ANALYTIC AND MAPPING PROPERTIES OF CERTAIN ANALYTIC FUNCTIONS WITH APPLICATIONS

Ph. D. Thesis

by

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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **ANALYTIC AND MAPPING PROPERTIES OF CERTAIN ANALYTIC FUNCTIONS WITH APPLICATIONS** in the partial fulfillment of the requirements for the award of the degree of **DOCTOR OF PHILOSOPHY** and submitted in the **DISCIPLINE OF MATHEMATICS, Indian Institute of Technology Indore,** is an authentic record of my own work carried out during the time period from July 2011 to May 2016 under the supervision of Dr. Swadesh Kumar Sahoo, Assistant Professor, Indian Institute of Technology Indore.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other institute.

Signature of the student with date

(SARITA AGRAWAL)

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This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

Signature of Thesis Supervisor with date

(Dr. SWADESH KUMAR SAHOO)

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SARITA AGRAWAL has successfully given her Ph.D. Oral Examination held on **October 24, 2016**.

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ABSTRACT

KEYWORDS: Analytic, univalent, starlike, convex and close-to-convex functions; convexity in the direction of the imaginary axis; continued fraction; g-fraction; Hausdorff moment sequence; q-difference operator; Gauss and basic hypergeometric functions; q-starlike functions; order of starlikeness; order of q-starlikeness; Bieberbach's conjecture; infinite product; uniform convergence; Herglotz representation; probability measure; convex hull; uniformly starlike and uniformly convex functions; spiral functions; coefficient functional; Zalcman's conjecture; John domains; pre-Schwarzian derivative; Schwarzian derivative; the Nehari class; odd functions; radius of convexity; partial sums or sections.

We consider basic hypergeometric functions introduced by Heine. We study mapping properties of certain ratios of basic hypergeometric functions having shifted parameters and show that they map the domains of analyticity onto domains convex in the direction of the imaginary axis. In order to investigate these mapping properties, some useful identities are obtained in terms of basic hypergeometric functions. In addition, we find conditions under which the basic hypergeometric functions are in a q-close-to-convex family.

For every $q \in (0, 1)$ and $0 \leq \alpha < 1$ we define a class of analytic functions, the so-called q-starlike functions of order α, on the open unit disk. We study this class of functions and explore some inclusion properties with the well-known class of starlike functions of order α. We discuss the Herglotz representation formula for analytic functions $zf'(z)/f(z)$ when $f(z)$ is q-starlike of order α. As an application we also discuss the Bieberbach conjecture problem for the q-starlike functions of order α.

We consider certain subfamilies, of the family of univalent functions in the open unit disk, defined by means of sufficient coefficient conditions for univalency. In this thesis, we study the problem of the well-known conjecture of Zalcman consisting of a generalized

coefficient functional, the so-called generalized Zalcman conjecture problem, for functions belonging to those subfamilies. We estimate the bounds associated with the generalized coefficient functional and show that the estimates are sharp.

we study some necessary conditions for bounded John domains associated with functions in Nehari-type classes. The series of preparatory results, which are applications of certain initial value problems, consist of sharp estimations of pre-Schwarzian derivatives of functions belonging to the Nehari-type classes. In the sequel, we also see that a solution of a complex differential equation has a special form in terms of ratio of hypergeometric functions resulting to an integral representation. Finally, we attempt to study univalent functions f in the unit disk $\mathbb{D}$ such that $f(\mathbb{D})$ are unbounded John domains and state some related open problems.

We consider the class of all analytic and locally univalent functions f of the form $f(z) = z + \sum_{n=2}^{\infty} a_{2n-1} z^{2n-1}$, $|z| < 1$, satisfying the condition

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > -\frac{1}{2}.$$

We show that every section $s_{2n-1}(z) = z + \sum_{k=2}^{n} a_{2k-1} z^{2k-1}$, of f, is convex in the disk $|z| < \sqrt{2}/3$. We also prove that the radius $\sqrt{2}/3$ is best possible, i.e. the number $\sqrt{2}/3$ cannot be replaced by a larger one.

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NOTATION

Symbols

$\operatorname{diam} A$	euclidean diameter of the set A
$\mathcal{A}$	class of normalized analytic functions in $\mathbb{D}$
$A \subset B$	A is a subset of B
$(a)_n$	the Pochhammer symbol or shifted factorial
$(a;q)_n$	the Watson symbol or the q-shifted factorial
$\mathbb{C}$	complex plane
$\overline{\mathbb{C}}$	the extended complex plane $\mathbb{C} \cup \{\infty\}$ or the Riemann sphere
$\mathcal{C}$	class of convex functions
$\mathcal{C}(\beta)$	class of convex functions of order $\beta,\ 0 \leq \beta < 1$
$co(\mathcal{C})$	convex hull of $\mathcal{C}$
$\mathbb{D}$	the open unit disk $\{z \in \mathbb{C} : \lvert z\rvert < 1\}$
$D_q f$	the q-difference operator
$\mathbb{D}_r$	$\{z \in \mathbb{D} : \lvert z\rvert < r\}$
$F(a,b;c;z)$	hypergeometric (or Gauss hypergeometric) function
$f \prec g$	f is subordinate to g
$\overline{G}$	closure of G
$\mathcal{H}(\mathbb{D})$	the set of all analytic (or holomorphic) functions in $\mathbb{D}$
$\mathcal{K}$	class of close-to-convex functions
$\mathcal{K}_q$	q-close-to-convex functions
$\ell(\gamma[z,w])$	the Euclidean length of γ joining z to w
$\operatorname{Log} z$	the principal value of the logarithmic function $\log z$ for $z \neq 0$
$\operatorname{Re} z$	real part of z
$\mathcal{S}$	class of univalent functions
$\mathcal{S}^*$	class of starlike functions

$\mathcal{S}^*(\beta)$	class of starlike functions of order $\beta,\ 0 \le \beta < 1$
S_f	pre-Schwarzian derivative of f
s_n	n-th section/partial sum
$\mathcal{S}_p^\nu(\beta)$	the class of ν-spiral-like functions of order $\beta,\ 0 \le \beta < 1$
$\mathcal{S}_q^*$	q-starlike functions
$\mathcal{S}_q^*(\alpha)$	q-starlike functions of order $\alpha,\ 0 \le \alpha < 1$
T_f	pre-Schwarzian derivative of f
$\mathcal{UCV}$	class of uniformly convex functions
$\mathcal{UST}$	class of uniformly starlike functions
Γ	gamma function
Γ_q	q-gamma function
$\Phi[a, b; c; q, z]$	basic hypergeometric function

CHAPTER 1

INTRODUCTION

This thesis is based on some research work on analytic function theory carried out at IIT Indore. The purpose of this chapter is to give some basic definitions, notations, and some preliminaries that provide a background for latter chapters. We begin this chapter with the definitions of some well-known special functions and their q-analogs.

1.1. Special functions and q-analogs

Most of the special functions appear as solutions of differential equations or integrals of elementary functions. For example, one of the power series solutions of the hypergeometric differential equation

$$z(1-z)w''(z) + [c - (a+b+1)z]w'(z) - abw(z) = 0$$

about the regular singular point $z = 0$ is obtained by

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n,$$

where $(a)_0 = 1$, $(a)_n = a(a+1)\cdots(a+n-1)$ is the *Pochhammer symbol* and $c \neq 0, -1, -2, \ldots$. This series is called the hypergeometric series which converges absolutely for $|z| < 1$. The converging function is called the *(Gaussian) hypergeometric function* denoted by $F(a, b; c; z)$.

The following well-known derivative formula is useful:

$$\frac{d}{dz} F(a, b; c; z) = \frac{ab}{c} F(a+1, b+1; c+1; z).$$

For more details on this topic we refer [**7, 87, 104**]. While it is true, both historically and practically, that the special functions and their applications arise primarily in mathematical physics, they do have many other uses in both pure and applied mathematics. One can find its applications in number theory also. Historical background says that Heine

[**33**] defined a basic quantity

$$a_q = \frac{1-q^a}{1-q},$$

where q and a are real or complex numbers so that $a_q \to a$ as $q \to 1$. Using this concept Heine defined the basic analog (or q-analog) of hypergeometric functions for $|q| < 1$ called the basic hypergeometric functions. In $\mathbb{D}$, the *basic hypergeometric function* (*Heine hypergeometric function*), for $|q| < 1$, is defined by

$$\Phi[a, b; c; q, z] = \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(c;q)_n (q;q)_n} z^n,$$

where $(a;q)_0 = 1$, $(a;q)_n = (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1})$ for $n \geq 1$, called the Watson symbol, and a, b, c are real or complex parameters with $(c;q)_n \neq 0$. The function $z\Phi[a, b; c; q, z]$ is called the *shifted basic hypergeometric function.* The limit

$$\lim_{q \to 1^-} \frac{(q^a;q)_n}{(q;q)_n} = \frac{a(a+1)\cdots(a+n-1)}{n!}$$

says that, with the substitution $a \mapsto q^a$, the Heine hypergeometric function takes to the Gaussian hypergeometric function $F(a, b; c; z)$ when q approaches 1^-. The following relation is useful in this context:

$$(1-a)(aq;q)_n = (a;q)_n(1-aq^n) = (a;q)_{n+1}. \tag{1.1}$$

For basic properties of Heine's hypergeometric series, we refer to [**7, 25, 104**]. After that many more developments were carried out in this direction. For instance, Jackson in [**38**] developed the concept of q-difference equation as basic analog of the ordinary difference equation. For $0 < q < 1$, the *q-difference operator* denoted as $D_q f$ and is defined by the equation

$$(D_q f)(z) = \frac{f(z) - f(qz)}{z(1-q)}, \quad z \neq 0, \quad (D_q f)(0) = f'(0). \tag{1.2}$$

It is evident that, when $q \to 1^-$, the difference operator $D_q f$ converges to the ordinary differential operator $Df = df/dz = f'$. The operator $D_q f$ plays an important role in the theory of special functions, subclasses of univalent functions, and quantum physics (see for instance [**6, 23, 24, 37, 47, 104**]). Practically every branch of normal function theory has been extended to the basic number field. Most of the actual applications of the q-concepts have occurred in the field of pure mathematics. This motivates us to study, for instance, the analytic and geometric behaviour of the basic hypergeometric functions.

1.2. Univalent functions

The theory of univalent functions is a classical subject, born around the turn of the century, yet it remains an active field of current research. A function f is said to be *univalent* in a domain D if for any two distinct points z_1 and z_2 in D, $f(z_1) \neq f(z_2)$. In other words we can say that a function is univalent if it provides a one-to-one (or injective) mapping onto its image. The function f is said to be *locally univalent* at a point $z_0 \in \mathbb{D}$ if it is univalent in some neighborhood of z_0. For analytic functions f, the condition $f'(z_0) \neq 0$ is equivalent to local univalence at z_0. In view of the *Riemann Mapping Theorem* in classical complex analysis, the unit disk $\mathbb{D}$ is usually considered as a standard domain. If g is univalent in $\mathbb{D}$ and has a power series representation $g(z) = b_0 + \sum_{n=1}^{\infty} b_n z^n$ which is convergent in $\mathbb{D}$, then $f(z) = (g(z) - b_0))/b_1$ has the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.3}$$

where $a_n = b_n/b_1$, is also convergent in $\mathbb{D}$ and vice-versa. We call the function f having form (1.3) as *normalized function* and denote the class of normalized analytic function in $\mathbb{D}$ of the form (1.3) by $\mathcal{A}$. Observe that a function $f \in \mathcal{A}$ has the relation $f(0) = 0 = f'(0) - 1$. The theory of univalent functions is largely concerned with the family $\mathcal{S}$ of univalent functions $f \in \mathcal{A}$.

One of the classical problems in univalent function theory is the famous Bieberbach conjecture problem. Bieberbach first proved that the second coefficient a_2 of a function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}$ has the sharp upper bound 2. Sharpness can be seen from the Koebe function $k(z) = z/(1-z)^2$. Based on this, Bieberbach proposed a conjecture in 1916 stating that "If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}$, then $|a_n| \leq n$ for $n \geq 2$". It was natural to suspect this due to the Koebe function as it plays the role of the extremal function for the class $\mathcal{S}$. The conjecture was a long standing open problem for function theorists. Initially, the conjecture was proved for first few coefficients and for some subclasses of $\mathcal{S}$. Subsequently, many useful conjectures such as Robertson's conjecture and Zalcman's conjecture were investigated to prove the Bieberbach conjecture. Finally, it was de Branges who settled the conjecture of Bieberbach in 1985. For more details about this conjecture we refer [**13, 21, 98**].

One of our focuses in this thesis is to investigate the Bieberbach conjecture problem in a more general setting linking to q-theory through well-known subclasses of $\mathcal{S}$. We call the connection between analytic function theory to its q-analog as the *q-function theory*. Now, we give definitions of some subclasses of $\mathcal{S}$ which are defined by natural geometric conditions.

A domain $D \subset \mathbb{C}$ is said to be *starlike with respect to a point* $z_0 \in D$ if the line segment joining z_0 to every other point $z \in D$ lies entirely in D. D is said to be *starlike* if it is starlike with respect to origin. A function $f \in \mathcal{S}$ is said to be starlike if $f(\mathbb{D})$ is a starlike domain. Analytically, a function $f \in \mathcal{A}$ is *starlike* ($f \in \mathcal{S}^*$) if

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad z \in \mathbb{D}.$$

A domain D is said to be *convex* if it is starlike with respect to each of its points; that is, if the line segment joining any two points of D lies entirely in D. A function $f \in \mathcal{S}$ is said to be convex if $f(\mathbb{D})$ is a convex domain. Analytically, $f \in \mathcal{A}$ is *convex* ($f \in \mathcal{C}$) if

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \quad z \in \mathbb{D}.$$

A function $f \in \mathcal{A}$ is called *close-to-convex* ($f \in \mathcal{K}$) if there exist a real number $\theta \in (-\pi/2, \pi/2)$ and a function $g \in \mathcal{S}^*$ such that

$$\mathrm{Re}\left(e^{i\theta}\frac{zf'(z)}{g(z)}\right) > 0, \quad z \in \mathbb{D}.$$

It is well-known that

$$\mathcal{C} \subset \mathcal{S}^* \subset \mathcal{K} \subset \mathcal{S}.$$

In 1990, Ismail et al. [**37**] introduced a link between starlike functions and the q-difference operator by introducing a q-analog of the starlike functions. We call these functions as *q-starlike functions*. By [**37**], a function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{S}_q^*$ if

$$\left|\frac{z(D_qf)(z)}{f(z)} - \frac{1}{1-q}\right| \leq \frac{1}{1-q}, \quad z \in \mathbb{D}.$$

Observe that as $q \to 1^-$ the closed disk $|w - (1-q)^{-1}| \leq (1-q)^{-1}$ becomes the right-half plane $\mathrm{Re}\, w \geq 0$ and the class $\mathcal{S}_q^*$ reduces to $\mathcal{S}^*$. For updated research work in function theory related to q-analysis, readers can refer to [**3, 37, 86, 92, 97**].

We are interested to consider a q-analog of starlike functions of order α and study the Bieberbach conjecture problem through the Herglotz representation theorem for the same class. It is now appropriate to recall the Herglotz Representation Theorem for functions with positive real part. The class of such functions is defined by the collection

$$\mathcal{P} = \left\{ p \in \mathcal{H}(\mathbb{D}) : p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \text{ and } \mathrm{Re}\,(p(z)) > 0 \right\}.$$

The Herglotz Representation Theorem [**21, 28, 80**]. *Let $p \in \mathcal{P}$. Then there exists an increasing function μ supported on the unit circle, with $\int_{|\sigma|=1} \mathrm{d}\mu(\sigma) = 1$ such that*

$$p(z) = \int_{|\sigma|=1} \frac{1+\sigma z}{1-\sigma z} \mathrm{d}\mu(\sigma).$$

The remarkable fact is that the converse of this theorem is also true. Likewise the Koebe function for the class $\mathcal{S}$, the Möbius function

$$l(z) = 1 + 2\sum_{n=1}^{\infty} z^n = \frac{1+z}{1-z}$$

plays a central role for the class $\mathcal{P}$.

1.3. The Zalcman conjecture

It is well-known that for $f \in \mathcal{S}$, $|a_2^2 - a_3| \leq 1$, see [**80**, Theorem 1.5]. At the end of 1960's, Zalcman made a conjecture that each $f \in \mathcal{S}$ satisfies the inequality

$$|a_n^2 - a_{2n-1}| \leq (n-1)^2, \quad n \geq 2 \tag{1.4}$$

with equality for the Koebe function $k(z) = z/(1-z)^2$. One of the main aims of the Zalcman conjecture was to prove the Bieberbach conjecture, using the famous Hayman Regularity Theorem (see [**21**, Theorem 5.6, pp. 163]). Although the Bieberbach conjecture is proved after several years, the conjecture of Zalcman remains open. Thus similar approaches for the proof of the Bieberbach conjecture, the conjecture of Zalcman is under consideration in many directions, in particular it is investigated through many subclasses of univalent functions. This motivates us to consider the problem of Zalcman in a more general setting.

1.4. Schwarzian and pre-Schwarzian derivatives

The *Schwarzian derivative* of a locally univalent meromorphic function $f : \mathbb{D} \to \overline{\mathbb{C}}$ is defined by

$$S_f(z) = T_f'(z) - \frac{1}{2}T_f^2(z)$$

at each point z where f is analytic, and $S_f(z) = S_{1/f}(z)$ at the poles of f. Here, the quantity $T_f(z) = f''(z)/f'(z)$ is known as the *pre-Schwarzian derivative* of f or the logarithmic derivative of f'.

There are certain domains in geometric function theory with nice geometric interpretations, e.g. John domains, uniform domains, etc. Here we concentrate only on John domains. *John domains* in the Euclidean n-space $\mathbb{R}^n$ were introduced by John [**39**] in connection with his work on elasticity. The term "John domain" is due to Martio and Sarvas [**64**] while they were studying injectivity theorems for plane and space. Initially, John domains were defined for bounded domains. Later on, Näkki and Väisälä in [**72**] introduced unbounded John domains and studied its several characterizations. However, in this thesis, we consider the case when domains are bounded. Bounded John domains are characterized by the following geometric fact: a bounded domain $D \subset \mathbb{C}$ is a John domain if and only if there is a constant $a > 0$ such that for every *crosscut* C of D the inequality

$$\operatorname{diam} H \leq a \operatorname{diam} C$$

holds for any one of the components H of $D \setminus C$. Here "diam" denotes the Euclidean diameter. Note that a simply connected John domain is called a *John disk*. For example, we can verify that outward cusp type bounded domains are not John disks. For several other geometric properties of John domains, reader can refer to [**45**]. John disks are understood in the sense of a one-sided *quasidisks* (A quasidisk is the image of the unit disk or half plane under a K-quasiconformal mapping of $\overline{\mathbb{C}}$; see for instance [**26**]). However, John disks differ essentially from the quasidisks in their behavior under Möbius transformations. Quasidisks are invariant under Möbius transformations, but John disks do not satisfy this mapping property in general. In fact, Näkki and Väisälä [**72**] proved that a domain is quasidisk if and only if all its Möbius images are John disks.

Chuaqui et al. in [**19**] have given a nice geometric interpretation of the John domains that are images of the open unit disk under functions satisfying Schwarzian derivative univalence criteria. Since a very few work in this direction have been done, we motivated to continue in studying such results in a more general setting and see that how they are appearing in the study of differential equations and special functions. In this connection, we prove a series of preliminary results consisting of sharp estimations of pre-Schwarzian derivatives of functions belonging to a family of functions considered by Nehari and discuss some necessary conditions for John domains, which are indeed image of the open unit disk under certain analytic functions. Grönwall's inequality is used in estimating the pre-Schwarzian derivatives. It is now appropriate to recall

Grönwall's inequality: Let I denote an interval of the real line of the form $[a,\infty)$ or $[a,b]$ or $[a,b)$ with $a<b$. Let β and u be real valued continuous functions defined on I. If u is differentiable in the interior I^0 of I (the interval without the end points a and possibly b) and satisfies the differential inequality

$$u'(t)\le u(t)\beta(t),\quad t\in I^0,$$

then u is bounded by the solution of the differential equation $u'(t)=u(t)\beta(t)$:

$$u(t)\le u(a)\exp\left(\int_a^t \beta(s)\,ds\right)$$

for all $t\in I$.

1.5. Radius Problem

Radius problem is one of the finest problems in univalent function theory. But this problem is studied in different senses. One of them is the radius of convexity. It is well-known that every convex function in the open unit disk $\mathbb{D}:=\{z\in\mathbb{C}:\,|z|<1\}$ is univalent, but the converse is not true in general. So, it was reasonable to find the largest subdisk $|z|<r$ of $\mathbb{D}$ in which the univalent functions are convex. Similar type of radius problems studied in the literature are the radius of starlikeness, the radius of close-to-convexity, etc.; see [**20**]. Furthermore, radius problems are also investigated for the sections of functions in $\mathcal{S}$ and its subclasses.

The Taylor polynomial $s_n(z)=s_n(f)(z)$ of f in $\mathcal{A}$, defined by,

$$s_n(z)=z+\sum_{k=2}^{n}a_kz^k$$

is called the n-th *section/partial sum* of f.

Odd univalent functions and classical problems of univalent function theory such as (successive) coefficient bounds, inverse functions, etc. are quite interesting and found throughout in the literature. In fact, an application of the Cauchy-Schwarz inequality shows that the conjecture of Robertson: $1+|c_3|^2+|c_5|^2+\cdots+|c_{2n-1}|^2 \leq n$, $n\geq 2$, for each odd function $f(z)=z+c_3z^3+c_5z^5+\cdots$ of $\mathcal{S}$, implies the Bieberbach conjecture [**21**]. The problem of finding the radius of univalence of sections of f in $\mathcal{S}$ was first initiated by Szegö. According to the theorem of Szegö [**21**, p. 243-246], every section $s_n(z)$ of $f\in\mathcal{S}$ is univalent in a subdisk of $\mathbb{D}$ having best possible radius 1/4. This and many other recent works in this direction motivate us to find the largest radius of convexity of partial sums of odd functions belonging to a close-to-convex family.

1.6. Structure of the thesis

The first chapter of the thesis covers almost all the preliminaries for the remaining chapters. The main theme is described in Chapter 2 to 6 whereas **Chapter 7** gives the concluding remarks and future directions to work out.

In **Chapter 2**, we obtain an integral representation of ratio of basic hypergeometric functions using the Hausdorff moment sequence which gives the region of analyticity of the ratio functions and its mapping properties.

One of the results in this chapter is the following:

Theorem 1.1. *For $q\in(0,1)$ suppose that a,b,c are non-negative real numbers satisfying $0\leq q(b-c)\leq 1-cq$ and $0<a-c\leq 1-c$. Then there exists a non-decreasing function $\mu:[0,1]\to[0,1]$ with $\mu(1)-\mu(0)=1$ such that*

$$\frac{z\Phi[a,bq;cq;q,qz]}{\Phi[a,b;c;q,qz]}=\int_0^1\frac{z}{1-tz}\,d\mu(t)$$

which is analytic in the cut-plane $\mathbb{C}\setminus[1,\infty]$ and maps both the unit disk and the half-plane $\{z\in\mathbb{C}:\operatorname{Re} z<1\}$ univalently onto domains convex in the direction of the imaginary axis.

We also show that the family of q-close-to-convex functions is non-empty by obtaining some conditions for basic hypergeometric functions to be in that class.

With the help of the difference operator D_qf, a q-analog of close-to-convex functions is studied in [**86, 97**].

Definition 1.2. A function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{K}_q$, the class of *q-close-to-convex functions*, if there exists $g \in \mathcal{S}^*$ such that

$$\left| \frac{z(D_q f)(z)}{g(z)} - \frac{1}{1-q} \right| \leq \frac{1}{1-q}, \quad z \in \mathbb{D}.$$

As $q \to 1^-$, the class $\mathcal{K}_q$ reduces to the class $\mathcal{K}$.

Theorem 1.3. *If* $a, b < 1$,

$$\begin{aligned} T_1(a,b) &= \min\left\{ab, ab + \frac{aq+bq-q-2ab+ab/q}{2(1-q)}, ab + \frac{aq+bq-q-2ab+ab/q}{(1-q)}\right. \\ &\qquad \left. + \frac{a+b-q-ab/q}{(1-q)}\right\} \end{aligned}$$

and c *satisfies either*

$$c \leq T_1(a,b) \tag{1.5}$$

or $c = ab$ *with*

$$ab \geq \frac{aq+bq-q}{2-1/q}, aq+bq+a+b-2q \leq 2ab \text{ and } \frac{\Gamma_q(\log_q ab)}{\Gamma_q(\log_q a)\Gamma_q(\log_q b)} \leq 2 \tag{1.6}$$

then $z\Phi[a,b;c;q,z] \in \mathcal{K}_q$ *with the starlike function* $g(z) = z/(1-z)$.

