively. However, as a direct consequence of Theorem 6.37 and Theorem 6.38, we obtain the following.

Theorem 6.40. *Let $\Lambda \in \{\mathrm{I}, \mathrm{I(E)}, \mathrm{I(P)}, \mathrm{I(C)}, \mathrm{I(CP)}\}$ be a modal system. Then, we can decide for a given wff $\alpha \in \mathcal{L}(\Box, \Box_\emptyset)$, whether $\vdash_\Lambda \alpha$.*

In the rest of the section, we will provide the finite model property for the semantics of Section 6.6 with respect to various classes of auxiliary models. We will handle the classes of auxiliary models with and without the property P1 separately.

6.7.1. Classes of auxiliary models without the property P1

We follow the standard filtration technique (cf. [7]) with natural modifications to the definitions. Let Σ denote a finite sub-formula closed subset of $\mathcal{L}(\Box, \Box_\emptyset)$. Let $\mathcal{M} := (\mathcal{F}, m, g)$, $\mathcal{F} := (\mathbb{S}, R, R_\emptyset)$, be an auxiliary model. We define an equivalence relation $\equiv_\Sigma$ on $\mathbb{S}$ as follows.

$s \equiv_\Sigma s'$, if and only if for all $\beta \in \Sigma \cup \mathcal{D}, \mathcal{M}, s \mid\models \beta$ if and only if $\mathcal{M}, s' \mid\models \beta$.

Definition 6.41 (Filtration model)**.** Let us consider an auxiliary model $\mathcal{M} := (\mathcal{F}, m, g)$ and Σ as above.

- We define an auxiliary model $\mathcal{M}^f := (\mathcal{F}^f, m^f, g^f)$, $\mathcal{F}^f := (\mathbb{S}^f, R^f, R^f_\emptyset)$, where
  - $\mathbb{S}^f := \{[s] : s \in \mathbb{S}\}, [s]$ is the equivalence class of s with respect to the equivalence relation $\equiv_\Sigma$;
  - $([s], [s']) \in R^f$ if and only if there exist $s_1 \in [s]$ and $s_2 \in [s']$ such that $(s_1, s_2) \in R$;
  - $([s], [s']) \in R^f_\emptyset$ if and only if there exists $s_1 \in [s]$ and $s_2 \in [s']$ such that $(s_1, s_2) \in R_\emptyset$;
  - $m^f(p) := \{[s] : s \in m(p)\}$;
  - $g^f(a, v) := \{[s] : s \in g(a, v)\}$.
- For $\Psi \in \{\Upsilon, \Upsilon_{\mathsf{e}}, \Upsilon_{\mathsf{p}}\}$, we define the model $\mathcal{M}^\Psi := (\mathcal{F}^\Psi, m^f, g^f)$, $\mathcal{F}^\Psi := (\mathbb{S}^f, R^\Psi, R^{f^*}_\emptyset)$, where
  - $R^{f^*}_\emptyset$ is the transitive closure of $R^f_\emptyset$;

– $R^\Psi := \begin{cases} R^f & \text{if } \Psi = \Upsilon \\ R^{f^*} & \text{if } \Psi \in \{\Upsilon_{\mathsf{e}}, \Upsilon_{\mathsf{p}}\}; \end{cases}$
where R^{f^*} is the transitive closure of R^f.

Note that g^f is well defined as for all $s' \in [s]$, $s \in g(a, v)$ if and only if $s' \in g(a, v)$ for all $(a, v) \in \mathcal{D}$.

Proposition 6.42. *Let $\Psi \in \{\Upsilon, \Upsilon_{\mathsf{e}}, \Upsilon_{\mathsf{p}}\}$. If $\mathcal{M} \in \Psi$, then $\mathcal{M}^\Psi \in \Psi$.*

Proposition 6.43. *The domain $\mathbb{S}^f$ of the auxiliary model $\mathcal{M}^\Psi$ contains at most $2^{|\Sigma \cup \mathcal{D}|}$ elements.*

Proof. Define the map $\Xi : \mathbb{S}^f \to 2^{\Sigma \cup \mathcal{D}}$ such that

$$\Xi([s]) = \{\beta \in \Sigma \cup \mathcal{D} : \mathcal{M}, s \mid\models \beta\}.$$

Since Ξ is injective, $|\mathbb{S}^f|$ is less than or equal to $2^{|\Sigma \cup \mathcal{D}|}$.
The following result can be proved by induction on the number of connectives in β.

Proposition 6.44 (Filtration Theorem)**.** *For all wffs $\beta \in \Sigma \cup \mathcal{D}$, and all elements $s \in \mathbb{S}$,*

$$\mathcal{M}, s \mid\models \beta \iff \mathcal{M}^\Psi, [s] \mid\models \beta.$$

Using Proposition 6.42, 6.43 and 6.44, we finally obtain the following result.

Theorem 6.45 (Finite Model Property)**.** *Let $\Psi \in \{\Upsilon, \Upsilon_{\mathsf{e}}, \Upsilon_{\mathsf{p}}\}$. Let α be a wff and Σ be the set of all sub-wffs of α. If α is satisfiable in the class Ψ, then it is satisfiable in a finite auxiliary model belonging to the class Ψ with at most $2^{|\Sigma \cup \mathcal{D}|}$ elements.*

6.7.2. Classes of auxiliary models with the property P1

The standard filtration technique is not applicable in this case as the filtration model in Definition 6.41 does not preserve the property P1. Therefore, we use the technique given in [13]. Let us first note the following theorem.

Theorem 6.46. *Let (Λ, Ψ) be a tuple consisting of a modal system Λ and a class Ψ of auxiliary models from the same row of Table 6.9. Then, for any wff $\alpha \in \mathcal{L}(\Box, \Box_\emptyset)$, α is satisfiable in the class Ψ if and only if α is satisfiable in the canonical auxiliary model $\mathcal{M}^\Lambda$.*

Consider the following sets corresponding to a wff α.

$$\Theta'_\alpha = \{\beta : \beta \text{ is a sub-wff of } \alpha\};$$

$$\Theta^\neg_\alpha = \Theta'_\alpha \cup \{\neg\beta : \beta \in \Theta'_\alpha\};$$

$$\Theta^\wedge_\alpha \text{ is } \Theta^\neg_\alpha \text{ together with all finite conjunctions of distinct elements of } \Theta^\neg_\alpha;$$

$$\Theta(\alpha) = \Theta^\wedge_\alpha \cup \{\Diamond_\emptyset\beta : \beta \in \Theta^\wedge_\alpha\};$$

$$\Theta^{\Diamond_\emptyset}_\alpha = \Theta(\alpha) \setminus \Theta^\wedge_\alpha.$$

Note that all these sets are finite and closed under sub-wffs. Up to equivalence, $\Theta^\neg_\alpha$ is closed under negation, $\Theta^\wedge_\alpha$ under $\wedge$ and $\Theta(\alpha)$ under $\Diamond_\emptyset$. We will prove the following result.

