GEOMETRIC PROPERTIES OF THE CASSINIAN METRIC

Ph. D. Thesis

by

MANAS RANJAN MOHAPATRA

DISCIPLINE OF MATHEMATICS

INDIAN INSTITUTE OF TECHNOLOGY INDORE

JANUARY 2017

GEOMETRIC PROPERTIES OF THE CASSINIAN METRIC

A THESIS

submitted in partial fulfillment of the
requirements for the award of the degree

of

DOCTOR OF PHILOSOPHY

by

MANAS RANJAN MOHAPATRA

DISCIPLINE OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY INDORE

JANUARY 2017

INDIAN INSTITUTE OF TECHNOLOGY INDORE
CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **GEOMETRIC PROPERTIES OF THE CASSINIAN METRIC** in the partial fulfillment of the requirements for the award of the degree of **DOCTOR OF PHILOSOPHY** and submitted in the **DISCIPLINE OF MATHEMATICS, Indian Institute of Technology Indore,** is an authentic record of my own work carried out during the time period from July 2011 to December 2016 under the supervision of Dr. Swadesh Kumar Sahoo, Associate Professor, Indian Institute of Technology Indore.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other institute.

Signature of the student with date
(MANAS RANJAN MOHAPATRA)

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

Signature of Thesis Supervisor with date
(Dr. SWADESH KUMAR SAHOO)

MANAS RANJAN MOHAPATRA has successfully given his Ph.D. Oral Examination held on **May 26, 2017**.

Signature of Thesis Supervisor
Date:

Convener, DPGC
Date:

Sign. of PSPC Member #1
Date:

Sign. of PSPC Member #2
Date:

Sign. of External Examiner
Date:

ACKNOWLEDGEMENTS

I wish to express my sincere gratitude to my supervisor Dr. Swadesh Kumar Sahoo for introducing me to the world of hyperbolic-type metrics and for suggesting me a very interesting topic for the thesis. He had always made himself available to guide and support me in the research. His advices, care and moral support will always be remembered. I am also grateful to his family members for their affection and the atmosphere that they have given me during my residence life at Silver Springs. Lots of love for the sweet little Tani and Bitu.

My warm and sincere thanks to Professor Saminathan Ponnusamy (Head, ISI Chennai Centre) for his inspiration, encouragement and the valuable suggestions that he has given me during my visit to IIT Kharagpur and ISI Chennai. I thank Dr. Zair Ibragimov (Associate Professor, California State University, Fullerton, USA) for his valuable comments on my research work and for being a collaborator to our research. Also, I thank Dr. Riku Klén (a post-doctoral researcher at the Turku Centre for Biotechnology, University of Turku, Finland) and Dr. Xiaohui Zhang (Zhejiang Sci-Tech University, Hangzhou, China) for their contributions and useful discussions on the topic.

I acknowledge Professor Matti Vuorinen (Department of Mathematics and Statistics, University of Turku) and Dr. Antti Rasila (Senior University Lecturer, Aalto University, Helsinki, Finland) for the useful discussions during the international conference ICGFTA 2014 at IIT Kharagpur. I must take this opportunity to thank Dr. Manzi Huang and Professor Xiantao Wang (Hunan Normal University, Changsha, China) for the useful discussions during their visit at IIT Indore. I thank the examiners of this thesis for their valuable comments which makes the thesis at its level best.

It is my pleasure to thank my PSPC members, Dr. Niraj Kumar Shukla and Dr. Santosh Kumar Vishvakarma for reviewing my research progress periodically and for their helpful suggestions/comments. I wish to thank Professor Pradeep Mathur, Director IIT Indore and the Head, Discipline of Mathematics for the academic support and the facilities provided to carry out the research work at IIT Indore. I am thankful to all the faculty members of the Discipline of Mathematics for their inspirational words. Special thanks to my university teacher Professor Jagannath Patel who unknowingly motivated me as a research fellow in complex analysis and my graduation teacher Prafull Chandra Das for making me understand what mathematical analysis is all about.

I gratefully acknowledge the Ministry of Human Resource Development, India for their financial support to carry out my research work at IIT Indore.

I thank Dr. Navneet Lal Sharma, Dr. Sarita Agrawal, Dr. Nijjwal Karak, Vibhuti and Arstu for the helpful discussions within the research group of geometric function theory. I am thankful to my colleagues Ali bhai, Nitin, Prince, Linia, Rupsha Di, Harish and Anupam for their understanding and co-operation. I must mention that the time spent with you guys is a best part of my life. Special thanks to my friends Sanju and Anu for being a part of this journey.

Last but not the least, my deep appreciation to my parents for their support and understanding and to the almighty GOD for helping me to choose this perfect path.

<u>List of Publications</u>

1. Ibragimov Z., Mohapatra M. R., Sahoo S. K., Zhang X.-H. (2017), *Geometry of the Cassinian metric and its inner metric*, Bull. Malays. Math. Sci. Soc., **40** (1), 361–372.
2. Klén R., Mohapatra M. R., Sahoo S. K. (2016), *Geometric properties of the Cassinian metric*, Math. Nachr., `DOI: 10.1002/mana.201600117`.
3. Mohapatra M. R., Sahoo S. K., *Mapping properties of a scale invariant Cassinian metric and a Gromov hyperbolic metric*, Bull. Aust. Math. Soc., Accepted.
4. Mohapatra M. R., Sahoo S. K., *A Gromov hyperbolic metric vs the hyperbolic and other related metrics*, Submitted.

ABSTRACT

KEYWORDS: The hyperbolic metric, the Cassinian metric, the distance ratio metric, a Gromov hyperbolic metric, hyperbolic-type metrics, inner metric, Möbius transformation, distortion (quasi-invariance) property, metric balls, inclusion property, starlikeness and convexity properties, quasiconformal mappings, modulus of continuity.

In this thesis we obtain various inequalities between the so-called Cassinian metric and other well-known hyperbolic-type metrics. For this, comparison of a scale invariant Cassinian metric with a Gromov hyperbolic metric and other hyperbolic-type metrics plays a significant role. We discuss the local convexity property of the Cassinian metric balls and their inclusion relations with other hyperbolic-type metric balls by fixing centre common to each pair of metric balls. The metric ball inclusion properties of the scale invariant Cassinian metric and the Gromov hyperbolic metric with other hyperbolic-type metric balls are also interpreted. We study the distortion property of the Cassinian metric under Möbius transformations of the unit ball onto itself and under Möbius transformations of a punctured ball onto another punctured ball. Hence, the distortion property of the scale invariant Cassinian metric under Möbius transformations of a punctured ball onto another punctured ball is also natural to discuss. We also discuss the quasi-invariance property of the scale invariant Cassinian metric and the Gromov hyperbolic metric under quasiconformal mappings of $\mathbb{R}^n$, $n \geq 2$. Finally, we estimate the modulus of continuity for the identity mapping on a bounded domain equipped with the Cassinian metric onto the same domain with the Euclidean metric.

LIST OF METRICS USED WITH THEIR SYMBOLS

Metric	**Symbol**	**Page #**
The absolute ratio metric (Seittenranta's metric)	δ_D	7
The Apollonian metric	α_D	32
The Cassinian metric	c_D	4
The distance ratio metric Vuorinen version Gehring and Osgood version	 $\tilde{j}_D$ j_D	6
The Gromov hyperbolic metric	u_D	5
The half apollonian metric	η_D	8
The hyperbolic metric of $\mathbb{B}^n$	$\rho_{\mathbb{B}^n}$	6
The inner Cassinian metric	$\tilde{c}_D$	20
The quasihyperbolic metric	k_D	7
The scale-invariant Cassinian metric	$\tilde{\tau}_D$	4
The triangular ratio metric	s_D	8
The visual angle metric	v_D	8

TABLE OF CONTENTS

ACKNOWLEDGEMENTS iii

ABSTRACT vii

LIST OF METRICS USED WITH THEIR SYMBOLS ix

LIST OF FIGURES xiii

Chapter 1 INTRODUCTION 1

1.1 The Cassinian metric 4

1.2 A Gromov hyperbolic metric 5

1.3 Hyperbolic-type metrics 5

1.4 Structure of the thesis 8

Chapter 2 COMPARISONS OF THE CASSINIAN METRIC WITH OTHER RELATED METRICS 15

2.1 Comparison with the hyperbolic metric of $\mathbb{B}^n$ 15

2.2 Comparison with the distance ratio metric 16

2.3 Comparison with the Seittenranta metric 19

2.4 The inner Cassinian metric 20

Chapter 3 COMPARISON WITH A GROMOV HYPERBOLIC METRIC 23

3.1 The Gromov hyperbolic metric u_D 23

3.2 The u_D-metric vs other hyperbolic-type metrics 30

3.3 The u_D-metric vs the hyperbolic metric of $\mathbb{B}^n$ 33

Chapter 4 THE CASSINIAN METRIC BALLS 35

4.1 Starlikeness property 36

4.2 Convexity property 38

4.3 Inclusion properties 39

Chapter 5 MAPPING PROPERTIES 51

5.1 Distortion properties under Möbius transformations 51

5.2 Quasi-invariance properties under quasiconformal mappings 57

5.3 Modulus of continuity 60

Chapter 6 SUMMARY AND FUTURE DIRECTIONS 69

BIBLIOGRAPHY 71

INDEX 77

LIST OF FIGURES

1.1	The shaded region is the the c_{D_0}-metric ball $B_{c_{D_0}}(e_1, \log 3)$ which is not starlike.	11
1.2	The Cassinian disks $B_{c_{\mathbb{H}^2}}(e_2, R)$ with radii $R = 0.6, 0.8$ and 1.0.	11
4.1	The left figure describes the Cassinian disks $B_{c_{\mathbb{R}^2\setminus\{0\}}}(e_1, R)$ with radii $R = 0.3, 0.6, 0.9$ respectively, and the right figure describes the Cassinian disk $B_{c_{\mathbb{R}^2\setminus\{0\}}}(e_1, R)$ with radius $R = 1.1$. The shaded regions are the interior of the disks.	36
5.1	The maximal Cassinian oval with foci at x and y is not larger than the maximal Cassinian oval with foci at x' and y'.	62

CHAPTER 1

INTRODUCTION

This thesis is based on the research work carried out at IIT Indore. The purpose of this chapter is to give some motivations and background knowledge about the research problems discussed in this thesis. Also, this chapter stands as preliminaries for the upcoming chapters.

The concept of distance is a basic mechanism used in the whole human experience. In everyday life it usually means some degree of closeness of two physical objects or ideas, i.e., length, time interval, gap, rank difference, while the term *metric* is often used as a standard tool for measurement. In mathematics, we understand that a metric (distance) d on a non-empty set X is a function from $X \times X$ to the set of non-negative real numbers satisfying the following conditions:

- $d(x, y) \geq 0$ for all $x, y \in X$ (Positivity)
- $d(x, y) = 0 \iff x = y$
- $d(x, y) = d(y, x)$ (Symmetric)
- $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, z, y \in X$ (Triangle Inequality).

The set X together with the distance function d is called a *metric space* and is denoted by (X, d). The notion of metric was first introduced by M. Fréchet [14] in his PhD thesis in the year 1906. F. Hausdorff also studied metric spaces in 1914 as a special case of an infinite topological space. However, it was K. Menger, who, in 1928 introduced metric spaces in geometry [56] (see also [62] for more historical background). The notion of metric got adopted very quickly among function theorists. For example, in topology for defining open sets, in functional analysis for defining norms, etc. Now-a-days metric is considered as an essential tool in many areas of mathematics. For more on metrics, we refer [12].

In nineteenth century, the concept of the hyperbolic geometry came to the picture. The hyperbolic geometry satisfies all the postulates of Euclid except the parallel postulate which states that *given a straight line and a point (not on the line), there exists a* *unique straight line passing through the given point and not intersecting the given line.* In hyperbolic sense, there are infinite number of hyperbolic lines passing through the given point and not intersecting the given line. The hyperbolic metric enables one to formulate the Schwarz-Pick lemma in a simple way [58] (see, also [6]). Ever since the introduction of the hyperbolic geometry there were many attempts to extend the hyperbolic metric to more general settings. For instance, it can be extended to simply-connected planar domains having at least three boundary points with the help of the classical Riemann Mapping Theorem. The absence of the Riemann Mapping Theorem in higher dimensions (i.e, in $\mathbb{R}^n$, $n \geq 3$) leads to define the hyperbolic metric only on balls and on half spaces. However, there are metrics developed in subdomains of $\mathbb{R}^n$ which coincide with the hyperbolic metric either in balls or in half spaces. This development leads to many other metrics defined in subdomains of $\mathbb{R}^n$ and they are very close to the hyperbolic metric and its generalizations in several senses. A common property among all these metrics is that they are evaluated at some of the boundary points of the domain induced by these metrics. Due to this reason, probably, they are named as *hyperbolic-type metrics*.

In the literature, there are many well-known domains which are characterized in terms of metric inequalities. For example, uniform domains, John domains, quasidisks, etc, are characterized in terms of inequalities associated with certain hyperbolic-type metrics, see for instance [15–17,53]. Saying differently, the comparison of metrics reveals the geometry of the domain. This motivated us to compare the so-called *Cassinian metric* with other related hyperbolic-type metrics.

It is well-known that the geometric structure of a metric space can be viewed from the geometric structure of the fundamental element, namely, the metric balls associated with the metric. Hence it is reasonable to study the metric balls and their inclusion relations with other metric balls by fixing the centre common to each pair of the metric balls.

A significant part of geometric function theory is to study the behaviour of distances under certain classes of mappings, namely, the Möbius class, the quasiconformal class, etc. Möbius invariants have special role in function theory [1, 2, 66]. If a metric is not invariant under *Möbius transformations*, we must study its quasi-invariance property. The quasi-invariance property under *quasiconformal mappings* is hence natural to study.