Similar to the class $\mathcal{K}_q$, the q-analog of the class of starlike functions (*called the class of q-starlike functions*), denoted by $\mathcal{S}_q^*$, is defined in [**37**]. In **Chapter 3** we consider the class of *q-starlike functions of order* α by

Definition 1.4. A function $f \in \mathcal{A}$ is said to *belong to the class* $\mathcal{S}_q^*(\alpha)$, $0 \leq \alpha < 1$, if

$$\left| \frac{\dfrac{z(D_q f)(z)}{f(z)} - \alpha}{1-\alpha} - \frac{1}{1-q} \right| \leq \frac{1}{1-q}, \quad z \in \mathbb{D}.$$

The following is the Herglotz representation of functions belonging to the class $\mathcal{S}_q^*(\alpha)$:

Theorem 1.5. *Let* $f \in \mathcal{A}$. *Then* $f \in \mathcal{S}_q^*(\alpha)$ *if and only if there exists a probability measure* μ *supported on the unit circle such that*

$$\frac{zf'(z)}{f(z)} = 1 + \int_{|\sigma|=1} \sigma z F_{q,\alpha}'(\sigma z) \mathrm{d}\mu(\sigma)$$

where

$$F_{q,\alpha}(z) = \sum_{n=1}^{\infty} \frac{(-2)\left(\ln \frac{q}{1-\alpha(1-q)}\right)}{1-q^n} z^n, \quad z \in \mathbb{D}. \tag{1.7}$$

This helps us to prove the Bieberbach conjecture for the class $\mathcal{S}_q^*(\alpha)$ in the following form:

Theorem 1.6. *Let*

$$G_{q,\alpha}(z) := z\,\exp[F_{q,\alpha}(z)] = z + \sum_{n=2}^{\infty} c_n z^n. \tag{1.8}$$

Then $G_{q,\alpha} \in \mathcal{S}_q^*(\alpha)$. Moreover, if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}_q^*(\alpha)$, then $|a_n| \leq c_n$ with equality holding for all n if and only if f is a rotation of $G_{q,\alpha}$.

Chapter 4 deals with estimations of the generalized Zalcman coefficient functional for some functions generated through sufficient conditions of univalence in terms of Taylor coefficients. We consider the class

$$\mathcal{H} = \left\{ f \in \mathcal{A} : f(z) = z + \sum_{n=2}^{\infty} a_n z^n \text{ and } \sum_{n=2}^{\infty} r(n)|a_n| \leq 1, r(n) > 0 \text{ for } n \geq 2 \right\} \tag{1.9}$$

and prove the following theorem:

Theorem 1.7. *Let $\lambda > 0$ and $n = 2, 3, \ldots$. For $f \in \mathcal{H}$, we have*

$$|\lambda a_n^2 - a_{2n-1}| \leq \max\left\{ \frac{\lambda}{r(n)^2}, \frac{1}{r(2n-1)} \right\}.$$

Equality holds if and only if

$$f(z) = \begin{cases} z + \dfrac{\alpha}{r(2n-1)} z^{2n-1} & \text{for } \lambda \leq \dfrac{r(n)^2}{r(2n-1)}, \\ z + \dfrac{\alpha}{r(n)} z^n & \text{for } \lambda \geq \dfrac{r(n)^2}{r(2n-1)}, \end{cases}$$

where α is a complex number such that $|\alpha| = 1$.

Chapter 5 studies on a family of functions defined by

$$\mathcal{N}_\alpha(k) = \left\{ f \in \mathcal{A} : (1 - |z|^2)^\alpha |S_f(z)| \leq k,\ f''(0) = 0, \alpha \geq 0 \text{ and } k \geq 0 \right\} \tag{1.10}$$

and

$$\mathcal{M}_2(k) = \left\{ f \in \mathcal{A} : (1 - |z|^2)^2 |S_f(z)| \leq k,\ f''(0) = \sqrt{4 - 2k}, 0 \leq k \leq 2 \right\}, \tag{1.11}$$

where $S_f(z)$ is the *Schwarzian derivative* of f. We call the families $\mathcal{N}_\alpha(k)$ and $\mathcal{M}_2(k)$, the Nehari-type classes of functions.

We obtain the following integral representation as a solution to a differential equation that lies in $\mathcal{N}_1(k)$ (put $\alpha = 1$ in (1.10)):

Theorem 1.8. *The solution of the differential equation*

$$w'(z) = \frac{1}{2}w^2(z) + \frac{k}{1-z^2}$$

can be represented by $w(z) = \displaystyle\int_0^1 \frac{kz}{1-tz^2}\,d\mu(t)$*, where* $\mu(t) : [0,1] \to [0,1]$ *is a non-decreasing function with* $\mu(1) - \mu(0) = 1$*.*

We prove sharp estimations for the pre-Schwarzian derivative of functions in the Nehari-type classes. One of our results is of the following form:

Lemma 1.9. *If* $f \in \mathcal{M}_2(k)$*,* $0 \le k \le 2$*, then*

$$|T_f(z)| \le \frac{2|z| + \sqrt{4-2k}}{1-|z|^2}.$$

Equality holds at a single $z \neq 0$ *if and only if* f *is a suitable rotation of* $F_0(z)$*, where*

$$F_0(z) = \frac{e^{\sqrt{4-2k}\tanh^{-1}(z)} - 1}{\sqrt{4-2k}}.$$

As an application of this, we prove that

Theorem 1.10. *Let* $f \in \mathcal{M}_2(k)$*,* $0 \le k < 2$*. Then*

$$\limsup_{|z|\to 1}(1-|z|^2)\mathrm{Re}\,(zT_f(z)) < 2 + \sqrt{4-2k}.$$

Note that the image of the unit disk $\mathbb{D}$ under maps from the class $\mathcal{M}_2(k)$ are John domains.

Chapter 6 is about the radius problem where we focus on a subclass of $\mathcal{K}$, denoted by $\mathcal{L}$, of the class of all locally univalent *odd* functions f such that

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > -\frac{1}{2}, \quad z \in \mathbb{D}. \tag{1.12}$$

Here, we find the largest radius in which all the sections of odd functions in the close-to-convex family are convex. We prove that the radius $\sqrt{2}/3$ is the best possible. Indeed, we have

Theorem 1.11. *Every section of a function in* $\mathcal{L}$ *is convex in the disk* $|z| < \sqrt{2}/3$*. The radius* $\sqrt{2}/3$ *cannot be replaced by a greater one.*

CHAPTER 2

GEOMETRIC PROPERTIES OF BASIC HYPERGEOMETRIC FUNCTIONS

In this chapter we show that the functions

$$\frac{z\Phi[a,bq;cq;q,z]}{\Phi[a,b;c;q,z]} \left(\text{or } \frac{z\Phi[aq,bq;cq;q,z]}{\Phi[aq,b;c;q,z]}\right), \frac{z\Phi[aq,b;c;q,z]}{\Phi[a,b;c;q,z]} \left(\text{or } \frac{z\Phi[aq,bq;cq;q,z]}{\Phi[a,bq;cq;q,z]}\right)$$

and

$$\frac{z\Phi[aq,bq;cq;q,z]}{\Phi[a,b;c;q,z]}$$

are analytic in a cut plane and map both the unit disk and a half-plane univalently onto domains convex in the direction of the imaginary axis. We also discuss the q-close-to-convexity property of the basic shifted hypergeometric functions $z\Phi[a,b;c;q,z]$.

Results of this chapter are published in: Agrawal S., Sahoo S. K. (2014), Geometric properties of basic hypergeometric functions, J. Difference Equ. Appl., **20**(11), 1502–1522.

2.1. Motivation and preliminaries

The study of univalent functions on a simply connected domain can be confined to the study of these functions onto the unit disk $\mathbb{D}$. The classes of convex, starlike, and close-to-convex functions defined in $\mathbb{D}$ have been studied extensively and numerous applications to various problems in complex analysis and related topics have been found. Part of this development is the study of subclasses of the class of univalent functions, more general than the classes of convex, starlike, and close-to-convex functions. A number of geometric characterizations of such functions in terms of image of the unit disk is extensively studied by several authors. Background knowledge in this theory can be found from standard books in geometric function theory (see for instance, [**21**]). In this connection, our main aim is to study certain geometric properties of basic hypergeometric functions introduced by Heine [**34**]. Motivation behind this comes from mapping properties of the Gaussian hypergeometric functions studied in [**50**] in terms of convexity properties of shifted hypergeometric functions in the direction of the imaginary axis. One of the key tools to study this geometric property was the continued fraction of Gauss and a theorem of Wall concerning a characterization of Hausdorff moment sequences by means of (continued) g-fractions [**108**]. More background on mapping properties of the Gaussian hypergeometric functions can be found in [**32, 69, 81, 85, 101**].

2.2. Continued fractions and mapping properties

In this section, we mainly concentrate on mapping properties of functions of the form

$$\frac{z\Phi[aq,bq;cq;q,z]}{\Phi[a,b;c;q,z]} \quad \text{or} \quad \frac{\Phi[aq,bq;cq;q,z]}{\Phi[a,b;c;q,z]}.$$

First we collect some useful identities on basic hypergeometric functions. Further, analytic properties of the continued fraction of Gauss and Wall's characterization of Hausdorff moment sequences by means of (continued) g-fractions [**108**] are used as important tools, and finally, the following lemma has been used to derive the results.

Lemma 2.1. [**50, 66**] *Let $\mu : [0,1] \to [0,1]$ be non-decreasing with $\mu(1)-\mu(0)=1$. Then the function*

$$z \mapsto \int_0^1 \frac{z}{1-tz}\, d\mu(t)$$

is analytic in the cut-plane $\mathbb{C} \setminus [1,\infty]$ and maps both the unit disk and the half-plane $\{z \in \mathbb{C} : \operatorname{Re} z < 1\}$ univalently onto domains convex in the direction of the imaginary axis.

Here, a domain $D \subset \mathbb{C}$ is called *convex in the direction of the imaginary axis* [**89, 93**] if the intersection of D with any line parallel to the imaginary axis is either empty or a line segment. As an application of Lemma 2.1, subject to some ranges for the real parameters a, b, c, it is proved by Küstner in [**50**] that the hypergeometric function $z \mapsto F(a,b;c;z)$ as well as the shifted function $z \mapsto zF(a,b;c;z)$ each maps both the unit disk $\mathbb{D}$ and the half-plane $\{z \in \mathbb{C} : \operatorname{Re} z < 1\}$ univalently onto domains convex in the direction of the imaginary axis. Moreover, he obtained similar properties of images under ratios of hypergeometric functions having shifted parameters. For instance, see Figure 2.1 for description of such a function. In order to use analytic properties of the continued fraction of Gauss, certain identities on the Gaussian hypergeometric functions

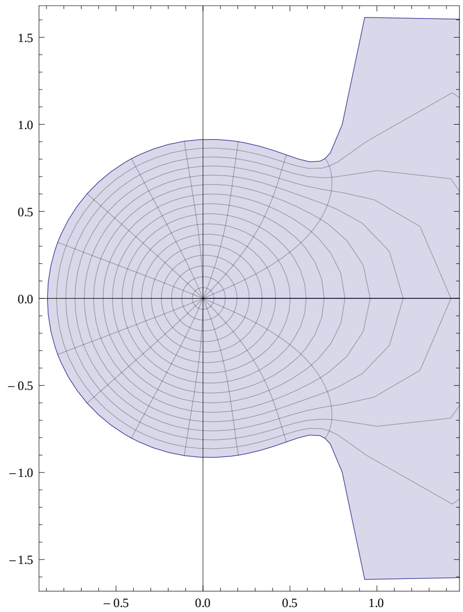


FIGURE 2.1. The image of the disk $|z| < r$ ($r = 0.999$) under the mapping $zF(a+1, b; c; z)/F(a, b; c; z)$, when $a = 0$, $b = 0.0199$, $c = 0.1$.

were crucial to consider. In this context, it is also important to collect similar relations for basic hypergeometric functions. One such relation is obtained in [**37**] and we also use that relation in our proofs.

Lemma 2.2. *The basic hypergeometric function of Heine* $\Phi[a, b; c; q, z]$ *satisfies the identities*

(a) $\Phi[a, b; c; q, z] - \Phi[a, bq; cq; q, z] = \dfrac{(1-a)(c-b)}{(1-c)(1-cq)} z\Phi[aq, bq; cq^2; q, z]$;

(b)

$$\begin{aligned}\Phi[aq, b; c; q, z] - \Phi[a, b; c; q, z] &= \frac{a(1-b)}{(1-c)} z\Phi[aq, bq; cq; q, z] \\ &= \frac{a}{1-a}(\Phi[a, b; c; q, z] - \Phi[a, b; c; q, qz]).\end{aligned}$$

Proof. The identity stated in **(a)** is established by Lorentzen and Waadeland (see [**59**, 3.2.1]). For the proof of **(b)** we make use of the identities given in (1.1). We obtain

$$\begin{aligned}\Phi[aq, b; c; q, z] - \Phi[a, b; c; q, z] &= \sum_{n=0}^{\infty} \frac{(aq; q)_n (b; q)_n}{(c; q)_n (q; q)_n} \left[1 - \frac{1-a}{1-aq^n}\right] z^n \\ &= \sum_{n=0}^{\infty} \frac{(aq; q)_{n+1} (b; q)_{n+1}}{(c; q)_{n+1} (q; q)_{n+1}} \frac{a(1-q^{n+1})}{(1-aq^{n+1})} z^{n+1}.\end{aligned}$$

Now, we use the relation (1.1) and obtain the difference

$$\begin{aligned}\Phi[aq,b;c;q,z]-\Phi[a,b;c;q,z] &= \sum_{n=0}^{\infty}\frac{a(aq;q)_n(1-b)(bq;q)_n}{(1-c)(cq;q)_n(q;q)_n}z^{n+1}\\ &= \frac{a(1-b)}{(1-c)}z\,\Phi[aq,bq;cq;q,z].\end{aligned}$$

Finally, the identity

$$\frac{a}{1-a}\left(\Phi[a,b;c;q,z]-\Phi[a,b;c;q,qz]\right)=\frac{a(1-b)}{(1-c)}z\Phi[aq,bq;cq;q,z]$$

follows from a similar identity obtained in [**37**]. $\square$

The following subsections deal with mapping properties discussed above. In particular, we generalize certain results of Küstner [**50**]. Now we recall Wall's theorem which is useful in this context.

Lemma 2.3. *(Wall's Theorem)* [**108**, Theorem 69.2] *The moment problem for the interval* $(0,1)$

$$\mu_i=\int_0^1 t^i\,d\mu(t),\quad i=1,2,\cdots,$$

has a solution if and only if the power series

$$1+\mu_1 z+\mu_2 z^2+\cdots$$

has a continued fraction expansion of the form

$$\frac{1}{1-}\;\frac{(1-g_1)g_2z}{1-}\;\frac{(1-g_2)g_3z}{1-}\;\frac{(1-g_3)g_4z}{1-\dots},$$

where $0\le g_i\le 1$, $i=1,2,\cdots$.

2.2.1. The ratio $\dfrac{z\Phi[a,bq;cq;q,z]}{\Phi[a,b;c;q,z]}$ or $\dfrac{z\Phi[aq,bq;cq;q,z]}{\Phi[aq,b;c;q,z]}$

Figure 2.2 visualizes the behaviour of the image domain of the disk $|z|<0.998$ under the map $z\Phi[a,bq;cq;q,z]/\Phi[a,b;c;q,z]$ when $a=0.9$, $b=0.7$, $c=0.6$, $q=0.8$. This shows that the map $z\Phi[a,bq;cq;q,z]/\Phi[a,b;c;q,z]$ in general does not take the unit disk onto convex domains in all the directions. Theorem 1.1 obtains conditions on the parameters a,b,c for which the image domain is convex in the direction of the imaginary axis. We now prove Theorem 1.1.

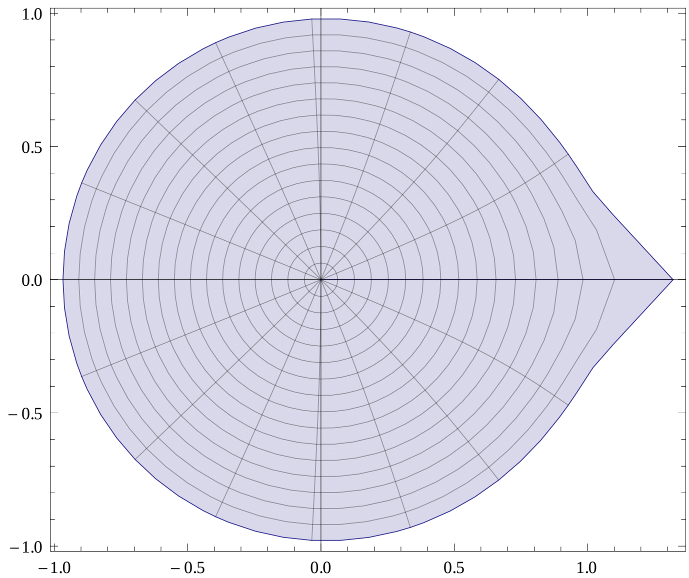


FIGURE 2.2. The image of the disk $|z| < 0.998$ under the mapping $z\Phi[a,bq;cq;q,z]/\Phi[a,b;c;q,z]$, when $a = 0.9$, $b = 0.7$, $c = 0.6$, $q = 0.8$.

Proof of Theorem 1.1. First of all we find the continued fraction of the ratio $z\phi_1/\phi_0$, where $\phi_1 = \Phi[a,bq;cq;q,z]$ and $\phi_0 = \Phi[a,b;c;q,z]$. Consider the iteration

$$\phi_{i-1} - \phi_i = d_i z \phi_{i+1}, \quad i = 1, 2, 3, \ldots \tag{2.1}$$

where d_i's are to be computed for each i. Rewrite this iteration in the form

$$\frac{\phi_i}{\phi_{i-1}} = \frac{1}{1 + d_i z \dfrac{\phi_{i+1}}{\phi i}}, \quad i = 1, 2, 3, \ldots. \tag{2.2}$$

Starting with $i = 1$, the relation (2.2) yields the following continued fraction for ϕ_1/ϕ_0:

$$\frac{\phi_1}{\phi_0} = \frac{1}{1 + d_1 z \dfrac{\phi_2}{\phi_1}} = \frac{1}{1+}\,\frac{d_1 z}{1+}\; d_2 z \frac{\phi_3}{\phi_2} = \frac{1}{1+}\,\frac{d_1 z}{1+}\,\frac{d_2 z}{1+}\; d_3 z \frac{\phi_4}{\phi_3}.$$

Continuing in this manner, it leads to the continued fraction

$$\frac{\phi_1}{\phi_0} = \frac{\Phi[a,bq;cq;q,z]}{\Phi[a,b;c;q,z]} = \frac{1}{1+}\,\frac{d_1 z}{1+}\,\frac{d_2 z}{1+}\,\frac{d_3 z}{1 + \ldots}. \tag{2.3}$$

We now calculate the values of d_i for all i. First, to find d_1, we use Lemma 2.2**(a)** and see that

$$\phi_0 - \phi_1 = \Phi[a,b;c;q,z] - \Phi[a,bq;cq;q,z] = \frac{(1-a)(c-b)}{(1-c)(1-cq)} z\, \Phi[aq,bq;cq^2;q,z].$$

Comparing with (2.1), for $i = 1$, we get

$$d_1 = \frac{(1-a)(c-b)}{(1-c)(1-cq)} \quad \text{and} \quad \phi_2 = \Phi[aq, bq; cq^2; q, z].$$

A similar computation as in Lemma 2.2**(a)** gives

$$\phi_1 - \phi_2 = \Phi[a, bq; cq; q, z] - \Phi[aq, bq; cq^2; q, z] = \frac{(1-bq)(cq-a)}{(1-cq)(1-cq^2)} z\, \Phi[aq, bq^2; cq^3; q, z].$$

Again by comparing with (2.1), for $i = 2$, we get

$$d_2 = \frac{(1-bq)(cq-a)}{(1-cq)(1-cq^2)} \quad \text{and} \quad \phi_3 = \Phi[aq, bq^2; cq^3; q, z].$$

By a similar technique one can compute

$$d_3 = q\,\frac{(1-aq)(cq-b)}{(1-cq^2)(1-cq^3)} \quad \text{and} \quad d_4 = q\,\frac{(1-bq^2)(cq^2-a)}{(1-cq^3)(1-cq^4)}.$$

Therefore, inductively we obtain

$$d_{2n+1} = q^n\,\frac{(1-aq^n)(cq^n-b)}{(1-cq^{2n})(1-cq^{2n+1})}, \quad \text{for } n \geq 0$$

and

$$d_{2n} = q^{n-1}\,\frac{(1-bq^n)(cq^n-a)}{(1-cq^{2n-1})(1-cq^{2n})}, \quad \text{for } n \geq 1.$$

In order to apply the notion of the Hausdorff moment sequences by means of (continued) g-fractions, a technique used in [**50**], we first rewrite (2.3) in the form

$$\frac{\Phi[a, bq; cq; q, z]}{\Phi[a, b; c; q, z]} = \frac{1}{1-}\;\frac{b_1 z}{1-}\;\frac{b_2 z}{1-}\;\frac{b_3 z}{1-\ldots}.$$

(*Remark. A similar form of the continued fraction of the reciprocal of the function* $\Phi[a, bq; cq; q, z]/\Phi[a, b; c; q, z]$ *was first done by Heine himself* [**35**] *as it is noted in* [**59**, pp 320]).

Then we get

$$b_{2n+1} = q^n\,\frac{(1-aq^n)(b-cq^n)}{(1-cq^{2n})(1-cq^{2n+1})}, \quad \text{for } n \geq 0$$

and

$$b_{2n} = q^{n-1}\,\frac{(1-bq^n)(a-cq^n)}{(1-cq^{2n-1})(1-cq^{2n})}, \quad \text{for } n \geq 1.$$

Now by replacing z by qz, we have

$$\frac{\Phi[a, bq; cq; q, qz]}{\Phi[a, b; c; q, qz]} = \frac{1}{1-}\;\frac{b_1 qz}{1-}\;\frac{b_2 qz}{1-}\;\frac{b_3 qz}{1-\ldots} = \frac{1}{1-}\;\frac{a_1 z}{1-}\;\frac{a_2 z}{1-}\;\frac{a_3 z}{1-\ldots}$$

where $a_i = b_i q$ with

$$a_{2n+1} = q^{n+1}\,\frac{(1-aq^n)(b-cq^n)}{(1-cq^{2n})(1-cq^{2n+1})} \quad \text{for } n \geq 0,$$

and

$$a_{2n} = q^n \frac{(1-bq^n)(a-cq^n)}{(1-cq^{2n-1})(1-cq^{2n})} \qquad \text{for } n \geq 1.$$

Set $a_i = (1-g_i)g_{i+1}$ for each i. Then, the ratio $\Phi[a, bq; cq; q, qz]/\Phi[a, b; c; q, qz]$ has the continued fraction (also called a g-fraction)

$$\frac{\Phi[a, bq; cq; q, qz]}{\Phi[a, b; c; q, qz]} = \frac{1}{1-} \; \frac{(1-g_1)g_2 z}{1-} \; \frac{(1-g_2)g_3 z}{1-} \; \frac{(1-g_3)g_4 z}{1-\dots}$$

in terms of the moment sequence $< g_i >$ given by

$$g_{2n+1} = q^n \left(\frac{a - cq^n}{1 - cq^{2n}} \right), \qquad n \geq 0$$

and

$$g_{2n} = q^n \left(\frac{b - cq^{n-1}}{1 - cq^{2n-1}} \right), \qquad n \geq 1.$$

Note that the moment sequence $< g_i >$ should satisfy the relation $0 \leq g_i \leq 1$, when we apply Lemma 2.3. By hypothesis, it is clear that $0 \leq g_1, g_2 \leq 1$. Using this, it is now easy to verify the relation $0 \leq g_i \leq 1$ for all i. Indeed, since $b \geq c > cq^{n-1}$ and $1 \geq cq > cq^{2n-1}$, we get the lower bound for g_{2i}. Next, as $bq - cq < 1 - cq$, we have $bq < 1$ and hence $bq^n < 1$ which implies $bq^n - cq^{2n-1} < 1 - cq^{2n-1}$, leading to the upper bound for g_{2i}. The bounds for g_{2i+1} can be proved similarly. Hence, there exists a non-decreasing function $\mu : [0, 1] \to [0, 1]$ such that $\mu(1) - \mu(0) = 1$ and

$$\frac{\Phi[a, bq; cq; q, qz]}{\Phi[a, b; c; q, qz]} = \int_0^1 \frac{1}{1 - tz} \mathrm{d}\mu(t). \tag{2.4}$$

This conclude the proof of our theorem. $\square$

Corollary 2.4. *For $q \in (0, 1)$ suppose that a, b, c are non-negative real numbers satisfying $0 \leq q(b - c) \leq 1 - cq$ and $0 < a - c \leq 1 - c$. Then there exists a non-decreasing function $\mu : [0, 1] \to [0, 1]$ with $\mu(1) - \mu(0) = 1$ such that*

$$\frac{z\Phi[a, bq; cq; q, z]}{\Phi[a, b; c; q, z]} = \int_0^1 \frac{qz}{q - tz} d\mu(t)$$

which is analytic in the cut-plane $\mathbb{C} \setminus [q, \infty]$ and maps both the unit disk and the half-plane $\{z \in \mathbb{C} : \operatorname{Re} z < q\}$ univalently onto domains convex in the direction of the imaginary axis.