Theorem 6.47 (Finite Model Property). *Let $\Psi \in \{\Upsilon_{\mathsf{c}}, \Upsilon_{\mathsf{cp}}\}$ be a class of auxiliary models. For a wff $\alpha \in \mathcal{L}(\Box, \Box_\emptyset)$, if α is satisfiable in the class Ψ, then it is satisfiable in a finite auxiliary model $\mathcal{M}^f_\Lambda \in \Psi$ with at most $2^{|\Theta(\alpha)|}$ elements.*

Note that, due to Theorem 6.46, Theorem 6.47 follows directly from the following result.

Theorem 6.48. *Let (Λ, Ψ) be a tuple consisting of a modal system $\Lambda \in \{\text{I(C)}, \text{I(CP)}\}$ and a class $\Psi \in \{\Upsilon_{\mathsf{c}}, \Upsilon_{\mathsf{cp}}\}$ of auxiliary models from the same row of Table 6.9. For a wff $\alpha \in \mathcal{L}(\Box, \Box_\emptyset)$, if α is satisfiable in the canonical auxiliary model $\mathcal{M}^\Lambda$, then it is satisfiable in a finite auxiliary model $\mathcal{M}^f_\Lambda \in \Psi$ with at most $2^{|\Theta(\alpha)|}$ elements.*

The remaining part of the section consists of a proof sketch of the above theorem. Although the sets $\Theta'_\alpha, \Theta^\neg_\alpha, \Theta^\wedge_\alpha, \Theta(\alpha), \Theta^{\Diamond_\emptyset}_\alpha$ depends on the wff α, in the rest of the section we will drop α in these notations to avoid excessive notation.

Let Δ be a finite set of wffs, and let s be any map from a super-set of Δ into two points set $\{0, 1\}$. Define

$$\Delta_s = \bigwedge\{\beta \in \Delta : s(\beta) = 1\} \wedge \bigwedge\{\neg\beta : \beta \in \Delta, s(\beta) = 0\}.$$

We note that for all $\beta \in \Delta$, $\vdash_\Lambda \Delta_s \to \beta$, or $\vdash_\Lambda \Delta_s \to \neg\beta$.

We also note the following useful facts for all s (cf. [13]).

$$\vdash_\Lambda \Theta_s \leftrightarrow (\Theta_s^{\Diamond_\emptyset} \wedge \Theta_s^{\wedge}), \tag{6.32}$$

$$\vdash_\Lambda \Theta_s^{\wedge} \leftrightarrow \Theta'_s, \text{ provided } \Theta_s \text{ is } \Lambda\text{-consistent}, \tag{6.33}$$

$$\vdash_\Lambda \Theta_s^{\Diamond_\emptyset} \wedge \Diamond_\emptyset \Theta_s^{\wedge} \to \Diamond_\emptyset(\Theta_s^{\Diamond_\emptyset} \wedge \Theta_s^{\wedge}). \tag{6.34}$$

For a finite set Δ of wffs and $\Gamma \in \mathbb{M}_\Lambda$, we will write Δ_Γ for Δ_{1_Γ}, where $1_\Gamma : \mathcal{L}(\Box, \Box_\emptyset) \to \{0,1\}$ is the characteristic function of Γ. That is,

$$1_\Gamma(\alpha) := \begin{cases} 1, & \text{if } \alpha \in \Gamma; \\ 0, & \text{otherwise.} \end{cases}$$

Consider the relation $\equiv_\Theta$ on $\mathbb{M}_\Lambda$ defined as follows:

$$\Gamma \equiv_\Theta \Delta \text{ if and only if } \Theta_\Gamma = \Theta_\Delta.$$

It is not difficult to obtain the following.

Proposition 6.49. *1.* $\Gamma \equiv_\Theta \Delta$ *if and only if* $\Gamma \cap \Theta = \Delta \cap \Theta$.

2. $\equiv_\Theta$ *is an equivalence relation.*

Now we are in a position to define the filtration model.

Definition 6.50 (Filtration Model)**.** Consider the auxiliary model $\mathcal{M}_\Lambda^f := (\mathcal{F}_\Lambda^f, m^f, g^f)$, $\mathcal{F}_\Lambda^f := (\mathbb{M}_\Lambda^f, \mathrm{R}_\Box^f, \mathrm{R}_{\Box_\emptyset}^f)$, where

- $\mathbb{M}_\Lambda^f := \{[\Gamma] : \Gamma \in \mathbb{M}_\Lambda\}$, $[\Gamma]$ is the equivalence class of Γ with respect to equivalence relation $\equiv_\Theta$;
- $([\Gamma], [\Delta]) \in \mathrm{R}_\Box^f$ if and only if there exist $\Gamma' \in [\Gamma]$ and $\Delta' \in [\Delta]$ such that $(\Gamma', \Delta') \in \mathrm{R}_\Box^\Lambda$;
- $([\Gamma], [\Delta]) \in \mathrm{R}_{\Box_\emptyset}^f$ if and only if there exist $\Gamma' \in [\Gamma]$ and $\Delta' \in [\Delta]$ such that $(\Gamma', \Delta') \in \mathrm{R}_{\Box_\emptyset}^\Lambda$;
- $m^f(p) := \{[\Gamma] : \Gamma \in m^\Lambda(p)\}$ for $p \in PV$;
- $g^f(a, v) := \{[\Gamma] : \Gamma \in g^\Lambda(a, v)\}$ for $(a, v) \in \mathcal{D}$.

Definition 6.51. A set Δ of wffs is said to be Λ *strongly closed under* $\Diamond_\emptyset$ if the following holds for all maps s and t whose domains contain Δ and co-domain is the set $\{0,1\}$:

$$\text{If } \Delta_s \wedge \Diamond_\emptyset \Delta_t \text{ is } \Lambda\text{-consistent, then } \vdash_\Lambda \Delta_s \to \Diamond_\emptyset \Delta_t.$$

Let us recall that Θ is used to denote the set $\Theta(\alpha)$. We note the following results.