A *modulus of continuity* is a function $\omega : [0, \infty] \to [0, \infty]$ used to measure quantitatively the uniform continuity of functions. Let (X_j, d_j), $j = 1, 2$, be metric spaces. A function $f: X_1 \to X_2$ admits ω as *modulus of continuity* if and only if for all $x, y \in X_1$, $d_2(f(x), f(y)) \le \omega(d_1(x, y))$. We also call such functions as *uniform continuous with modulus of continuity* ω (or ω*-uniformly continuous*). For instance, for $k > 0$, the modulus $\omega(t) = kt$ describes the k-Lipschitz functions, the moduli $\omega(t) = kt^\alpha$, $\alpha > 0$, describe the Hölder continuity, and so on. To simplify matters in this topic, we always assume that $\omega(t)$ is an increasing homeomorphism. Conversely, any uniformly continuous function $f : (X_1, d_1) \to (X_2, d_2)$ admits a modulus of continuity $\omega_f(t)$ defined by

$$\omega_f(t) = \sup\{d_2(f(x), f(y)) : d_1(x, y) = t\}, \quad x, y \in X_1.$$

The quasiconformal counter part of the Schwarz lemma [1] says that quasiconformal mappings of the unit ball are uniformly continuous with respect to the hyperbolic metric. Hence, we are also interested to study the *uniform continuity*, precisely to estimate the *modulus of continuity*, of identity mappings on bounded domains equipped with the Cassinian metric onto the same domain with the Euclidean metric.

Unless otherwise stated, throughout this thesis, D denotes an arbitrary, proper subdomain (open connected set) of the Euclidean space $\mathbb{R}^n$. We write $D \subsetneq \mathbb{R}^n$. We denote the punctured space $\mathbb{R}^n \setminus \{p\}$ by D_p and the twice-punctured space $\mathbb{R}^n \setminus \{p, q\}$ by $D_{p,q}$. The Euclidean distance between $x, y \in \mathbb{R}^n$ is denoted by $|x - y|$. The standard Euclidean norm of a point $x \in \mathbb{R}^n$ is denoted by $|x|$. Given $x \in \mathbb{R}^n$ and $r > 0$, the open ball centred at x and of radius r is denoted by $B(x, r) = \{y \in \mathbb{R}^n : \ |x - y| < r\}$. We set $\mathbb{B}^n = B(0, 1)$. The upper half space of $\mathbb{R}^n$ is defined by $\mathbb{H}^n := \{(x_1, x_2, \ldots, x_n) : x_n > 0\}$. The closed line segment between two points x and y in $\mathbb{R}^n$ is denoted by $[x, y]$. Given $x \in D$, the distance $d(x)$ from x to the boundary ∂D of D is given by

$$d(x) = \inf\big\{|x - \xi| : \ \xi \in \partial D\big\}.$$

For real numbers r and s, we set $r \vee s = \max\{r, s\}$ and $r \wedge s = \min\{r, s\}$. In a metric space (D, d), we use the terminology d*-metric*, for the distance function associated with the metric d.

1.1. The Cassinian metric

In 2009, Ibragimov introduced the notion of the *Cassinian metric* [37], $c_D(x,y)$, defined for $x,y\in D\subsetneq\mathbb{R}^n$ by

$$c_D(x,y) := \sup_{p\in\partial D}\frac{|x-y|}{|x-p|\,|p-y|}.$$

Geometrically, it can be defined by means of maximal Cassinian ovals in the similar fashion as the Apollonian metric is defined in terms of maximal Apollonian balls [5]. This metric was introduced in a desire to express the notion of convexity in Möbius invariant fashion. Note that the Cassinian metric space is not complete in general. However, if D is a domain with $\infty\notin\partial D$, then the Cassinian metric space is complete. Also, the c_D-metric is not invariant under *Möbius transformations*. Ibragimov in [37] conjectured that the *Cassinian isometries* are restrictions of Euclidean isometries (orthogonal transformations as well as the translations of $\mathbb{R}^n$) if the underlying domain has at least two finite boundary points. Here we mean by a Cassinian isometry, a homeomorphism $f: D\to\overline{\mathbb{R}^n}$ satisfying

$$c_{f(D)}(f(x),f(y)) = c_D(x,y), \qquad \text{for all } x,y\in D.$$

However, it can be seen from the definition that the c_D-metric is not invariant under scaling maps. That is, the c_D-metric is not invariant under the maps of the form $f(x)=kx$, $k\in\mathbb{R}$. It was natural to expect a scale invariant version of the c_D-metric which is introduced very recently by Ibragimov in [40]. For more on the c_D-metric, we refer [20, 37, 42, 49]. Note that a more general form of this metric was considered by P. Hästö (see [21, Lemma 6.1]).

For $x,y\in D\subsetneq\mathbb{R}^n$, a *scale invariant Cassinian metric*, $\tilde{\tau}_D(x,y)$, is defined by

$$\tilde{\tau}_D(x,y) = \log\left(1+\sup_{p\in\partial D}\frac{|x-y|}{\sqrt{|x-p||p-y|}}\right).$$

An interesting part of this metric is that many properties in arbitrary domains are revealed in the setting of once-punctured spaces. For example, $\tilde{\tau}_D(x,y)$ is a metric in an arbitrary domain $D\subsetneq\mathbb{R}^n$ if it is a metric on once-punctured spaces. Both the c_D-metric and the $\tilde{\tau}_D$-metric are monotone with respect to domains. That is, if $D\subsetneq D'$, then $m_D(x,y)\geq m_{D'}(x,y)$ for all $x,y\in D$ and $m_D\in\{c_D,\tilde{\tau}_D\}$. Another interesting part is that the $\tilde{\tau}_D$-metric is Möbius invariant only in once-punctured spaces [40, Lemma 2.1]. However, isometries of this metric are not studied in the literature yet.

1.2. A Gromov hyperbolic metric

Gromov in 1987 introduced the notion of an abstract hyperbolic space [19]. Let (D, d) be a metric space and $x, y, z \in D$. The *Gromov product* of x and y with respect to z is defined by the formula

$$(x|y)_z = \frac{1}{2}\left[d(x,z) + d(y,z) - d(x,y)\right].$$

The metric space (D, d) is said to be *Gromov hyperbolic* if there exists $\beta \geq 0$ such that

$$(x|y)_w \geq (x|z)_w \wedge (z|y)_w - \beta$$

for all $x, y, z, w \in D$. We also say that D is β-hyperbolic. Equivalently, the metric space (D, d) is called Gromov hyperbolic if and only if there exist a constant $\beta > 0$ such that

$$d(x,z) + d(y,w) \leq (d(x,w) + d(y,z)) \vee (d(x,y) + d(z,w)) + 2\beta$$

for all points $x, y, z, w \in D$. Note that Gromov hyperbolicity is preserved under quasi-isometries. That means it is preserved under the mappings $f : (D, d_1) \to (f(D), d_2)$ satisfying

$$\frac{1}{\lambda} d_1(x,y) - k \leq d_2(f(x), f(y)) \leq \lambda d_1(x,y) + k, \quad x, y \in D,$$

where $\lambda \geq 1$ and $k \geq 0$. Literature on Gromov hyperbolicity are available in [8, 19, 25, 38, 39, 65].

A Gromov hyperbolic metric, the so-called u_D*-metric*, defined by

$$u_D(x,y) = 2\log\left(\frac{|x-y| + d(x) \vee d(y)}{\sqrt{d(x)\,d(y)}}\right), \quad x, y \in D \subsetneq \mathbb{R}^n,$$

is introduced by Ibragimov in [38]. This metric hyperbolizes (in the sense of Gromov) locally compact non-complete metric spaces without changing its quasiconformal geometry, that is, the quasiconformal geometry of the original metric space behaves similar if it is equipped with the u_D-metric. Note that the u_D-metric can be defined on any metric space, however, this does not satisfy the domain monotone property, in general.

1.3. Hyperbolic-type metrics

Recall that the hyperbolic metric enables us to formulate the classical Schwarz-Pick lemma in a simple way which states that analytic functions of the unit disk decreases hyperbolic distances. Due to absence of the Riemann Mapping Theorem in higher dimensions (i.e. in $\mathbb{R}^n$, $n \geq 3$), the hyperbolic metric is defined only on balls and half spaces. However, there are several metrics developed in arbitrary subdomains of $\mathbb{R}^n$ which coincide with the hyperbolic metric of balls or with the hyperbolic metric of half spaces, and some of them are not. They are named as hyperbolic-type metrics. In this section, we begin with the definition of the hyperbolic metric of the unit ball followed by other well-known hyperbolic-type metrics.

The hyperbolic metric

The *hyperbolic metric*, $\rho_{\mathbb{B}^n}(x, y)$, of the unit ball $\mathbb{B}^n$ is given by

$$\rho_{\mathbb{B}^n}(x, y) = \inf_{\gamma} \int_{\gamma} \frac{2|\mathrm{d}z|}{1 - |z|^2},$$

where the infimum is taken over all rectifiable curves $\gamma \subset \mathbb{B}^n$ joining x and y. Note that for all $x, y \in \mathbb{B}^n$, the $\rho_{\mathbb{B}^n}$-metric has the following simpler form:

$$\sinh\left(\frac{\rho_{\mathbb{B}^n}(x, y)}{2}\right) = \frac{|x - y|}{\sqrt{(1 - |x|^2)(1 - |y|^2)}} \tag{1.1}$$

(see, for example, [4, p. 40]).

The distance ratio metric

The *distance ratio metric* [66], $\tilde{j}_D(x, y)$, is defined for $x, y \in D \subsetneq \mathbb{R}^n$ by

$$\tilde{j}_D(x, y) = \log\left(1 + \frac{|x - y|}{d(x) \wedge d(y)}\right).$$

It is a slight modification of the original distance ratio metric, $j_D(x, y)$, introduced by Gehring and Osgood in [17], defined for $x, y \in D \subsetneq \mathbb{R}^n$ by

$$j_D(x, y) = \frac{1}{2} \log\left(1 + \frac{|x - y|}{d(x)}\right)\left(1 + \frac{|x - y|}{d(y)}\right).$$

The $\tilde{j}_D$-metric and the j_D-metric are related by: $(1/2)\tilde{j}_D \leq j_D \leq \tilde{j}_D$. Note that for $x, y \in D_0$ with $|x| = |y|$, $j_{D_0}(x, y) = \tilde{j}_{D_0}(x, y) = u_{D_0}(x, y)$. The j_D-metric is also quasi-isometric to the u_D-metric. Indeed, we have

$$2j_D(x, y) \leq u_D(x, y) \leq 2j_D(x, y) + 2\log 2, \quad x, y \in D$$

see, for instance [38, Theorem 3.1]. The $\tilde{j}_D$-metric space is complete. The $\tilde{j}_D$-metric is monotone with respect to domains, and also invariant under similarity mappings. That is, the $\tilde{j}_D$-metric is invariant under the mappings $f : \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$ satisfying $f(\infty) = \infty$ and

$|f(x)-f(y)| = c|x-y|, c>0$, for all $x,y \in \mathbb{R}^n$. For more literature on the $\tilde{j}_D$-metric we refer [67].

The quasihyperbolic metric

The *quasihyperbolic metric* [18], $k_D(x,y)$, is defined by

$$k_D(x,y) = \inf_{\gamma \in \Gamma(x,y)} \int_\gamma \frac{|dz|}{d(z)},$$

where $\Gamma(x,y)$ denotes the family of rectifiable curves joining x and y in D. The quasihyperbolic metric was introduced by Gehring and Palka in 1976 and subsequently studied by Gehring and Osgood (see [17,18]) as a generalization of the hyperbolic metric of the upper half plane to arbitrary proper subdomains of $\mathbb{R}^n$. Indeed, we have $k_{\mathbb{H}^n}(x,y) = \rho_{\mathbb{H}^n}(x,y)$ for all $x,y \in \mathbb{H}^n$. The k_D-metric is the inner metric of the $\tilde{j}_D$-metric [64] and in the unit ball they also coincide along radial directions, that is, $k_{\mathbb{B}^n}(x,y) = \tilde{j}_{\mathbb{B}^n}(x,y)$ for $x,y \in \mathbb{B}^n$ with $y = tx, t > 0$. It is also important here to record that the k_D-metric and the $\tilde{j}_D$-metric are equivalent only in uniform domains (see [17] for details). This is one of the reasons why comparisons of hyperbolic-type metrics are important in this theory. Isometries of the k_D-metric are studied by Hästö in [23].

The Seittenranta metric

Seittenranta in [59] defined a Möbius invariant metric, namely the *absolute ratio metric* (also called *the Seittenranta metric*), denoted by $\delta_D(x,y)$, and is defined for $x,y \in D \subsetneq \mathbb{R}^n$ by

$$\delta_D(x,y) = \log(1 + m_D(x,y)),$$

where

$$m_D(x,y) = \sup_{a,b \in \partial D} \frac{|x-y||a-b|}{|x-a||y-b|}.$$

Note that the quantity $m_D(x,y)$ does not define a metric, whereas the quantity δ_D defines a metric. The δ_D-metric is a generalization of the hyperbolic metric of the unit ball $\mathbb{B}^n$ onto proper subdomains of $\mathbb{R}^n$. In fact, $\delta_{\mathbb{B}^n}(x,y) = \rho_{\mathbb{B}^n}(x,y)$ for all $x,y \in \mathbb{B}^n$. Uniform domains are also characterized by comparing the δ_D-metric and *Ferrand's metric* [59].

The visual angle metric

The *visual angle metric*, $v_D(x,y)$, introduced in [48], is defined by

$$v_D(x,y) = \sup\{\angle(x,z,y) : z \in \partial D\}, \quad x, y \in D.$$

The v_D-metric is similarity invariant. It is defined on a domain $D \subsetneq \mathbb{R}^n$ whose boundary is not a proper subset of a line. This metric does not coincide with the hyperbolic metric of $\mathbb{B}^n$ (or $\mathbb{H}^n$), but it is comparable (see, [48, Theorem 1.1]).

The triangular ratio metric

The *triangular ratio metric* [11], $s_D(x,y)$, defined on $D \subsetneq \mathbb{R}^n$ by

$$s_D(x,y) = \sup_{p \in \partial D} \frac{|x-y|}{|x-p|+|p-y|}, \quad x, y \in D.$$

Geometrically, the triangular ratio metric can be viewed by taking the maximal ellipse in D with foci at x and y in the similar fashion as the Cassinian metric [37] is defined. For more details on the s_D-metric we refer [11]. Note that this metric also does not coincide with the hyperbolic metric.

The half-apollonian metric

The *half-apollonian metric* [28], $\eta_D(x,y)$, is defined for $D \subsetneq \mathbb{R}^n$ by

$$\eta_D(x,y) = \sup_{p \in \partial D} \left| \log \frac{|x-p|}{|y-p|} \right|, \quad x, y \in D.$$

The η_D-metric was introduced by Hästö and Linden in [28]. This metric is bilipschitz equivalent to the so-called *Apollonian metric*, introduced by Beardon in [5], as a generalization of the hyperbolic metric of the unit ball as well as the upper half space onto arbitrary subdomains of $\mathbb{R}^n$. The η_D-metric can be defined in arbitrary domains in Euclidean space and has the advantages of being easy to calculate and estimate. It is shown in [28] that in many cases the isometries of the η_D-metric are similarity mappings.