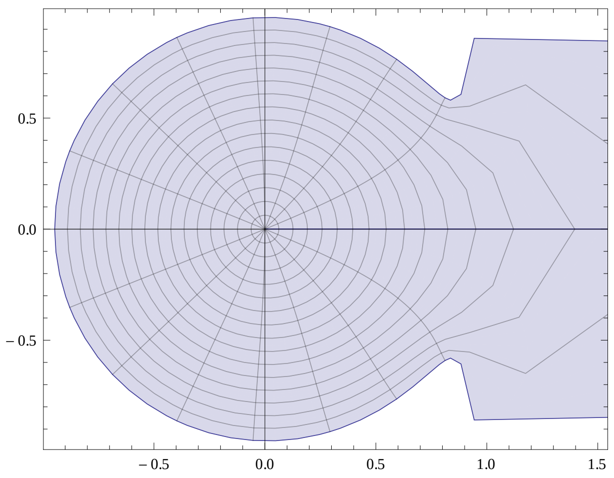


FIGURE 2.3. The image of the disk $|z| < 0.999$ under the mapping $\Phi[aq, b; c; q, z]/\Phi[a, b; c; q, z]$, when $a = 0.99$, $b = 0.998$, $c = 0.98$, $q = 0.9$.

Proof. Replacing z by z/q in (2.4), we have

$$\frac{\Phi[a, bq; cq; q, z]}{\Phi[a, b; c; q, z]} = \int_0^1 \frac{q}{q - tz} \mathrm{d}\mu(t).$$

Thus, the assertion of our corollary follows. □

Remark 2.5. If we substitute a by aq in (2.4), we get the same integral expression for the ratio $z\Phi[aq, bq; cq; q, z]/\Phi[aq, b; c, q, z]$. Moreover, if we substitute a by q^a, b by q^b and c by q^c, we obtain a result of Küstner (see [**50**, Theorem 1.5]) in the limiting sense when $q \to 1^-$.

2.2.2. The ratio $\dfrac{z\Phi[aq, b; c; q, z]}{\Phi[a, b; c; q, z]}$ or $\dfrac{z\Phi[aq, bq; cq; q, z]}{\Phi[a, bq; cq; q, z]}$

Figure 2.3 visualizes the behaviour of the image domain of the disk $|z| < 0.999$ under the map $z\Phi[aq, b; c; q, z]/\Phi[a, b; c; q, z]$ when $a = 0.99$, $b = 0.998$, $c = 0.98$, $q = 0.9$. This shows that the map $z\Phi[aq, b; c; q, z]/\Phi[a, b; c; q, z]$ in general does not take the unit disk onto domains convex in all directions. The following result obtains conditions on the parameters a, b, c for which the image domain is convex in the direction of the imaginary axis.

Theorem 2.6. *For $q \in (0, 1)$ suppose that a, b, c are non-negative real numbers satisfying $0 \le 1 - aq \le 1 - cq$ and $0 < 1 - b \le 1 - c$. Then there exists a non-decreasing function $\mu : [0, 1] \to [0, 1]$ with $\mu(1) - \mu(0) = 1$ such that*

$$\frac{z\Phi[aq, b; c; q, z]}{\Phi[a, b; c; q, z]} = \int_0^1 \frac{z}{1 - tz} d\mu(t)$$

which is analytic in the cut-plane $\mathbb{C}\setminus[1,\infty]$ and maps both the unit disk and the half-plane $\{z \in \mathbb{C} : \operatorname{Re} z < 1\}$ univalently onto domains convex in the direction of the imaginary axis.

Proof. In order to find the continued fraction of the ratio $\Phi[aq,b;c;q,z]/\Phi[a,b;c;q,z]$, let us first consider the continued fraction of the ratio $\Phi[a,bq;cq;q,z]/\Phi[a,b;c;q,z]$ obtained in the proof of Theorem 1.1. Now, by replacing a by aq, we get the continued fraction of $\Phi[aq,bq;cq;q,z]/\Phi[aq,b;c;q,z]$, say,

$$\frac{\Phi[aq,bq;cq;q,z]}{\Phi[aq,b;c;q,z]} = \frac{1}{1-}\,\frac{c_1 z}{1-}\,\frac{c_2 z}{1-}\,\frac{c_3 z}{1-\ldots}$$

where

$$c_{2n+1} = q^n\,\frac{(1-aq^{n+1})(b-cq^n)}{(1-cq^{2n})(1-cq^{2n+1})}, \qquad \text{for } n \geq 0$$

and

$$c_{2n} = q^n\,\frac{(1-bq^n)(a-cq^{n-1})}{(1-cq^{2n-1})(1-cq^{2n})} \qquad \text{for } n \geq 1.$$

Now, by Lemma 2.2**(b)**, we have

$$\Phi[aq,b;c;q,z] - \Phi[a,b;c;q,z] = \frac{a(1-b)}{(1-c)} z\, \Phi[aq,bq;cq;q,z].$$

Simplifying this, we get

$$\frac{\Phi[a,b;c;q,z]}{\Phi[aq,b;c;q,z]} = 1 - \frac{a(1-b)}{(1-c)} z\,\frac{\Phi[aq,bq;cq;q,z]}{\Phi[aq,b;c;q,z]}.$$

This implies

$$\frac{\Phi[aq,b;c;q,z]}{\Phi[a,b;c;q,z]} = \frac{1}{1-\dfrac{a(1-b)}{(1-c)}z\,\dfrac{\Phi[aq,bq;cq;q,z]}{\Phi[aq,b;c;q,z]}} = \frac{1}{1-}\,\frac{\dfrac{a(1-b)z}{(1-c)}}{1-}\,\frac{c_1 z}{1-}\,\frac{c_2 z}{1-}\,\frac{c_3 z}{1-\ldots},$$

where c_i's are defined as above. Rewriting this continued fraction by means of continued g-fractions of the form

$$\frac{\Phi[aq,b;c;q,z]}{\Phi[a,b;c;q,z]} = \frac{1}{1-}\,\frac{(1-g_0)g_1 z}{1-}\,\frac{(1-g_1)g_2 z}{1-}\,\frac{(1-g_2)g_3 z}{1-\ldots},$$

we get

$$g_{2n} = \frac{1-aq^n}{1-cq^{2n-1}} \qquad \text{for } n \geq 1$$

and

$$g_{2n+1} = \frac{1-bq^n}{1-cq^{2n}} \qquad \text{for } n \geq 0$$

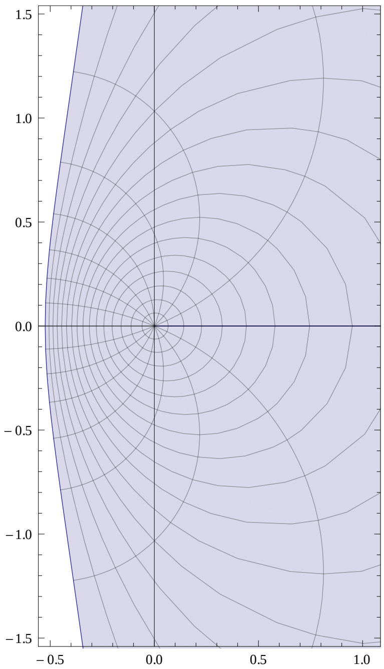


FIGURE 2.4. The image of the disk $|z| < 0.999$ under the mapping $\Phi[aq, bq; cq; q, z]/\Phi[a, b; c; q, z]$, when $a = 0.99$, $b = 0.998$, $c = 0.98$, $q = 0.9$.

with $g_0 = 1 - a$. By a similar technique as in the proof of Theorem 1.1, one can show by using the hypothesis that $0 \le g_i \le 1$ for all i. Hence, Wall's theorem shows that there exists a non-decreasing function $\mu : [0, 1] \to [0, 1]$ such that $\mu(1) - \mu(0) = 1$ and

$$\frac{\Phi[aq, b; c; q, z]}{\Phi[a, b; c; q, z]} = \int_0^1 \frac{1}{1 - tz} \mathrm{d}\mu(t). \tag{2.5}$$

Thus, the assertion of our theorem follows. $\square$

Remark 2.7. If we substitute b by bq and c by cq in (2.5), we get the same integral expression for the ratio $z\Phi[aq, bq; cq; q, z]/\Phi[a, bq; cq; q, z]$. Moreover, if we substitute a by q^a, b by q^b and c by q^c, and apply the limit as $q \to 1^-$, we obtain a result of Küstner (see [**50**, Theorem 1.5]).

2.2.3. The Ratio $\dfrac{z\Phi[aq, bq; cq; q, z]}{\Phi[a, b; c; q, z]}$

Figure 2.4 visualizes the behaviour of the image domain of the disk $|z| < 0.999$ under the map $z\Phi[aq, bq; cq; q, z]/\Phi[a, b; c; q, z]$ when $a = 0.99$, $b = 0.998$, $c = 0.98$, $q = 0.9$. The following result obtains conditions on the parameters a, b, c for which the image domain will be convex in the direction of the imaginary axis.

Theorem 2.8. *For $q \in (0,1)$ suppose that a, b, c are non-negative real numbers satisfying $0 \le 1 - aq \le 1 - cq$ and $0 < 1 - b \le 1 - c$. Then there exists a non-decreasing function $\mu : [0,1] \to [0,1]$ with $\mu(1) - \mu(0) = 1$ such that*

$$\frac{z\Phi[aq, bq; cq; q, z]}{\Phi[a, b; c; q, z]} = \frac{1}{a} \int_0^1 \frac{z}{1 - tz} d\mu(t)$$

which is analytic in the cut-plane $\mathbb{C} \setminus [1, \infty]$ and maps both the unit disk and the half-plane $\{z \in \mathbb{C} : \operatorname{Re} z < 1\}$ univalently onto domains convex in the direction of the imaginary axis.

Proof. From the difference equation of Lemma 2.2**(b)** and Theorem 2.6, we have

$$\begin{aligned} \frac{z\Phi[aq, bq; cq; q, z]}{\Phi[a, b; c; q, z]} &= \frac{(1-c)}{a(1-b)} \left[\frac{\Phi[aq, b; c; q, z]}{\Phi[a, b; c; q, z]} - 1 \right] \\ &= \frac{(1-c)}{a(1-b)} \left[\int_0^1 \frac{1}{1 - tz} d\mu_0(t) - 1 \right], \end{aligned}$$

for some non-decreasing function $\mu_0 : [0,1] \to [0,1]$ with $\mu_0(1) - \mu_0(0) = 1$. Define

$$\mu_1(t) := \frac{1}{g_1} \int_0^t s \, d\mu_0(s)$$

for $g_1 = (1-b)/(1-c) > 0$ as in the proof of Theorem 2.6. It follows from [**50**, Remark 3.2] that

$$\frac{\Phi[aq, b; c; q, z]}{\Phi[a, b; c; q, z]} = \int_0^1 \frac{1}{1 - tz} d\mu_0(t) = 1 + g_1 \int_0^1 \frac{z}{1 - tz} d\mu_1(t)$$

where μ_1 is also a non-decreasing self-mapping of $[0,1]$ with $\mu_1(1) - \mu_1(0) = 1$. Finally, we get

$$\frac{z\Phi[aq, bq; cq; q, z]}{\Phi[a, b; c; q, z]} = \frac{(1-c)}{a(1-b)} \, g_1 \int_0^1 \frac{z}{1 - tz} d\mu_1(t) = \frac{1}{a} \int_0^1 \frac{z}{1 - tz} d\mu_1(t)$$

and thus, Lemma 2.1 proves the conclusion of our theorem. □

Remark 2.9. If we substitute a by q^a, b by q^b and c by q^c, then as $q \to 1^-$, we get the result of Küstner [**50**, Theorem 1.5] for the ratio $zF(a+1, b+1; c+1; z)/F(a, b; c; z)$ of the Gaussian hypergeometric functions. This function has also the similar mapping properties.

2.3. The q-close-to-convexity property

The q-close-to-convex functions are analytically characterized by the fact that $|g(z)+f(qz)-f(z)|/|g(z)|\leq 1$ for all $z\in\mathbb{D}$ (see [**97**, Lemma 3.1]). It shows that if the function $g(z)$ vanishes at z then z has to be zero, else the quotient $(g(z)+f(qz)-f(z))/g(z)$ would have a pole at $z=0$. However, one can see that if the function $g(z)$ has a zero of, say order r, at $z_0=0$ and $f'(z)$ has a zero of order at least $r-1$ there, then the quotient does not have a pole at $z=z_0$.

We recall the following lemma from [**85**] concerning a sufficient condition for the shifted Gaussian hypergeometric functions $zF(a,b;c;z)$ to be in $\mathcal{K}$.

Lemma 2.10. [**85**, Theorem 2.1] *Define* $T_1(a,b):=\max\{a+b,a+b+(ab-1)/2,2ab\}$ *for* $a,b>0$. *Suppose that* c *satisfies either* $c\geq T_1(a,b)$ *or* $c=a+b$ *with*

$$ab\geq 1,\quad a+b\leq 2ab \ \ \textit{and}\ \ \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\leq 2.$$

Then $zF(a,b;c;z)$ *is close-to-convex with* $g(z)=z/(1-z)$.

A number of problems on the convexity, starlikeness, and close-to-convexity properties of the Gaussian hypergeometric functions is investigated in [**32, 81, 85, 101**]. In fact, a large number of open problems on the starlikeness of hypergeometric functions are remained unsolved. Our objective in this section is to extend Lemma 2.10 associated with the shifted basic hypergeometric function $z\Phi[a,b;c;q,z]$. Theorem 1.3 in this direction improves a result obtained in [**86**]. For its proof we use the following result, a generalization of a result by MacGregor [**63**, Theorem 1], recently obtained in [**97**].

Lemma 2.11. [**97**] *Let* $\{A_n\}$ *be a sequence of real numbers such that* $A_1=1$ *and for all* $n\geq 1$, *define* $B_n=A_n(1-q^n)/(1-q)$. *Suppose that*

$$1\geq B_2\geq\cdots\geq B_n\geq\cdots\geq 0,$$

or,

$$1\leq B_2\leq\cdots\leq B_n\leq\cdots\leq 2$$

holds. Then $f(z)=z+\displaystyle\sum_{n=2}^{\infty}A_nz^n\in\mathcal{K}_q$ *with* $g(z)=z/(1-z)$.

The concept of q-gamma function is useful in this setting. The *q-gamma function* is denoted by $\Gamma_q(z)$ and is defined by the formula

$$\Gamma_q(z) = \frac{(q;q)_\infty}{(q^z;q)_\infty}(1-q)^{1-z}, \quad |q|<1,$$

where $(a;q)_\infty = \lim_{n\to\infty}(a;q)_n = \prod_{k=0}^{\infty}(1-aq^k)$. Here, the principal values of q^z and $(1-q)^{1-z}$ are chosen. Then $\Gamma_q(z)$ becomes a meromorphic function with obvious poles at $z = -n \pm 2\pi ik/\log q$, where k and n are non-negative integers. When $q \to 1^-$, the q-gamma function coincides with the gamma function Γ. For interesting properties of q-gamma function, reader can refer to [**7**].

The following limit formula is also used in the proof of Theorem 1.3.

Lemma 2.12. *For $0<q<1$ and the real parameters a, b, c, we have*

$$\lim_{n\to\infty} \frac{(q^a,q)_n(q^b,q)_n}{(q^c,q)_n(q,q)_n} = (1-q)^{c-a-b+1}\frac{\Gamma_q(c)}{\Gamma_q(a)\Gamma_q(b)}.$$

Proof. It suffices to show

$$\frac{\Gamma_q(a)\Gamma_q(b)}{\Gamma_q(c)} \lim_{n\to\infty} \frac{(q^a,q)_n(q^b,q)_n}{(q^c,q)_n(q,q)_n} = (1-q)^{c-a-b+1}.$$

Now,

$$\begin{aligned}
&\frac{\Gamma_q(a)\Gamma_q(b)}{\Gamma_q(c)} \lim_{n\to\infty} \frac{(q^a,q)_n(q^b,q)_n}{(q^c,q)_n(q,q)_n} \\
&= \lim_{n\to\infty}\left[\frac{(q,q)_n(1-q)^{1-a}(1-q^{n+1})^a(q,q)_n(1-q)^{1-b}(1-q^{n+1})^b(q^c,q)_{n+1}}{(q^a,q)_{n+1}(q^b,q)_{n+1}(q,q)_n(1-q)^{1-c}(1-q^{n+1})^c}\right] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\left[\frac{(q^a,q)_n(q^b,q)_n}{(q^c,q)_n(q,q)_n}\right] \\
&= \lim_{n\to\infty} \frac{(1-q)^{c-a-b+1}(q^c,q)_{n+1}(q^a,q)_n(q^b,q)_n}{(1-q^{n+1})^{c-a-b}(q^c,q)_n(q^a,q)_{n+1}(q^b,q)_{n+1}} \\
&= \lim_{n\to\infty} \frac{(1-q)^{c-a-b+1}(1-q^{n+c})}{(1-q^{n+1})^{c-a-b}(1-q^{n+a})(1-q^{n+b})} \\
&= (1-q)^{c-a-b+1}.
\end{aligned}$$

This completes the proof of our lemma. □

Remark 2.13. Taking $q \to 1^-$, the limit expression in Lemma 2.12, coincides with the well-known fact

$$\lim_{n\to\infty} \frac{(a)_n(b)_n}{(c)_n n!} = \begin{cases} \dfrac{\Gamma(c)}{\Gamma(a)\Gamma(b)} & \text{for } c+1 = a+b \\ 0 & \text{for } c+1 > a+b \\ \infty & \text{for } c+1 < a+b \end{cases}$$

described in [**85**].

We next use the limiting value

$$\lim_{n\to\infty} \frac{(a,q)_n(b,q)_n}{(c,q)_n(q,q)_n} = (1-q)^{\log_q c - \log_q a - \log_q b + 1} \frac{\Gamma_q(\log_q c)}{\Gamma_q(\log_q a)\Gamma_q(\log_q b)}$$

which can be easily verified with the substitutions $q^a \to a$, $q^b \to b$ and $q^c \to c$. Note that $\log_q$ represents the ordinary logarithm with base q.

Proof of Theorem 1.3. Let $f(z) = z\Phi[a, b; c; q, z]$. Then $f \in \mathcal{A}$ and is of the form $f(z) = z + \sum_{n=2}^{\infty} A_n z^n$, where

$$A_1 = 1, \quad A_n = \frac{(a,q)_{n-1}(b,q)_{n-1}}{(c,q)_{n-1}(q,q)_{n-1}}, \qquad \text{for } n \geq 2.$$

From the definition of A_n, we observe the recurrence relation:

$$A_{n+1} = \frac{(1-aq^{n-1})(1-bq^{n-1})}{(1-cq^{n-1})(1-q^n)} A_n.$$

First we will treat the situation covered by formula (1.5) by showing that $\{((1-q^n)/(1-q))A_n\}$ is a decreasing sequence of positive real numbers. For this we compute

$$\begin{aligned}
&\left(\frac{1-q^n}{1-q}\right) A_n - \left(\frac{1-q^{n+1}}{1-q}\right) A_{n+1} \\
&= \left(\frac{1-q^n}{1-q}\right) A_n - \left(\frac{1-q^{n+1}}{1-q}\right) \frac{(1-aq^{n-1})(1-bq^{n-1})}{(1-cq^{n-1})(1-q^n)} A_n \\
&= \frac{A_n[(1-q^n)^2(1-cq^{n-1}) - (1-q^{n+1})(1-aq^{n-1})(1-bq^{n-1})]}{(1-cq^{n-1})(1-q^n)(1-q)} \\
&= \frac{A_n}{\left(\dfrac{1-cq^{n-1}}{1-q}\right)\left(\dfrac{1-q^n}{1-q}\right)} X(n)
\end{aligned}$$

where

$$X(n) = \frac{1}{(1-q)^3}\left[(1-q^n)^2(1-cq^{n-1}) - (1-q^{n+1})(1-aq^{n-1})(1-bq^{n-1})\right].$$

On simplification, we have

$$X(n) = q^{n-1}\left\{\left(\frac{1-q^n}{1-q}\right)^2\left(\frac{ab-c}{1-q}\right)+\left(\frac{1-q^n}{1-q}\right)\left(\frac{aq+bq-q-2ab+ab/q}{(1-q)^2}\right)\right.$$
$$\left.+\left(\frac{a+b-q-ab/q}{(1-q)^2}\right)\right\}.$$

Therefore, to prove the first part, it is sufficient to show that $X(n)$ is non-negative. Note that the condition (1.5) implies $c \leq ab$ and so the coefficient of the factor $((1-q^n)/(1-q))^2$ in the above expression of $X(n)$ is non-negative. Thus, for all $n \geq 1$, we can write

$$\begin{aligned} X(n) &\geq q^{n-1}\left[\left(2\left(\frac{1-q^n}{1-q}\right)-1\right)\left(\frac{ab-c}{1-q}\right)+\left(\frac{1-q^n}{1-q}\right)\right. \\ &\qquad \left.\left(\frac{aq+bq-q-2ab+ab/q}{(1-q)^2}\right)+\left(\frac{a+b-q-ab/q}{(1-q)^2}\right)\right] \\ &= q^{n-1}\left[\left(\frac{1-q^n}{1-q}\right)\left\{2\left(\frac{ab-c}{1-q}\right)+\left(\frac{aq+bq-q-2ab+ab/q}{(1-q)^2}\right)\right\}\right. \\ &\qquad \left.+\left(\frac{a+b-q-ab/q}{(1-q)^2}\right)-\left(\frac{ab-c}{1-q}\right)\right] = Y(n), \text{ say.} \end{aligned}$$

By equation (1.5), we have $c \leq ab+(aq+bq-q-2ab+ab/q)/(2(1-q))$. So, the coefficient of $(1-q^n)/(1-q)$ in the expression of $Y(n)$ is non-negative and hence we obtain

$$X(n) \geq Y(n) \geq Y(1) = \left(\frac{ab-c}{1-q}\right)+\left(\frac{aq+bq-q-2ab+ab/q}{(1-q)^2}\right)$$
$$+\left(\frac{a+b-q-ab/q}{(1-q)^2}\right).$$

Again, by (1.5), we get $Y(1) \geq 0$. This argument proves that if $c \leq T_1(a,b)$ then the function $z\Phi[a,b;c;q,z] \in \mathcal{K}_q$ with the starlike function $g(z) = z/(1-z)$.