Proposition 6.52. *Θ is a finite set which contains α and which is Λ strongly closed under $\Diamond_\emptyset$.*

Proof. Obviously Θ is a finite set containing α. So, it remains to prove that Θ is Λ strongly closed under $\Diamond_\emptyset$. Suppose $\Theta_s \wedge \Diamond_\emptyset\Theta_t$ is Λ-consistent and we show $\vdash_\Lambda \Theta_s \to \Diamond_\emptyset\Theta_t$.
Claim 1: s and t agree on all wffs of $\Theta^{\Diamond_\emptyset}$.
Let us first prove this claim. If s and t do not agree on all wffs of $\Theta^{\Diamond_\emptyset}$, then there exists $\Diamond_\emptyset\psi \in \Theta^{\Diamond_\emptyset}$ such that $s(\Diamond_\emptyset\psi) \neq t(\Diamond_\emptyset\psi)$. Without loss of generality, suppose $s(\Diamond_\emptyset\psi) = 1$ and $t(\Diamond_\emptyset\psi) = 0$. This gives

$$\vdash_\Lambda \Theta_s \to \Diamond_\emptyset\psi \text{ and} \tag{6.35}$$

$$\vdash_\Lambda \Theta_t \to \neg\Diamond_\emptyset\psi. \tag{6.36}$$

From (6.36), we obtain

$$\vdash_\Lambda \Diamond_\emptyset\Theta_t \to \neg\Box_\emptyset\Diamond_\emptyset\psi. \tag{6.37}$$

Also, from axiom $B(\Box_\emptyset)$ and $4(\Box_\emptyset)$, we have

$$\vdash_\Lambda \neg\Box_\emptyset\Diamond_\emptyset\psi \to \neg\Diamond_\emptyset\psi. \tag{6.38}$$

From (6.37) and (6.38), we obtain

$$\vdash_\Lambda \Diamond_\emptyset\Theta_t \to \neg\Diamond_\emptyset\psi. \tag{6.39}$$

Combining (6.35) and (6.39), we get $\vdash_\Lambda \Diamond_\emptyset\Theta_t \wedge \Theta_s \to \neg\Diamond_\emptyset\psi \wedge \Diamond_\emptyset\psi$. This shows that $\Diamond_\emptyset\Theta_t \wedge \Theta_s$ is not Λ-consistent, a contradiction. Hence s and t agree on all wffs of $\Theta^{\Diamond_\emptyset}$.
Claim 2: $\vdash_\Lambda \Theta_s \to {\Theta_t}^{\Diamond_\emptyset}$.
From (6.32), we get

$$\vdash_\Lambda \Theta_s \to {\Theta_s}^{\Diamond_\emptyset}. \tag{6.40}$$

Since s and t agrees on $\Theta^{\Diamond_\emptyset}$, by (6.40), we obtain the claim. Claim 3: Θ_t is Λ-consistent. Suppose not, then

$$\begin{aligned} & \vdash_\Lambda \neg\Theta_t \\ \Longrightarrow\ & \vdash_\Lambda \Box_\emptyset\neg\Theta_t \text{ (using N}(\Box_\emptyset)) \\ \Longrightarrow\ & \vdash_\Lambda \neg\Diamond_\emptyset\Theta_t \\ \Longrightarrow\ & \Diamond_\emptyset\Theta_t \text{ is not } \Lambda\text{-consistent.} \end{aligned}$$

This is a contradiction to the fact that $\Theta_s \wedge \Diamond_\emptyset\Theta_t$ is Λ-consistent. Hence we obtained the Claim 3.

Claim 4: $\Diamond_\emptyset\Theta'_t$ is a conjunct of $\Theta_t^{\Diamond_\emptyset}$.

Note that each conjunct of Θ'_t is a conjunct of Θ_t. We also have $\Diamond_\emptyset\Theta'_t \in \Theta$ as $\Theta'_t \in \Theta^\wedge$. If $\Diamond_\emptyset\Theta'_t$ is not a conjunct of $\Theta_t^{\Diamond_\emptyset}$, then, since $\Diamond_\emptyset\Theta'_t \in \Theta$, $\neg\Diamond_\emptyset\Theta'_t$ becomes a conjunct of $\Theta_t^{\Diamond_\emptyset}$. Thus, we obtain $\neg\Diamond_\emptyset\Theta'_t \wedge \Theta'_t$ as a conjunct of Θ_t and hence, we get

$$\begin{aligned} & \vdash_\Lambda \Theta_t \to \neg\Diamond_\emptyset\Theta'_t \wedge \Theta'_t \\ \Longrightarrow\ & \vdash_\Lambda \Diamond_\emptyset\Theta_t \to \Diamond_\emptyset(\neg\Diamond_\emptyset\Theta'_t \wedge \Theta'_t) \\ \Longrightarrow\ & \vdash_\Lambda \Diamond_\emptyset\Theta_t \to \Diamond_\emptyset\neg\Diamond_\emptyset\Theta'_t \wedge \Diamond_\emptyset\Theta'_t \\ \Longrightarrow\ & \vdash_\Lambda \Diamond_\emptyset\Theta_t \to \neg\Diamond_\emptyset\Theta'_t \wedge \Diamond_\emptyset\Theta'_t \ (\because \vdash_\Lambda \Diamond_\emptyset\neg\Diamond_\emptyset\gamma \to \neg\Diamond_\emptyset\gamma) \\ \Longrightarrow\ & \vdash_\Lambda \Diamond_\emptyset\Theta_t \to \bot. \\ \Longrightarrow\ & \Diamond_\emptyset\Theta_t \text{ is not } \Lambda\text{-consistent.} \end{aligned}$$

This is not possible and hence $\Diamond_\emptyset\Theta'_t$ is a conjunct of $\Theta_t^{\Diamond_\emptyset}$. This completes the proof of Claim 4.

Claim 5: $\vdash_\Lambda \Theta_t^{\Diamond_\emptyset} \to \Diamond_\emptyset\Theta_t$.

From (6.32) and (6.34), we have

$$\vdash_\Lambda \Theta_t^{\Diamond_\emptyset} \wedge \Diamond_\emptyset\Theta_t^\wedge \to \Diamond_\emptyset\Theta_t. \tag{6.41}$$

Since Θ_t is Λ-consistent (Claim 3), from (6.33), we get

$$\vdash_\Lambda \Diamond_\emptyset\Theta_t^\wedge \leftrightarrow \Diamond_\emptyset\Theta'_t \tag{6.42}$$

From (6.41) and (6.42), we obtain

$$\vdash_\Lambda \Theta_t^{\Diamond_\emptyset} \wedge \Diamond_\emptyset \Theta'_t \to \Diamond_\emptyset \Theta_t. \tag{6.43}$$

Since $\Diamond_\emptyset \Theta'_t$ is a conjunct of $\Theta_t^{\Diamond_\emptyset}$ (Claim 4), from (6.43), we get

$$\vdash_\Lambda \Theta_t^{\Diamond_\emptyset} \to \Diamond_\emptyset \Theta_t. \tag{6.44}$$

This completes the proof of Claim 5.