1.4. Structure of the thesis

The thesis consists of six chapters. **Chapter 1** covers almost all the definitions, motivations and background knowledge for problems discussed in the remaining chapters and **Chapter 6** deals with concluding remarks with some open problems for further study.

In **Chapter 2**, we compare the Cassinian metric with the hyperbolic metric of the unit ball, and with some of the metrics defined in Section 1.3. First we obtain the following relationship between the $c_{\mathbb{B}^n}$-metric and the $\rho_{\mathbb{B}^n}$-metric.

Theorem 1.1. *For $x, y \in \mathbb{B}^n$, we have*

$$\sinh\left(\frac{\rho_{\mathbb{B}^n}(x,y)}{2}\right) \le c_{\mathbb{B}^n}(x,y). \tag{1.2}$$

The following relationship holds true between the $\tilde{j}_D$-metric and the c_D-metric in proper subdomains D of $\mathbb{R}^n$.

Theorem 1.2. *Let D be a proper subdomain of $\mathbb{R}^n$ and let $x, y \in D$. Then*

$$\tilde{j}_D(x,y) \le \big(|x-y| + (d(x) \wedge d(y))\big) c_D(x,y).$$

The δ_D-metric and the c_D-metric are comparable in the following way:

Theorem 1.3. *Let $D \subsetneq \overline{\mathbb{R}^n}$ be any domain. Then for $x, y \in D$*

$$c_D(x,y) \le \frac{e^{\delta_D(x,y)} - 1}{d(y)}.$$

Equality holds for $D = D_p$.

In **Chapter 3**, we compare the u_D-metric and the $\tilde{\tau}_D$-metric with other hyperbolic-type metrics. We also show that a domain equipped with the $\tilde{\tau}_D$-metric is hyperbolic in the sense of Gromov by comparing the u_D-metric and the $\tilde{\tau}_D$-metric. We prove that

Theorem 1.4. *Let $D \subsetneq \mathbb{R}^n$ be any domain with $\partial D \neq \emptyset$ and $x, y \in D$. Then*

$$2\tilde{\tau}_D(x,y) \le u_D(x,y).$$

Equality holds whenever $d(x) = |x-p| = |y-p| = d(y)$ for some $p \in \partial D$. Moreover, there exists no constant $k \ge 0$ such that

$$u_D(x,y) \le 2\tilde{\tau}_D(x,y) + k$$

for all $x, y \in D$ unless D is a one-punctured space. If D is a once-punctured space, then

$$u_D(x,y) \le 2\tilde{\tau}_D(x,y) + 2\log 2$$

and the inequality is sharp.

Theorem 1.4 shows that the metric space $(D_p, \tilde{\tau}_{D_p})$ is Gromov hyperbolic (or β-hyperbolic) with $\beta = \log 2$. Moreover, in this chapter we compare the u_D-metric with

- the $\tilde{j}_D$-metric
- the j_D-metric
- the $\rho_{\mathbb{B}^n}$-metric.

Chapter 4 deals with the local convexity properties of the c_D-metric balls and their inclusion relations with other hyperbolic-type metric balls. Precisely, we obtain the maximal radius for which the c_D-metric ball is *convex*. We say that $D \subsetneq \mathbb{R}^n$ is *starlike with respect to* x if each line segment from x to $y \in D$ is contained completely in D. The domain D is *strictly starlike with respect to* x if D is bounded and each ray from x meets the boundary of D at exactly one point. The domain D is called convex (strictly convex) if it is starlike (strictly starlike) with respect to all points in D.

Note that there are non-starlike domains in which the Cassinian metric balls are not starlike (see Figure 1.1). But we observe that the c_D-metric balls are starlike in starlike domains-a fact-described in the following form:

Theorem 1.5. *Let $r > 0$. If $D \subsetneq \mathbb{R}^n$ is a starlike domain with respect to $x \in D$, then $B_{c_D}(x, r)$ is starlike with respect to x.*

We also study the convexity property of the Cassinian metric balls in small radii. In this regard we prove the following result.

Theorem 1.6. *Let $D \subsetneq \mathbb{R}^n$ be any arbitrary domain. Then $B_{c_D}(x, R)$ is convex for $R \leq \sup\{1/|x - z_i| : z_i \in \partial D\}$, and is strictly convex for $R < \sup\{1/|x - z_i| : z_i \in \partial D\}$.*

In arbitrary domains, the Cassinian balls are convex for small radii. But they need not be convex for any radius. Even in convex domains, the Cassinian balls need not be convex for any radius. For example, in the upper half plane $\mathbb{H}^2 := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$, if we choose $x = i/2$, then $B_{c_D}(x, r)$ need not be convex for all $r > 0$. See Figure 1.2. This discussion leads to the following conjectures.

Conjecture 1.7. *There exists $r_0 (\approx 0.85)$ such that $B_{c_{\mathbb{H}^2}}(x, r)$ is convex for all $x \in \mathbb{H}^2$ and $r \in (0, r_0 d_{\mathbb{H}^2}(x)]$.*

Conjecture 1.8. *In a bounded convex domain D, $B_{c_D}(x, r)$ is convex for all $x \in D$ and $r > 0$.*

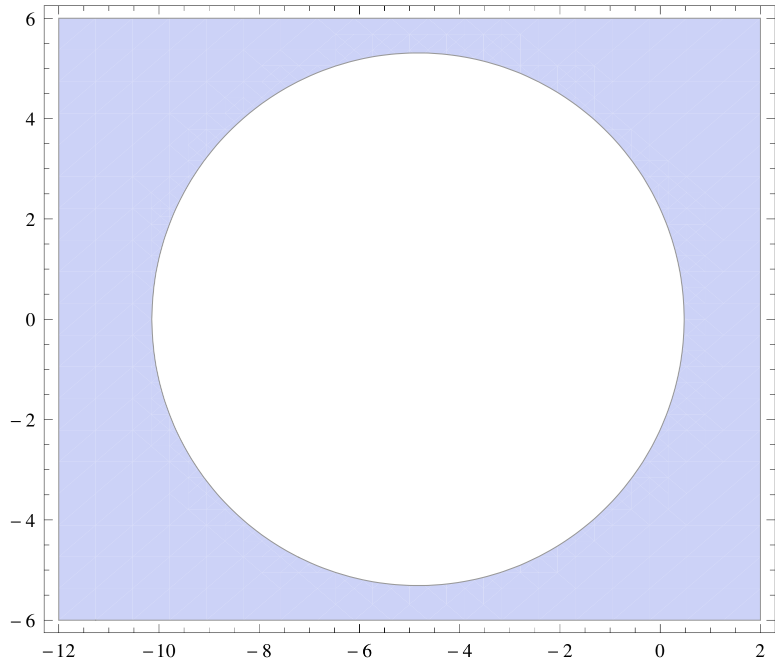

FIGURE 1.1. The shaded region is the the c_{D_0}-metric ball $B_{c_{D_0}}(e_1, \log 3)$ which is not starlike.

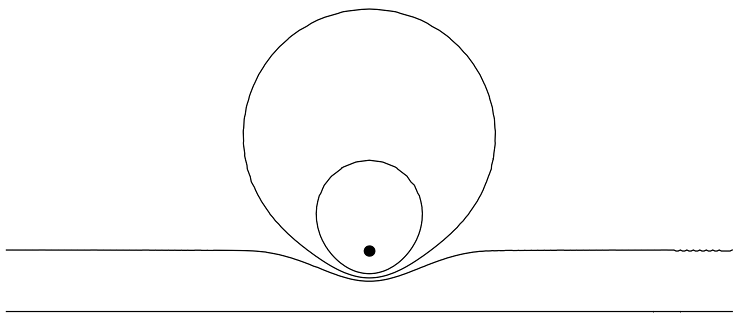

FIGURE 1.2. The Cassinian disks $B_{c_{\mathbb{H}^2}}(e_2, R)$ with radii $R = 0.6, 0.8$ and 1.0.

We also discuss the problem of the following type in this chapter.

Given $x \in D \subsetneq \mathbb{R}^n$ *and* $t > 0$*, we find optimal radii* $r, R > 0$ *depending only on* x *and* t *such that*

$$B_d(x, r) \subset B_{m_D}(x, t) \subset B_d(x, R),$$

where $m_D \in \{c_D, \tilde{\tau}_D, u_D\}$ *and* d *is any other hyperbolic-type metric different from* m_D*.*

In **Chapter 5**, we discuss the quasi-invariance (distortion) properties for the c_D-metric and the $\tilde{\tau}_D$-metric under *Möbius transformations* as well as the quasi-invariance properties for the $\tilde{\tau}_D$-metric and the u_D-metric under *quasiconformal mappings*. First, we obtain the distortion of the c_D-metric under Möbius transformations of the unit ball.

Theorem 1.9. *Let ϕ be a Möbius transformation with $\phi(\mathbb{B}^n) = \mathbb{B}^n$. Then*

$$\frac{1-|\phi(0)|}{1+|\phi(0)|} c_{\mathbb{B}^n}(x,y) \le c_{\mathbb{B}^n}\big(\phi(x),\phi(y)\big) \le \frac{1+|\phi(0)|}{1-|\phi(0)|} c_{\mathbb{B}^n}(x,y)$$

for all $x, y \in \mathbb{B}^n$. The equalities in both sides can be attained.

We also prove that the same distortion constant holds true for the c_D-metric under Möbius transformations of a punctured ball onto another punctured ball. Indeed, we prove that

Theorem 1.10. *Let $a \in \mathbb{B}^n$ and $f : \mathbb{B}^n \setminus \{0\} \to \mathbb{B}^n \setminus \{a\}$ be a Möbius transformation with $f(0) = a$. Then for $x, y \in \mathbb{B}^n \setminus \{0\}$ we have*

$$\frac{1-|a|}{1+|a|} c_{\mathbb{B}^n\setminus\{0\}}(x,y) \le c_{\mathbb{B}^n\setminus\{a\}}(f(x),f(y)) \le \frac{1+|a|}{1-|a|} c_{\mathbb{B}^n\setminus\{0\}}(x,y).$$

The equalities in both sides can be attained.

Similar to the distortion property of the c_D-metric under Möbius transformations of $\mathbb{B}^n$, the $\tilde{\tau}_D$-metric exhibits same distortion constants under Möbius transformations of a punctured ball onto another punctured ball. Note that the distortion of the $\tilde{\tau}_D$-metric under Möbius transformations of $\mathbb{B}^n$ is proved in [40]. We prove that

Theorem 1.11. *Let $a \in \mathbb{B}^n$ and $f : \mathbb{B}^n \setminus \{0\} \to \mathbb{B}^n \setminus \{a\}$ be a Möbius transformation with $f(0) = a$. Then for $x, y \in \mathbb{B}^n \setminus \{0\}$ we have*

$$\frac{1-|a|}{1+|a|} \tilde{\tau}_{\mathbb{B}^n\setminus\{0\}}(x,y) \le \tilde{\tau}_{\mathbb{B}^n\setminus\{a\}}(f(x),f(y)) \le \frac{1+|a|}{1-|a|} \tilde{\tau}_{\mathbb{B}^n\setminus\{0\}}(x,y).$$

Also, we discuss the quasi-invariance property of the $\tilde{\tau}_D$-metric and the u_D-metric under quasiconformal mappings. Quasiconformal mappings in $\mathbb{R}^n$ can be characterized in many equivalent ways. A comprehensive treatment of different definitions is given in [10]. While definitions are equivalent, there are subtle differences between them when we consider the parameter K in the notion of K-quasiconformality. It is appropriate here to mention that we adopt the metric definition of K-quasiconformality introduced by Väisälä in [63]. Recall that a homeomorphism $f : D \to f(D) \subsetneq \mathbb{R}^n$, $D \subsetneq \mathbb{R}^n$, is said to be K-quasiconformal for some K $(1 \le K < \infty)$ if

$$H(f,x) = \limsup_{r\to 0} \frac{\max\{|f(x)-f(y)| : |x-y| = r\}}{\min\{|f(x)-f(y)| : |x-y| = r\}} \le K, \qquad \text{for } x \in D, r \in (0, d(x)).$$

Here $H(f,x)$ is called the *linear dilatation of* f *at* $x \in D$. Moreover, if f is K-quasiconformal then $\sup_{x\in D} H(f,x) \le c(n,K) < \infty$, where $c(n,K)$ is a constant depending upon the dimension of the space n and K. It can easily be verified that L-bilipschitz mappings are L^2-quasiconformal (see Section 5.2 for detailed discussion). We prove that

Theorem 1.12. *For $n \ge 1$ and $K \ge 1$, if $f : \mathbb{R}^n \to \mathbb{R}^n$ is a K-quasiconformal mapping of $\mathbb{R}^n$ which maps $D \subsetneq \mathbb{R}^n$ onto $D' \subsetneq \mathbb{R}^n$ then there exists a constant C_1 depending only on n and K such that*

$$\tilde{\tau}_{D'}(f(x), f(y)) \le C_1 \max\{\tilde{\tau}_D(x,y), \tilde{\tau}_D(x,y)^\alpha\}$$

for all $x, y \in D$, where $\alpha = K^{1/(1-n)}$.

Theorem 1.13. *For $n \ge 1$ and $K \ge 1$, if $f : \mathbb{R}^n \to \mathbb{R}^n$ is a K-quasiconformal mapping of $\mathbb{R}^n$ which maps $D \subsetneq \mathbb{R}^n$ onto $D' \subsetneq \mathbb{R}^n$ then there exists a constant C_2 depending only on n and K such that*

$$u_{D'}(f(x), f(y)) \le C_2 \max\{u_D(x,y), u_D(x,y)^\alpha\}$$

for all $x, y \in D$, where $\alpha = K^{1/(1-n)}$.