To cover the situation of formula (1.6), we are going to show that $\{((1-q^n)/(1-q))A_n\}$ is a non-decreasing sequence and has a limit less than or equal to 2. From (1.6), we note that $c = ab$ and $ab \geq (aq+bq-q)/(2-q^{-1})$. So, by the hypothesis (1.6), we obtain

$$X(n) = Y(n) \leq Y(1) = \left(\frac{aq+bq-q-2ab+ab/q}{(1-q)^2}\right)+\left(\frac{a+b-q-ab/q}{(1-q)^2}\right) \leq 0.$$

Now, we have to show that the limiting value of $A_n(1-q^n)/(1-q)$ is less than or equal to 2. Write $c = ab$ and

$$\begin{aligned}
\left(\frac{1-q^n}{1-q}\right) A_n &= \left(\frac{1-q^n}{1-q} - 1\right) A_n + A_n \\
&= \frac{q(1-q^{n-1})}{1-q} \frac{(a,q)_{n-1}(b,q)_{n-1}}{(c,q)_{n-1}(q,q)_{n-1}} + \frac{(a,q)_{n-1}(b,q)_{n-1}}{(c,q)_{n-1}(q,q)_{n-1}} \\
&= \frac{q}{1-q} \frac{(a,q)_{n-2}(1-aq^{n-2})(b,q)_{n-2}(1-bq^{n-2})}{(c,q)_{n-2}(1-cq^{n-2})(q,q)_{n-2}} + \frac{(a,q)_{n-1}(b,q)_{n-1}}{(c,q)_{n-1}(q,q)_{n-1}}.
\end{aligned}$$

Taking limit as $n \to \infty$ on both the sides, we have

$$\lim_{n\to\infty} \left(\frac{1-q^n}{1-q}\right) A_n = \frac{q}{1-q} \lim_{n\to\infty} \frac{(a,q)_{n-2}(b,q)_{n-2}}{(c,q)_{n-2}(q,q)_{n-2}} + \lim_{n\to\infty} \frac{(a,q)_{n-1}(b,q)_{n-1}}{(c,q)_{n-1}(q,q)_{n-1}}.$$

From Remark 2.13, we have

$$\begin{aligned}
\lim_{n\to\infty} \left(\frac{1-q^n}{1-q}\right) A_n &= \frac{q}{1-q}(1-q)^{\log_q c - \log_q a - \log_q b + 1} \frac{\Gamma_q(\log_q c)}{\Gamma_q(\log_q a)\Gamma_q(\log_q b)} \\
&\quad + (1-q)^{\log_q c - \log_q a - \log_q b + 1} \frac{\Gamma_q(\log_q c)}{\Gamma_q(\log_q a)\Gamma_q(\log_q b)}.
\end{aligned}$$

Using $c = ab$, the above expression reduces to

$$\begin{aligned}
\lim_{n\to\infty} \left(\frac{1-q^n}{1-q}\right) A_n &= q \frac{\Gamma_q(\log_q ab)}{\Gamma_q(\log_q a)\Gamma_q(\log_q b)} + (1-q) \frac{\Gamma_q(\log_q ab)}{\Gamma_q(\log_q a)\Gamma_q(\log_q b)} \\
&= \frac{\Gamma_q(\log_q ab)}{\Gamma_q(\log_q a)\Gamma_q(\log_q b)}.
\end{aligned}$$

The conclusion follows from (1.6) and Lemma 2.11. $\square$

Corollary 2.14. *Let $a, b < 1/q$ and $(1-a)(1-b) \neq 0$. If c satisfies either $c \leq T_1(aq, bq)/q$ where $T_1(a, b)$ is defined in Theorem 1.3, or*

$$c = abq \text{ with } abq \geq \max\left\{\frac{aq+bq-1}{2-(1/q)}, \frac{aq+bq+a+b-2}{2}\right\}$$

$$\text{and } \frac{\Gamma_q(\log_q abq^2)}{\Gamma_q(\log_q aq)\Gamma_q(\log_q bq)} \leq 2$$

then $z(D_q\phi)(z)$ is q-close-to-convex in $\mathbb{D}$, where $\phi(z) = \Phi[a, b; c; q, z]$.

Proof. Some simple calculation gives the q-differentiation of $\phi(z)$ in the following form:

$$\frac{(1-a)(1-b)}{(1-c)(1-q)} z\Phi[aq, bq; cq; q, z] = z(D_q\phi)(z)$$

i.e.
$$z\Phi[aq, bq; cq; q, z] = \frac{(1-c)(1-q)}{(1-a)(1-b)} z(D_q\phi)(z).$$
Apply this identity in Theorem 1.3 and deduce that the function $z\Phi[aq, bq; cq; q, z]$ is in $\mathcal{K}_q$ with the starlike function $z/(1-z)$. Therefore, the conclusion of our corollary follows. $\square$

CHAPTER 3

A GENERALIZATION OF STARLIKE FUNCTIONS OF ORDER ALPHA

The aim of this chapter is to investigate the Bieberbach conjecture problem for q-starlike functions of order α.

This chapter is based on the paper: Agrawal S., Sahoo S. K. (2015), A generalization of starlike functions of order alpha, Hokkaido Math. J., Accepted, `arXiv:1404.3988 [math.CV]`.

3.1. Introduction and Main Results

As we discussed in Chapter 1, many techniques were developed to give a partial solution to the Bieberbach conjecture. One of the important techniques is the *Herglotz representation theorem* for univalent functions with positive real part. The conjecture was considered in many special cases. In one direction, it was considered for certain subclasses of univalent functions like starlike, convex, close-to-convex, typically real functions, etc. The concept of order for the starlike and convex was also introduced, which are the subclasses of the class of starlike and convex functions respectively, and the conjecture was proved in these subclasses. In other direction, many conjectures which imply the Bieberbach conjecture were discussed; namely, the Zalcman conjecture, the Robertson conjecture, the Littlewood-Paley conjecture, etc. Finally, the full conjecture for univalent functions was settled by de Branges in 1985 [**13**].

We aim to introduce a class of q-starlike functions of order α and prove the Bieberbach type problem for the same class. In particular, we also discuss several other basic properties on the order of q-starlike functions.

A function $f \in \mathcal{A}$ is called starlike of order α, $0 \leq \alpha < 1$, if

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad z \in \mathbb{D}.$$

We use the notation $\mathcal{S}^*(\alpha)$ for the class of starlike functions of order α. Set $\mathcal{S}^* := \mathcal{S}^*(0)$, the class of all starlike functions.

One way to generalize the starlike functions of order α is to replace the derivative function f' by the q-difference operator $D_q f$, which is defined in (1.2), and replace the right-half plane $\{w : \operatorname{Re} w > \alpha\}$ by a suitable domain in the definition of the starlike functions of order α. The appropriate definition turned out to be Definition 1.4. The following is the equivalent form of Definition 1.4.

$$f \in \mathcal{S}_q^*(\alpha) \iff \left| \frac{z(D_q f)(z)}{f(z)} - \frac{1-\alpha q}{1-q} \right| \le \frac{1-\alpha}{1-q}.$$

Observe that as $q \to 1^-$ the closed disk $|w-(1-\alpha q)(1-q)^{-1}| \le (1-\alpha)(1-q)^{-1}$ becomes the right-half plane $\operatorname{Re} w \ge \alpha$ and the class $\mathcal{S}_q^*(\alpha)$ reduces to $\mathcal{S}^*(\alpha)$, $0 \le \alpha < 1$. In particular, when $\alpha = 0$, the class $\mathcal{S}_q^*(\alpha)$ coincides with the class $\mathcal{S}_q^* := \mathcal{S}_q^*(0)$, which was studied in [**37**] and also in recent years [**3, 86, 92, 97**]. In words we call $\mathcal{S}_q^*(\alpha)$, the class of *q-starlike functions of order α*.

The first main theorem of this chapter describes the Herglotz representation for functions belonging to the class $\mathcal{S}_q^*(\alpha)$ in the form of a Poisson-Stieltjes integral which is stated in Theorem 1.5 (see Herglotz Representation Theorem for analytic functions with positive real part in [**21**, pp. 22]).

Remark 3.1. When q approaches 1, Theorem 1.5 leads to the Herglotz Representation Theorem for starlike functions of order α (see for instance [**28**, Problem 3, pp. 172]). Note that the coefficients of the function $F_{q,\alpha}$ are all positive.

Our second main theorem is Theorem 1.6. This theorem concerns about the Bieberbach conjecture problem for functions in $\mathcal{S}_q^*(\alpha)$. The extremal function is also explicitly obtained in terms of exponential of the function $F_{q,\alpha}(z)$. This exponential form generalizes the Koebe function $k_\alpha(z) = z/(1-z)^{2(1-\alpha)}$, $z \in \mathbb{D}$. That is, when $q \to 1^-$, the exponential form $G_{q,\alpha}(z) := z \exp[F_{q,\alpha}(z)]$ representing the extremal function for the class $\mathcal{S}_q^*(\alpha)$ turns into the Koebe function $k_\alpha(z)$.

Remark 3.2. When q approaches 1, Theorem 1.6 leads to the Bieberbach conjecture for starlike functions of order α (see for instance [**28**, Theorem 2, pp. 140]).

In [**37**], the authors have obtained the Herglotz representation for functions of the class $\mathcal{S}_q^*$ in the following form:

Theorem A. [**37**, Theorem 1.15] *Let $f \in \mathcal{A}$. Then $f \in \mathcal{S}_q^*$ if and only if there exists a probability measure μ supported on the unit circle such that*

$$\frac{zf'(z)}{f(z)} = 1 + \int_{|\sigma|=1} \sigma z F_q^{'}(\sigma z) \mathrm{d}\mu(\sigma)$$

where

$$F_q(z) = \sum_{n=1}^{\infty} \frac{-2\ln q}{1-q^n} z^n, \quad z \in \mathbb{D}.$$

Also they have proved the Bieberbach conjecture problem for q-starlike functions in the following form:

Theorem B. [**37**, Theorem 1.18] *Let*

$$G_q(z) := z \exp[F_q(z)] = z + \sum_{n=2}^{\infty} c_n z^n.$$

Then $G_q \in \mathcal{S}_q^*$. Moreover, if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}_q^*$, then $|a_n| \leq c_n$ with equality holding for all n if and only if f is a rotation of G_q.

Remark 3.3. Note that Theorem 1.5 and Theorem 1.6 are respectively generalizations of Theorem A and Theorem B.

3.2. Properties of the class $\mathcal{S}_q^*(\alpha)$

As a matter of fact, the following proposition says that a function f in $\mathcal{S}_q^*(\alpha)$ can be obtained in terms of a function g in $\mathcal{S}_q^*$. The proof is obvious and it follows from the definition of $\mathcal{S}_q^*(\alpha)$, $0 \leq \alpha < 1$.

Proposition 3.4. *Let $f \in \mathcal{S}_q^*(\alpha)$. Then there exists a unique function $g \in \mathcal{S}_q^*$ such that*

$$\frac{\dfrac{z(D_q f)(z)}{f(z)} - \alpha}{1-\alpha} = \frac{z(D_q g)(z)}{g(z)} \quad or \quad \frac{f(qz) - \alpha q f(z)}{(1-\alpha)f(z)} = \frac{g(qz)}{g(z)}. \tag{3.1}$$

holds. Similarly, for a given function $g \in \mathcal{S}_q^*$ there exists $f \in \mathcal{S}_q^*(\alpha)$ satisfying the above relation. Uniqueness follows trivially.

Next, we present a easy characterization of functions in the class $\mathcal{S}_q^*(\alpha)$. This shows that if $f \in \mathcal{S}_q^*(\alpha)$ then $f(z) = 0$ implies $z = 0$, otherwise $f(qz)/f(z)$ would have a pole at a zero of $f(z)$ with least nonzero modulus.

Theorem 3.5. *Let $f \in \mathcal{A}$. Then $f \in \mathcal{S}_q^*(\alpha)$ if and only if*

$$\left|\frac{f(qz)}{f(z)} - \alpha q\right| \le 1-\alpha, \quad z \in \mathbb{D}.$$

Proof. The proof can be easily obtained from the fact

$$\frac{z(D_q f)(z)}{f(z)} = \left(\frac{1}{1-q}\right)\left(1 - \frac{f(qz)}{f(z)}\right)$$

and the definition of $\mathcal{S}_q^*(\alpha)$. □

The next result is an immediate consequence of Theorem 3.5.

Corollary 3.6. *The class $\mathcal{S}_q^*(\alpha)$ satisfies the inclusion relation*

$$\bigcap_{q<p<1} \mathcal{S}_p^*(\alpha) \subset \mathcal{S}_q^*(\alpha) \quad \text{and} \quad \bigcap_{0<q<1} \mathcal{S}_q^*(\alpha) = \mathcal{S}^*(\alpha).$$

Proof. The inclusions

$$\bigcap_{q<p<1} \mathcal{S}_p^*(\alpha) \subset \mathcal{S}_q^*(\alpha) \;\text{ and }\; \bigcap_{0<q<1} \mathcal{S}_q^*(\alpha) \subset \mathcal{S}^*(\alpha)$$

clearly hold. It remains to show that

$$\mathcal{S}^*(\alpha) \subset \bigcap_{0<q<1} \mathcal{S}_q^*(\alpha)$$

holds. For this, we let $f \in \mathcal{S}^*(\alpha)$. Then it is enough to show that $f \in \mathcal{S}_q^*(\alpha)$ for all $q \in (0,1)$. Since $f \in \mathcal{S}^*(\alpha)$ there exists a unique $g \in \mathcal{S}^*$ satisfying

$$\frac{\dfrac{zf'(z)}{f(z)} - \alpha}{1-\alpha} = \frac{zg'(z)}{g(z)}, \quad |z| < 1.$$

Since $\mathcal{S}^* = \cap_{0<q<1}\mathcal{S}_q^*$, it follows that $g \in \mathcal{S}_q^*$ for all $q \in (0,1)$. Thus, by Proposition 3.4 there exists a unique $h \in \mathcal{S}_q^*(\alpha)$ satisfying the identity (3.1) with $h(z) = f(z)$. The proof now follows immediately. □

We now define two sets and proceed to prepare some basic results which are being used to prove our main results in this section. They are

$$B_q = \{g : g \in \mathcal{H}(\mathbb{D}),\; g(0) = q \text{ and } g : \mathbb{D} \to \mathbb{D}\} \;\text{ and }\; B_q^0 = \{g : g \in B_q \text{ and } 0 \notin g(\mathbb{D})\}.$$

Now it is appropriate to recall the following lemma.

Lemma 3.7. [**37**, Lemma 2.1] *If $g \in B_q$ then the infinite product $\prod_{n=0}^{\infty}\{g(zq^n)/q\}$ converges uniformly on compact subsets of $\mathbb{D}$.*

Using Lemma 3.7, we prove the following lemma.

Lemma 3.8. *If $h \in B_q$ then the infinite product $\prod_{n=0}^{\infty}\{((1-\alpha)h(zq^n)+\alpha q)/q\}$ converges uniformly on compact subsets of $\mathbb{D}$.*

Proof. We set $(1-\alpha)h(z)+\alpha q = g(z)$. Since $h \in B_q$, it easily follows that $g \in B_q$. By Lemma 3.7, the conclusion of our lemma follows. $\square$

Lemma 3.9. *If $h \in B_q^0$ then the infinite product $\prod_{n=0}^{\infty}\{((1-\alpha)h(zq^n)+\alpha q)/q\}$ converges uniformly on compact subsets of $\mathbb{D}$ to a nonzero function in $\mathcal{H}(\mathbb{D})$ with no zeros. Furthermore, the function*

$$f(z) = \frac{z}{\prod_{n=0}^{\infty}\{((1-\alpha)h(zq^n)+\alpha q)/q\}} \tag{3.2}$$

belongs to $\mathcal{S}_q^*(\alpha)$ and $h(z) = ((f(qz)/f(z)) - \alpha q)/(1-\alpha)$.

Proof. The convergence of the infinite product is proved in Lemma 3.8. Since $h \in B_q^0$, we have $h(z) \neq 0$ in $\mathbb{D}$ and the infinite product does not vanish in $\mathbb{D}$. Thus, the function $f \in \mathcal{A}$ and we find the relation

$$\frac{f(qz)}{f(z)} = (1-\alpha)h(z)+\alpha q, \quad \text{equivalently} \quad \frac{\dfrac{f(qz)}{f(z)} - \alpha q}{1-\alpha} = h(z).$$

Since $h \in B_q^0$, we get $f \in \mathcal{S}_q^*(\alpha)$ and the proof of our lemma is complete. $\square$

We define two classes $B_{q,\alpha}$ and $B_{q,\alpha}^0$ by

$$B_{q,\alpha} = \left\{ g : g \in \mathcal{H}(\mathbb{D}),\ g(0) = \frac{q}{1-\alpha(1-q)} \text{ and } g : \mathbb{D} \to \mathbb{D} \right\}$$

and

$$B_{q,\alpha}^0 = \{g : g \in B_{q,\alpha} \text{ and } 0 \notin g(\mathbb{D})\}.$$

Lemma 3.10. *A function $g \in B_{q,\alpha}^0$ if and only if it has the representation*

$$g(z) = \exp\left\{ \left(\ln \frac{q}{1-\alpha(1-q)} \right) p(z) \right\}, \tag{3.3}$$

where $p(z)$ belongs to the class

$$\mathcal{P} = \{p : p \in \mathcal{H}(\mathbb{D}), p(0) = 1 \textit{ and } \operatorname{Re}(p(z)) > 0 \textit{ for } z \in \mathbb{D}\}.$$

Proof. For $g \in B^0_{q,\alpha}$, define the function $L(z) = \operatorname{Log} g(z)$. Then it is easy to show that the function $p(z) = \dfrac{L(z)}{\ln \frac{q}{1-\alpha(1-q)}} \in \mathcal{P}$ and satisfies (3.3). Conversely, if g is given by (3.3), then it is obvious that $g \in B^0_{q,\alpha}$. $\square$

Theorem 3.11. *The mapping $\rho : \mathcal{S}^*_q(\alpha) \to B^0_q$ defined by*

$$\rho(f)(z) = \frac{\dfrac{f(qz)}{f(z)} - \alpha q}{1-\alpha}$$

is a bijection.

Proof. For $h \in B^0_q$, define a mapping $\sigma : B^0_q \to \mathcal{A}$ by

$$\sigma(h)(z) = \frac{z}{\prod_{n=0}^{\infty}\{((1-\alpha)h(zq^n)+\alpha q)/q\}}.$$

It is clear from Lemma 3.9 that $\sigma(h) \in \mathcal{S}^*_q(\alpha)$ and $(\rho \circ \sigma)(h) = h$. Considering the composition mapping $\sigma \circ \rho$ we compute that

$$(\sigma \circ \rho)(f)(z) = \frac{z}{\prod_{n=0}^{\infty}\{(f(zq^{n+1})/qf(zq^n)\}} = \frac{z}{z/f(z)} = f(z).$$

Hence $\sigma \circ \rho$ and $\rho \circ \sigma$ are identity mappings and σ is the inverse of ρ, i.e. the map $\rho(f)$ is invertible. Hence $\rho(f)$ is a bijection. This completes the proof of our theorem. $\square$

3.3. Proof of the main theorems

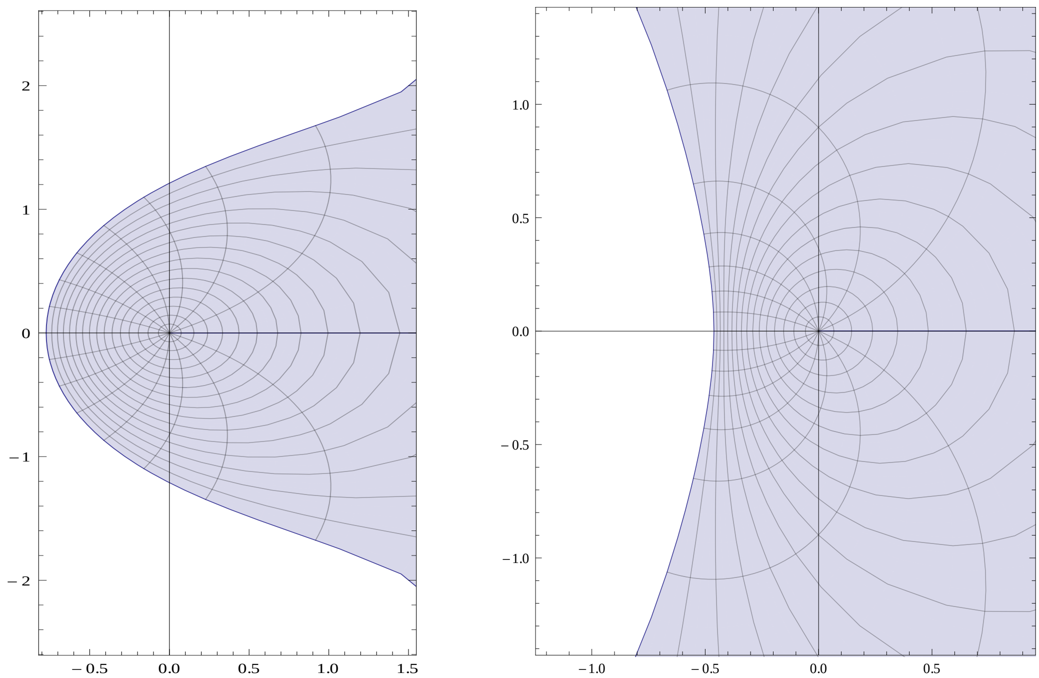
This section is devoted to the proofs of main theorems using the supplementary results proved in Section 3.2.

Proof of Theorem 1.5. For $0 < q < 1$ and $0 \le \alpha < 1$, let $F_{q,\alpha}$ be defined by (1.7). Geometry of $F_{q,\alpha}$ is described in Figure 3.1 for different ranges over the parameters q and α. Suppose that $f \in \mathcal{S}^*_q(\alpha)$. Then by Theorem 3.11 and Lemma 3.9, it is clear that f has the representation (3.2) with $h \in B^0_q$. The logarithmic derivative of f gives

$$\frac{zf'(z)}{f(z)} = 1 - \sum_{n=0}^{\infty} \frac{(1-\alpha)zq^n h'(zq^n)}{(1-\alpha)h(zq^n)+\alpha q}. \tag{3.4}$$

Now, let us assume that

$$g(z) = \frac{(1-\alpha)h(z)+\alpha q}{1-\alpha(1-q)}.$$



Graph of $F_{5/6,1/2}(z)$, $|z| < 1$ Graph of $G_{5/6,1/2}(z)$, $|z| < 1$

FIGURE 3.1. Graphs of the functions $F_{5/6,1/2}(z)$ and $G_{5/6,1/2}(z)$ for $|z| < 1$.

Clearly, $g \in B^0_{q,\alpha}$ and hence Lemma 3.10 guarantees that $g(z)$ has the representation (3.3). Taking the logarithmic derivative of g we have

$$\frac{zg'(z)}{g(z)} = \left(\ln \frac{q}{1-\alpha(1-q)}\right) zp'(z), \tag{3.5}$$

where $\operatorname{Re}(p(z)) \geq 0$. By Herglotz representation of $p(z)$, there exists a probability measure μ supported on the unit circle $|\sigma| = 1$ such that

$$zp'(z) = \int_{|\sigma|=1} 2\sigma z(1-\sigma z)^{-2} d\mu(\sigma). \tag{3.6}$$

Using (3.5) and (3.6) in (3.4), we have

$$\begin{aligned}
\frac{zf'(z)}{f(z)} &= 1 - 2\left(\ln \frac{q}{1-\alpha(1-q)}\right) \sum_{n=0}^{\infty} \int_{|\sigma|=1} \sigma z q^n (1-\sigma z q^n)^{-2} d\mu(\sigma) \\
&= 1 - 2\left(\ln \frac{q}{1-\alpha(1-q)}\right) \int_{|\sigma|=1} \left\{ \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} m\sigma^m z^m q^{mn} \right\} d\mu(\sigma) \\
&= 1 - 2\left(\ln \frac{q}{1-\alpha(1-q)}\right) \int_{|\sigma|=1} \left\{ \sum_{m=1}^{\infty} m\sigma^m z^m \frac{1}{1-q^m} \right\} d\mu(\sigma) \\
&= 1 + \int_{|\sigma|=1} \sigma z F'_{q,\alpha}(\sigma z) \mathrm{d}\mu(\sigma).
\end{aligned}$$

This completes the proof of our theorem. $\square$

Proof of Theorem 1.6. For $0 < q < 1$ and $0 \leq \alpha < 1$, let $G_{q,\alpha}$ be defined by (1.8). Geometry of the mapping $G_{q,\alpha}$ is described in Figure 3.1 for different ranges over the parameters q and α. As a special case to Theorem 1.5, when the measure has a unit mass, it is clear that $G_{q,\alpha} \in \mathcal{S}_q^*(\alpha)$. Let $f \in \mathcal{S}_q^*(\alpha)$. Then by Theorem 3.11, there exists a function $h \in B_q^0$ such that $h(z) = ((f(qz)/f(z)) - \alpha q)/(1-\alpha)$. Now set $g(z) = ((1-\alpha)h(z) + \alpha q)/(1-\alpha(1-q))$. It is clear that $g(0) = q/(1-\alpha(1-q))$ and $|g(z)| < 1$. Since $\mathcal{S}_q^*(\alpha) \subseteq \mathcal{S}_q^*$, it follows that $0 \notin g(\mathbb{D})$. If not, then $f(qz)/f(z) = 0$ for some $z \in \mathbb{D}$, which is a contradiction to the fact that $f(qz)/f(z) \in B_q^0$ for every $z \in \mathbb{D}$ [**37**, Theorem 1.13]. Hence, $g \in B_{q,\alpha}^0$. By Lemma 3.10, $g(z)$ has the representation (3.3) and on solving we get,

$$\frac{f(qz)}{f(z)} = (1-\alpha(1-q)) \exp\left\{\left(\ln \frac{q}{1-\alpha(1-q)}\right) p(z)\right\}.$$

Define the function $\phi(z) = \operatorname{Log}\{f(z)/z\}$ and set

$$\phi(z) = \operatorname{Log} \frac{f(z)}{z} = \sum_{n=1}^{\infty} \phi_n z^n. \tag{3.7}$$

On solving, we get

$$\ln \frac{q}{1-\alpha(1-q)} + \phi(qz) = \phi(z) + \left(\ln \frac{q}{1-\alpha(1-q)}\right) p(z).$$

This implies

$$\phi_n = p_n \left(\ln \frac{q}{1-\alpha(1-q)}\right) / (q^n - 1).$$

Since $|p_n| \leq 2$, we have

$$|\phi_n| \leq \frac{(-2)\left(\ln \frac{q}{1-\alpha(1-q)}\right)}{1-q^n}.$$

From this inequality, together with the expression of $G_{q,\alpha}(z)$ and (3.7), the conclusion follows. $\square$

CHAPTER 4

ON COEFFICIENT FUNCTIONALS ASSOCIATED WITH THE ZALCMAN CONJECTURE

This chapter covers the sharp estimates of the Zalcman functionals including the form proposed by Ma in [**61**] for some subclasses of $\mathcal{S}$. The main results of this chapter are proved in Section 4.2.