From claim 2 and (6.44), we have

$$\vdash_\Lambda \Theta_s \to \Diamond_\emptyset \Theta_t.$$

This completes the proof of the proposition. □

Proposition 6.53. *1. If $\Diamond\Theta_\Gamma \in \Delta'$, then there exists $\Gamma' \in [\Gamma]$ such that $(\Delta', \Gamma') \in \mathrm{R}_\Box^\Lambda$ and $\Theta_\Gamma \in \Gamma'$.*

2. If $\Diamond_\emptyset\Theta_\Gamma \in \Delta'$, then there exists $\Gamma' \in [\Gamma]$ such that $(\Delta', \Gamma') \in \mathrm{R}_{\Box_\emptyset}^\Lambda$ and $\Theta_\Gamma \in \Gamma'$.

Proof. We only provide the proof for Item 2 as Item 1 can be proved in a similar way. Let $\Diamond_\emptyset\Theta_\Gamma \in \Delta'$. By Existence Lemma 6.15, we obtain Γ' such that $(\Delta', \Gamma') \in \mathrm{R}_{\Box_\emptyset}^\Lambda$ and $\Theta_\Gamma \in \Gamma'$. So, it remains to show that $\Gamma' \in [\Gamma]$, that is, $\Theta_\Gamma = \Theta_{\Gamma'}$. Since Θ is finite set, let us assume $\Theta = \{\alpha_1, \alpha_2, \ldots, \alpha_m, \alpha_{m+1}, \ldots, \alpha_n\}$. Suppose β is a conjunct of $\Theta_{\Gamma'}$ and we show that β is a conjunct of Θ_Γ. Note that either (i) β is α_i for some $i \in \{1, 2, \ldots, n\}$, or (ii) β is $\neg\alpha_i$ for some $i \in \{1, 2, \ldots, n\}$. Let us first consider the case when β is some α_i. It is enough to show that $\beta \in \Gamma$. Suppose not, then $\neg\alpha_i \in \Gamma$ and this implies $\neg\alpha_i$ is a conjunct of Θ_Γ. Therefore, since $\Theta_\Gamma \in \Gamma'$, we obtain $\neg\alpha_i \in \Gamma'$. This is not possible as $\alpha_i \in \Gamma'$. Hence $\beta \in \Gamma$. Similarly we can show that β is a conjunct of Θ_Γ when β is $\neg\alpha_i$ for some i.

Next, we assume β to be a conjunct of Θ_Γ and we show that β is a conjunct of $\Theta_{\Gamma'}$. Since $\Theta_\Gamma \in \Gamma'$, in this case we obtain $\beta \in \Gamma'$. Hence, by the definition of $\Theta_{\Gamma'}$, it follows that β is a conjunct of $\Theta_{\Gamma'}$.

Thus, we have shown that $\Theta_\Gamma = \Theta_{\Gamma'}$, and hence $\Gamma' \in [\Gamma]$. □

Proposition 6.54. *The following are equivalent:*

1. $([\Gamma], [\Delta]) \in \mathrm{R}_{\Box_\emptyset}^f$;
2. $\Theta_\Gamma \wedge \Diamond_\emptyset\Theta_\Delta$ *is consistent;*

3. $\vdash_\Lambda \Theta_\Gamma \to \Diamond_\emptyset \Theta_\Delta$.

Proof. (1) $\to$ (2): Suppose $([\Gamma],[\Delta]) \in \mathrm{R}^f_{\Box_\emptyset}$, then there exist $\Gamma' \in [\Gamma]$ and $\Delta' \in [\Delta]$ such that $(\Gamma', \Delta') \in \mathrm{R}^\Lambda_{\Box_\emptyset}$. Now $\Gamma' \in [\Gamma]$ and definition of Θ_Γ gives $\Theta_\Gamma \in \Gamma'$, similarly $\Theta_\Delta \in \Delta'$. Since $\Theta_\Delta \in \Delta'$ and $(\Gamma', \Delta') \in \mathrm{R}^\Lambda_{\Box_\emptyset}$, $\Diamond_\emptyset \Theta_\Delta \in \Gamma'$. Hence, we have $\Theta_\Gamma \wedge \Diamond_\emptyset \Theta_\Delta \in \Gamma'$ which means $\Theta_\Gamma \wedge \Diamond_\emptyset \Theta_\Delta$ is Λ-consistent as Γ' is a maximal consistent set.
(2) $\to$ (3) holds by Proposition 6.52.
(3) $\to$ (1): Suppose $\vdash_\Lambda \Theta_\Gamma \to \Diamond_\emptyset \Theta_\Delta$ and we show $([\Gamma],[\Delta]) \in \mathrm{R}^f_{\Box_\emptyset}$. Using $\vdash_\Lambda \Theta_\Gamma \to \Diamond_\emptyset \Theta_\Delta$ and $\Theta_\Gamma \in \Gamma$, we obtain $\Diamond_\emptyset \Theta_\Delta \in \Gamma$. So, by Lemma 6.53, there exists a maximal consistent set $\Delta' \in [\Delta]$ such that $(\Gamma, \Delta') \in \mathrm{R}^\Lambda_{\Box_\emptyset}$ and $\Theta_\Delta \in \Delta'$, and hence $([\Gamma],[\Delta]) \in \mathrm{R}^f_{\Box_\emptyset}$. This completes the proof. $\Box$

Proposition 6.55. *The following are equivalent:*

1. $([\Gamma],[\Delta]) \in \mathrm{R}^f_{\Box}$*;*
2. $\Theta_\Gamma \wedge \Diamond \Theta_\Delta$ *is* Λ*-consistent.*

Let $\mathcal{M}_1 := (\mathcal{F}_1, m_1, g_1)$ and $\mathcal{M}_2 := (\mathcal{F}_2, m_2, g_2)$ be two auxiliary models, where $\mathcal{F}_i := (\mathbb{S}_i, R^i, R^i_\emptyset)$, $i = 1, 2$. For $s \in \mathbb{S}_i$, let us use $th_{\mathcal{M}_i}(s)$ to denote the set $\{\gamma : \mathcal{M}_i, s \mid\models \gamma\}$ of wffs. Note that $th_{\mathcal{M}_i}(s)$ is a Λ-maximal consistent set.