Finally, we discuss modulus of continuity for an identity map associated with the Cassinian metric. Indeed, we aim to obtain the modulus of continuity for the identity map

$$id : (D, m_D) \to (D, |.|),$$

where m_D is either the c_D-metric, or the u_D-metric, or the $\tilde{\tau}_D$-metric. More precisely, we consider the problem to find a bound, as sharp as possible, for the modulus of continuity of the identity map defined above. In this regard, we prove that

Theorem 1.14. *Let $D \subsetneq \mathbb{R}^n$ be a domain with* $\operatorname{diam} D < \infty$ *and* $r = \sqrt{n/(2n+2)}\operatorname{diam} D$. *Then we have*

$$c_D(x,y) \ge \frac{4|x-y|}{4-|x-y|^2} \ge \frac{|x-y|}{r}$$

for all distinct $x, y \in D$ *with equality in the first step when* $D = B^n(z,r)$ *and* $z = (x+y)/2$. *In particular, the modulus of continuity for* $id : (D, c_D) \to (D, |.|)$ *is* $\omega(t) = rt$.

Theorem 1.15. *Let* $D \subsetneq \mathbb{R}^n$ *be a domain with* $\operatorname{diam} D < \infty$ *and* $r = \sqrt{n/(2n+2)}\operatorname{diam} D$. *Then we have*

$$\tilde{\tau}_D(x,y) \geq \log\left(1 + \frac{2|x-y|}{\sqrt{4r^2 - |x-y|^2}}\right) \geq c\,\frac{|x-y|}{r}$$

for all distinct $x, y \in D$ *with equality in the first step when* $D = B^n(z,r)$ *and* $z = (x+y)/2$. *Here* $c\,(\approx 0.76)$ *is the solution of the equation*

$$(4-t^2)(2t + \sqrt{4-t^2})\log\left(1 + \frac{2t}{\sqrt{4-t^2}}\right) - 8t = 0.$$

In particular, the modulus of continuity for $id : (D, \tilde{\tau}_D) \to (D, |.|)$ *is* $\omega(t) = rt/c$.

CHAPTER 2

COMPARISONS OF THE CASSINIAN METRIC WITH OTHER RELATED METRICS

This chapter is devoted to find upper and lower bounds for the Cassinian metric in terms of other hyperbolic-type metrics defined in Chapter 1. We begin with the comparison of the Cassinian metric and the hyperbolic metric of the unit ball $\mathbb{B}^n$.

2.1. Comparison with the hyperbolic metric of $\mathbb{B}^n$

In this section, we provide the proof of the relation between the $c_{\mathbb{B}^n}$-metric and the $\rho_{\mathbb{B}^n}$-metric which is stated in Theorem 1.1. We recall the statement of the theorem.

Theorem 2.1. *(see also, Theorem 1.1) For $x, y \in \mathbb{B}^n$, we have*

$$\sinh\left(\frac{\rho_{\mathbb{B}^n}(x,y)}{2}\right) \le c_{\mathbb{B}^n}(x,y).$$

Proof. Without loss of generality, we may assume that $|y| \ge |x|$. It is trivial that the inequality (1.2) holds for $y = 0$, since $x = 0$ in this case also. Hence we assume that $y \neq 0$. It is easy to see that

$$\begin{aligned} \inf_{z \in \partial\mathbb{B}^n} |x-z||y-z| &\le \left|x - \frac{y}{|y|}\right| \left|y - \frac{y}{|y|}\right| \\ &= (1-|y|)\left|x - \frac{y}{|y|}\right| \\ &\le (1-|y|)(1+|x|) \\ &\le \sqrt{(1-|x|^2)(1-|y|^2)}, \end{aligned}$$

where the last inequality follows since $|x| \le |y|$. Now, the formula (1.1) easily yields

$$\sinh\left(\frac{\rho_{\mathbb{B}^n}(x,y)}{2}\right) \le \frac{|x-y|}{\inf\limits_{z\in\partial\mathbb{B}^n} |x-z||y-z|} = c_{\mathbb{B}^n}(x,y).$$

Hence the proof is complete. $\square$

Remark 2.2. The inequality in (1.2) is sharp in the following sense. For 0 and $x \in \mathbb{B}^n \setminus \{0\}$, we use the formulae

$$c_{\mathbb{B}^n}(0,x) = \frac{|0-x|}{\left|\frac{x}{|x|}-0\right|\left|\frac{x}{|x|}-x\right|} = \frac{|x|}{1-|x|} \tag{2.1}$$

(see [37, Example 3.9(B)]) and (1.1). It follows that

$$\frac{\sinh\left(\frac{\rho_{\mathbb{B}^n}(0,x)}{2}\right)}{c_{\mathbb{B}^n}(0,x)} = \frac{1-|x|}{\sqrt{1-|x|^2}}$$

approaches 1 as x approaches 0.

It is well known that $\sinh x \geq x$ for all $x \geq 0$. This leads to

Corollary 2.3. *For $x, y \in \mathbb{B}^n$, we have the following sharp inequality*

$$\rho_{\mathbb{B}^n}(x,y) \leq 2c_{\mathbb{B}^n}(x,y).$$

2.2. Comparison with the distance ratio metric

The relationship between the c_D-metric and the $\tilde{j}_D$-metric is stated in Theorem 1.2. Here we recall the statement and provide its proof.

Theorem 2.4. *(see also, Theorem 1.2) Let D be a proper subdomain of $\mathbb{R}^n$ and let $x, y \in D$. Then*

$$\tilde{j}_D(x,y) \leq \big(|x-y| + (d(x) \wedge d(y))\big)c_D(x,y).$$

Proof. We may assume that $d(x) \wedge d(y) = d(x)$. Choose $z \in \partial D$ such that $d(x) = |x-z|$. By the triangle inequality, we have that

$$\inf_{p \in \partial D} |x-p||y-p| \leq |x-z||y-z| \leq d(x)(|x-y| + d(x)),$$

and

$$\begin{aligned} c_D(x,y) &\geq \frac{|x-y|}{d(x)(|x-y|+d(x))} \\ &\geq \frac{1}{|x-y|+d(x)} \log\left(1 + \frac{|x-y|}{d(x)}\right) \\ &= \frac{1}{|x-y|+d(x)} \tilde{j}_D(x,y). \end{aligned}$$

This completes the proof of our theorem. □

Corollary 2.5. *For $x, y \in \mathbb{B}^n$, we have*

$$\tilde{j}_{\mathbb{B}^n}(x,y) \le (1+|x|\wedge|y|)c_{\mathbb{B}^n}(x,y) \le 2c_{\mathbb{B}^n}(x,y).$$

In particular,

$$k_{\mathbb{B}^n}(0,x) = \tilde{j}_{\mathbb{B}^n}(0,x) \le c_{\mathbb{B}^n}(0,x).$$

Proof. Since $x, y \in \mathbb{B}^n$, by Theorem 1.2 we observe that

$$\begin{aligned}
|x-y| + (d(x)\wedge d(y)) &\le |x|+|y|+((1-|x|)\wedge(1-|y|)) \\
&= |x|+|y|+1-(|x|\vee|y|) \\
&= 1+(|x|\wedge|y|).
\end{aligned}$$

The desired inequalities follow. □

The following lemma shows that the $c_{\mathbb{B}^n}$-metric and the $\tilde{j}_{\mathbb{B}^n}$-metric are reversely comparable when points are lying entirely in sub-disks of $\mathbb{B}^n$.

Lemma 2.6. *For all $x, y \in \mathbb{B}^n$ with $|x|\vee|y| \le \lambda < 1$ we have*

$$c_{\mathbb{B}^n}(x,y) \le \frac{1}{(1-\lambda)^2}\tilde{j}_{\mathbb{B}^n}(x,y). \tag{2.2}$$

Proof. Without loss of generality, we assume that $|y| = |x|\vee|y| \le \lambda$. For any $w \in \partial\mathbb{B}^n$, we have

$$|x-w||w-y| \ge (1-\lambda)^2,$$

and hence,

$$(1-\lambda)^2 c_{\mathbb{B}^n}(x,y) \le |x-y|. \tag{2.3}$$

Now,

$$\begin{aligned}
\tilde{j}_{\mathbb{B}^n}(x,y) &= \log\left(1+\frac{|x-y|}{d_{\mathbb{B}^n}(x)\wedge d_{\mathbb{B}^n}(y)}\right) = \log\left(1+\frac{|x-y|}{1-|y|}\right) \\
&\ge \frac{\dfrac{2|x-y|}{1-|y|}}{2+\dfrac{|x-y|}{1-|y|}} \quad \left(\because \log(1+t) \ge \frac{2t}{2+t} \text{ for } t>0\right) \\
&= \frac{2|x-y|}{2-2|y|+|x-y|} \ge |x-y| \ge (1-\lambda)^2 c_{\mathbb{B}^n}(x,y),
\end{aligned}$$

where the last two inequalities follow from the inequalities $|x-y| \le 2|y|$ and (2.3) respectively. □

The next lemma describes the relations between the $c_{\mathbb{B}^n}$-metric and the $v_{\mathbb{B}^n}$-metric.

Lemma 2.7. *The following inequalities hold.*

1. *For $x, y \in \mathbb{B}^n$ we have*

$$\frac{v_{\mathbb{B}^n}(x,y)}{2} \le \tan \frac{v_{\mathbb{B}^n}(x,y)}{2} \le c_{\mathbb{B}^n}(x,y).$$

2. *For all $x, y \in \mathbb{B}^2$ with $|x| \vee |y| \le \lambda < 1$ we have*

$$c_{\mathbb{B}^2}(x,y) \le \frac{2(3+\lambda^2)}{3(1-\lambda^2)(1-\lambda)^2} v_{\mathbb{B}^2}(x,y).$$

Proof. Combining the inequality [48, Theorem 3.11]

$$\tan \frac{v_{\mathbb{B}^n}(x,y)}{2} \le \sinh \frac{\rho_{\mathbb{B}^n}(x,y)}{2}$$

and from the inequality (1.1), we have that

$$\tan \frac{v_{\mathbb{B}^n}(x,y)}{2} \le c_{\mathbb{B}^n}(x,y).$$

It is clear that

$$\frac{v_{\mathbb{B}^n}(x,y)}{2} \le \tan \frac{v_{\mathbb{B}^n}(x,y)}{2}.$$

This proves the first part.

For the proof of the second part, we combine the inequality [11, Theorem 3.9]

$$j_{\mathbb{B}^2}(x,y) \le \frac{2(3+\lambda^2)}{3(1-\lambda^2)} v_{\mathbb{B}^2}(x,y)$$

with (2.2), and we obtain

$$c_{\mathbb{B}^2}(x,y) \le \frac{2(3+\lambda^2)}{3(1-\lambda^2)(1-\lambda)^2} v_{\mathbb{B}^2}(x,y).$$

Thus, the proof of our lemma is complete. □

We also consider the quantity p_D,

$$p_D(x,y) = \frac{|x-y|}{\sqrt{|x-y|^2 + 4d(x)d(y)}}.$$

Note that the quantity p_D, which was first considered in [11], does not define a metric (see [11, Remark 3.1]). However, it has a nice connection with the hyperbolic metric, $\rho_{\mathbb{H}^2}$, of the upper half-plane $\mathbb{H}^2$. Namely,

$$p_{\mathbb{H}^2}(z_1,z_2) = \tanh \frac{\rho_{\mathbb{H}^2}(z_1,z_2)}{2} = \frac{|z_1-z_2|}{|z_1-\bar{z}_2|}, \quad z_1, z_2 \in \mathbb{H}^2,$$

where $\bar{z}_2$ is the reflection of z_2 with respect to the real line $\mathbb{R}$ (see [11]). Hence it is natural to ask whether the quantity p_D is comparable with hyperbolic-type metrics such as, the c_D-metric, in more general domains D or not. One such example is given in the next theorem which compares the c_D-metric with the quantity p_D.

Theorem 2.8. *Let $x, y \in D \subsetneq \mathbb{R}^n$. Then*

$$p_D(x,y) \le \sqrt{2}\Big(d(x) \wedge d(y)\Big) c_D(x,y).$$

Proof. Fix $x, y \in D$ and let $s = d(x) \wedge d(y)$. Then

$$\begin{aligned}
p_D(x,y) &= \frac{|x-y|}{\sqrt{|x-y|^2 + 4d(x)d(y)}} \le \frac{|x-y|}{\sqrt{|x-y|^2 + (2s)^2}} \\
&\le \frac{\sqrt{2}|x-y|}{|x-y| + 2s} \le \frac{\sqrt{2}|x-y|}{|x-y| + s} \\
&= \sqrt{2}\Big(d(x) \wedge d(y)\Big) \frac{|x-y|}{\Big(d(x) \wedge d(y)\Big)\Big(|x-y| + \big(d(x) \wedge d(y)\big)\Big)} \\
&= \sqrt{2}\Big(d(x) \wedge d(y)\Big) \left[\frac{|x-y|}{d(x)\big(|x-y| + d(x)\big)} \vee \frac{|x-y|}{d(y)\big(|x-y| + d(y)\big)}\right] \\
&\le \sqrt{2}\Big(d(x) \wedge d(y)\Big) c_D(x,y),
\end{aligned}$$

where the second inequality follows from [1, 1.58 (13)] and the last inequality follows from [37, Lemma 3.4]. □

Remark 2.9. Observe that if we take the domain D in Theorem 2.8 to be the unit ball $\mathbb{B}^n$, then we can see that $p_D(x,y) \le \sqrt{2} c_D(x,y)$. In fact, if D is a bounded domain in $\mathbb{R}^n$, then $p_D(x,y) \le \big(\operatorname{diam}(D)/\sqrt{2}\big) c_D(x,y)$.

2.3. Comparison with the Seittenranta metric

Recall that Seittenranta introduced a Möbius invariant metric in arbitrary proper subdomains of $\mathbb{R}^n$ which agrees with the hyperbolic metric in the unit ball $\mathbb{B}^n$. The comparison between the c_D-metric with the δ_D-metric is stated in the Theorem 1.3. Here we first recall the statement Theorem 1.3 and then provide its proof. This result shows that the Cassinian metric can be written as a lower bound to the Seittenranta metric.

Theorem 2.10. *(see also, Theorem 1.3) Let $D \subsetneq \overline{\mathbb{R}^n}$ be any domain. Then for $x, y \in D$*

$$c_D(x,y) \leq \frac{e^{\delta_D(x,y)} - 1}{d(y)}.$$

Equality holds for $D = D_p$.