The results of this chapter are from the article: Agrawal S., Sahoo S. K., On coefficient functionals associated with the Zalcman conjecture, Under review.

4.1. Preliminaries and Main results

The Zalcman conjecture problem (1.4) has been studied for several well-known subclasses of the class $\mathcal{S}$. For example, in [**15**], Brown and Tsao proved that (1.4) holds for the class $\mathcal{T}$ of typically real functions and the class $\mathcal{S}^*$ of starlike functions. In [**60**], Ma proved the Zalcman conjecture for the class $\mathcal{K}$ of close-to-convex functions when $n \geq 4$. Readers can refer to, for instance, [**1, 48, 49, 54**] and references therein for more information on this topic. A generalized version of Zalcman's inequality, in terms of the so-called *generalized coefficient functional* $\lambda a_n^2 - a_{2n-1}$, $\lambda > 0$, has been considered in [**1, 15, 22, 54**].

In [**61**], Ma proposed a generalized version of the Zalcman conjecture as follows: for $f \in \mathcal{S}$,

$$|a_n a_m - a_{n+m-1}| \leq (n-1)(m-1) \quad (n, m = 2, 3, \ldots)$$

and proved that this holds for starlike functions and univalent functions with real coefficients.

In this paper, we establish sharp estimates of the Zalcman conjecture in the form proposed by Ma in [**61**] for some subclasses of $\mathcal{S}$. Consequently, we obtain sharp estimates of the results proved in [**22**] for remaining ranges of λ.

We use the concept of *convex hull of a set*, but mainly for the set $\mathcal{C}$ of convex functions. Denote by $co(\mathcal{C})$, the *convex hull of* $\mathcal{C}$ and its closure is denoted by $\overline{co(\mathcal{C})}$ in the topology of uniform convergence on compact subsets of $\mathbb{D}$.

A function $f \in \mathcal{A}$ is said to be *starlike of order* $\beta\,(0 \le \beta < 1)$ if $\operatorname{Re}\{zf'(z)/f(z)\} > \beta$ and denote the class of starlike functions of order β by $\mathcal{S}^*(\beta)$. Similarly, a function $f \in \mathcal{A}$ is said to be *convex of order* $\beta\,(0 \le \beta < 1)$ if $\operatorname{Re}\{1 + zf''(z)/f'(z)\} > \beta$ and denote the class of convex functions of order β by $\mathcal{C}(\beta)$. Clearly, functions in the classes $\mathcal{S}^*(\beta)$ and $\mathcal{C}(\beta)$ are univalent in $\mathbb{D}$. Moreover $\mathcal{S}^*(0) = \mathcal{S}^*$ and $\mathcal{C}(0) = \mathcal{C}$.

A function f is said to be *uniformly starlike* in $\mathbb{D}$ if f is starlike and has the property that for every circular arc γ contained in $\mathbb{D}$, with center $\zeta \in \mathbb{D}$, the arc $f(\gamma)$ is starlike with respect to $f(\zeta)$. We denote by $\mathcal{UST}$, the class of all uniformly starlike functions. Similarly, we say a convex function f in $\mathbb{D}$ is *uniformly convex* if for each circular arc γ in $\mathbb{D}$ with center η in $\mathbb{D}$, the image arc $f(\gamma)$ is convex. Denote the class of all uniformly convex functions by $\mathcal{UCV}$, see [**29, 30**]. We call a function $f \in \mathcal{A}$ be *ν-spiral-like of order* $\beta, 0 \le \beta < 1$, if there is a real number $\nu\,(-\pi/2 < \nu < \pi/2)$ such that $\operatorname{Re}[e^{i\nu}\{zf'(z)/f(z)\}] > \beta\cos\nu$ for $z \in \mathbb{D}$. We denote by $\mathcal{S}_p^\nu(\beta)$, the class of ν-spiral-like functions of order β, see [**51**]. More literature on spiral-like functions can be found in [**5, 56, 71**].

Recently, in [**22**], Efraimidis and Vukotić have studied the generalized Zalcman coefficient functional for the subclasses, $\overline{co(\mathcal{C})}$, $\mathcal{R}$ and H of $\mathcal{S}$, where the classes $\mathcal{R}$ and H are respectively known as *the Noshiro-Warschawski class* and *the Hurwitz class*, defined by

$$\mathcal{R} = \{f \in \mathcal{A} : \operatorname{Re} f'(z) > 0\}$$

and

$$H = \left\{ f \in \mathcal{A} : f(z) = z + \sum_{n=2}^{\infty} a_n z^n \text{ and } \sum_{n=2}^{\infty} n|a_n| \le 1 \right\}.$$

A well-known fact is that

$$H \subset \mathcal{R} \cap \mathcal{S}^* \subseteq \mathcal{S},$$

where the inclusion relation $H \subset \mathcal{R}$ is explained in [**22**]. Now we recall

Theorem A. [**22**, Theorem 3] *Let* $0 < \lambda \le 2$. *If* $f \in \overline{co(\mathcal{C})}$, *then* $|\lambda a_n^2 - a_{2n-1}| \le 1$ *for all* $n \ge 2$. *For any fixed* n *and* $\lambda < 2$, *equality holds only for the functions of the following* *form (and for their rotations):*

$$f(z) = \sum_{k=1}^{2n-2} m_k \frac{z}{1 - e^{i\theta_k} z}, \tag{4.1}$$

where $0 \leq m_k \leq 1$, $\theta_k = \frac{(2k+1)\pi}{2n-2}$, *and*

$$\sum_{k=1}^{n-1} m_{2k} = \sum_{k=1}^{n-1} m_{2k-1} = \frac{1}{2}.$$

Theorem B. [**22**, Theorem 4] *If* $0 < \lambda \leq 4/3$ *and* $f \in \mathcal{R}$, *then for all* $n \geq 2$ *we have*

$$|\lambda a_n^2 - a_{2n-1}| \leq \frac{2}{2n-1}.$$

For $\lambda < 4/3$ *and for any fixed* $n \geq 2$, *equality holds only for the functions of the following form (and for their rotations):*

$$F(z) = -z + 2\int_0^z \frac{f(t)}{t} dt = -z - \sum_{k=1}^{2n-2} 2m_k e^{-i\theta_k} \log(1 - e^{i\theta_k} z)$$

where $f(z), m_k$, *and* θ_k *are given by* (4.1).

Theorem C. [**22**, Theorem 6] *If* $\lambda > 0$ *and* $f \in H$, *then for each* $n \geq 2$ *we have*

$$|\lambda a_n^2 - a_{2n-1}| \leq \max\left\{\frac{\lambda}{n^2}, \frac{1}{2n-1}\right\}.$$

Equality holds if and only if

$$f(z) = \begin{cases} z + \dfrac{\alpha}{2n-1} z^{2n-1} & \text{for } \lambda \leq \dfrac{n^2}{2n-1}, \\ z + \dfrac{\alpha}{n} z^n & \text{for } \lambda \geq \dfrac{n^2}{2n-1}, \end{cases}$$

where α *is a complex number such that* $|\alpha| = 1$.

We intend to extend Theorems A and B in terms of the generalized Zalcman conjecture, in the form suggested by Ma in [**61**], for the classes $\overline{co(\mathcal{C})}$ and $\mathcal{R}(\beta) := \{f \in \mathcal{A} : \operatorname{Re} f'(z) > \beta\}$ respectively, where $\beta \in [0, 1)$. Note that $\mathcal{R} = \mathcal{R}(0)$.

Theorem 4.1. *If* $f \in \overline{co(\mathcal{C})}$, *then*

$$|\lambda a_n a_m - a_{n+m-1}| \leq \lambda - 1,$$

where $n, m = 2, 3, \ldots$ *and* $\lambda \in [2, \infty)$. *Equality holds for the function* $l(z) = z/(1-z)$ *and its rotations.*

Theorem 4.2. *If $f \in \mathcal{R}(\beta)$, then*

$$|\lambda a_n a_m - a_{n+m-1}| \leq \frac{4\lambda(1-\beta)^2}{nm} - \frac{2(1-\beta)}{n+m-1},$$

where $n, m = 2, 3, \ldots$ and $\lambda \in \left[\frac{nm}{(1-\beta)(n+m-1)}, \infty\right)$. Equality holds for the function $m(z) = -2(1-\beta)\ln(1-z) - z(1-2\beta)$ and its rotations.

4.1.1. The class $\mathcal{H}$

We consider the class $\mathcal{H}$ defined in (1.9) with some restrictions on $r(n)$ such that $\mathcal{H}$ is a subclass of $\mathcal{S}$. For example,

- If $r(n) = (n-\beta)/(1-\beta)$, then $\mathcal{H} \subset \mathcal{S}^*(\beta) \subset \mathcal{S}$ [**102**]. In particular, for $\beta = 0$ we have $\mathcal{H} = H$, the Hurwitz class.
- If $r(n) = n(n-\beta)/(1-\beta)$, then $\mathcal{H} \subset \mathcal{C}(\beta) \subset \mathcal{S}$ [**102**].
- If $r(n) = 3n-2$, then $\mathcal{H} \subset \mathcal{UST} \subset \mathcal{S}$ [**44**].
- If $r(n) = n(2n-1)$, then $\mathcal{H} \subset \mathcal{UCV} \subset \mathcal{S}$ [**44**].
- If $r(n) = n/(1-\beta)$, then $\mathcal{H} \subset \mathcal{R}(\beta) \subset \mathcal{S}$.
- If $r(n) = 1 + [(n-1)/(1-\beta)]\sec\nu$, then $\mathcal{H} \subset \mathcal{S}_p^{\nu}(\beta) \subset \mathcal{S}$ [**51**].

In all these classes $\beta \in [0, 1)$.

Our main result for the class $\mathcal{H}$ is stated in Theorem 1.7. We remark that for the choice $r(n) = n$, Theorem 1.7 turns into Theorem C. Indeed, our proof is much simpler than the proof of [**22**, Theorem 6].

4.2. Proof of the main results

This section is devoted to the proof of our main results. The following lemmas are useful.

Lemma A. [**58**, Lemma 1] *Let $\mu(\theta)$ be a probability measure on $[0, 2\pi]$. Then*

$$|b_{n-1}b_{m-1} - b_{n+m-2}| \leq 2 \qquad (n, m = 2, 3, \ldots),$$

where $b_n = 2\int_0^{2\pi} e^{in\theta}\mathrm{d}\mu(\theta)$.

Lemma 4.3. *Let $\lambda \in \mathbb{C}$, $\mu(\theta)$ be a probability measure on $[0, 2\pi]$, and for some function $s(n) > 0$, write $a_n = s(n)\int_0^{2\pi} e^{i(n-1)\theta}\mathrm{d}\mu(\theta) = s(n)b_{n-1}/2$ where b_n is same as in Lemma A.*

Then

$$|\lambda a_n a_m - a_{n+m-1}| \leq \left|\lambda - \frac{2s(n+m-1)}{s(n)s(m)}\right| s(n)s(m) + s(n+m-1),$$

for $n, m = 2, 3, \ldots$.

Proof. Putting the values of a_n, a_m, a_{n+m-1} and by using Lemma A we get

$$\begin{aligned} & |\lambda a_n a_m - a_{n+m-1}| \\ = & \left|\left(\lambda - \frac{2s(n+m-1)}{s(n)s(m)}\right) s(n)\frac{b_{n-1}}{2}s(m)\frac{b_{m-1}}{2} + \frac{s(n+m-1)}{2}(b_{n-1}b_{m-1} - b_{n+m-2})\right| \\ \leq & \left|\lambda - \frac{2s(n+m-1)}{s(n)s(m)}\right| s(n)s(m) + s(n+m-1). \end{aligned}$$

The proof of our lemma is complete. □

Remark 4.4. Lemma 4.3 helps us to estimate the generalized Zalcman coefficient functional $\lambda a_n a_m - a_{n+m-1}$ for several classes of functions in $\mathcal{S}$, where the coefficients a_n are of the form $s(n)\int_0^{2\pi} e^{i(n-1)\theta}\mathrm{d}\mu(\theta)$ and these lead to extremal functions whose series representations are of the form $z + \sum_{n=2}^{\infty} s(n)z^n$, for instance, see [**61**] and the present paper.

Proof of Theorem 4.1. By a well-known result from [**14**], there is a unique probability measure μ on $[0, 2\pi]$, such that

$$f(z) = \int_0^{2\pi} \frac{z}{1 - e^{i\theta}z}\mathrm{d}\mu(\theta)$$

for all f in $\overline{co(\mathcal{C})}$. Comparing the n-th coefficients of the series expansion of f and of the geometric series expansion of the right hand side, it can easily be seen that

$$a_n = \int_0^{2\pi} e^{i(n-1)\theta}\mathrm{d}\mu(\theta), \quad n \geq 2.$$

From Lemma 4.3, we can see that $s(n) = 1$ and hence we get

$$|\lambda a_n a_m - a_{n+m-1}| \leq |\lambda - 2| + 1 = \lambda - 1.$$

The sharpness can easily be verified using the function $l(z)$ stated in the statement. □

Here we remark that

Remark 4.5. For $0 < \lambda < 2$,

$$|\lambda a_n a_m - a_{n+m-1}| \leq \begin{cases} \lambda + 1, & 0 < \lambda \leq 1 \\ 3 - \lambda, & 1 \leq \lambda < 2. \end{cases}$$

However, the bounds are not sharp. The case for $n = m$ is done in [**22**] where the sharp bound is obtained.

Proof. Proof for the case $0 < \lambda \leq 1$ comes from the triangle inequality and the case $1 \leq \lambda \leq 2$ follows from the proof of Theorem 4.1. $\square$

Putting $m = n$ in Theorem 4.1, we get

Corollary 4.6. *If* $f \in \overline{co(\mathcal{C})}$, *then*

$$|\lambda a_n^2 - a_{2n-1}| \leq \lambda - 1,$$

where $n = 2, 3, \dots$ *and* $\lambda \in [2, \infty)$. *Equality holds for the function* $l(z) = z/(1-z)$ *and its rotations.*

Remark 4.7. We can also prove Corollary 4.6 by using the same technique as in [**22**]. Indeed, from the proof of [**22**, Theorem 3] we have

$$|\lambda a_n^2 - a_{2n-1}| \leq (\lambda - 2) \int_0^{2\pi} \cos^2((n-1)\theta) \mathrm{d}\mu(\theta) + 1 \leq \lambda - 1,$$

where the second inequality follows from the fact that $\cos\theta \leq 1$ for $0 \leq \theta \leq 2\pi$.

Proof of Theorem 4.2. By the Herglotz representation theorem for functions with positive real part [**21**, 1.9], there is a unique probability measure μ on $[0, 2\pi]$ such that

$$\frac{f'(z) - \beta}{1 - \beta} = \int_0^{2\pi} \frac{1 + e^{i\theta}z}{1 - e^{i\theta}z} \mathrm{d}\mu(\theta)$$

or, equivalently,

$$1 + \sum_{n=2}^{\infty} n a_n z^{n-1} = 1 + (1 - \beta) \sum_{n=2}^{\infty} 2 \int_0^{2\pi} e^{in\theta} \mathrm{d}\mu(\theta) z^n.$$

Comparing the coefficients, we obtain

$$a_n = \frac{2(1-\beta)}{n} \int_0^{2\pi} e^{i(n-1)\theta} \mathrm{d}\mu(\theta), \quad n \geq 2.$$

From Lemma 4.3, we can see that $s(n) = 2(1-\beta)/n$ and hence we get

$$\begin{aligned}|\lambda a_n a_m - a_{n+m-1}| &\leq \left|\lambda - \frac{nm}{(1-\beta)(n+m-1)}\right| \frac{4(1-\beta)^2}{nm} + \frac{2(1-\beta)}{n+m-1} \\ &= \frac{4\lambda(1-\beta)^2}{nm} - \frac{2(1-\beta)}{n+m-1}.\end{aligned}$$

The sharpness can easily be verified using the given function $m(z)$ stated in the hypothesis of the theorem. □

Remark 4.8. For $0 < \lambda < \dfrac{nm}{(1-\beta)(n+m-1)}$,

$$|\lambda a_n a_m - a_{n+m-1}| \leq \begin{cases} \dfrac{4\lambda(1-\beta)^2}{nm} + \dfrac{2(1-\beta)}{n+m-1}, & 0 < \lambda \leq \dfrac{nm}{2(1-\beta)(n+m-1)} \\ \dfrac{6(1-\beta)}{n+m-1} - \dfrac{4\lambda(1-\beta)^2}{nm}, & \dfrac{nm}{2(1-\beta)(n+m-1)} \leq \lambda \\ & \quad \leq \dfrac{nm}{(1-\beta)(n+m-1)}. \end{cases}$$

However, the bounds are not sharp. The case for $n = m$ and $\beta = 0$ is done in [**22**] where the sharp bound is obtained.

Proof. The first part follows from the triangle inequality and the second part follows from the proof of Theorem 4.2. □

In particular when $m = n$, Theorem 4.2 leads to

Corollary 4.9. *If $f \in \mathcal{R}(\beta)$ and $\lambda \in \left[\frac{n^2}{(2n-1)(1-\beta)}, \infty\right)$, then*

$$|\lambda a_n^2 - a_{2n-1}| \leq \frac{4\lambda(1-\beta)^2}{n^2} - \frac{2(1-\beta)}{2n-1},$$

where $n = 2, 3, \ldots$. Equality holds for the function $m(z) = -2(1-\beta)\ln(1-z) - z(1-2\beta)$ and its rotations.

Remark 4.10. An alternative proof of Corollary 4.9 can be done by the same technique as in [**22**]. Indeed, from the proof of [**22**, Theorem 4] we have

$$\begin{aligned}|\lambda a_n^2 - a_{2n-1}| &\leq \left(\frac{4\lambda(1-\beta)^2}{n^2} - \frac{4(1-\beta)}{2n-1}\right) \int_0^{2\pi} \cos^2((n-1)\theta) \mathrm{d}\mu(\theta) + \frac{2(1-\beta)}{2n-1} \\ &\leq \frac{4\lambda(1-\beta)^2}{n^2} - \frac{2(1-\beta)}{2n-1},\end{aligned}$$

where the second inequality follows from the fact that $\cos\theta \leq 1$ for $0 \leq \theta \leq 2\pi$.

Remark 4.11. From Remark 4.10, it is clear that for $f \in \mathcal{R}(\beta)$ and for $0 < \lambda \leq 4/3(1-\beta)$,

$$|\lambda a_n^2 - a_{2n-1}| \leq \frac{2(1-\beta)}{2n-1}.$$

Equality holds for the functions of the form $G(z) = (1-\beta)F(z) + \beta z$ and its rotations, where $F(z)$ is the same function defined in Theorem B.

To prove the generalized Zalcman problem for $\mathcal{H}$, we need the following lemma which is a similar form of [**22**, Lemma 5].

Lemma 4.12. *Let $\lambda > 0, n \geq 2, q(n), q(2n-1) > 0$, and consider the triangular region*

$$P = \{(u,v) \in \mathbb{R}^2 : u, v \geq 0, q(n)u + q(2n-1)v \leq 1\}$$

in the uv-plane. Then

$$\max_{(u,v)\in P} (\lambda u^2 + v) = \max\left\{\frac{\lambda}{q(n)^2}, \frac{1}{q(2n-1)}\right\},$$

and the maximum is attained only at $(u,v) = (0, 1/q(2n-1))$ and $(u,v) = (1/q(n), 0)$.

Proof. The function $F(u,v) = \lambda u^2 + v$ is readily seen to have no critical points, so its maximum on the compact set P is achieved on the boundary ∂P. Clearly, $F(0,v) \leq \frac{1}{q(2n-1)}$ while $F(u,0) \leq \frac{\lambda}{q(n)^2}$.

Finally, on the third piece of the boundary of P we have $q(n)u + q(2n-1)v = 1$. Hence the function F on that piece can be seen as a function of one variable leading to

$$F(u,v) = g(u) = \lambda u^2 + \frac{1-q(n)u}{q(2n-1)}.$$

Since $g''(u) = 2\lambda > 0$, the above function cannot achieve its maximum within the interval $\left[0, \frac{1}{q(n)}\right]$. Hence, the maximum value can only be achieved at one of the end points of this interval and since

$$g(0) = \frac{1}{q(2n-1)}, \quad g\left(\frac{1}{q(n)}\right) = \frac{\lambda}{q(n)^2},$$

the assertion follows. □

Remark 4.13. *Lemma* 4.12 can also be proved using graphical solution methods from Linear Programming. Since the conditions $u, v \geq 0$, and $q(n)u + q(2n-1)v \leq 1$ give a convex triangular region with the vertices $(0,0), (0, 1/q(2n-1))$ and $(1/q(n), 0)$, by the graphical solution method, the maximum value can only be achieved at one of these vertices.

Proof of Theorem 1.7. By the definition of the class $\mathcal{H}$, the ordered pair $(|a_n|, |a_{2n-1}|)$ belongs to P, where P is defined in Lemma 4.12. Hence by using Lemma 4.12, we have

$$|\lambda a_n^2 - a_{2n-1}| \leq \lambda |a_n|^2 + |a_{2n-1}| \leq \max\left\{\frac{\lambda}{r(n)^2}, \frac{1}{r(2n-1)}\right\}.$$

It can easily be seen that one way implication is true for the equality. For the converse part, we must have $r(n)|a_n| + r(2n-1)|a_{2n-1}| = 1$. Together with the definition of $\mathcal{H}$, it follows that the rotated function must be of the form $f_c(z) = z + A_n z^n + A_{2n-1} z^{2n-1}$, where, $f_c(z) = \overline{c} f(cz), |c| = 1$, a rotation of a function f in $\mathcal{S}$, $A_n = c^{n-1} a_n$, and similarly $A_{2n-1} = c^{2n-2} a_{2n-1}$. Further inspection of the case of equality in Lemma 4.12 and the values of λ readily yields that one of the coefficients A_n, A_{2n-1} must be zero and the more precise form of these functions follows immediately. This completes the proof of the theorem. $\square$

CHAPTER 5

NEHARI'S UNIVALENCE CRITERIA, JOHN DOMAINS AND APPLICATIONS

In this chapter, using techniques from differential equations, we establish the bound for the pre-Schwarzian derivative whenever the bound for the Schwarzian derivative is known. In addition, we give necessary conditions for John domains in terms of the pre-Schwarzian derivative. In Section 5.2, we prove sharp estimates for the pre-Schwarzian derivatives for functions in Nehari-type classes. These estimates are used to prove our main results established in Section 5.3.

Contents of this chapter are from the article: Agrawal S., Sahoo S. K., Nehari's univalence criteria, John domains and applications, Under review.