Consider the model $\mathcal{M}_1 \oplus \mathcal{M}_2 := (\mathcal{F}, m, g)$, $\mathcal{F} := (\mathbb{S}, R, R_\emptyset)$, where

- $\mathbb{S}$ is a disjoint union of $\mathbb{S}_1$ and $\mathbb{S}_2$;
- $(s,t) \in R$ if and only if $(s,t) \in R^1$ or $(s,t) \in R^2$ or there exist $s_1 \in \mathbb{S}_1$ and $s_2 \in \mathbb{S}_2$ such that $(s, s_1) \in R^1$ and $(s_2, t) \in R^2$, and

$$\Theta \cap th_{\mathcal{M}_1}(s_1) = \Theta \cap th_{\mathcal{M}_2}(s_2); \tag{6.45}$$

- $(s,t) \in R_\emptyset$ if and only if $(s,t) \in R^1_\emptyset$ or $(s,t) \in R^2_\emptyset$;
- $m(p) := m_1(p) \cup m_2(p)$ for $p \in PV$;
- $g(a,v) := g_1(a,v) \cup g_2(a,v)$ for $(a,v) \in \mathcal{D}$.

Note that an equivalent way to write (6.45) is $\Theta_{th_{\mathcal{M}_1}(s_1)} = \Theta_{th_{\mathcal{M}_2}(s_2)}$. We will require the following two lemmas for the class Υ_{cp}.

Lemma 6.56. *If* $\mathcal{M}_1, \mathcal{M}_2 \in \Upsilon_{\mathsf{cp}}$*, then* $\mathcal{M}_1 \oplus \mathcal{M}_2 \in \Upsilon_{\mathsf{cp}}$.

Proof. Since $R^1_\emptyset$ and $R^2_\emptyset$ are equivalence relations, equivalence of $R_\emptyset$ follows directly from its definition. Reflexivity of R is also obvious. So, we prove the transitivity of R. Let us take $(s,t) \in R$, $(t,r) \in R$ and we show that $(s,r) \in R$. If $s,t,r \in \mathbb{S}_1$ or $s,t,r \in \mathbb{S}_2$, then we obtain $(s,r) \in R$ as R^1 and R^2 are transitive relations. The non-trivial case is when $s \in \mathbb{S}_1, r \in \mathbb{S}_2$ and t belongs to $\mathbb{S}_1$ or $\mathbb{S}_2$. Let us consider the case when t belongs to $\mathbb{S}_2$. Since $(s,t) \in R$, there exist $u \in \mathbb{S}_1, v \in \mathbb{S}_2$ such that $(s,u) \in R^1, (v,t) \in R^2$ and $\Theta \cap th_{\mathcal{M}_1}(u) = \Theta \cap th_{\mathcal{M}_2}(v)$. Since R^2 is transitive and $(v,t) \in R^2, (t,r) \in R^2$, we obtain $(v,r) \in R^2$. Thus, we obtain $(s,u) \in R^1, (v,r) \in R^2$ and $\Theta \cap th_{\mathcal{M}_1}(u) = \Theta \cap th_{\mathcal{M}_2}(v)$ and this gives $(s,r) \in R$. We can similarly prove $(s,r) \in R$ when t belongs to $\mathbb{S}_1$.

It remains to show that the auxiliary model $\mathcal{M}_1 \oplus \mathcal{M}_2$ has property P1 to complete the proof of the lemma. So, let us assume that $(s,t) \in R_\emptyset$, $(t,r) \in R$ and we show that there exists $d' \in \mathbb{S}$ such that $(s,d') \in R$ and $(d',r) \in R_\emptyset$ (cf. Figure 6.6). If $s,t,r \in \mathbb{S}_1$ or $s,t,r \in \mathbb{S}_2$, then we obtain such a d' using the fact that $\mathcal{M}_1$ and $\mathcal{M}_2$ have property P1. So, let us consider the case when $s,t \in \mathbb{S}_1$ and $r \in \mathbb{S}_2$. Since $(t,r) \in R$, there exist $c \in \mathbb{S}_1$ and $d \in \mathbb{S}_2$ such that $(t,c) \in R^1, (d,r) \in R^2$, and $\Theta \cap th_{\mathcal{M}_1}(c) = \Theta \cap th_{\mathcal{M}_2}(d)$. Applying P1 property for $\mathcal{M}_1$, we obtain $e \in \mathbb{S}_1$ such that $(s,e) \in R^1$, $(e,c) \in R^1_\emptyset$. Since $(e,c) \in R^1_\emptyset$ and $R^1_\emptyset$ is symmetric relation, we obtain $(c,e) \in R^1_\emptyset$. This, in turn, gives

$$(th_{\mathcal{M}_1}(c), th_{\mathcal{M}_1}(e)) \in \mathrm{R}^{\Lambda}_{\Box_\emptyset}. \tag{6.46}$$

Let us use Θ_c and Θ_e to denote wffs $\Theta_{th_{\mathcal{M}_1}(c)}$ and $\Theta_{th_{\mathcal{M}_1}(e)}$, respectively. From (6.46) and Proposition 6.54, we obtain

$$\vdash_\Lambda \Theta_c \rightarrow \Diamond_\emptyset \Theta_e. \tag{6.47}$$

Since $\mathcal{M}_1, c \Vdash \Theta_c$ and $\Theta \cap th_{\mathcal{M}_1}(c) = \Theta \cap th_{\mathcal{M}_2}(d)$, we get $\mathcal{M}_2, d \Vdash \Theta_c$. Therefore, from (6.47), we obtain

$$\mathcal{M}_2, d \Vdash \Diamond_\emptyset \Theta_e. \tag{6.48}$$

Thus, there exists $c' \in \mathbb{S}_2$ such that $(d,c') \in R^2_\emptyset$ and $\mathcal{M}_2, c' \Vdash \Theta_e$. From $\mathcal{M}_2, c' \Vdash \Theta_e$, we obtain $\Theta \cap th_{\mathcal{M}_2}(c') = \Theta \cap th_{\mathcal{M}_1}(e)$. Since $\mathrm{R}^2_\emptyset$ is symmetric, we have $(c',d) \in R^2_\emptyset$. Further, we also have $(d,r) \in R^2$ and hence using P1 property of $\mathcal{M}_2$, we obtain a $d' \in \mathbb{S}_2$ such that $(c',d') \in R^2$, $(d',r) \in R^2_\emptyset$. We thus have $(s,e) \in R^1, (c',d') \in R^2$ and $\Theta \cap th_{\mathcal{M}_2}(c') = \Theta \cap th_{\mathcal{M}_1}(e)$, and therefore we get $(s,d') \in R$. We also have $(d',r) \in R_\emptyset$ as $(d',r) \in R^2_\emptyset$ (cf. Figure 6.6). This completes the proof.