Proof. Let $p \in \partial D$ such that

$$c_D(x,y) = \frac{|x-y|}{|x-p||p-y|}.$$

Choose $q \in \partial D$ such that $|p-q| \geq |y-q|$. Now,

$$\begin{aligned} c_D(x,y) &= \frac{|x-y|}{|x-p||p-y|} \\ &= \frac{|x-y||p-q|}{|x-p||y-q|} . \frac{|y-q|}{|p-q||p-y|} \\ &\leq \frac{m_D(x,y)}{d(y)}. \end{aligned}$$

Hence we get

$$\delta_D(x,y) = \log(1 + m_D(x,y)) \geq \log(1 + d(y).c_D(x,y))$$

and the proof is complete. For the sharpness, consider the punctured domain D_p and $x, y \in D_p$ with $|x-p| \leq |y-p|$. It is clear that $\delta_{D_p}(x,y) = \tilde{j}_{D_p}(x,y) = \log(1+|x-y|/|x-p|)$ and hence the sharpness follows. □

2.4. The inner Cassinian metric

Let $D \subsetneq \mathbb{R}^n$ and γ be a rectifiable curve in D. We define the Cassinian length of γ as

$$c_D(\gamma) = \sup \sum_{i=0}^{n-1} c_D(\gamma(t_i), \gamma(t_{i+1})),$$

where the supremum is taken over all partitions $(t_i)_{i=1}^n$ of $I = [a, b]$ with $t_1 = a$ and $t_n = b$. Then the inner Cassinian metric is defined as

$$\tilde{c}_D(x,y) = \inf_\gamma c_D(\gamma) = \inf_\gamma \int_\gamma \frac{|\mathrm{dz}|}{d(z)^2},$$

where the infimum is taken over all rectifiable curves $\gamma \subset D$ connecting x and y (see [37]). First, we establish the monotonicity property of the inner Cassinian metric.

Lemma 2.11. *The inner Cassinian metric is monotonic with respect to domains. That is, if $D \subset D'$, then $\tilde{c}_{D'}(x,y) \leq \tilde{c}_D(x,y)$ for all $x, y \in D$.*

Proof. Given $x, y \in D$, we have

$$\tilde{c}_D(x,y) = \inf_\gamma c_D(\gamma),$$

where the infimum is taken over all rectifiable curves $\gamma \subset D$ connecting x and y. Since the Cassinian metric is monotonic ([37, Corollary 3.2]), $c_D(\gamma) \geq c_{D'}(\gamma)$ for all such γ and, consequently,

$$\inf_\gamma c_D(\gamma) \geq \inf_\gamma c_{D'}(\gamma).$$

Since each such γ also connects x and y in D', we have

$$\tilde{c}_{D'}(x,y) = \inf_\gamma c_{D'}(\gamma) \leq \inf_\gamma c_D(\gamma) = \tilde{c}_D(x,y),$$

completing the proof. □

Next, we compute the inner Cassinian metrics in some special cases.

Example 2.12. For the punctured space $D = D_0$, the inner Cassinian metric $\tilde{c}_D$ is same as the Cassinian metric c_D and is given by the formula

$$\tilde{c}_D(x,y) = c_D(x,y) = \frac{|x-y|}{|x||y|}.$$

To see this, let $f(\xi) = \xi/|\xi|^2$ be the inversion about the unit sphere $\mathbb{S}^{n-1}(0,1) = \{\xi \in \mathbb{R}^n\colon\ |\xi| = 1\}$. Then $f(D) = D$ and that f is an isometry between (D, c_D) and $(D, |-|)$, where $|-|$ is the Euclidean distance in D (see, [37, Example 3.9(A)]). Since the inner metric of the Euclidean metric in D is the same as the Euclidean metric itself and since (D, c_D) is isometric to $(D, |-|)$, we conclude that $(D, \tilde{c}_D)$ is isometric to $(D, |-|)$. Hence $\tilde{c}_D$ is same as the Cassinian metric c_D. In particular, it follows from [4, (3.1.5)] that

$$\tilde{c}_D(x,y) = c_D(x,y) = |f(x) - f(y)| = \frac{|x-y|}{|x||y|}$$

for all $x, y \in D$, as required.

Example 2.13. For each $x \in \mathbb{B}^n$, we have

$$\tilde{c}_{\mathbb{B}^n}(0,x) = c_{\mathbb{B}^n}(0,x) = \frac{|x|}{1-|x|}.$$

It follows from [37, Theorem 3.8] that the line segment $[0,x]$ is a Cassinian geodesic so that its Cassinian length is equal to $c_{\mathbb{B}^n}(0,x)$. That is,

$$c_{\mathbb{B}^n}([0,x]) = c_{\mathbb{B}^n}(0,x) = \frac{|x|}{1-|x|},$$

where the last equality is derived in (2.1). Therefore,

$$c_{\mathbb{B}^n}(0,x) \leq \tilde{c}_{\mathbb{B}^n}(0,x) = \inf_{\gamma} c_{\mathbb{B}^n}(\gamma) \leq c_{\mathbb{B}^n}([0,x]).$$

Hence $\tilde{c}_{\mathbb{B}^n}(0,x) = c_{\mathbb{B}^n}(0,x)$, as required.

Finally, we end this chapter with the following lemma which is-in one-way-an easy consequence of Lemma 2.11 and Example 2.13. Remaining part of the proof will follow from the concept of arc-length parametrization: let γ be a rectifiable curve joining x and y in D and the Euclidean length of γ is $\ell(\gamma) = \lambda$. Then the arc-length parametrization of γ is $\gamma^\circ : [0,\lambda] \to \gamma$ such that $\gamma^\circ(0) = x$, $\gamma^\circ(\lambda) = y$, and $\ell(\gamma[x,\gamma^\circ(t)]) = t$ for all $t \in [0,\lambda]$.

Lemma 2.14. *Given $x \in D \setminus \{\infty\}$, we have*

$$\frac{|x-y|}{d(x)(d(x)+|x-y|)} \leq \tilde{c}_D(x,y) \leq \frac{|x-y|}{d(x)(d(x)-|x-y|)}$$

for all $y \in D$ with $|x-y| < d(x)$.

Proof. Set $B = B(x,d(x))$. Then as in Example 2.13 we obtain

$$\tilde{c}_B(x,y) = \frac{|x-y|}{d(x)(d(x)-|x-y|)}$$

for any $y \in B$. Now the right hand side inequality follows from Lemma 2.11.

For the left hand side inequality, let γ be a rectifiable curve joining x and y in D. Let γ° be an arc-length parametrization of γ. Now, for each $t \in [0,\lambda]$ we have

$$d(\gamma^\circ(t)) \leq d(x) + |\gamma^\circ(t) - x| \leq d(x) + \ell(\gamma[x,\gamma^\circ(t)]) = d(x) + t.$$

Again, the inner Cassinian length of γ satisfies

$$\tilde{c}_D(\gamma) = \int_\gamma \frac{|dz|}{d(z)^2} \geq \int_0^\lambda \frac{dt}{(d(x)+t)^2} \geq \frac{|x-y|}{d(x)(d(x)+|x-y|)}.$$

This completes the proof of our corollary. $\square$

CHAPTER 3

COMPARISON WITH A GROMOV HYPERBOLIC METRIC

This chapter deals with the comparison of the u_D-metric with the $\tilde{\tau}_D$-metric and their relationship with other hyperbolic-type metrics. We begin with the comparison of the u_D-metric with the $\tilde{\tau}_D$-metric.

3.1. The Gromov hyperbolic metric u_D

Recall that the u_D-metric is defined for $x, y \in D \subsetneq \mathbb{R}^n$ by

$$u_D(x,y) = 2\log\left(\frac{|x-y| + d(x) \vee d(y)}{\sqrt{d(x)\, d(y)}}\right).$$

Some of the interesting characteristics of the u_D-metric are

(i) it generalizes the distance ratio metric, the hyperbolic cone metric, and the hyperbolic metric of hyperspaces [39];

(ii) it is quasi-isometric to the quasihyperbolic metric of uniform metric spaces [39].

The main results of this section are explained in Table 3.1.

First, we prove the comparison of the u_D-metric with the $\tilde{\tau}_D$-metric stated in Theorem 1.4. Note that, in [40] Ibragimov proved that the $\tilde{\tau}_D$-metric is Gromov-hyperbolic (β-hyperbolic) with the constant $\beta = \log 3$ by comparing the $\tilde{\tau}_D$-metric and the $\tilde{j}_D$-metric. Here we improve the constant $\beta = \log 2$ by comparing the $\tilde{\tau}_D$-metric with the u_D-metric.

Theorem 3.1. *(see also, Theorem 1.4) Let $D \subsetneq \mathbb{R}^n$ be any domain with $\partial D \neq \emptyset$ and $x, y \in D$. Then*

$$2\tilde{\tau}_D(x,y) \le u_D(x,y).$$

Equality holds whenever $d(x) = |x-p| = |y-p| = d(y)$ for some $p \in \partial D$. Moreover, there exists no constant $k \ge 0$ such that

$$u_D(x,y) \le 2\tilde{\tau}_D(x,y) + k$$

	Comparison with the $\tilde{\tau}_D$-metric	Comparison with the u_D-metric
j_D	$\frac{1}{2}j_D \leq \tilde{\tau}_D \leq j_D$ [40, Theorems 5.1,5.4]	$2j_D \leq u_D \leq 4j_D$ [Theorem 3.7]
$\tilde{j}_D$	$\frac{1}{2}\tilde{j}_D \leq \tilde{\tau}_D \leq \tilde{j}_D$ [40, Lemma 4.1, Theorem 4.3]	$\tilde{j}_D \leq u_D \leq 2\tilde{j}_D$ [Theorem 3.5]
$\rho_{\mathbb{B}^n}$	$\tilde{\tau}_{\mathbb{B}^n} \leq \rho_{\mathbb{B}^n}$ [40, Theorem 6.1]	$\frac{1}{2}\rho_{\mathbb{B}^n} \leq u_{\mathbb{B}^n} \leq 4\rho_{\mathbb{B}^n}$ [Theorem 3.17]
$\tilde{\tau}_D$	-	$2\tilde{\tau}_D \leq u_D \leq 4\tilde{\tau}_D$ [Theorem 3.4]

TABLE 3.1. Comparisons of the $\tilde{\tau}_D$-metric and the u_D-metric with other hyperbolic-type metrics

for all $x, y \in D$ unless D is a one-punctured space. If D is a once-punctured space, then

$$u_D(x,y) \leq 2\tilde{\tau}_D(x,y) + 2\log 2$$

and the inequality is sharp.

Proof. Without loss of generality assume that $d(x) \geq d(y)$. For $x, y \in D$, the relation $|x-p||y-p| \geq d(x)d(y)$ clearly holds for all $p \in \partial D$. Then we have

$$\begin{aligned} u_D(x,y) &= 2\log\frac{|x-y|+d(x)}{\sqrt{d(x)d(y)}} \geq 2\log\left(1+\frac{|x-y|}{\sqrt{d(x)d(y)}}\right) \\ &\geq 2\log\left(1+\sup_{p\in\partial D}\frac{|x-y|}{\sqrt{|x-p||p-y|}}\right) = 2\tilde{\tau}_D(x,y), \end{aligned}$$

where the first inequality follows from the fact that $d(x)/d(y) \geq 1$. It is clear that if $d(x) = |x-p| = |y-p| = d(y)$ for some $p \in \partial D$, then both the above inequalities turn into an equality and hence the sharpness part is proved.

To prove the second part, suppose that D has more than one boundary point and $k \geq 0$ such that $u_D(x,y) \leq 2\tilde{\tau}_D(x,y) + k$ for all $x, y \in D$. Since the space (D, u_D) is

β-hyperbolic in D and $2\tilde{\tau}_D(x,y) \le u_D(x,y) \le 2\tilde{\tau}_D(x,y) + k$, we conclude that the space $(D, \tilde{\tau}_D)$ is also β-hyperbolic in D, contradicting [40, Remark 4.4].

The translation invariance of the $\tilde{\tau}_D$-metric and the u_D-metric allows us to take the punctured space to be D_0 without any loss to generality. Again we assume that $|x| \ge |y|$. To show the third part, it is sufficient to show

$$\frac{|x-y|+|x|}{\sqrt{|x||y|}} \le 2\left(1 + \frac{|x-y|}{\sqrt{|x||y|}}\right),$$

or, equivalently,

$$\frac{|x|-|x-y|}{\sqrt{|x||y|}} \le 2.$$

The hypothesis $|x| \ge |y|$ along with the triangle inequality yields

$$\frac{|x|-|x-y|}{\sqrt{|x||y|}} \le \frac{|y|}{\sqrt{|x||y|}} \le 2.$$

To prove the sharpness, let $y = e_1$ and $x = te_1, t > 1$. Then

$$\lim_{t\to\infty} u_D(x,y) - 2\tilde{\tau}_D(x,y) = \lim_{t\to\infty} 2\log\frac{2t-1}{t+\sqrt{t}-1} = 2\log 2.$$

Hence the proof is complete. $\square$

Next, we aim to prove the other way of comparison. That is, to find a constant k such that $u_D \le k\tilde{\tau}_D$. First, we prove this result in once-punctured spaces and then we extend this to arbitrary proper subdomains of $\mathbb{R}^n$. Next result shows that in punctured spaces the constant $k = 4$.

Lemma 3.2. *Let $x, y \in D_0$. Then*

$$u_{D_0}(x,y) \le 4\tilde{\tau}_{D_0}(x,y).$$

Proof. Without loss of generality assume that $|x| \ge |y|$. To prove the required inequality, it suffices to show that

$$\frac{|x-y|+|x|}{\sqrt{|x||y|}} \le \left(1 + \frac{|x-y|}{\sqrt{|x||y|}}\right)^2$$

or, equivalently,

$$\frac{|x|-|x-y|}{\sqrt{|x||y|}} \le 1 + \frac{|x-y|^2}{|x||y|}.$$

This holds true, because

$$\frac{|x|-|x-y|}{\sqrt{|x||y|}} \le \frac{|y|}{\sqrt{|x||y|}} \le 1 \le 1 + \frac{|x-y|^2}{|x||y|},$$

completing the proof of our lemma. $\square$

We now prove that the conclusion of Lemma 3.2 still holds if we replace the once-punctured space by twice-punctured spaces.