5.1. Introduction and preliminaries

There are a number of sufficient conditions available in the literature for a function to be univalent in $\mathbb{D}$ and most of them are very far from necessary conditions. However, there are a few of them which are also close to necessity. One such example is about the Nehari condition. From this fact, the well-known Nehari class is generated and it is associated with the Schwarzian derivative of functions (see [**21, 73, 74**]). Moreover, sufficient conditions for starlikeness and convexity in terms of Schwarzian derivatives are studied in [**46**].

For $\alpha \geq 0$ and $k \geq 0$, we consider the class $\mathcal{N}_\alpha(k)$ defined in (1.10). The set $\mathcal{N}_2(2)$, called the Nehari class, is intensively studied by Chuaqui, Osgood and Pommerenke in [**19**]. Due to [**18**, Lemma 1], if $f \in \mathcal{N}_2(k)$ then $(1-|z|^2)|T_f(z)| \leq k|z|$, for $0 \leq k \leq 2$. However, the constant k in this case is not best possible. This result is indeed improved and discussed in Section 5.2 of this chapter. Except for the cases $\alpha = 2, k = 2; \alpha = 1, k = 4$ and $\alpha = 0, k = \pi^2/2$, all mapping considered in the Schwarzian classes $\mathcal{N}_\alpha(k)$ have images that are quasidisks, that is, John disks whose complements are also John disks. This follows from [**27**, Theorem 6], that if $|S_f(z)| \leq \rho(z)$ is a sufficient condition for univalence in the disk, then $|S_f(z)| \leq t\rho(z)$ for some $0 \leq t < 1$ which guarantees that the images are quasidisks. Furthermore, in the cases $\alpha = 1, k = 4$ and $\alpha = 0, k = \pi^2/2$, the images will also be quasidisks as soon as they are Jordan domains.

For functions $f \in \mathcal{N}_2(2)$, John domains $f(\mathbb{D})$ are characterized by means of the pre-Schwarzian derivative in [**19, 31**]. In this article we approach to investigate similar characterizations for functions in $\mathcal{N}_\alpha(k)$ for some choices of α and k, wherever applicable. This was one of the problems suggested by Chuaqui, et al. in [**19**]. But we fail to provide a complete solution to this problem, however, a partial solution is obtained in this chapter. In fact, we could manage to find only necessary condition for functions in $\mathcal{N}_\alpha(k)$ (whenever applicable) satisfying the condition that $f(\mathbb{D})$ is a John domain in the form

$$\limsup_{|z|\to 1}(1-|z|^2)\mathrm{Re}\,(zT_f(z)) < c \tag{5.1}$$

for some positive constant c. In this chapter, the so-called, *Nehari-type classes* are constructed by considering the well-known sufficient conditions for univalency of type (1.10) in terms of Schwarzian and pre-Schwarzian derivatives. They are $\mathcal{N}_0(\pi^2/2)$, $\mathcal{N}_1(4)$, $\mathcal{N}_2(2)$,

$$P = \left\{f \in \mathcal{A} : (1-|z|^2)\,|T_f(z)| \leq 1\right\}, \quad \text{and} \quad B = \left\{f \in \mathcal{A} : (1-|z|^2)\,|zT_f(z)| \leq 1\right\}.$$

5.2. Preparatory results

This section deals with sharp estimates of pre-Schwarzian derivative of functions f belonging to the family $\mathcal{N}_2(k)$, $\mathcal{N}_0(k)$, $\mathcal{N}_1(k)$.

The following result is a generalization of [**18**, Lemma 1]. Note that this idea was originally proposed by Chuaqui and Osgood (see [**18**, pp. 660-662]), but it was not precisely estimated whereas an optimal bound for $|T_f|$, $f \in \mathcal{N}_2(k)$, was proved. Here we provide the sharp estimation of $|T_f|$, $f \in \mathcal{N}_2(k)$, precisely.

Lemma 5.1. *If $f \in \mathcal{N}_2(k)$, $0 \leq k \leq 2$, then*

$$|T_f(z)| \leq \frac{2|z| - 2\beta^2 A_k(|z|)}{1-|z|^2},$$

where $A_k(z) = \dfrac{1}{\beta}\dfrac{(1+z)^\beta - (1-z)^\beta}{(1+z)^\beta + (1-z)^\beta}$ *with* $\beta = \sqrt{1-(k/2)}$. *Equality holds at a single $z \neq 0$ if and only if f is a suitable rotation of $A_k(z)$.*

Proof. A simple computation gives

$$T_f'(z) = \frac{1}{2}T_f^2(z) + S_f(z), \quad T_f(0) = 0.$$

Now, consider the initial value problem

$$w'(x) = \frac{1}{2}w^2(x) + \frac{k}{(1-x^2)^2}, \quad w(0) = 0$$

on $(-1,1)$. Note that it is satisfied by $w(x) = \dfrac{2x - 2\beta^2 A_k(x)}{1-x^2}$. We shall show that $|T_f(z)| \leq w(|z|)$.

Fix z_0 with $|z_0| = 1$, and let

$$\psi(\tau) = |T_f(\tau z_0)|, \quad 0 \leq \tau < 1.$$

It is evident that the zeros of $\psi(\tau)$ are isolated unless $f(z) \equiv z$. Away from these zeros, $\psi(\tau)$ is differentiable and satisfies $\psi'(\tau) \leq |T_f'(\tau z_0)|$. Since $(1-\tau^2)^2|S_f(\tau z_0)| \leq k$ we obtain

$$\frac{d}{d\tau}(\psi(\tau) - w(\tau)) \leq |T_f'(\tau z_0)| - w'(\tau) \leq \frac{1}{2}(|T_f(\tau z_0)|^2 - w^2(\tau)) = \frac{1}{2}(\psi(\tau) - w(\tau))(\psi(\tau) + w(\tau)).$$

The initial condition $\psi(0) - w(0) = 0$, with the Grönwall inequality, tells us that $\psi(\tau) - w(\tau) \leq 0$ and hence the required inequality follows.

Finally, one can easily see that $A_k(z) \in \mathcal{N}_2(k)$ and the equality

$$|T_{A_k}(z)| = \frac{2|z| - 2\beta^2 A_k(|z|)}{1-|z|^2}$$

holds for some z. □

We observe that $w(x) = (2x + \sqrt{4-2k})/(1-x^2)$ is also a solution of the differential equation

$$w'(x) = \frac{1}{2}w^2(x) + \frac{k}{(1-x^2)^2}$$

on $(-1,1)$. Note that $w(0) = \sqrt{4-2k}$. This motivates us to define a class $\mathcal{M}_2(k)$ similar to $\mathcal{N}_2(k)$ (refer (1.11)). Note that for $k = 2$, the class $\mathcal{M}_2(k)$ coincides with the Nehari class $\mathcal{N}_2(2)$. Now we give a result similar to Lemma 5.1 for the class $\mathcal{M}_2(k)$ where the bound obtained is more simpler than Lemma 5.1.

Lemma 5.2. *If $f \in \mathcal{M}_2(k)$, $0 \leq k \leq 2$, then*

$$|T_f(z)| \leq \frac{2|z| + \sqrt{4-2k}}{1-|z|^2}.$$

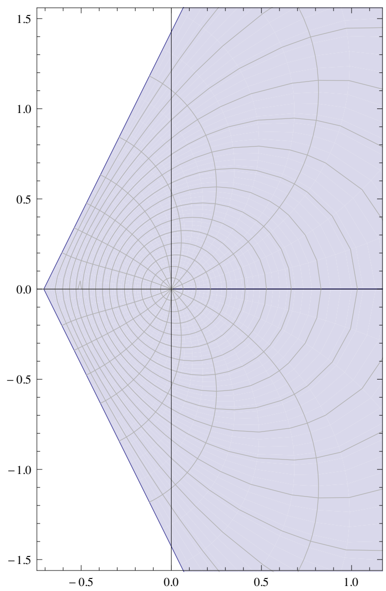


FIGURE 5.1. Graph of the function $F_0(z)$ with $k = 1$.

Equality holds at a single $z \neq 0$ if and only if f is a suitable rotation of $F_0(z)$, where

$$F_0(z) = \frac{e^{\sqrt{4-2k}\tanh^{-1}(z)} - 1}{\sqrt{4-2k}}.$$

Proof. A simple computation gives

$$T_f'(z) = \frac{1}{2}T_f^2(z) + S_f(z), \quad T_f(0) = \sqrt{4-2k}.$$

Now, consider the initial value problem

$$w'(x) = \frac{1}{2}w^2(x) + \frac{k}{(1-x^2)^2}, \quad w(0) = \sqrt{4-2k}$$

on $(-1, 1)$. Note that it is satisfied by $w(x) = (2x+\sqrt{4-2k})/(1-x^2)$. Now it is enough to prove that $|T_f(z)| \leq w(|z|)$. This can be proved similar to the proof given in Lemma 5.1.

Finally, one can easily see that $F_0(z) \in \mathcal{N}_2(k)$ and the equality

$$|T_{F_0}(z)| = \frac{2|z| + \sqrt{4-2k}}{1-|z|^2}$$

holds. $\square$

The particular choice of $k = 2$ in Lemma 5.1 and Lemma 5.2 imply the following well-known result.

Corollary 5.3. [**18**, Lemma 1] *If $f \in \mathcal{N}_2(2)$ then*

$$|T_f(z)| \leq \frac{2|z|}{1-|z|^2}.$$

Equality holds at a single $z \neq 0$ if and only if f is a rotation of

$$\frac{1}{2} \ln \frac{1+z}{1-z}.$$

Similarly, the next result is stated as follows:

Lemma 5.4. *If $f \in \mathcal{N}_0(k)$, $0 \leq k \leq \pi^2/2$, then*

$$|T_f(z)| \leq \sqrt{2k} \tan\left(\sqrt{\frac{k}{2}}|z|\right).$$

Equality holds at a single $z \neq 0$ if and only if f is a rotation of $F_1(z)$, where

$$F_1(z) = \sqrt{\frac{2}{k}} \tan\left(\sqrt{\frac{k}{2}} z\right).$$

Proof. A simple computation gives

$$T_f'(z) = \frac{1}{2} T_f^2(z) + S_f(z), \quad T_f(0) = 0.$$

Now, consider the initial value problem

$$w'(x) = \frac{1}{2} w^2(x) + k, \quad w(0) = 0$$

on $(-1,1)$. Clearly $w(x) = \sqrt{2/k} \tan\left(\sqrt{k/2}\, x\right)$ is a solution of the initial value problem. It suffices to show that $|T_f(z)| \leq w(|z|)$.

Fix z_0, $|z_0| = 1$, and let

$$\psi(\tau) = |T_f(\tau z_0)|, \quad 0 \leq \tau < 1.$$

The zeros of $\psi(\tau)$ are isolated unless $f(z) \equiv z$. Away from these zeros $\psi(\tau)$ is differentiable and $\psi'(\tau) \leq |T_f'(\tau z_0)|$. Since $|S_f(\tau z_0)| \leq k$ it follows that

$$\frac{d}{d\tau}(\psi(\tau)-w(\tau)) \leq |T_f'(\tau z_0)|-w'(\tau) \leq \frac{1}{2}(|T_f(\tau z_0)|^2-w^2(\tau)) = \frac{1}{2}(\psi(\tau)-w(\tau))(\psi(\tau)+w(\tau)).$$

As $\psi(0) - w(0) = 0$, by Grönwall's inequality we prove that $\psi(\tau) - w(\tau) \leq 0$ and hence the required inequality follows.

One can easily see that the equality holds for $F_1(z)$ defined in the statement of the lemma. $\square$

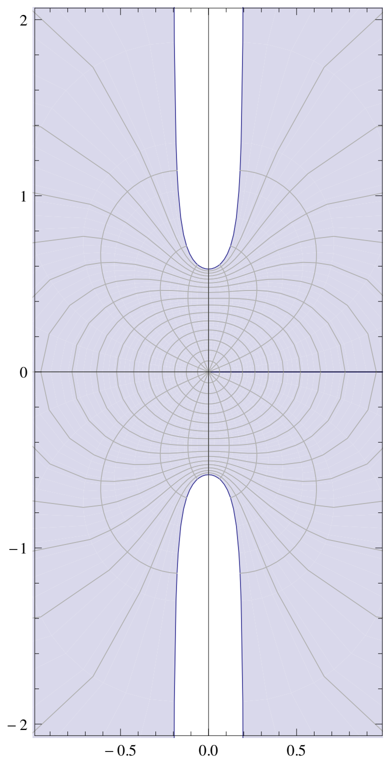


FIGURE 5.2. Graph of the function $F_1(z)$ with $k = \pi^2/2$.

Corollary 5.5. *If $f \in \mathcal{N}_0(\pi^2/2)$ then*

$$|T_f(z)| \leq \pi \tan\left(\frac{\pi}{2}|z|\right).$$

The equality holds at a single $z \neq 0$ if and only if f is a rotation of

$$\frac{2}{\pi} \tan\left(\frac{\pi}{2} z\right).$$

Next, we present a similar result for functions in the class $\mathcal{N}_1(k)$. Since we use the same technique and it involves solution of a differential equation, as a supplementary result we see that the solution of the differential equation is a ratio of Gaussian hypergeometric functions.

Proof of Theorem 1.8. Let the solution of the differential equation

$$w'(z) = \frac{1}{2} w^2(z) + \frac{k}{1 - z^2}$$

be of the form $w(z) = -2u'(z)/u(z)$. Then $u(z)$ is a solution of the second order linear differential equation

$$u'' + \frac{k}{2(1 - z^2)} u = 0.$$

It can be verified that this differential equation is satisfied by

$$u(z) = F((-1/4)(1+\sqrt{1+2k}), (1/4)(-1+\sqrt{1+2k}); 1/2; z^2), \quad |z|<1.$$

Note that the series solution method can also produce two linearly independent solutions where the above hypergeometric representation of $u(z)$ is one of them. Hence, the required solution is

$$w(z) = kz\left(\frac{F((-1/4)(-3+\sqrt{1+2k}), (1/4)(3+\sqrt{1+2k}); 3/2; z^2)}{F((-1/4)(1+\sqrt{1+2k}), (1/4)(-1+\sqrt{1+2k}); 1/2; z^2)}\right).$$

The conclusion follows from [**108**, Theorem 69.2]. ☐

Now we can estimate the pre-Schwarzian derivative of a function f in $\mathcal{N}_1(k)$.

Lemma 5.6. *If $f \in \mathcal{N}_1(k)$, $0 \le k \le 4$, then*

$$|T_f(z)| \le k|z|\left(\frac{F((-1/4)(-3+\sqrt{1+2k}), (1/4)(3+\sqrt{1+2k}); 3/2; |z|^2)}{F((-1/4)(1+\sqrt{1+2k}), (1/4)(-1+\sqrt{1+2k}); 1/2; |z|^2)}\right).$$

Equality holds at a single $z \neq 0$ if and only if f is a rotation of $F_2(z)$, where

$$F_2(z) = \int_0^z \frac{1}{(F((-1/4)(1+\sqrt{1+2k}), (1/4)(-1+\sqrt{1+2k}); 1/2; t^2))^2}\, dt.$$

Proof. An easy computation gives that

$$T_f'(z) = \frac{1}{2}T_f^2(z) + S_f(z), \quad T_f(0) = 0.$$

Consider the initial value problem

$$w'(x) = \frac{1}{2}w^2(x) + \frac{k}{1-x^2}, \quad w(0) = 0$$

on $(-1,1)$. Use Theorem 1.8 and proceed in the same manner as in the proof of Lemma 5.4. We can easily show that $|T_f(z)| \le w(|z|)$. Equality can also be verified easily by considering the function $F_2(z)$ defined in the statement. ☐

As a consequence of Lemma 5.6, we obtain

Corollary 5.7. *If $f \in \mathcal{N}_1(4)$ then*

$$|T_f(z)| \le \frac{4|z|}{1-|z|^2}.$$

Equality holds at a single $z \neq 0$ if and only if f is a rotation of

$$\frac{1}{4}\left(\frac{2z}{1-z^2} + \ln\frac{1+z}{1-z}\right).$$

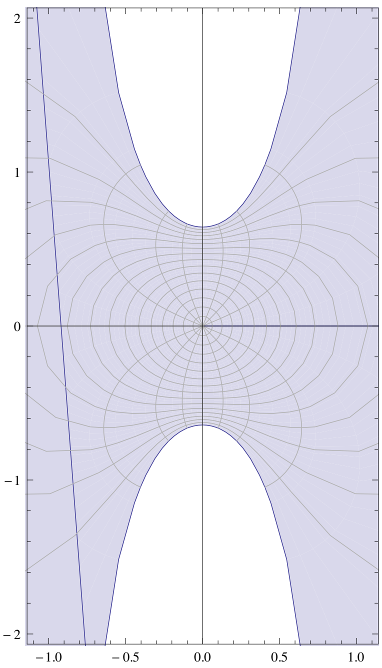


FIGURE 5.3. Graph of the function $F_2(z)$ with $k = 4$.

5.3. Schwarzian derivative and John domains

This section is devoted to the study of functions in Nehari-type classes. We shall apply the following characterization of John domains $f(\mathbb{D})$ to find necessary conditions, in the form (5.1), for functions f belonging to the Nehari-type classes discussed in Section 5.1.

Lemma 5.8. [**19**, Lemma 2] *Let f be analytic and univalent in $\mathbb{D}$. Then $f(\mathbb{D})$ is a John domain if and only if there exists $0 < x < 1$ such that*

$$\sup_{|\zeta|=1} \sup_{r<1} \frac{(1-\rho^2)|f'(\rho\zeta)|}{(1-r^2)|f'(r\zeta)|} < 1, \quad \rho = \frac{x+r}{1+xr}.$$

The following lemma is proved in [**19**] for the class $\mathcal{N}_2(2)$.

Lemma 5.9. [**19**, Theorem 4] *Let $f \in \mathcal{N}_2(2)$ and $f(\mathbb{D})$ be a bounded John domain. Then*

$$\limsup_{|z|\to 1} (1-|z|^2)\mathrm{Re}\,(zT_f(z)) < 2.$$

Remark 5.10. From [**18**, Lemma 1] it is clear that for all bounded mappings,

$$\limsup_{|z|\to 1} (1-|z|^2)^2|S_f(z)| \le k \Rightarrow \limsup_{|z|\to 1} (1-|z|^2)|T_f(z)| \le k.$$

From this we conclude that, for $f \in \mathcal{N}_2(k)$, $0 \le k \le 2$,

$$\limsup_{|z|\to 1}(1-|z|^2)\mathrm{Re}\,(zT_f(z)) \le k.$$

By the similar argument, from Lemma 5.1, we can say that

$$\limsup_{|z|\to 1}(1-|z|^2)\mathrm{Re}\,(zT_f(z)) \le 2-\sqrt{4-2k},$$

for all bounded mappings in $\mathcal{N}_2(k)$, $0 \le k \le 2$. Here the bound $2-\sqrt{4-2k}$ improves the bound k but the same proof method given in [**19**, Theorem 4] is not working to get the exact analog of Lemma 5.9 for the class $\mathcal{N}_2(k)$, $0 \le k < 2$.

Now we give the proof of Theorem 1.10. We use the similar technique which is used to prove Lemma 5.9.

Proof of Theorem 1.10. From [**27**, Theorem 6], it is clear that $f(\mathbb{D})$ is a John domain. By Lemma 1.9 we get

$$|T_f(z)| \le \frac{2|z|+\sqrt{4-2k}}{1-|z|^2},$$

and, with $|z|=r$,

$$\begin{aligned} |T_f'(z)| &= \left|S_f(z)+\frac{1}{2}T_f^2(z)\right| \\ &\le \frac{k}{(1-r^2)^2}+\frac{1}{2}\left(\frac{2r+\sqrt{4-2k}}{1-r^2}\right)^2 \\ &= \frac{d}{dr}\left(\frac{2r+\sqrt{4-2k}}{1-r^2}\right). \end{aligned}$$

We prove the theorem by contradiction method. Suppose that the required inequality does not hold. That is, $\exists$ a sequence $z_m \in \mathbb{D}$ with $|z_m| \to 1$ such that

$$\limsup_{|z_m|\to 1}(1-|z_m|^2)\mathrm{Re}\,(z_mT_f(z_m)) \ge 2+\sqrt{4-2k}.$$

Now choose a subsequence $z_{m_l}(=z_n)$ of z_m with $|z_n| \to 1$ such that

$$(1-|z_n|^2)\mathrm{Re}\,(z_nT_f(z_n)) \to 2+\sqrt{4-2k} \tag{5.2}$$

holds. Let $x \in (0,1)$ be fixed. Set $z_n = \rho_n\zeta_n$, $|\zeta_n| = 1$, and $r_n = (\rho_n - x)/(1-x\rho_n)$. The above upper bound for T_f' leads to

$$|\mathrm{Re}\,(\zeta_nT_f(z_n)) - \mathrm{Re}\,(\zeta_nT_f(r\zeta_n))| \le \int_r^{\rho_n} |T_f'(t\zeta_n)|dt \le \frac{2\rho_n+\sqrt{4-2k}}{1-\rho_n^2} - \frac{2r+\sqrt{4-2k}}{1-r^2}$$

or,

$$-\mathrm{Re}\,(\zeta_n T_f(r\zeta_n)) \le \frac{2\rho_n + \sqrt{4-2k}}{1-\rho_n^2} - \frac{2r+\sqrt{4-2k}}{1-r^2} - \mathrm{Re}\,(\zeta_n T_f(z_n))$$

or,

$$\begin{aligned} -\frac{1-r^2}{r}\mathrm{Re}\,(\zeta_n T_f(r\zeta_n)) &\le -\frac{\sqrt{4-2k}+2r}{r} \\ &\quad + \frac{1-r^2}{1-\rho_n^2}\frac{1}{r\rho_n}\left[(\sqrt{4-2k}+2\rho_n)\rho_n - (1-\rho_n^2)\mathrm{Re}\,(z_n T_f(z_n))\right]. \end{aligned}$$

If $r_n \le r \le \rho_n$ then

$$\frac{1-r^2}{1-\rho_n^2} \le \frac{1+x}{1-x}$$

and

$$-\frac{\sqrt{4-2k}+2r}{r} \le -\frac{\sqrt{4-2k}+2\rho_n}{\rho_n}.$$

Hence,

$$\begin{aligned} &(2+\sqrt{4-2k}) - \frac{1-r^2}{r}\mathrm{Re}\,(\zeta_n T_f(r\zeta_n)) \\ &\quad \le \left(\frac{1+x}{1-x}\right)\left(2+\sqrt{4-2k} - (1-|z_n|^2)\mathrm{Re}\,(z_n T_f(z_n))\right). \end{aligned}$$

Therefore, by the assumption (5.2) we get

$$\left|(2+\sqrt{4-2k}) - \frac{1-r^2}{r}\mathrm{Re}\,(\zeta_n T_f(r\zeta_n))\right| < \epsilon$$

for all $n \ge n_0(\epsilon, x)$.

From the above estimations we get

$$\begin{aligned} \log\frac{(1-r_n^2)|f'(r_n\zeta_n)|}{(1-\rho_n^2)|f'(\rho_n\zeta_n)|} &= \int_{r_n}^{\rho_n}\left(\frac{2r}{1-r^2} - \mathrm{Re}\,(\zeta_n T_f(r\zeta_n))\right)dr \\ &< \int_{r_n}^{\rho_n}\frac{\epsilon}{1-r^2}dr - \sqrt{4-2k}\int_{r_n}^{\rho_n}\frac{r}{1-r^2}dr \\ &< \int_{r_n}^{\rho_n}\frac{\epsilon}{1-r^2}dr = \epsilon h_{\mathbb{D}}(r_n\zeta_n, \rho_n\zeta_n) = \epsilon h_{\mathbb{D}}(0,x), \end{aligned}$$

for $n \ge n_0$. Here, $h_{\mathbb{D}}(\cdot,\cdot)$ denotes the usual hyperbolic distance of the unit disk $\mathbb{D}$. Thus,

$$\frac{(1-\rho_n^2)|f'(\rho_n\zeta_n)|}{(1-r_n^2)|f'(r_n\zeta_n)|} > e^{-\epsilon h_{\mathbb{D}}(0,x)}.$$

But since $\rho_n = (r_n+x)/(1+xr_n)$, the last inequality contradicts to Lemma 5.8. $\square$

Remark 5.11. One can ask similar questions when $f \in \mathcal{N}_0(k)$, $0 \le k \le \pi^2/2$ and $f \in \mathcal{N}_1(k)$, $0 \le k \le 4$. Indeed, we notice that the quantity

$$\limsup_{|z|\to 1}(1-|z|^2)\mathrm{Re}\,(zT_f(z))$$

vanishes due to [**18**, Lemma 1].