□

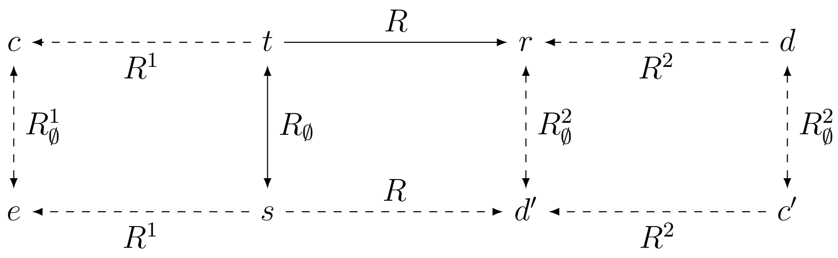

Figure 6.6

Lemma 6.57. *Let $\mathcal{M}_1, \mathcal{M}_2 \in \Upsilon_{\mathsf{cp}}$. For each wff $\psi \in \Theta$, each $i \in \{1, 2\}$, and each $s \in \mathbb{S}_i$, we have*

$$\mathcal{M}_i, s \Vdash \psi \textit{ if and only if } \mathcal{M}_1 \oplus \mathcal{M}_2, s \Vdash \psi.$$

Proof. We use induction on the number of connectives in the wff $\psi \in \Theta$. Obviously, the result holds when ψ is a propositional variable or a descriptor. We only provide the arguments when ψ is of the form $\Box_\emptyset$ or $\Box$. Boolean cases can be proved easily.

Let us first consider the case when ψ is of the form $\Box_\emptyset\phi$. Then, we have

$$\begin{aligned} &\mathcal{M}_i, s \Vdash \Box_\emptyset\phi \\ \iff &\mathcal{M}_i, t \Vdash \phi \text{ for all } t \in \mathbb{S}_i \text{ with } (s,t) \in R^i_\emptyset \\ \iff &\mathcal{M}_1 \oplus \mathcal{M}_2, t \Vdash \phi \text{ for all } t \in \mathbb{S} \text{ with } (s,t) \in R_\emptyset \\ \iff &\mathcal{M}_1 \oplus \mathcal{M}_2, s \Vdash \Box_\emptyset\phi. \end{aligned}$$

Next, consider the case when ψ is of the form $\Box\phi$. Let us first assume that $\mathcal{M}_1, s \Vdash \Box\phi$ and we show that $\mathcal{M}_1 \oplus \mathcal{M}_2, s \Vdash \Box\phi$. Let us take $t \in \mathbb{S}$ with $(s,t) \in R$ and we show that $\mathcal{M}_1 \oplus \mathcal{M}_2, t \Vdash \phi$. If $t \in \mathbb{S}_1$, then we can proceed in the same way as above to obtain $\mathcal{M}_1 \oplus \mathcal{M}_2, t \Vdash \phi$. So, let $t \in \mathbb{S}_2$. Since $(s,t) \in R$, there exists $u \in \mathbb{S}_1$ and $v \in \mathbb{S}_2$ such that $(s,u) \in R^1, (v,t) \in R^2$ and $\Theta \cap th_{\mathcal{M}_1}(u) = \Theta \cap th_{\mathcal{M}_2}(v)$. Since $\mathcal{M}_1, s \Vdash \Box\phi$, we obtain $\mathcal{M}_1, s \Vdash \Box\Box\phi$ ($\because R^1$ is transitive) and hence $\mathcal{M}_1, u \Vdash \Box\phi$ ($\because (s,u) \in R^1$). Using $\Theta \cap th_{\mathcal{M}_1}(u) = \Theta \cap th_{\mathcal{M}_2}(v)$ and $\Box\phi \in \Theta$, we get $\mathcal{M}_2, v \Vdash \Box\phi$ and hence $\mathcal{M}_2, t \Vdash \phi$. Therefore, by induction hypothesis, we obtain $\mathcal{M}_1 \oplus \mathcal{M}_2, t \Vdash \phi$ and hence $\mathcal{M}_1 \oplus \mathcal{M}_2, s \Vdash \Box\phi$.

Next, consider $\mathcal{M}_2, s \Vdash \Box\phi$, and we show that $\mathcal{M}_1 \oplus \mathcal{M}_2, s \Vdash \Box\phi$. Let $(s,t) \in R$. Then, as $s \in \mathbb{S}_2$, we must have $(s,t) \in R^2$ and hence we get $\mathcal{M}_2, t \Vdash \phi$. Therefore, by induction hypothesis, we get $\mathcal{M}_1 \oplus \mathcal{M}_2, t \Vdash \phi$.

The reverse implication is from the direct sum to one of its component and therefore it follows directly. $\square$

Proposition 6.58. *Let* (Λ, Ψ) *be a tuple consisting of a modal system* $\Lambda \in \{\mathrm{I(C)}, \mathrm{I(CP)}\}$ *and a class* $\Psi \in \{\Upsilon_{\mathsf{c}}, \Upsilon_{\mathsf{cp}}\}$ *of auxiliary models from the same row of Table 6.9. Then* $\mathcal{M}^f_\Lambda \in \Psi$*.*

Proof. We provide the proof when Λ is the modal system I(CP).
Claim 1: $\mathrm{R}^f_{\square_\emptyset}$ is an equivalence relation.
The proof of reflexivity of $\mathrm{R}^f_{\square_\emptyset}$ is not difficult, we omit it. Let us assume $([\Gamma], [\Delta]) \in \mathrm{R}^f_{\square_\emptyset}$ and we show $([\Delta], [\Gamma]) \in \mathrm{R}^f_{\square_\emptyset}$. Since $([\Gamma], [\Delta]) \in \mathrm{R}^f_{\square_\emptyset}$, there exist $\Gamma' \in [\Gamma]$ and $\Delta' \in [\Delta]$ such that $(\Gamma', \Delta') \in \mathrm{R}^\Lambda_{\square_\emptyset}$. Using symmetry of $\mathrm{R}^\Lambda_{\square_\emptyset}$, we obtain $(\Delta', \Gamma') \in \mathrm{R}^\Lambda_{\square_\emptyset}$ and hence $([\Delta], [\Gamma]) \in \mathrm{R}^f_{\square_\emptyset}$. For transitivity, let us take $([\Gamma], [\Delta]) \in \mathrm{R}^f_{\square_\emptyset}, ([\Delta], [\Delta_0]) \in \mathrm{R}^f_{\square_\emptyset}$ and we show $([\Gamma], [\Delta_0]) \in \mathrm{R}^f_{\square_\emptyset}$. Using Proposition 6.54, we get

$$\vdash_\Lambda \Theta_\Gamma \to \lozenge_\emptyset \Theta_\Delta \text{ and} \tag{6.49}$$

$$\vdash_\Lambda \Theta_\Delta \to \lozenge_\emptyset \Theta_{\Delta_0}. \tag{6.50}$$