Lemma 3.3. *Let $x, y \in D_{p,q}$. Then*

$$u_{D_{p,q}}(x,y) \leq 4\tilde{\tau}_{D_{p,q}}(x,y).$$

Proof. Suppose that $x, y \in D_{p,q}$. If $d(x) = |x-p|$ and $d(y) = |y-p|$ (or $d(x) = |x-q|$ and $d(y) = |y-q|$), then the proof follows from Lemma 3.2. Hence, without loss of generality, we assume $d(x) = |x-p|, d(y) = |y-q|$, and $|x-p| \geq |y-q|$. Note that $\tilde{\tau}_{D_{p,q}}(x,y) = \tilde{\tau}_{D_p}(x,y) \vee \tilde{\tau}_{D_q}(x,y)$. Hence, to prove our claim, it is enough to establish the inequality $u_{D_{p,q}}(x,y) \leq 4\tilde{\tau}_{D_q}(x,y)$. That is to prove the inequality

$$\frac{|x-y|+|x-p|}{\sqrt{|x-p||y-q|}} \leq \left(1 + \frac{|x-y|}{\sqrt{|x-q||y-q|}}\right)^2.$$

Let $|x-y| = a|y-q|$, where $a > 0$. From the assumption we know that

$$|x-p| \leq |x-q| \leq |x-y| + |y-q|. \tag{3.1}$$

Now

$$\begin{aligned} \frac{|x-y|+|x-p|}{\sqrt{|x-p||y-q|}} &= \frac{|x-y|}{\sqrt{|x-p||y-q|}} + \frac{|x-p|}{\sqrt{|x-p||y-q|}} \\ &\leq \frac{|x-y|}{|y-q|} + \sqrt{\frac{|x-p|}{|y-q|}} \\ &\leq \frac{|x-y|}{|y-q|} + \sqrt{\frac{|x-y|}{|y-q|} + 1} \\ &\leq a + \sqrt{a+1}. \end{aligned} \tag{3.2}$$

We obtain from (3.1) that

$$\frac{2|x-y|}{\sqrt{|x-q||y-q|}} \geq \frac{2|x-y|}{\sqrt{1+a}|y-q|} = \frac{2a}{\sqrt{1+a}}. \tag{3.3}$$

Next we divide the proof into two cases.

Case 1. $a < 1$.

It follows from (3.2) and (3.3) that

$$\begin{aligned}\left(1+\frac{|x-y|}{\sqrt{|x-q||y-q|}}\right)^2-\frac{|x-y|+|x-p|}{\sqrt{|x-p||y-q|}} &> \frac{2|x-y|}{\sqrt{|x-q||y-q|}}+1-\frac{|x-y|+|x-p|}{\sqrt{|x-p||y-q|}}\\ &\geq \frac{2a}{\sqrt{1+a}}+1-a-\sqrt{a+1}>0\end{aligned}$$

Case 2. $a \geq 1$.

From (3.1) we have

$$\frac{|x-y|^2}{|x-q||y-q|} \geq \frac{|x-y|^2}{(|x-y|+|y-q|)|y-q|} = \frac{a^2}{1+a}. \tag{3.4}$$

Then it follows from (3.2), (3.3) and (3.4) that

$$\begin{aligned}\left(1+\frac{|x-y|}{\sqrt{|x-q||y-q|}}\right)^2-\frac{|x-y|+|x-p|}{\sqrt{|x-p||y-q|}} &\geq \frac{2a}{\sqrt{1+a}}+\frac{a^2}{1+a}+1-a-\sqrt{a+1}\\ &\geq 0.\end{aligned}$$

This completes the proof of our lemma. $\square$

Theorem 1.4 and Lemma 3.3 jointly yield the following relationship between the $\tilde{\tau}_D$-metric and the u_D-metric in arbitrary domains $D \subsetneq \mathbb{R}^n$.

Theorem 3.4. *Let $x, y \in D \subsetneq \mathbb{R}^n$. Then*

$$2\tilde{\tau}_D(x,y) \leq u_D(x,y) \leq 4\tilde{\tau}_D(x,y).$$

Both the inequalities are sharp.

Proof. The first inequality is proved in Theorem 1.4. Now, we prove the second inequality. Suppose that $p, q \in \partial D$ such that $d(x) = |x-p|$ and $d(y) = |y-q|$. Clearly, $D \subset D_{p,q}$ and $u_D(x,y) = u_{D_{p,q}}(x,y)$. Now,

$$u_D(x,y) = u_{D_{p,q}}(x,y) \leq 4\tilde{\tau}_{D_{p,q}}(x,y) \leq 4\tilde{\tau}_D(x,y),$$

where the first inequality follows from Lemma 3.3 and the second inequality follows from the monotonicity property of the $\tilde{\tau}_D$-metric [40, p. 2].

The sharpness of the first inequality is given in Theorem 1.4. For the sharpness of the second inequality we consider the unit ball $\mathbb{B}^n$. Choose the points x and y such that $y = -x$. Now,

$$u_{\mathbb{B}^n}(x,-x) = 2\log\left(\frac{1+|x|}{1-|x|}\right) \text{ and } \tilde{\tau}_{\mathbb{B}^n}(x,-x) = \log\left(1+\frac{2|x|}{\sqrt{1-|x|^2}}\right).$$

It follows that

$$\lim_{|x|\to 1}\frac{u_{\mathbb{B}^n}(x,-x)}{4\tilde{\tau}_{\mathbb{B}^n}(x,-x)}=\lim_{|x|\to 1}\frac{(2|x|+\sqrt{1-|x|^2})^2}{(1+|x|)^2}=1.$$

Hence the proof is complete. $\square$

Recall that

$$\frac{1}{2}\tilde{j}_D(x,y)\le\tilde{\tau}_D(x,y)\le\tilde{j}_D(x,y) \tag{3.5}$$

holds true for $D\subsetneq\mathbb{R}^n$ (see [40, Theorem 4.2, 4.3]). Both the inequalities are sharp. The proof of the sharpness part of the left hand side inequality is done by the method of contradiction in [40]. Here we give a precise example to prove the sharpness part of the left hand side inequality..

Consider the unit ball $\mathbb{B}^n$ and $x,y\in\mathbb{B}^n$ with $y=-x$. Now we see that

$$\tilde{\tau}_{\mathbb{B}^n}(x,-x)=\log\left(1+\frac{2|x|}{\sqrt{1-|x|^2}}\right)\ \text{and}\ \tilde{j}_{\mathbb{B}^n}(x,-x)=\log\left(1+\frac{2|x|}{1-|x|}\right).$$

It follows that

$$\lim_{|x|\to 1}\frac{\tilde{j}_{\mathbb{B}^n}(x,-x)}{2\tilde{\tau}_{\mathbb{B}^n}(x,-x)}=\lim_{|x|\to 1}\frac{2|x|+\sqrt{1-|x|^2}}{2}=1. \tag{3.6}$$

By Theorem 3.4 and (3.5) we have

$$\tilde{j}_D(x,y)\le 2\tilde{\tau}_D(x,y)\le u_D(x,y)$$

and also

$$u_D(x,y)\le 4\tilde{\tau}_D(x,y)\le 4\tilde{j}_D(x,y).$$

Hence we have the following relationship between the $\tilde{j}_D$-metric and the u_D-metric.

Theorem 3.5. *For $D\subsetneq\mathbb{R}^n$ we have*

$$\tilde{j}_D(x,y)\le u_D(x,y)\le 4\tilde{j}_D(x,y).$$

The first inequality is sharp.

Proof. For the sharpness part, consider the domain $D=\mathbb{R}^n\setminus\{-e_1,e_1\}$. Choose $x=0$ and $y=te_2, t>1$. Then $\tilde{j}_D(0,te_2)=\log(1+t)$ and

$$u_D(0,te_2)=2\log\left(\frac{t+\sqrt{1+t^2}}{(1+t^2)^{1/4}}\right)=\log\left(\frac{1+2t^2+2t\sqrt{1+t^2}}{\sqrt{1+t^2}}\right).$$

Now, we see that

$$\lim_{t\to\infty}\frac{\tilde{j}_D(0,te_2)}{u_D(0,te_2)} = \lim_{t\to\infty}\frac{\log(1+t)}{\log\left(\dfrac{1+2t^2+2t\sqrt{1+t^2}}{\sqrt{1+t^2}}\right)}$$
$$= \lim_{t\to\infty}\frac{(1+t^2)(1+2t^2+2t\sqrt{1+t^2})}{(1+t)(4t(1+t^2)+2(1+t^2)^{3/2}-t-2t^3)} = 1.$$

This completes the proof of our theorem. □

Remark 3.6. The constant 4 in the right hand side inequality of Theorem 3.5 can't be replaced by 2 due to the fact that

$$u_D(x,y) \le 2\tilde{j}_D(x,y) \iff |x-y|^2 \ge d(x)d(y)$$

for every $x, y \in D$, which is not true in general.

Now, we compare the u_D-metric with the j_D-metric in arbitrary subdomains of $\mathbb{R}^n$.

Theorem 3.7. *Let $D \subsetneq \mathbb{R}^n$ be arbitrary. Then for $x, y \in D$ we have*

$$2j_D(x,y) \le u_D(x,y) \le 4j_D(x,y).$$

The first inequality becomes equality when $d(x) = d(y)$.

Proof. The first inequality is proved in [38, Theorem 3.1]. From the definitions of the j_D-metric and the u_D-metric, it follows that $2j_D(x,y) = u_D(x,y)$ whenever $d(x) = d(y)$. Now, we shall prove the second inequality. Without loss of generality we assume that $d(x) \ge d(y)$ for $x, y \in D \subsetneq \mathbb{R}^n$. To show our claim, it is enough to prove that

$$\frac{|x-y|+d(x)}{\sqrt{d(x)d(y)}} \le \left(1+\frac{|x-y|}{d(x)}\right)\left(1+\frac{|x-y|}{d(y)}\right),$$

or, equivalently,

$$d(x)d(y) \le (d(y)+|x-y|)^2$$

which is true by the triangle inequality. The proof is complete. □

	Comparison with $\tilde{\tau}_D$	Comparison with u_D
δ_D	$\frac{1}{4}\delta_D(x,y) \leq \tilde{\tau}_D(x,y) \leq \delta_D(x,y)$ [Theorem 3.8]	$\frac{\delta_D}{2} \leq u_D \leq 4j_D$ [Corollary 3.9]
c_D	$c_D(x,y) \leq \dfrac{e^{4\tilde{\tau}_D(x,y)}-1}{d(y)}$ [Corollary 3.10]	$c_D(x,y) \leq \dfrac{e^{2u_D(x,y)}-1}{d(y)}$ [Corollary 3.10]
s_D	$(\log 3)s_D \leq \tilde{\tau}_D$ [Theorem 3.11]	$(\log 9)s_D \leq u_D$ [Corollary 3.13]
η_D	$\frac{1}{2}\eta_D \leq \tilde{\tau}_D \leq \log(2+e^{\eta_D})$ [41]	$\eta_D \leq u_D \leq 4\log(2+e^{\eta_D})$ [Lemma 3.16]

TABLE 3.2. More comparisons of the $\tilde{\tau}_D$-metric and the u_D-metric with other hyperbolic-type metrics

3.2. The u_D-metric vs other hyperbolic-type metrics

In this section, we consider the δ_D-metric, the s_D-metric and the η_D-metric and compare them with the $\tilde{\tau}_D$-metric and the u_D-metric. Main results of this section are stated in Table 3.2.

Recall that the δ_D-metric is bilipschitz equivalent to the $\tilde{j}_D$-metric. Indeed we have

$$\tilde{j}_D \leq \delta_D \leq 2\tilde{j}_D, \tag{3.7}$$

see, for instance [59, p. 525]. Hence (3.5) along with (3.7) yield the following inequality between the δ_D-metric and the $\tilde{\tau}_D$-metric.

Theorem 3.8. *Let $x, y \in D \subsetneq \mathbb{R}^n$. Then the following holds true:*

$$\frac{1}{4}\delta_D(x,y) \leq \tilde{\tau}_D(x,y) \leq \delta_D(x,y).$$

The second inequality is sharp.

Proof. The proof of the inequality follows directly from (3.5) and (3.7). For the sharpness of the second inequality, consider the domain D_0 and $x, y \in D_0$ with $y = -x$. Then $\tilde{\tau}_{D_0}(x, -x) = \log 3 = \delta_{D_0}(x, -x)$. $\square$

Theorem 3.4 and Theorem 3.8 together yield the following.

Corollary 3.9. *Let $x, y \in D \subsetneq \mathbb{R}^n$. Then we have*

$$\frac{1}{2}\delta_D(x, y) \leq u_D(x, y) \leq 4\delta_D(x, y).$$

Hence, as a consequence to Theorem 1.3, we have

Corollary 3.10. *Let $x, y \in D \subsetneq \mathbb{R}^n$. Then we have*

$$c_D(x, y) \leq \frac{e^{4\tilde{\tau}_D(x,y)} - 1}{d(y)} \text{ and } c_D(x, y) \leq \frac{e^{2u_D(x,y)} - 1}{d(y)}.$$

Proof. The first inequality follows from Theorem 1.3 and Lemma 3.8 whereas the second inequality follows from Theorem 1.3 and Corollary 3.9. $\square$

Now, we compare the $\tilde{\tau}_D$-metric with the s_D-metric.

Theorem 3.11. *Let $D \subsetneq \mathbb{R}^n$ and $x, y \in D$. Then*

$$\tilde{\tau}_D(x, y) \geq (\log 3) s_D(x, y).$$

The inequality is sharp.

Proof. From AM-GM inequality it follows that

$$\frac{1}{\sqrt{|x-p||y-p|}} \geq \frac{2}{|x-p| + |y-p|}.$$

Now,

$$\begin{aligned}
\tilde{\tau}_D(x, y) &\geq \log\left(1 + \frac{|x-y|}{\sqrt{|x-p||y-p|}}\right) \\
&\geq \log\left(1 + \frac{2|x-y|}{|x-p| + |y-p|}\right) \\
&\geq \frac{|x-y|}{|x-p| + |y-p|}\log 3
\end{aligned}$$

holds for all $p \in \partial D$. Here the last inequality follows from the well known Bernoulli's inequality:

$$\log(1 + ax) \geq a\log(1 + x) \quad \text{for } a \in (0, 1), x > 0.$$

In particular, we have $\tilde{\tau}_D(x,y) \geq (\log 3) s_D(x,y)$. To prove the sharpness, consider the domain D_0 and $x, y \in D_0$ with $y = -x$. Then $\tilde{\tau}_D(x,-x) = \log 3$ and $s_D(x,-x) = 1$. $\square$

Remark 3.12. Combining Theorem 3.11 and [40, Theorem 4.3], one can obtain Theorem 3.3 of [11].