Theorem 5.12. *Let $f \in P \cup B$ and $f(\mathbb{D})$ be a bounded John domain. Then*

$$\limsup_{|z|\to 1}(1-|z|^2)\mathrm{Re}\,(zT_f(z)) < \frac{13}{4}.$$

Proof. If $f \in P$ then

$$|T_f(z)| \le \frac{1}{1-|z|^2}$$

holds, and with $|z| = r$,

$$\begin{aligned} |T_f'(z)| &= \left|S_f(z) + \frac{1}{2}T_f^2(z)\right| \\ &\le \frac{6}{(1-r^2)^2} + \frac{1}{2}\frac{1}{(1-r^2)^2} = \frac{13/2}{(1-r^2)^2} \\ &= \frac{d}{dr}\left(\frac{13}{2}\left[\frac{r}{2(1-r^2)} + \frac{1}{4}\ln\left(\frac{1+r}{1-r}\right)\right]\right). \end{aligned}$$

Suppose on contrary that the required relation does not hold. Then there exists a sequence $z_n \in \mathbb{D}$ with $|z_n| \to 1$ (with the similar explanation given in the proof of Theorem 1.10) such that

$$(1-|z_n|^2)\mathrm{Re}\,(z_nT_f(z_n)) \to 13/4 \tag{5.3}$$

holds.

Let $x \in (0,1)$ be fixed, set $z_n = \rho_n\zeta_n$, $|\zeta_n| = 1$, and $r_n = (\rho_n - x)/(1 - x\rho_n)$. We estimate

$$\begin{aligned} |\mathrm{Re}\,(\zeta_nT_f(z_n)) - \mathrm{Re}\,(\zeta_nT_f(r\zeta_n))| \le{}& \left(\frac{13}{2}\left[\frac{\rho_n}{2(1-\rho_n^2)} + \frac{1}{4}\ln\left(\frac{1+\rho_n}{1-\rho_n}\right)\right]\right) \\ &- \left(\frac{13}{2}\left[\frac{r}{2(1-r^2)} + \frac{1}{4}\ln\left(\frac{1+r}{1-r}\right)\right]\right) \end{aligned}$$

or,

$$\begin{aligned} -\mathrm{Re}\,(\zeta_nT_f(r\zeta_n)) \le{}& \left(\frac{13}{2}\left[\frac{\rho_n}{2(1-\rho_n^2)} + \frac{1}{4}\ln\left(\frac{1+\rho_n}{1-\rho_n}\right)\right]\right) \\ &- \left(\frac{13}{2}\left[\frac{r}{2(1-r^2)} + \frac{1}{4}\ln\left(\frac{1+r}{1-r}\right)\right]\right) - \mathrm{Re}\,(\zeta_nT_f(z_n)) \end{aligned}$$

or,

$$-\frac{1-r^2}{r}\mathrm{Re}\,(\zeta_n T_f(r\zeta_n)) \le -\left(\frac{1-r^2}{r}\right)\left(\frac{13}{2}\left[\frac{r}{2(1-r^2)}+\frac{1}{4}\ln\left(\frac{1+r}{1-r}\right)\right]\right)$$
$$+\frac{1-r^2}{1-\rho_n^2}\frac{1}{r\rho_n}\left(\frac{13}{2}\left[\frac{\rho_n^2}{2}+\frac{1}{4}(\rho_n)(1-\rho_n^2)\ln\left(\frac{1+\rho_n}{1-\rho_n}\right)\right)-(1-\rho_n^2)\mathrm{Re}\,(z_n T_f(z_n))\right].$$

It is evident that

$$\frac{1-r^2}{1-\rho_n^2} \le \frac{1+x}{1-x}$$

holds true if $r_n \le r \le \rho_n$. It follows that

$$\frac{13}{4}-\frac{1-r^2}{r}\mathrm{Re}\,(\zeta_n T_f(r\zeta_n)) \le \left(\frac{1+x}{1-x}\right)\left(\frac{13}{4}-(1-|z_n|^2)\mathrm{Re}\,(z_n T_f(z_n))\right).$$

Therefore, (5.3) leads to

$$\left|\frac{13}{4}-\frac{1-r^2}{r}\mathrm{Re}\,(\zeta_n T_f(r\zeta_n))\right| < \epsilon \quad \text{ for all } n \ge n_0(\epsilon, x).$$

The above estimations yield

$$\begin{aligned}
\log\frac{(1-r_n^2)|f'(r_n\zeta_n)|}{(1-\rho_n^2)|f'(\rho_n\zeta_n)|} &= \int_{r_n}^{\rho_n}\left(\frac{2r}{1-r^2}-\mathrm{Re}\,(\zeta_n T_f(r\zeta_n))\right)dr \\
&< \int_{r_n}^{\rho_n}\frac{\epsilon}{1-r^2}dr-\frac{5}{4}\int_{r_n}^{\rho_n}\frac{r}{1-r^2}dr \\
&< \int_{r_n}^{\rho_n}\frac{\epsilon}{1-r^2}dr = \epsilon h_{\mathbb{D}}(r_n\zeta_n, \rho_n\zeta_n) = \epsilon h_{\mathbb{D}}(0,x),
\end{aligned}$$

for $n \ge n_0$. This is equivalent to

$$\frac{(1-\rho_n^2)|f'(\rho_n\zeta_n)|}{(1-r_n^2)|f'(r_n\zeta_n)|} > e^{-\epsilon h_{\mathbb{D}}(0,x)}.$$

This contradicts Lemma 5.8, since $\rho_n = (r_n+x)/(1+xr_n)$.

Secondly, if $f \in B$ then we have

$$|T_f(z)| \le \frac{1}{|z|(1-|z|^2)},$$

and, with $|z| = r$,

$$\begin{aligned}
|T_f'(z)| &= \left|S_f(z)+\frac{1}{2}T_f^2(z)\right| \\
&\le \frac{6}{(1-r^2)^2}+\frac{1}{2}\frac{1}{r^2(1-r^2)^2} = \frac{12r^2+1}{2r^2(1-r^2)^2} \\
&= \frac{d}{dr}\left[-\frac{1}{2r}+\frac{13r}{4(1-r^2)}+\frac{15}{8}\ln\left(\frac{1+r}{1-r}\right)\right].
\end{aligned}$$

The rest of the proof follows by contradiction method similarly. □

Theorem 5.13. *Let $f \in P \cup B$ and $f(\mathbb{D})$ be a bounded John domain which is also convex. Then*

$$\limsup_{|z|\to 1}(1-|z|^2)\mathrm{Re}\ (zT_f(z)) < \frac{5}{4}.$$

Proof. Since f is convex, it is well-known that

$$|S_f(z)| \le \frac{2}{(1-|z|^2)^2}.$$

Then the proof follows in the same way as discussed in the proof of Theorem 5.12. □

Remark 5.14. We believe that the upper bounds 13/4 and 5/4 obtained in Theorems 5.12 and 5.13 can be improved.

CHAPTER 6

RADIUS OF CONVEXITY OF PARTIAL SUMS OF ODD FUNCTIONS

Our objective in this chapter is to find the largest disk $|z| < r$ (in $\mathbb{D}$) in which every section $s_{2n-1}(z) = z + \sum_{k=2}^{n} a_{2k-1} z^{2k-1}$, of $f \in \mathcal{L}$ (defined by (1.12) in Chapter 1), is convex; that is, s_{2n-1} satisfies

$$\mathrm{Re}\left(1 + \frac{z s''_{2n-1}(z)}{s'_{2n-1}(z)}\right) > 0.$$

This chapter is based on the article: Agrawal S., Sahoo S. K., Radius of convexity of partial sums of odd functions in the close-to-convex family, Under review.

6.1. Introduction and Main Result

Note that a subclass denoted by $\mathcal{F}$, of the class, $\mathcal{K}$, of close-to-convex functions, consisting of all locally univalent functions $f \in \mathcal{A}$ satisfying the condition (1.12) was considered in [**83**]. In this chapter, we consider functions from $\mathcal{F}$ that have odd Taylor coefficients and denote it by $\mathcal{L}$. Clearly, a function $f \in \mathcal{L}$ will have the Taylor series expansion $f(z) = z + \sum_{n=2}^{\infty} a_{2n-1} z^{2n-1}$. The function $f_0(z) = z/\sqrt{1-z^2}$ plays the role of an extremal function for $\mathcal{L}$; see for instance [**70**, p. 68, Theorem 2.6i].

Note that the following inclusion relations hold:

$$\mathcal{L} \subsetneq \mathcal{F} \subsetneq \mathcal{K} \subsetneq \mathcal{S}.$$

The fact that functions in $\mathcal{F}$ are close-to-convex may be obtained as a consequence of the result due to Kaplan (see [**21**, p. 48, Theorem 2.18]). In [**83**], Ponnusamy et al. have shown that every section of a function in the class $\mathcal{F}$ is convex in the disk $|z| < 1/6$ and the radius $1/6$ is the best possible. They conjectured that every section of functions in the family $\mathcal{F}$ is univalent and close-to-convex in the disk $|z| < 1/3$. This conjecture has been recently settled by Bharanedhar and Ponnusamy in [**12**, Theorem 1].

The problem of finding the radius of univalence of sections of f in $\mathcal{S}$ was first initiated by Szegö in 1928. According to the Szegö theorem [**21**, Section 8.2, p. 243-246], every section $s_n(z)$ of a function $f \in \mathcal{S}$ is univalent in the disk $|z| < 1/4$; see [**106**] for the original paper. The radius $1/4$ is best possible and can be verified from the second partial sum of the Koebe function $k(z) = z/(1-z)^2$. Determining the exact (largest) radius of univalence r_n of $s_n(z)$ ($f \in \mathcal{S}$) remains an open problem. However, many other related problems on sections have been solved for various geometric subclasses of $\mathcal{S}$, eg. the classes $\mathcal{S}^*$, $\mathcal{C}$ and $\mathcal{K}$ of starlike, convex and close-to-convex functions, respectively (see Duren [**21**, §8.2, p.241–246], [**28, 91, 94, 99**] and the survey articles [**36, 88**]). In [**62**], MacGregor considered the class

$$\mathcal{R} = \{f \in \mathcal{A} : \mathrm{Re}\,(f'(z)) > 0, z \in \mathbb{D}\}$$

and proved that the partial sums $s_n(z)$ of $f \in \mathcal{R}$ are univalent in $|z| < 1/2$, where the radius $1/2$ is best possible. On the other hand, in [**103**], Singh obtained the best radius, $r = 1/4$, of convexity for sections of functions in the class $\mathcal{R}$. The reader can refer to [**79**] for related information. Radius of close-to-convexity of sections of close-to-convex functions is obtained in [**67**].

By the argument principle, it is clear that the n-th section $s_n(z)$ of an arbitrary function in $\mathcal{S}$ is univalent in each fixed compact subdisk $\overline{\mathbb{D}_r} := \{z \in \mathbb{D} : |z| \leq r\}(r < 1)$ of $\mathbb{D}$ provided that n is sufficiently large. In this way one can get univalent polynomials in $\mathcal{S}$ by setting $p_n(z) = \frac{1}{r}s_n(rz)$. Consequently, the set of all univalent polynomials is dense in the topology of locally uniformly convergence in $\mathcal{S}$. The radius of starlikeness of the partial sums $s_n(z)$ of $f \in \mathcal{S}^*$ was obtained by Robertson in [**91**]; (see also [**100**, Theorem 2]) in the following form:

Theorem A. [**91**] *If $f \in \mathcal{S}$ is either starlike, convex, typically-real, or convex in the direction of imaginary axis, then there is an N such that, for $n \geq N$, the partial sum $s_n(z)$ has the same property in $\mathbb{D}_r := \{z \in \mathbb{D} : |z| < r\}$, where $r \geq 1 - 3(\log n)/n$.*

However, Ruscheweyh in [**95**] proved a stronger result by showing that the partial sums $s_n(z)$ of f are indeed starlike in $\mathbb{D}_{1/4}$ for functions f belonging not only to $\mathcal{S}$ but also to the closed convex hull of $\mathcal{S}$. Robertson [**91**] further showed that sections of the Koebe function $k(z)$ are univalent in the disk $|z| < 1 - 3n^{-1}\log n$ for $n \geq 5$, and that the constant 3 cannot be replaced by a smaller constant. However, Bshouty and Hengartner [**16**] pointed out that the Koebe function is not extremal for the radius of univalency of the partial sums of $f \in \mathcal{S}$. A well-known theorem by Ruscheweyh and Sheil-Small [**96**] on convolution allows us to conclude immediately that if f belongs to $\mathcal{C}, \mathcal{S}^*$, or $\mathcal{K}$, then its n-th section is respectively convex, starlike, or close-to-convex in the disk $|z| < 1 - 3n^{-1} \log n$, for $n \geq 5$. Silverman in [**100**] proved that the radius of starlikeness for sections of functions in the convex family $\mathcal{C}$ is $(1/2n)^{1/n}$ for all n. We suggest the reader to refer [**83, 94, 99, 106**] and recent articles [**75, 76, 77, 78**] for further interest on this topic. It is worth recalling that radius properties of harmonic sections have recently been studied in [**42, 52, 53, 55, 84**].

Our main objective in this chapter is to prove the following theorem which is also stated in the Theorem 1.11.

Main Theorem. *Every section of a function in $\mathcal{L}$ is convex in the disk $|z| < \sqrt{2}/3$. The radius $\sqrt{2}/3$ cannot be replaced by a greater one.*

This observation is also explained geometrically in Figure 6.1 by considering the third partial sum, $s_{3,0}$, of the extremal function f_0.

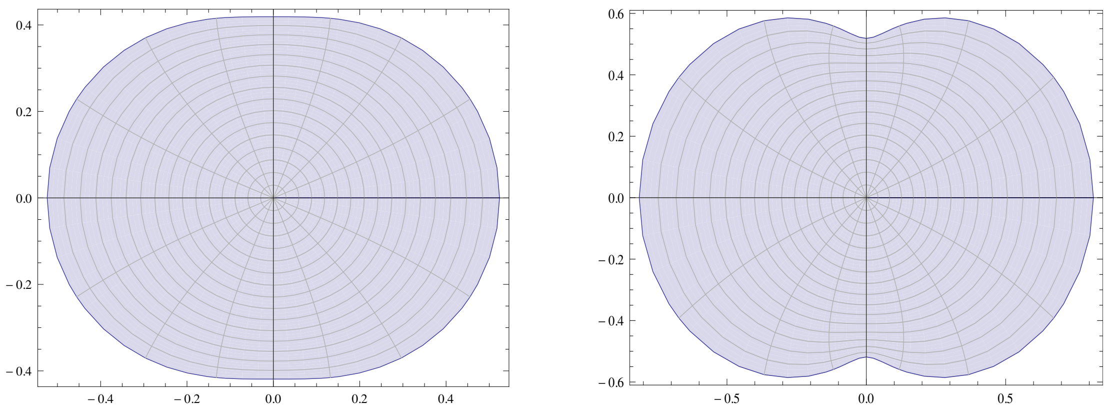


FIGURE 6.1. The first figure shows convexity of the image domain $s_{3,0}(z)$ for $|z| < \sqrt{2}/3$ and the second figure shows non-convexity of the image domain $s_{3,0}(z)$ for $|z| < 2/3 =: r_0$ ($r_0 > \sqrt{2}/3$).

6.2. Preparatory results

In this section we derive some useful results to prove our main theorem.

Lemma 6.1. *If $f(z) = z + \sum_{n=2}^{\infty} a_{2n-1} z^{2n-1} \in \mathcal{L}$, then the following estimates are obtained:*

(a) $|a_{2n-1}| \leq \frac{(2n-2)!}{2^{2n-2}(n-1)!^2}$ *for $n \geq 2$. The equality holds for*

$$f_0(z) = \frac{z}{\sqrt{1-z^2}}$$

and its rotations.

(b) $\left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{3r^2}{1-r^2}$ *for $|z| = r < 1$. The inequality is sharp.*

(c) $\frac{1}{(1+r^2)^{3/2}} \leq |f'(z)| \leq \frac{1}{(1-r^2)^{3/2}}$ *for $|z| = r < 1$. The inequality is sharp.*

(d) *If $f(z) = s_{2n-1}(z) + \sigma_{2n-1}(z)$, with $\sigma_{2n-1}(z) = \sum_{k=n+1}^{\infty} a_{2k-1} z^{2k-1}$, then for $|z| = r < 1$ we have*

$$|\sigma'_{2n-1}(z)| \leq A(n,r) \quad \textit{and} \quad |z\sigma''_{2n-1}(z)| \leq B(n,r),$$

where

$$A(n,r) = \sum_{k=n+1}^{\infty} \frac{(2k-1)!}{2^{2k-2}(k-1)!^2} r^{2k-2} \quad \textit{and} \quad B(n,r) = \sum_{k=n+1}^{\infty} \frac{(2k-2)(2k-1)!}{2^{2k-2}(k-1)!^2} r^{2k-2}.$$

The ratio test guarantees that both the series are convergent.

Proof. **(a)** Set

$$p(z) = 1 + \frac{2}{3}\left(\frac{zf''(z)}{f'(z)}\right). \tag{6.1}$$

Clearly, $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ is analytic in $\mathbb{D}$ and $\operatorname{Re}(p(z)) > 0$ there. So, by Carathéodory's Lemma, we obtain that $|p_n| \leq 2$ for all $n \geq 1$. Putting the series expansions for $f'(z)$, $f''(z)$ and $p(z)$ in (6.1) we get

$$\sum_{n=2}^{\infty} (2n-1)(2n-2) a_{2n-1} z^{2n-1} = \frac{3}{2} \sum_{n=2}^{\infty} \left(\sum_{k=1}^{n-1} p_{2k-1}(2n-2k-1) a_{2n-2k-1} \right) z^{2n-2}$$

$$+ \frac{3}{2} \sum_{n=2}^{\infty} \left(\sum_{k=1}^{n-1} p_{2k}(2n-2k-1) a_{2n-2k-1} \right) z^{2n-1}.$$

Equating the coefficients of z^{2n-1} and z^{2n-2} on both sides, we obtain

$$\sum_{k=1}^{n-1} p_{2k-1}(2n-2k-1) a_{2n-2k-1} = 0$$

and

$$(6.2) \qquad (2n-1)(2n-2)a_{2n-1} = \frac{3}{2}\sum_{k=1}^{n-1} p_{2k}(2n-2k-1)a_{2n-2k-1}, \qquad \text{for all } n \geq 2.$$

Hence,

$$(6.3) \qquad |a_{2n-1}| \leq \frac{3}{(2n-1)(2n-2)}\sum_{k=1}^{n-1}(2k-1)|a_{2k-1}|.$$

For $n = 2$, we can easily see that $|a_3| \leq 1/2$, and for $n = 3$, we have

$$|a_5| \leq \frac{3}{20}(1 + 3|a_3|) \leq \frac{3}{8}.$$

Now, we can complete the proof by the method of induction. Therefore, if we assume $|a_{2k-1}| \leq \frac{(2k-2)!}{2^{2k-2}(k-1)!^2}$ for $k = 2, 3, \ldots, n-1$, then we deduce from (6.3) that

$$|a_{2n-1}| \leq \frac{3}{(2n-1)(2n-2)}\sum_{k=1}^{n-1}\frac{(2k-1)!}{2^{2k-2}(k-1)!^2}.$$

The induction principle tells us to show that

$$|a_{2n-1}| \leq \frac{(2n-2)!}{2^{2n-2}(n-1)!^2}.$$

It suffices to show that

$$\frac{3}{(2n-1)(2n-2)}\sum_{k=1}^{n-1}\frac{(2k-1)!}{2^{2k-2}(k-1)!^2} = \frac{(2n-2)!}{2^{2n-2}(n-1)!^2}$$

or,

$$\sum_{k=1}^{n-1}\frac{3(2k-1)!}{2^{2k-2}(k-1)!^2} = \frac{(2n-2)(2n-1)!}{2^{2n-2}(n-1)!^2}.$$

Again, we prove this by the method of induction. It can easily be seen that for $k = 1$ it is true. Assume that it is true for $k = 2, 3, \ldots, n-1$, then we have to prove that

$$\sum_{k=1}^{n}\frac{3(2k-1)!}{2^{2k-2}(k-1)!^2} = \frac{(2n)(2n+1)!}{2^{2n}(n)!^2},$$

which is easy to see, since

$$\sum_{k=1}^{n}\frac{3(2k-1)!}{2^{2k-2}(k-1)!^2} = \frac{(2n-2)(2n-1)!}{2^{2n-2}(n-1)!^2} + \frac{3(2n-1)!}{2^{2n-2}(n-1)!^2} = \frac{(2n)(2n+1)!}{2^{2n}(n)!^2}.$$

Hence, the proof is complete. For equality, it can easily be seen that

$$f_0(z) = \frac{z}{\sqrt{1-z^2}} = z + \sum_{n=2}^{\infty}\frac{(2n-2)!}{2^{2n-2}(n-1)!^2}z^{2n-1}$$

belongs to $\mathcal{L}$.

The image of the unit disk $\mathbb{D}$ under f_0 is shown in Figure 6.2 which indicates that $f_0(\mathbb{D})$ is not convex.

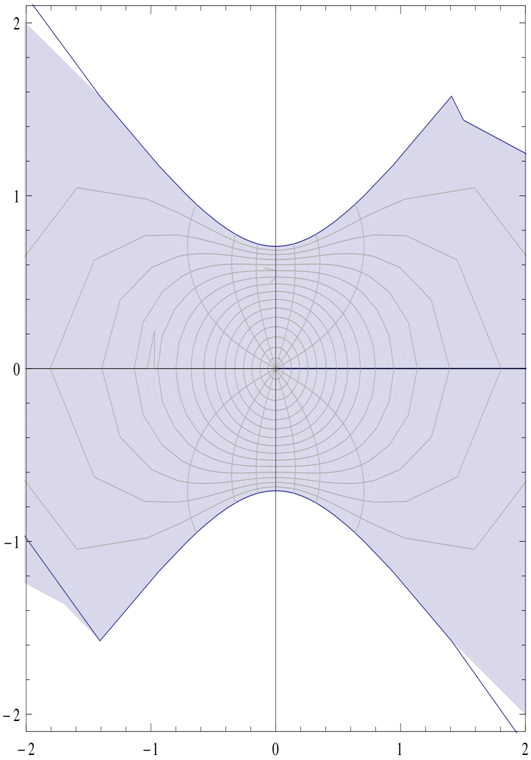


FIGURE 6.2. The image domain $f_0(\mathbb{D})$, where $f_0(z) = \frac{z}{\sqrt{1-z^2}}$.

(b) We see from the definition of $\mathcal{L}$ that

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+2z^2}{1-z^2}, \quad \text{i.e.,} \; \frac{zf''(z)}{f'(z)} \prec \frac{3z^2}{1-z^2} =: h(z),$$

where $\prec$ denotes the usual subordination (see for example [**21**, Chapter 6]). The poof of (b) now follows easily.

(c) Since

$$\frac{zf''(z)}{f'(z)} \prec h(z),$$

it follows by the well-known subordination result due to Suffridge [**105**] that

$$f'(z) \prec \exp\left(\int_0^z \frac{h(t)}{t}\mathrm{d}t\right) = \exp\left(3\int_0^z \frac{t}{1-t^2}\mathrm{d}t\right) = \frac{1}{(1-z^2)^{3/2}}.$$

Hence, the proof of (c) follows.