From (6.50), we obtain

$$\vdash_\Lambda \lozenge_\emptyset \Theta_\Delta \to \lozenge_\emptyset \lozenge_\emptyset \Theta_{\Delta_0}. \tag{6.51}$$

Using (Taut) on (6.49) and (6.51), we obtain

$$\vdash_\Lambda \Theta_\Gamma \to \lozenge_\emptyset \lozenge_\emptyset \Theta_{\Delta_0}. \tag{6.52}$$

From axiom $4(\square_\emptyset)$, we have

$$\vdash_\Lambda \lozenge_\emptyset \lozenge_\emptyset \Theta_{\Delta_0} \to \lozenge_\emptyset \Theta_{\Delta_0} \tag{6.53}$$

Using (Taut) on (6.52) and (6.53), we obtain

$$\vdash_\Lambda \Theta_\Gamma \to \lozenge_\emptyset \Theta_{\Delta_0}. \tag{6.54}$$

Therefore, by Proposition 6.54, we get $([\Gamma], [\Delta_0]) \in \mathrm{R}^f_{\square_\emptyset}$.
Claim 2: $\mathcal{M}^f_\Lambda$ has P1 property.
Suppose $([\Gamma], [\Delta]) \in \mathrm{R}^f_{\square_\emptyset}, ([\Delta], [\Delta_0]) \in \mathrm{R}^f_{\square}$ and we need to show that there exist $[\Gamma_0] \in \mathcal{M}^f_\Lambda$

such that $([\Gamma],[\Gamma_0]) \in \mathrm{R}^f_\Box$ and $([\Gamma_0],[\Delta_0]) \in \mathrm{R}^f_{\Box_\emptyset}$. Since $([\Gamma],[\Delta]) \in \mathrm{R}^f_{\Box_\emptyset}$ and $\mathrm{R}^f_{\Box_\emptyset}$ is symmetric, we get $([\Delta],[\Gamma]) \in \mathrm{R}^f_{\Box_\emptyset}$. Using Proposition 6.54, we obtain

$$\vdash_\Lambda \Theta_\Delta \to \Diamond_\emptyset \Theta_\Gamma. \tag{6.55}$$

Due to $([\Delta],[\Delta_0]) \in \mathrm{R}^f_\Box$, there exist a $\Delta' \in [\Delta]$ and a $\Delta'_0 \in [\Delta_0]$ such that $(\Delta',\Delta'_0) \in \mathrm{R}^\Lambda_\Box$. Since $\Theta_\Delta \in \Delta'$, by (6.55) we obtain $\Diamond_\emptyset\Theta_\Gamma \in \Delta'$. By the use of Lemma 6.53, we get an $\Gamma' \in [\Gamma]$ containing Θ_Γ such that $(\Delta',\Gamma') \in \mathrm{R}^\Lambda_{\Box_\emptyset}$. Again, using symmetry of $\mathrm{R}^\Lambda_{\Box_\emptyset}$, we get $(\Gamma',\Delta') \in \mathrm{R}^\Lambda_{\Box_\emptyset}$. We have $(\Gamma',\Delta') \in \mathrm{R}^\Lambda_{\Box_\emptyset}$ and $(\Delta',\Delta'_0) \in \mathrm{R}^\Lambda_\Box$, now using the fact that $\mathcal{M}^\Lambda \in \Upsilon_{\mathsf{cp}}$, there exist $\Gamma_0 \in \mathbb{M}_\Lambda$ such that $(\Gamma',\Gamma_0) \in \mathrm{R}^\Lambda_\Box$ and $(\Gamma_0,\Delta'_0) \in \mathrm{R}^\Lambda_{\Box_\emptyset}$. Hence, we get $([\Gamma],[\Gamma_0]) \in \mathrm{R}^f_\Box$ and $([\Gamma_0],[\Delta_0]) \in \mathrm{R}^f_{\Box_\emptyset}$.

Claim 3: $\mathrm{R}^f_\Box$ is reflexive and transitive relation.

Reflexivity of $\mathrm{R}^f_\Box$ is obvious. Let us prove transitivity. Let

$$([\Gamma],[\Delta]) \in \mathrm{R}^f_\Box \text{ and } ([\Delta],[\Delta_0]) \in \mathrm{R}^f_\Box \tag{6.56}$$

and we show that $([\Gamma],[\Delta_0]) \in \mathrm{R}^f_\Box$. Due to Proposition 6.55, it is enough to show that $\Theta_\Gamma \wedge \Diamond\Theta_{\Delta_0}$ is Λ-consistent wff. From (6.56) and Proposition 6.55, we obtain $\Theta_\Gamma \wedge \Diamond\Theta_\Delta$ and $\Theta_\Delta \wedge \Diamond\Theta_{\Delta_0}$ to be Λ-consistent wffs. For $i = 1,2$, let $\mathcal{M}_i := (\mathcal{F}_i, m_i, g_i)$, $\mathcal{F}_i := (W_i, R^i, R^i_\emptyset)$, be auxiliary models from the class Υ_{cp} such that

$$\mathcal{M}_1, s_1^* \Vdash \Theta_\Gamma \wedge \Diamond\Theta_\Delta \tag{6.57}$$

$$\mathcal{M}_2, s_2^* \Vdash \Theta_\Delta \wedge \Diamond\Theta_{\Delta_0}. \tag{6.58}$$

From (6.57), we obtain

$$\mathcal{M}_1, s_1^* \Vdash \Theta_\Gamma \text{ and}$$

$$\mathcal{M}_1, t_1 \Vdash \Theta_\Delta \text{ for some } t_1 \text{ such that } (s_1^*, t_1) \in R^1.$$