The following result is also a consequence of Theorem 3.11.

Corollary 3.13. *Let $D \subsetneq \mathbb{R}^n$. Then for all $x, y \in D$ we have*

$$s_D(x,y) \leq \frac{1}{\log 9} u_D(x,y).$$

Proof. The proof follows from Theorem 3.4 and Theorem 3.11. $\square$

It is now appropriate here to recall the definition of the Apollonian metric. The *Apollonian metric*, $\alpha_D(x,y)$, introduced by Beardon in [5], defined for $D \subsetneq \mathbb{R}^n$ by

$$\alpha_D(x,y) = \sup_{p,q \in \partial D} \log \left(\frac{|x-p||y-q|}{|x-q||y-p|} \right), \quad x, y \in D.$$

The α_D-metric is a generalization of the hyperbolic metric of $\mathbb{B}^n$ as well as the hyperbolic metric of the upper half-space $\mathbb{H}^n$. Note that the α_D-metric was first introduced by Barbilian [3] in 1934 and then rediscovered by Beardon in 1998. Recall the following result due to Seittenranta.

Lemma 3.14. *[59, Theorem 3.11] Let $D \subset \overline{\mathbb{R}^n}$ be an open set with* card $\partial D \geq 2$. *Then*

$$\alpha_D(x,y) \leq \delta_D(x,y) \leq \log(e^{\alpha_D} + 2)$$

The inequalities give the best possible bounds for the δ_D-metric expressed in terms of the α_D-metric only.

As a special case of Lemma 3.14, the following result holds true, which is also proved in [41].

Lemma 3.15. *Let $D \subsetneq \mathbb{R}^n$ and $x, y \in D$. Then*

$$\frac{1}{2}\eta_D(x,y) \leq \tilde{\tau}_D(x,y) \leq \log(2 + e^{\eta_D(x,y)}).$$

Both the inequalities are sharp.

Comparing Theorem 3.4 and Lemma 3.15 we have the following relationship between the η_D-metric and the $\tilde{\tau}_D$-metric.

Lemma 3.16. *Let $D \subsetneq \mathbb{R}^n$ and $x, y \in D$. Then*

$$\eta_D(x,y) \le u_D(x,y) \le 4\log(2 + e^{\eta_D(x,y)}).$$

Proof. Proof directly follows from Theorem 3.4 and Lemma 3.15. □

3.3. The u_D-metric vs the hyperbolic metric of $\mathbb{B}^n$

This section is devoted to the comparison of the u_D-metric and the ρ_D-metric when $D = \mathbb{B}^n$ and $D = \mathbb{H}^n$.

Next result establishes a relationship between the $u_{\mathbb{B}^n}$-metric and the $\rho_{\mathbb{B}^n}$-metric.

Theorem 3.17. *For all $x, y \in \mathbb{B}^n$ we have*

$$\frac{1}{2}\rho_{\mathbb{B}^n}(x,y) \le u_{\mathbb{B}^n}(x,y) \le 4\rho_{\mathbb{B}^n}(x,y)$$

Proof. From [1, p. 151] and from [67, p. 29] we have the following inequalities respectively:

$$\tilde{j}_{\mathbb{B}^n}(x,y) \le \rho_{\mathbb{B}^n}(x,y) \le 2\tilde{j}_{\mathbb{B}^n}(x,y) \text{ and } \frac{1}{2}\tilde{j}_D(x,y) \le j_D(x,y) \le \tilde{j}_D(x,y). \tag{3.8}$$

Now the proof of the theorem follows using Lemma 3.7. □

Observe that for the choice of points $x, y \in \mathbb{B}^n$ with $y = -x$,

$$u_{\mathbb{B}^n}(x,-x) = 2\log\left(\frac{1+|x|}{1-|x|}\right) = 2\rho_{\mathbb{B}^n}(0,x) = \rho_{\mathbb{B}^n}(x,-x).$$

This observation leads to the following conjecture.

Conjecture 3.18. *For $x, y \in \mathbb{B}^n$ we have $\rho_{\mathbb{B}^n}(x,y) \le u_{\mathbb{B}^n}(x,y) \le 2\rho_{\mathbb{B}^n}(x,y)$.*

Next theorem shows that the $u_{\mathbb{B}^n}$-metric and the $\rho_{\mathbb{B}^n}$-metric are satisfying the quasi-isometry property.

Theorem 3.19. *For all $x, y \in \mathbb{B}^n$, we have*

$$\rho_{\mathbb{B}^n}(x,y) - 2\log 2 \le u_{\mathbb{B}^n}(x,y) \le 2\rho_{\mathbb{B}^n}(x,y) + 2\log 2.$$

Proof. The right hand side inequality easily follows from (3.8) and [38, Theorem 3.1]. For the left hand side inequality we assume that $x, y \in \mathbb{B}^n$ with $|x| \le |y|$. It is clear that

$(1-|x|)(1-|y|) \leq (1-|x|^2)(1-|y|^2)$. Now with the help of formula (1.1) we have

$$
\begin{aligned}
u_{\mathbb{B}^n}(x,y) &= 2\log\left(\frac{|x-y|+1-|x|}{\sqrt{(1-|x|)(1-|y|)}}\right) \geq 2\log\left(1+\frac{|x-y|}{\sqrt{(1-|x|^2)(1-|y|^2)}}\right)\\
&= 2\log\left(1+\sinh\left(\frac{\rho_{\mathbb{B}^n}(x,y)}{2}\right)\right) \geq \rho_{\mathbb{B}^n}(x,y) - 2\log 2,
\end{aligned}
$$

where the first inequality follows from the fact that $(1-|x|)/(1-|y|) \geq 1$ and the second inequality follows from the fact that $1+\sinh(r) \geq e^r/2$ for $r \in \mathbb{R}^+$ (the set of positive real numbers). The proof is complete. $\square$

Now, we compare the $u_{\mathbb{H}^n}$-metric with the $\rho_{\mathbb{H}^n}$-metric. Note that for $x, y \in \mathbb{H}^n$, the $\rho_{\mathbb{H}^n}$-metric can be computed by the formula (see [4, p. 35])

$$
2\sinh\left(\frac{\rho_{\mathbb{H}^n}(x,y)}{2}\right) = \frac{|x-y|}{\sqrt{x_n y_n}}. \tag{3.9}
$$

Theorem 3.20. *For $x, y \in \mathbb{H}^n$ we have*

$$
\rho_{\mathbb{H}^n}(x,y) \leq u_{\mathbb{H}^n}(x,y).
$$

The inequality is sharp.

Proof. Suppose that $x, y \in \mathbb{H}^n$. Without loss of generality we assume that $x_n \geq y_n$. Now,

$$
u_{\mathbb{H}^n} = 2\log\left(\frac{|x-y|+x_n}{\sqrt{x_n y_n}}\right) \geq 2\log\left(2\sinh\left(\frac{\rho_{\mathbb{H}^n}(x,y)}{2}\right)+1\right) \geq \rho_{\mathbb{H}^n}(x,y),
$$

where the first inequality follows from (3.9) and the hypothesis. However, the second inequality follows from the fact that $1+2\sinh(r) = 1+e^r-e^{-r} \geq e^r$ for $r \in \mathbb{R}^+$. For sharpness, consider the points $x = te_2$ and $y = (1/t)e_2$ with $t > 1$. Then

$$
\rho_{\mathbb{H}^n}(te_2,(1/t)e_2) = 2\sinh^{-1}\left(\frac{t^2-1}{2t}\right) = 2\log t \text{ and } u_{\mathbb{H}^n}(te_2,(1/t)e_2) = 2\log\left(\frac{2t^2-1}{t}\right).
$$

Now taking the limits at $t \to \infty$ we get

$$
\lim_{t\to\infty}\frac{\rho_{\mathbb{H}^n}(te_2,(1/t)e_2)}{u_{\mathbb{H}^n}(te_2,(1/t)e_2)} = \lim_{t\to\infty}\frac{\log t}{\log\left(\frac{2t^2-1}{t}\right)} = 1.
$$

Hence completing the proof. $\square$

CHAPTER 4

THE CASSINIAN METRIC BALLS

In this chapter, we study the metric balls associated with the c_D-metric , the $\tilde{\tau}_D$-metric, and the u_D-metric. We define the metric ball as follows: let (D, d) be a metric space. Then the set

$$B_d(x, R) = \{z \in D : \, d(x, z) < R\}$$

is called the *d-metric ball* of the domain D. A metric ball with respect to the Cassinian metric is called a *Cassinian (metric) ball.*

We mainly focus on the starlikeness and convexity properties of the Cassinian metric balls. In addition, we focus on the following type problem:

Given $x \in (D, d_i) \subsetneq \mathbb{R}^n, i = 1, 2$ and $t > 0$, we find optimal radii $r, R > 0$ depending only on x and t such that

$$B_{d_1}(x, r) \subseteq B_{d_2}(x, t) \subseteq B_{d_1}(x, R).$$

First we describe the Cassinian ball of a domain D in terms of Cassinian balls of $\mathbb{R}^n \setminus \partial D$ fixing a centre in D.

Lemma 4.1. *Let $D \subsetneq \mathbb{R}^n$ and $x \in D$. Then*

$$B_{c_D}(x, R) = \cap_{z \in \partial D} B_{c_{D_z}}(x, R).$$

Proof. Suppose that $y \in \cap_{z \in \partial D} B_{c_{D_z}}(x, R)$. Then $c_{D_z}(x, y) < R$ for all $z \in \partial D$. Choose $z' \in \partial D$ such that

$$c_{D_{z'}}(x, y) = \sup_{z \in \partial D} c_{D_z}(x, y) = c_D(x, y).$$

As $z' \in \partial D$, it is clear that $c_D(x, y) < R$. Hence $B_{c_D}(x, R) \supseteq \cap_{z \in \partial D} B_{c_{D_z}}(x, R)$. Conversely, fix $y \in B_{c_D}(x, R)$ and $z \in \partial D$. Then

$$B_{c_D}(x, R) \subseteq B_{c_{D_z}}(x, R)$$

by the domain monotonicity property of the c_D-metric. Hence $B_{c_D}(x, R) \subseteq \cap_{z \in \partial D} B_{c_{D_z}}(x, R)$ and the proof is complete. $\square$

For the study of Cassinian balls in arbitrary domains, Lemma 4.1 underlies the importance of the properties of balls in punctured spaces.

4.1. Starlikeness property

There exists non starlike domains in which Cassinian balls are not starlike. For example, consider the punctured space D_0. Clearly, D_0 is not starlike with respect to $e_1 = (1, 0, 0, \ldots, 0) \in \mathbb{R}^n$. Note that

$$\partial B_{c_{D_0}}(e_1, R) = \{x \in \mathbb{R}^n : (1 - R^2)|x|^2 + 1 - 2\langle x, e_1\rangle = 0\},$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product of two vector elements. Thus, if $R = 1$ the Cassinian sphere $\partial B_{c_{D_0}}(e_1, R)$ describes the hyper-plane passing through $x = e_1/2$; if $R < 1$ then the Cassinian balls $B_{c_{D_0}}(e_1, R)$ are convex and hence starlike; if $R > 1$ then the Cassinian balls $B_{c_{D_0}}(e_1, R)$ are not starlike; see Fig. 4.1 for description of the Cassinian balls $B_{c_{D_0}}(e_1, R)$ (see also Lemma 4.4 for proof details). Hence, it is reasonable to study the starlikeness property of Cassinian metric balls in starlike domains.

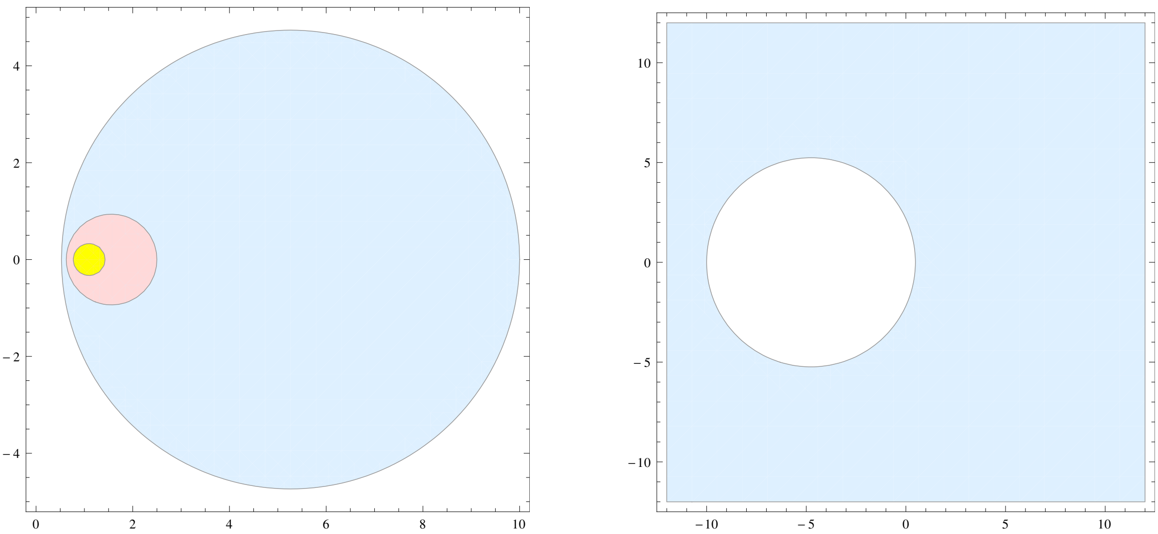

Figure 4.1. The left figure describes the Cassinian disks $B_{c_{\mathbb{R}^2\setminus\{0\}}}(e_1, R)$ with radii $R = 0.3, 0.6, 0.9$ respectively, and the right figure describes the Cassinian disk $B_{c_{\mathbb{R}^2\setminus\{0\}}}(e_1, R)$ with radius $R = 1.1$. The shaded regions are the interior of the disks.

The following lemma is useful in this setting.