(d) By (a), we see that

$$|\sigma'_{2n-1}(z)| \leq \sum_{k=n+1}^{\infty} (2k-1)|a_{2k-1}|r^{2k-2} \leq A(n,r).$$

and

$$|z\sigma''_{2n-1}(z)| \le \sum_{k=n+1}^{\infty} (2k-1)(2k-2)|a_{2k-1}|r^{2k-2} \le B(n,r).$$

The proof of our lemma is complete. □

6.3. Proof of the Main Theorem

For an arbitrary $f(z) = z + \sum_{n=2}^{\infty} a_{2n-1}z^{2n-1} \in \mathcal{L}$, we first consider its third section $s_3(z) = z + a_3z^3$ of f. Simple computations show

$$1 + \frac{zs_3''(z)}{s_3'(z)} = 1 + \frac{6a_3z^2}{1+3a_3z^2}.$$

By using Lemma 6.1(a), we have $|a_3| \le 1/2$ and hence

$$\mathrm{Re}\left(1 + \frac{zs_3''(z)}{s_3'(z)}\right) \ge 1 - \frac{6|a_3||z|^2}{1-3|a_3||z|^2} \ge 1 - \frac{3|z|^2}{1-\frac{3}{2}|z|^2}$$

which is positive for $|z| < \sqrt{2}/3$. Thus, $s_3(z)$ is convex in the disk $|z| < \sqrt{2}/3$. To show that the constant $\sqrt{2}/3$ is best possible, we consider the function $f_0(z)$ defined by

$$f_0(z) = \frac{z}{\sqrt{1-z^2}}.$$

We denote by $s_{3,0}(z)$, the third partial sum $s_3(f_0)(z)$ of $f_0(z)$ so that $s_{3,0}(z) = z + (1/2)z^3$ and hence, we find

$$1 + \frac{zs_{3,0}''(z)}{s_{3,0}'(z)} = \frac{2+9z^2}{2+3z^2}.$$

This shows that

$$\mathrm{Re}\left(1 + \frac{zs_{3,0}''(z)}{s_{3,0}'(z)}\right) = 0$$

when $z^2 = (-2/9)$ or $(-2/3)$ i.e., when $|z|^2 = (2/9)$ or $(2/3)$. Hence, the equality occurs.

Next, let us consider the case $n = 3$. Our aim in this case is to show that

$$\mathrm{Re}\left(1 + \frac{zs_5''(z)}{s_5'(z)}\right) = \mathrm{Re}\left(\frac{1+9a_3z^2+25a_5z^4}{1+3a_3z^2+5a_5z^4}\right) > 0$$

for $|z| < \sqrt{2}/3$. Since the real part $\mathrm{Re}\,([1+9a_3z^2+25a_5z^4]/[1+3a_3z^2+5a_5z^4])$ is harmonic in $|z| \le \sqrt{2}/3$, it suffices to check that

$$\mathrm{Re}\left(\frac{1+9a_3z^2+25a_5z^4}{1+3a_3z^2+5a_5z^4}\right) > 0$$

for $|z| = \sqrt{2}/3$. Also we see that

$$\mathrm{Re}\left(\frac{1+9a_3z^2+25a_5z^4}{1+3a_3z^2+5a_5z^4}\right) = 3 - \mathrm{Re}\left(\frac{2-10a_5z^4}{1+3a_3z^2+5a_5z^4}\right) \geq 3 - \left|\frac{2-10a_5z^4}{1+3a_3z^2+5a_5z^4}\right|$$

and, so by considering a suitable rotation of $f(z)$, the proof reduces to $z = \sqrt{2}/3$; this means that it is enough to prove

$$\frac{3}{2} > \left|\frac{81-20a_5}{81+54a_3+20a_5}\right|.$$

From (6.2), we have

$$a_3 = \frac{p_2}{4} \quad \text{and} \quad a_5 = \left(\frac{3}{40}\right)\left(\frac{3}{4}p_2^2 + p_4\right).$$

Since $|p_2| \leq 2$ and $|p_4| \leq 2$, it is convenient to rewrite the last two relations as

$$a_3 = \frac{\alpha}{2} \quad \text{and} \quad a_5 = \frac{3}{40}(3\alpha^2 + 2\beta)$$

for some $|\alpha| \leq 1$ and $|\beta| \leq 1$.

Substituting the values for a_3 and a_5, and applying the maximum principle in the last inequality, it suffices to show the inequality

$$\frac{3}{2}\left|81 + 27\alpha + \frac{9\alpha^2}{2} + 3\beta\right| > \left|81 - \frac{9\alpha^2}{2} - 3\beta\right|$$

for $|\alpha| = 1 = |\beta|$. Finally, by the triangle inequality, the last inequality follows if we can show that

$$9\left|9 + 3\alpha + \frac{\alpha^2}{2}\right| - 6\left|9 - \frac{\alpha^2}{2}\right| > 5$$

which is easily seen to be equivalent to

$$9\left|9\overline{\alpha} + 3 + \frac{\alpha}{2}\right| - 6\left|9\overline{\alpha} - \frac{\alpha}{2}\right| > 5$$

as $|\alpha| = 1$. Write $\mathrm{Re}\,\alpha = x$. It remains to show that

$$T(x) := 9\sqrt{18x^2 + 57x + \frac{325}{4}} - 6\sqrt{\frac{361}{4} - 18x^2} > 5$$

for $-1 \leq x \leq 1$. It suffices to show

$$9\sqrt{18x^2 + 57x + \frac{325}{4}} > 5 + 6\sqrt{\frac{361}{4} - 18x^2}.$$

Squaring both sides we have

$$2106x^2 + 4617x + \frac{13229}{4} > 60\left(\sqrt{\frac{361}{4} - 18x^2}\right).$$

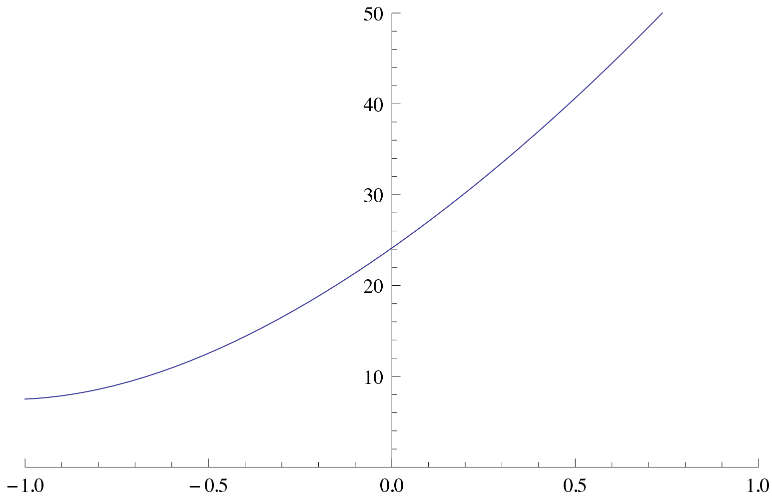


FIGURE 6.3. Graph of $T(x)$.

Again by squaring both sides we have

$$\left(2106x^2 + 4617x + \frac{13229}{4}\right)^2 > 3600\left(\frac{361}{4} - 18x^2\right).$$

After computing, it remains to show that $\phi(x) > 0$, where

$$\phi(x) = ax^4 + bx^3 + cx^2 + dx + e$$

and the coefficients are

$$a = 4435236, b = 19446804, c = 35311626, d = 30539146.5, e = 10613002.5625.$$

Here we see that $\phi^{(4)}(x) = 24a > 0$. Thus the function $\phi'''(x)$ is increasing in $-1 \leq x \leq 1$ and hence $\phi'''(x) \geq \phi'''(-1) = 10235160 > 0$. This implies $\phi''(x)$ is increasing. Hence $\phi''(x) \geq \phi''(-1) = 7165260 > 0$. Consequently, $\phi'(x)$ is increasing and we have $\phi'(x) \geq \phi'(-1) = 515362.5 > 0$. Finally we get, $\phi(x)$ is increasing and hence we have $\phi(x) > \phi(-1) = 373914.0625 > 0$. This completes the proof for $n = 3$.

We next consider the general case $n \geq 4$. It suffices to show that

$$\mathrm{Re}\left(1 + \frac{zs''_{2n-1}}{s'_{2n-1}}\right) > 0 \quad \text{for} \quad |z| = r$$

with $r = \sqrt{2}/3$ for all $n \geq 4$. From the maximum modulus principle, we shall then conclude that the last inequality holds for all $n \geq 4$

$$\mathrm{Re}\left(1 + \frac{zs''_{2n-1}}{s'_{2n-1}}\right) > 0$$

for $|z| < \sqrt{2}/3$. More generally, it remains to find the largest r so that the last inequality holds for all $n \geq 4$. By the same setting of $f(z)$ as in Lemma 6.1(d), it follows easily that

$$1 + \frac{zs''_{2n-1}}{s'_{2n-1}} = 1 + \frac{z(f''(z) - \sigma''_{2n-1}(z))}{f'(z) - \sigma'_{2n-1}(z)} = 1 + \frac{zf''(z)}{f'(z)} + \frac{\frac{zf''(z)}{f'(z)}\sigma'_{2n-1}(z) - z\sigma''_{2n-1}(z)}{f'(z) - \sigma'_{2n-1}(z)}$$

or,

$$\mathrm{Re}\left(1 + \frac{zs''_{2n-1}}{s'_{2n-1}}\right) \geq 1 - \left|\frac{zf''(z)}{f'(z)}\right| - \frac{\left|\frac{zf''(z)}{f'(z)}\right| |\sigma'_{2n-1}(z)| + |z\sigma''_{2n-1}(z)|}{|f'(z)| - |\sigma'_{2n-1}(z)|}.$$

Then by using Lemma 6.1, we obtain

$$\mathrm{Re}\left(1 + \frac{zs''_{2n-1}}{s'_{2n-1}}\right) \geq 1 - \frac{3r^2}{1-r^2} - \frac{\left(\frac{3r^2}{1-r^2}\right) A(n,r) + B(n,r)}{\frac{1}{(1+r^2)^{(3/2)}} - A(n,r)}.$$

Thus, we conclude that

$$\mathrm{Re}\left(1 + \frac{zs''_{2n-1}}{s'_{2n-1}}\right) > 0$$

provided

$$\frac{1-4r^2}{1-r^2} - \frac{(1+r^2)^{3/2}}{1-r^2}\left(\frac{3r^2A(n,r) + (1-r^2)B(n,r)}{1-(1+r^2)^{3/2}A(n,r)}\right) > 0,$$

or, equivalently

$$(1+r^2)^{3/2}\left(\frac{3r^2A(n,r) + (1-r^2)B(n,r)}{1-(1+r^2)^{3/2}A(n,r)}\right) < 1 - 4r^2.$$

We show that the above relation holds for all $n \geq 4$ with $r = \sqrt{2}/3$. The choice $r = \sqrt{2}/3$ brings the last inequality to the form

$$\left(\frac{11}{9}\right)^{3/2}\left(\frac{\frac{2}{3}A(n,\frac{\sqrt{2}}{3}) + \frac{7}{9}B(n,\frac{\sqrt{2}}{3})}{1-(\frac{11}{9})^{3/2}A(n,\frac{\sqrt{2}}{3})}\right) < \frac{1}{9}.$$

Set

$$C\left(n, \frac{\sqrt{2}}{3}\right) := 1 - \left(\frac{11}{9}\right)^{3/2} A\left(n, \frac{\sqrt{2}}{3}\right).$$

We shall prove that $C\left(n, \frac{\sqrt{2}}{3}\right) > 0$ for $n \geq 4$ i.e.,

$$A\left(n, \frac{\sqrt{2}}{3}\right) < \frac{27}{(11)^{3/2}}$$

and

$$A\left(n, \frac{\sqrt{2}}{3}\right) + B\left(n, \frac{\sqrt{2}}{3}\right) < \frac{27}{7 \times (11)^{3/2}} \quad \text{for } n \geq 4.$$

If the last inequality is proved, then automatically the previous one follows. Hence, it is enough to prove the last inequality. Now,

$$\begin{aligned} A(n,r)+B(n,r) &= \sum_{k=n+1}^{\infty} \frac{(2k-1)(2k-1)!}{2^{2k-2}(k-1)!^2}(r^2)^{k-1} \\ &\leq \sum_{k=5}^{\infty} \frac{(2k-1)(2k-1)!}{2^{2k-2}(k-1)!^2}(r^2)^{k-1} \\ &= \sum_{k=1}^{\infty} \frac{(2k-1)(2k-1)!}{2^{2k-2}(k-1)!^2}(r^2)^{k-1} - \sum_{k=1}^{4} \frac{(2k-1)(2k-1)!}{2^{2k-2}(k-1)!^2}(r^2)^{k-1} \\ &= \frac{1+2r^2}{(1-r^2)^{5/2}} - \left(1+\frac{9}{2}r^2+\frac{75}{8}r^4+\frac{245}{16}r^6\right). \end{aligned}$$

Substituting the value $r=\sqrt{2}/3$, we obtain

$$A\left(n,\frac{\sqrt{2}}{3}\right)+B\left(n,\frac{\sqrt{2}}{3}\right) \leq 0.076\cdots < 0.105\cdots = \frac{27}{7\times(11)^{3/2}}.$$

This completes the proof of our main theorem. $\square$

CHAPTER 7

CONCLUSION AND FUTURE DIRECTIONS

As we have already discussed in Chapter 1, Ismail et al. [**37**] first applied the q-function theory in geometric function theory. Later it is also studied in [**3, 86, 92, 97**]. It has provided important insight into the existing function theoretic structure as well as a number of problems in the current avenues in special functions. At the beginning of the last century, studies on q-difference equations appeared in intensive works especially by Jackson [**38**], Carmichael [**17**], Mason [**65**], Adams [**2**], Trjitzinsky [**107**], and later by others such as Poincaré, Picard, Ramanujan. Unfortunately, from the thirties up to the beginning of the eighties only non-significant interest in this area was investigated. Recently some research on this topic is carried out by Bangerezako [**10**]; see also references therein for other related work. The purpose of Chapter 2 and 3 is to develop q-function theory in some new point of view. Specifically, Chapter 2 gives an extension of the well-known Gaussian hypergeometric functions and close-to-convex functions to the q-function theory whereas Chapter 3 gives the concept of order of q-starlike functions.

In one hand, some of the important results of **Chapter 2** concern about analyticity in a cut plane and convexity in the direction of the imaginary axis of the basic hypergeometric functions of the form

$$\frac{z\Phi[dq, eq; fq; q, z]}{\Phi[a, b; c; q, z]},$$

where either $d = a/q, e = b, f = c$; or $d = a, e = b/q, f = c/q$; or $d = a, e = b, f = c$ satisfy some conditions. If we look at the first identity stated in Lemma 2.2, a similar mapping properties of the basic hypergeometric functions of the form

$$\frac{z\Phi[dq, eq; fq^2; q, z]}{\Phi[a, b; c; q, z]}$$

may be possible and interesting to obtain, where the constants d, e, f can be obtained in terms of the parameters a, b, c and q. Indeed, similar but other form of basic hypergeometric functions; for example having q^2 term with the first or (and) second parameter(s) may be investigated (see [**59**] for related identities). In general, generating functions of

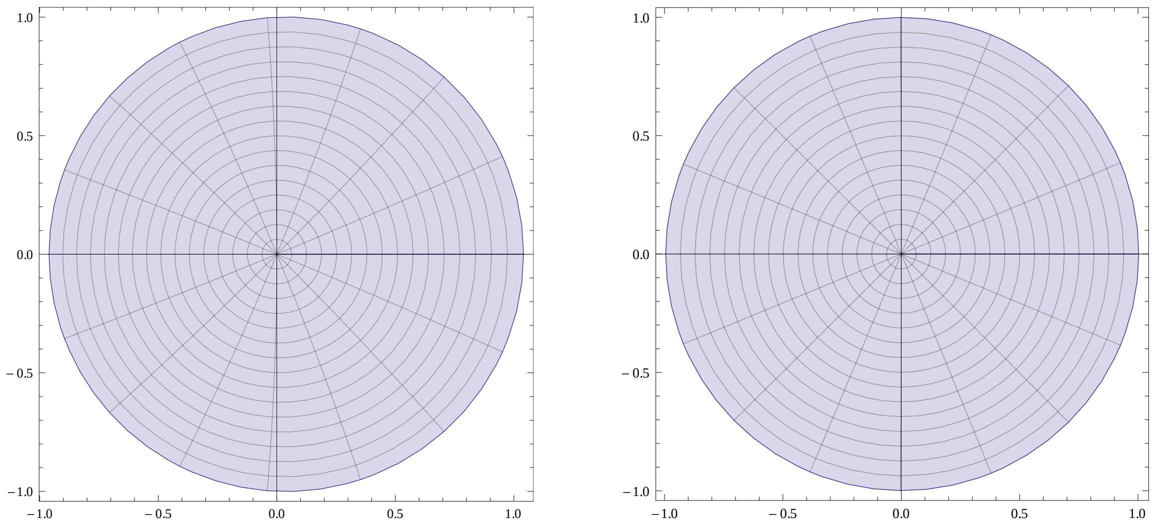


FIGURE 7.1. Description of $zF(0,2;c;z)/F(-1,2;c;z)$ that maps the disk $|z| < 0.999$ to a region close to the unit disk. The left region is computed for $c = 50$ and the right region is for $c = 500$.

this type with interesting geometric properties and their 2D and 3D graphical plots may be of interested. One disadvantage of work in this direction is that solving problems analytically may be sometimes difficult. However, it can be a challenge to describe the relevant image domains and find interesting problems to work in this direction. For example, image of the unit disk under the mapping $zF(0,2;c;z)/F(-1,2;c;z)$ converges to the unit disk when c is larger and larger (see Figures 7.1). Indeed, this is also easy to see from the definition of the basic hypergeometric functions when $c \to \infty$. So, can it be practically possible to analyze the behaviour of $zF(0,2;c;z)/F(-1,2;c;z)$ under certain ranges for the parameter c in terms of a and b? Moreover, one can investigate similar problems where the resultant image domains are special type of convex domains, in particular.

On the other hand, the main theorem in the latter section of Chapter 2 concerns with under which situation the normalized basic hypergeometric functions $z\Phi[a,b;c;q,z]$ belong to the class of q-close-to-convex family. This concept generalizes a result from [**85**]. However, in [**85**] a number of problems associated with convexity, starlikeness and close-to-convexity properties have been extensively studied. Therefore, it is natural to find conditions on the parameters a, b, c and q such that the normalized basic hypergeometric functions $z\Phi[a,b;c;q,z]$ and $z\Phi[a,b;c;q,z^2]$ both are q-close-to-convex as well as q-starlike.

Moreover, order of q-convexity and q-starlikeness can be studied in this setting (see [**4**] for the definition of order of q-starlikeness).

In conclusion, the results of Chapter 2 demonstrate that the computational framework in this setting helps us to generate special functions having interesting geometric properties. Further work in this direction will certainly bring a strong foundation between q-function theory and geometric function theory. It also opens up several avenues for future work which may lead to interesting dissertations.

As pointed out by Andrews in [**6**], there have been several applications of basic hypergeometric series to physics. Baker and Coon [**9**] have successfully utilized basic hypergeometric functions in a series of papers (in the *Physical Review* D) on particle physics. Also, Kampen [**43**] has applied basic hypergeometric functions to fluctuations in an electric circuit consisting of a condenser and a diode. Therefore, it will be extremely remarkable if some application of this development can be worked out connecting to quantum analysis, physics and related topics; see for instance [**6, 23, 24**].

One can think of many more research problems related to **Chapter 3**. For instance, q-analog of convexity of analytic functions in the unit disk and even more general in arbitrary simply connected domains may be interesting for researchers in this field. Recently, the concept of q-convexity for basic hypergeometric functions is considered in [**11**]. Bieberbach conjecture problem for q-close-to-convex functions is estimated optimally in a recent paper [**97**]. In fact sharpness of this result is still an open problem and concerning this, a conjecture is stated there. It would further be interesting to find the best possible $q \in (0,1)$ such that $\mathcal{S}_q^*(\alpha) \subset \mathcal{S}$.

The generalized Zalcman conjecture in the form proposed by Ma in [**61**] is still open for the classes $\overline{co(\mathcal{C})}$ and $\mathcal{R}(\beta)$ for $0 < \lambda < 2$ and $0 < \lambda < \frac{nm}{(1-\beta)(n+m-1)}$ respectively. It would also be interesting to investigate this problem for the class $\mathcal{H}$. In [**97**], some sufficient conditions are established for a function to be in the class $\mathcal{K}_q$. One can think of the Zalcman conjecture problem for the class $\mathcal{K}_q$ by taking some of those sufficient conditions. It would be more interesting if the Bieberbach conjecture problem can be solved by using the Zalcman conjecture problem for all the subclasses of $\mathcal{S}$ discussed in **Chapter 4** including the class $\mathcal{K}_q$. The q-analog of the Zalcman conjecture may be of interest and hence the Bieberbach conjecture.

Chapter 5 talks about necessary conditions for bounded John domains. Recall that Näkki and Väisälä in [**72**] introduced the notion of John domains when they are unbounded and also studied several characterizations of such domains. According to them, John domains are defined as follows:

Definition 7.1. A domain $D \subset \mathbb{C}$ is said to be a *John domain* if any pair of points $z_1, z_2 \in D$ can be joined by a rectifiable path $\gamma \subset D$ such that

$$\min\{\ell(\gamma[z_1, z]), \ell(\gamma[z, z_2])\} \leq c\,\mathrm{dist}\,(z, \partial D), \qquad \text{for all } z \in \gamma,$$

and for some constant $c > 0$, where $\ell(\gamma[z, z_i])$ denote the Euclidean length of γ joining z to z_i, $i = 1, 2$.

Note that when John domains are bounded, then Definition 7.1 is equivalent to the definition of John domains discussed in Section 5.1 (see [**72**]). One can check that the parallel strip $D_1 := \{z \in \mathbb{C} : |\mathrm{Im}\, z| < \pi/4\}$ and the two-sided slit domain D_2, the entire plane minus the two half-lines $-\infty < y \leq -1/2$ and $1/2 \leq y < \infty$, $y = \mathrm{Im}\, z$, are not John domains. But the half-planes and the Koebe domain are John domains.

In this context we are interested to introduce the notion of *John functions*. Motivation behind this comes from the definition of starlike and convex functions in $\mathbb{D}$. A starlike function is a conformal mapping of the unit disk onto a domain starlike with respect to the origin and a convex function is one which maps the unit disk conformally onto a convex domain. For the theory of starlike and convex functions, we refer to the standard books [**21, 28**]. For analytic functions f in $\mathbb{D}$, certain characterizations of John domains $f(\mathbb{D})$ have been studied in [**19, 31**], where functions were not necessarily assumed to be normalized and univalent (see for instance Lemma 5.8). It is also interesting to see what changes would come in the situation when analytic functions are normalized and univalent. This naturally leads to the concept of introducing John functions in $\mathbb{D}$.

Definition 7.2. A function $f \in \mathcal{S}$ is said to be a *John function* [1] if $f(\mathbb{D})$ is a John disk.

Clearly, f is bounded if and only if $f(\mathbb{D})$ is a bounded John disk. We also call such functions the bounded John functions. The functions $f_1(z) = (1/2)\mathrm{Log}\,[(1 + z)/(1 - z)]$ and $f_2(z) = z/(1 - z^2)$ respectively map the unit disk onto the parallel strip D_1 and

[1]The authors wish to call these functions "John functions" in honor of Professor Fritz John.

the two-sided slit domain D_2. Since D_1 and D_2 are not John domains, the functions f_1 and f_2 are not John functions. On the other hand, the functions $g_1(z) = z/(1-z)$ and $g_2(z) = z/(1-z)^2$ are John.

We conclude this section with the following future directional work.

The famous analytical characterization of the starlike and convex functions are respectively

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \quad \text{and} \quad \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \quad z \in \mathbb{D}. \tag{7.1}$$

It is discussed above that neither convex nor starlike functions are necessarily John functions and also the other way around implication fails. Therefore, although certain characterizations of John functions in different situations are studied in [**19, 31**] (see also Lemma 5.8), it would be interesting to find analytical characterizations of John functions similar to that of convex and starlike functions stated in (7.1).

As we know Bieberbach conjecture problem is one of the important problems in univalent function theory, naturally one can ask the similar questions for John functions. We denote by $\mathcal{J}$, the class of all John functions in $\mathbb{D}$. Since $\mathcal{J} \subset \mathcal{S}$, trivially $|a_n| \le n$ holds for $f(z) = z + a_2z^2 + a_3z^3 + \cdots \in \mathcal{J}$ and the bound is sharp since the Koebe function $g_2(z) \in \mathcal{J}$. We notice that for the function g_2, the image domain $g_2(\mathbb{D})$ is not a quasidisk. Therefore, it would be interesting to study the Bieberbach conjecture problem for functions $f \in \mathcal{S}$ such that the images $f(\mathbb{D})$ are quasidisks, since in this case the bound $|a_n| \le n$ is not sharp. Recall that quasidisks are nothing but simply connected uniform domains where a uniform domain [**64**] is a John domain satisfying

$$\ell(\gamma) \le c\,|z_1 - z_2|,$$

where γ and c are as in Definition 7.1.

On the other hand, one can introduce the q-analogs of the Schwarzian and pre-Schwarzian derivatives and find similar necessary conditions for John domains.

Chapter 6 deals with the radius of convexity of sections of odd functions belonging to a close-to-convex family.

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