Therefore, using Lemma 6.57, we get

$$\mathcal{M}_1 \oplus \mathcal{M}_2, s_1^* \Vdash \Theta_\Gamma \text{ and} \tag{6.59}$$

$$\mathcal{M}_1 \oplus \mathcal{M}_2, t_1 \Vdash \Theta_\Delta. \tag{6.60}$$

Similarly, from (6.58), we obtain

$$\mathcal{M}_1 \oplus \mathcal{M}_2, s_2^* \Vdash \Theta_\Delta \text{ and} \tag{6.61}$$

$$\mathcal{M}_1 \oplus \mathcal{M}_2, t_2 \Vdash \Theta_{\Delta_0} \text{ for some } t_2 \text{ such that } (s_2^*, t_2) \in R^2. \tag{6.62}$$

From (6.60) and (6.61), we get

$$\Theta \cap th_{\mathcal{M}_1 \oplus \mathcal{M}_2}(t_1) = \Theta \cap th_{\mathcal{M}_1 \oplus \mathcal{M}_2}(s_2^*). \tag{6.63}$$

We also have $(s_1^*, t_1) \in R^1, (s_2^*, t_2) \in R^2$ and therefore from (6.63), we get $(s_1^*, t_2) \in R$. Thus, from (6.59) and (6.62), we get $\mathcal{M}_1 \oplus \mathcal{M}_2, s_1^* \Vdash \Theta_\Gamma \wedge \Diamond\Theta_{\Delta_0}$. Due to Lemma 6.56, we also have $\mathcal{M}_1 \oplus \mathcal{M}_2 \in \Upsilon_{\mathsf{cp}}$. Thus we obtain $\Theta_\Gamma \wedge \Diamond\Theta_{\Delta_0}$ as a Λ-consistent wff. □

Remark 6.59. Note that we do not require the construction $\mathcal{M}_1 \oplus \mathcal{M}_2$ and Lemmas 6.56 and 6.57 in the above proof for the modal system I(C). In fact, the proof ends for this modal system with the proof of Claim 2.

Proposition 6.60 (Filtration Theorem)**.** *For all wffs $\beta \in \Theta$, and all elements $\Gamma \in \mathbb{M}_\Lambda$,*

$$\mathcal{M}^\Lambda, \Gamma \mid\models \beta \textit{ if and only if } \mathcal{M}_\Lambda^f, [\Gamma] \mid\models \beta.$$

Proposition 6.61. *The domain $\mathbb{M}_\Lambda^f$ of $\mathcal{M}_\Lambda^f$ contains at most $2^{|\Theta|}$ elements.*

Proof. Define the map $\Xi : \mathbb{M}_\Lambda^f \to 2^\Theta$, where

$$\Xi([\Gamma]) := \Gamma \cap \Theta.$$

Since Ξ is injective, $\mathbb{M}_\Lambda^f$ contains at most $2^{|\Theta|}$ elements. □

Theorem 6.48 now follows from Propositions 6.58, 6.60 and 6.61.

6.8. Conclusion

The possible world semantics of epistemic logic is extended to introduce the notion of *possible-worlds information systems* (PWIS), where each state is assigned an information system. We proposed a modal logic with semantics based on PWISs that can be used to reason about the approximation of concepts as well as knowledge of the agent. The essential issues of the proposed logic viz. sound and complete modal systems concerning various classes of models are also discussed. We have obtained the completeness theorem for all the classes of proposed models that do not have the constant domain restriction. The situation is not so nice for constant domain models. We are able to obtain the completeness theorem for the constant domain class Ω_{ce} where the relation R on states is equivalence, but our technique does not work for other classes of constant domain models. The decidability of the validity problem for the proposed constant domain classes of models are still open, although we have proved the decidability of the modal systems I(C), I(CP).

CHAPTER 7

SUMMARY

In literature, one can find several interpretations of the unary modal operator of the basic modal language. In Chapters 3 and 4, we have studied its interpretations based on structures inherited from rough set theory. Table 7.1 provides a summary of the results on axiomatization for these semantics.

In Chapter 5, we considered the modal language with two unary modal operators. The semantics of this modal language is defined over subset approximation structures. In the proposed semantics, one modal operator captures the lower approximation relative to subsets of the domain, whereas the other modal operator captures the quantification over subsets of the domain. Sound and complete modal systems for various classes of SASs are obtained. It will be interesting to determine the modal systems for the necessity and possibility approximation operators proposed in this chapter.

In order to study knowledge operator and approximation operators relative to different attributes, we considered a multi-modal language in Chapter 6. The semantics is based on the notion of possible-worlds information systems proposed in this dissertation. Modal systems for the classes of models Ω, Ω_{p}, Ω_{e}, and Ω_{ce} are obtained, but modal systems for the classes of models Ω_{c} and Ω_{cp} are still open.

We want to add here that in this dissertation, apart from axiomatization, a few other issues pertaining to the proposed semantics like invariance, definability, and decidability are also explored.

Interpretation of the unary modal operator	Classes of frames	Modal Systems	Remark
Strong lower approximation	M	K	-
	$\mathsf{M_r}$	T	-
	$\mathsf{M_s}$	B	-
	$\mathsf{M_t}$	K	-
	$\mathsf{M_{rs}}$	KTB	-
	$\mathsf{M_{rt}}$	T	-
	$\mathsf{M_e}$	KTB	-
	$\mathsf{M_{st}}$	Not known	Modal system lies between B and KB4
Weak lower approximation	M	EMN	-
	$\mathsf{M_r}$	EMNT	-
	$\mathsf{M_t}$	$\mathrm{EMN4}^0$	-
	$\mathsf{M_{rt}}$	EMNT4	-
	$\mathsf{M_s}$	Not known	Modal system lies between EMN and B
	$\mathsf{M_{rs}}$	Not known	Modal system lies between EMNT and KTB
	$\mathsf{M_e}$	Not known	Modal system lies between EMNT4 and S5
	$\mathsf{M_{st}}$	Not known	Modal system lies between $\mathrm{EMN4}^0$ and KB4
Lower approximation based on covering systems P_3, C_1 and C_{Gr}	-	ML_{C_1}	-
Lower approximation based on covering system P_4	-	$\mathbb{CLS}4B$	-

Table 7.1. Summary of the results

Lower approximation based on covering systems P_2, C_3, $C_*, C_-, C_\#, C_@, C_+$, and $C_\%$	-	Not known	-
Boundary operator based on covering systems C_2 and C_5	-	$\mathbb{CLS}4$	-
Boundary operator based on covering system P_1	-	$\mathbb{CLTB}$	-
Boundary operator based on covering systems P_2, C_3, $C_*, C_-, C_\#, C_@, C_+$, and $C_\%$	-	Not known	-

Table 7.1 continued

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