Lemma 4.2. *Let $r > 0$. For $x, y \in \mathbb{R}^n$ assume that $y' \in (x, y)$. Then $C(x, y'; |x-y'|r)$ is totally enclosed by $C(x, y; |x-y|r)$.*

Proof. As Cassinian ovals are symmetric, we may assume that $n = 2$, $x = 0$ and $y = 1$. Now for $y' \in (0,1)$, denote by $C_1 = C(0, y'; |y'|r)$ and $C_2 = C(0, 1; r)$. To verify C_1 is totally enclosed by C_2, it is sufficient to check $|\eta| \le |\zeta|$, where $\eta = C_1 \cap \{(u, 0) \subset \mathbb{R}^2 : u < 0\}$ and $\zeta = C_2 \cap \{(u, 0) \subset \mathbb{R}^2 : u < 0\}$. Now from the definition of the Cassinian oval, we have

$$|\zeta||\zeta - 1| = r \Rightarrow |\zeta| = \frac{-1 + \sqrt{1+4r}}{2}.$$

Again,

$$|\eta||\eta - y'| = r|y'| \Rightarrow |\eta| = \frac{-|y'| + \sqrt{|y'|^2 + 4r|y'|}}{2}.$$

Since $f(t) = (-t + \sqrt{t^2 + 4rt})/2$ is increasing in $t \in (0,1)$, the conclusion follows. $\square$

Now we recall the statement of Theorem 1.5 and provide its proof.

Theorem 4.3. *(see also, Theorem 1.5) Let $r > 0$. If $D \subsetneq \mathbb{R}^n$ is a starlike domain with respect to $x \in D$, then $B_{c_D}(x, r)$ is starlike with respect to x.*

Proof. On contrary, assume that $B_c(x, r)$ is not starlike with respect to x. Then there exists a point $y \in \partial B_c(x, r)$ such that the line segment $[x, y]$ intersects $\partial B_c(x, r)$ at some point y' different from y. Clearly, $c_D(x, y') = r$. Since D is starlike, $y' \in D$. Consider the maximal Cassinian ovals $C(x, y; r)$ and $C(x, y'; r)$ defined by

$$C(x, y; |x-y|/r) = \left\{ z \in \overline{D} : |x - z||z - y| = \frac{|x-y|}{r} \right\}$$

and

$$C(x, y'; |x-y'|/r) = \left\{ z' \in \overline{D} : |x - z'||z' - y| = \frac{|x-y'|}{r} \right\}$$

respectively. In one hand, by definition of $C(x, y'; |x-y'|/r)$, there exists a point $z'' \in C(x, y'; |x-y'|/r) \cap \partial D$. On the other hand, by Lemma 4.2, $C(x, y'; |x-y'|/r)$ is totally enclosed by $C(x, y; |x-y|/r)$. This leads to a contradiction. $\square$

4.2. Convexity property

Recall that the domain D is called convex (strictly convex) if it is starlike (strictly starlike) with respect to all points in D. First we study the convexity property of the Cassinian ball in punctured spaces.

Lemma 4.4. *Let $x \in D_p$. Then*

(a) *the Cassinian ball $B_{c_{D_p}}(x,R)$ is convex if and only if $R \in (0, 1/|x-p|]$.*

(b) *the Cassinian ball $B_{c_{D_p}}(x,R)$ is strictly convex if and only if $R \in (0, 1/|x-p|)$.*

Proof. Let $y \in \partial B_{c_{D_p}}(x,R)$. Definition of the Cassinian metric and a simple computation show that

$$c_{D_p}(x,y) = R \iff \left| y - \left(p + \frac{x-p}{1-R^2\,|x-p|^2} \right) \right| = \frac{R\,|x-p|^2}{1-R^2\,|x-p|^2}$$

for $R^2\,|x-p|^2 \neq 1$. This equation defines a Euclidean circle centred at $p + ((x-p)/(1-R^2\,|x-p|^2))$ with radius $R\,|x-p|^2/(1-R^2\,|x-p|^2)$. Let us denote this circle by Γ. Then the Cassinian ball $B_{c_{D_p}}(x,R)$ is convex if and only if the Euclidean ball enclosed by Γ is a convex region. This is possible only when the radius of Γ is at most the Euclidean distance between p and centre of Γ, else Γ may contain the boundary point p. Thus, the Cassinian ball $B_{c_{\mathbb{R}^n\setminus\{p\}}}(x,R)$ is convex if and only if

$$\frac{R|x-p|^2}{1-R^2|x-p|^2} \le \frac{|x-p|}{1-R^2|x-p|^2} \iff R \le \frac{1}{|x-p|}.$$

This concludes the proof of our lemma. $\square$

Theorem 4.5. *(see also, Theorem 1.6) Let $D \subsetneq \mathbb{R}^n$ be any arbitrary domain. Then $B_{c_D}(x,R)$ is convex for $R \le \sup\{1/|x-z_i| : z_i \in \partial D\}$, and is strictly convex for $R < \sup\{1/|x-z_i| : z_i \in \partial D\}$.*

Proof. The proof follows from the combination of Lemma 4.1 and Lemma 4.4, and the fact that the intersection of convex sets is convex. $\square$

In punctured spaces, Cassinian balls are convex with small radius, but the same is not true in general. The next result shows that this is not the case even in convex domain.

Proposition 4.6. *Let $r > 0$. There exists $x \in \mathbb{H}^n$ such that $B_{c_{\mathbb{H}^n}}(x,r)$ is not convex.*

Proof. It is sufficient to consider the case $n = 2$. For a given r we choose $x = i/r$ and consider Cassinian disk $B_{c_{\mathbb{H}^2}}(x, r)$ with radius r. To show that $B_{c_{\mathbb{H}^2}}(x, r)$ is not convex we choose two points y_1 and y_2 such that $c_{\mathbb{H}^2}(x, y_1) = r = c_{\mathbb{H}^2}(x, y_2)$ and $c_{\mathbb{H}^2}(x, (y_1 + y_2)/2) > r$.

We choose $y_1 = \frac{i}{2r}$. Now by the geometry of the Cassinian ovals

$$c_{\mathbb{H}^2}(x, y_1) = \frac{|x - y_1|}{|x - 0||0 - y_1|} = \frac{\dfrac{1}{2r}}{\left(\dfrac{1}{r}\right)\left(\dfrac{1}{2r}\right)} = r.$$

Let $y_2 = (2 + i)/r$. Again by the geometry of the Cassinian ovals

$$c_{\mathbb{H}^2}(x, y_2) = \frac{|x - y_2|}{|x - 1/r||1/r - y_2|} = \frac{\frac{2}{r}}{\sqrt{1/r^2 + 1/r^2}\sqrt{1/r^2 + 1/r^2}} = r.$$

Now $y_3 = (y_1 + y_2)/2 = \frac{1}{r} + \frac{3i}{4r}$ and we choose $z = \frac{2}{3r}$. We obtain

$$|x - y_3| = \frac{\sqrt{17}}{4r}, \quad |x - z| = \frac{\sqrt{13}}{3r}, \quad |z - y| = \frac{\sqrt{97}}{12r}$$

and thus

$$c_{\mathbb{H}^2}(x, y_3) \geq \frac{|x - y_3|}{|x - z||z - y_3|} = 9\sqrt{\frac{17}{1261}}r = (1.04498\ldots)r > r,$$

completing the proof. □

The proof of Proposition 4.6 suggests that the radius of convexity for the Cassinian balls $B_{c_{\mathbb{H}^2}}(x, r)$ in $\mathbb{H}^2$ depends on $d_{\mathbb{H}^2}(x)$ and r.

4.3. Inclusion properties

Recall that in this section we deal with the problems of the following type:

Given $x \in D \subsetneq \mathbb{R}^n$ and $t > 0$, we find optimal radii $r, R > 0$ depending only on x and t such that

$$B_d(x, r) \subset B_c(x, t) \subset B_d(x, R), \tag{4.1}$$

where d is a metric other than the Cassinian metric defined on D.

We begin with proving the relation (4.1) when d is the Euclidean metric.

Theorem 4.7. *Let $D \subsetneq \mathbb{R}^n$ and $x \in D$. Assume that $0 < t < \min\{d(x), 1/d(x)\}$. Then the following inclusion property holds:*

$$B^n(x,r) \subset B_c(x,t) \subset B^n(x,R)$$

where $r = \dfrac{t(d(x))^2}{1+td(x)}$ and $R = \dfrac{t(d(x))^2}{1-td(x)}$. The radii r and R are best possible. Moreover, $R/r \to 1$ as $t \to 0$.

Proof. It is clear that for $x, y \in D$,

$$\inf_{z\in\partial D} |x-z||z-y| \le d(x)(d(x)+|x-y|).$$

By the definition of the c_D-metric we have

$$c_D(x,y) = \frac{|x-y|}{\inf_{z\in\partial D}|x-z||z-y|} \ge \frac{|x-y|}{d(x)(d(x)+|x-y|)}$$

which implies

$$|x-y| \le \frac{c_D(x,y)(d(x))^2}{1-c_D(x,y)d(x)}.$$

Hence the second inclusion holds. Now we prove the first inclusion. Since $t \le d(x)$, $y \in B^n(x,t)$ implies $y \in B^n(x,d(x))$; and by the monotone property

$$c_D(x,y) \le c_{B^n(x,d(x))}(x,y) = \frac{|x-y|}{d(x)(d(x)-|x-y|)}.$$

In particular, if $y \in B^n(x,r)$ with $r = t(d(x))^2/(1+td(x))$, then $y \in B_c(x,t)$. Clearly, one can see that

$$\frac{R}{r} = \frac{1+td(x)}{1-td(x)} \to 1 \text{ as } t \to 0.$$

We finally show that radii r and R are best possible. For this, consider the domain D_a and $x \in D_a$. Let us denote by l the line through points a and x. We set $\{y_1, y_2\} = \partial B_c(x,t) \cap l$ with $|a-y_1| < |a-y_2|$. Now

$$c_{D_0}(x,y_1) = t = \frac{|x-y_1|}{|x-a|(|x-a|-|x-y_1|)}$$

implying

$$|x-y_1| = \frac{t|x-a|^2}{1+t|x-a|}$$

which shows that r is best possible. Similarly,

$$c_{D_0}(x,y_2) = t = \frac{|x-y_2|}{|x-a|(|x-a|+|x-y_2|)}$$

implying

$$|x-y_2| = \frac{t|x-a|^2}{1-t|x-a|}$$

which shows that R is best possible. This completes the proof of our theorem. $\square$

Now we move forward to discuss the inclusion relation (4.1) when $d = j_D$. Recall the following relation proved in [59]:

Lemma 4.8. *[59, Theorem 3.8] If $D \subsetneq \mathbb{R}^n$ is open, $x \in D$ and $t > 0$, then*

$$B^n(x,r) \subset B_{\tilde{j}}(x,t) \subset B^n(x,R)$$

where $r = (1-e^{-t})d(x)$ and $R = (e^t-1)d(x)$. The formulas for r and R are the best possible expressed in terms of t and $d(x)$ only.

In this connection, we prove

Theorem 4.9. *Let $D \subsetneq \mathbb{R}^n$, $x \in D$ and $0 < t < 1/d(x)$. Then the following holds:*

$$B_{\tilde{j}}(x,r) \subset B_c(x,t) \subset B_{\tilde{j}}(x,R),$$

where $r = \log\left(1 + \dfrac{td(x)}{1+td(x)}\right)$ and $R = \dfrac{td(x)}{1-td(x)}$. Moreover, $R/r \to 1$ as $t \to 0$.

Proof. We first prove the second inclusion. By [42, Theorem 3.4] we have

$$\tilde{j}_D(x,y) \le (|x-y| + d(x) \wedge d(y))c_D(x,y) \le (|x-y| + d(x))c_D(x,y)$$

and from Theorem 4.7,

$$c_D(x,y) < t \Rightarrow |x-y| < t(d(x))^2/(1-td(x)).$$

Now for $y \in B_c(x,r)$, using the above estimates we have, $\tilde{j}_D(x,y) < td(x)/(1-td(x))$. For the proof of the first inclusion we use Lemma 4.8 together with Theorem 4.7 to conclude that

$$\tilde{j}_D(x,y) < \log(1 + (rd(x))/(1+rd(x))) \Rightarrow c_D(x,y) < r.$$

By l'Hôspital rule it follows that $R/r \to 1$ as $t \to 0$. $\square$

The radii obtained in Theorem 4.9 can be improved in the special case if we choose the domain D_a. In this connection we prove

Theorem 4.10. *Let $x \in D_a$ and $0 < t < 1/(2|x-a|)$. Then the following holds:*

$$B_{\tilde{j}}(x,r) \subset B_c(x,t) \subset B_{\tilde{j}}(x,R),$$

where $r = \log(1+t|x-a|)$ and $R = \log\left(\dfrac{1-t|x-a|}{1-2t|x-a|}\right)$. The radius r is best possible. Moreover, $R/r \to 1$ as $t \to 0$.

Proof. Suppose that $y \in B_{\tilde{j}}(x,r)$. Then $\tilde{j}_D(x,y) < r$. On simplification, we get

$$|x-y| < t|x-a|(|x-a| \wedge |y-a|) \tag{4.2}$$

If $|x-a| \wedge |y-a| = |x-a|$, then

$$c_{D_a}(x,y) = \frac{|x-y|}{|x-a||y-a|} < \frac{t|x-a|}{|y-a|} \le t$$

where the first inequality follows from (4.2) and the last inequality follows from the fact that $|x-a| \le |y-a|$. Otherwise,

$$c_{D_a}(x,y) = \frac{|x-y|}{|x-a||y-a|} < \frac{t|x-a||y-a|}{|x-a||y-a|} = t$$

where the inequality follows from (4.2) and the first inclusion follows.

Now suppose that $c_{D_a}(x,y) < t$. This implies, by Theorem 4.7, that

$$|x-y| < t|x-a|^2/(1-t|x-a|).$$

If $|x-a| \wedge |y-a| = |x-a|$, then

$$\tilde{j}_{D_a}(x,y) < \log\left(1 + \frac{t|x-a|}{1-t|x-a|}\right) = \log\left(\frac{1}{1-t|x-a|}\right).$$

Otherwise

$$\tilde{j}_{D_a}(x,y) < \log\left(1 + \frac{t|x-a|^2}{|y-a|(1-t|x-a|)}\right) \le \log\left(\frac{1-t|x-a|}{1-2t|x-a|}\right),$$

where the second inequality follows from the fact that

$$|y-a| \ge |x-a| - |x-y| \ge \frac{|x-a|(1-2t|x-a|)}{1-t|x-a|}.$$

Now,

$$R = \max\left\{\log\left(\frac{1}{1-t|x-a|}\right), \log\left(\frac{1-t|x-a|}{1-2t|x-a|}\right)\right\} = \log\left(\frac{1-t|x-a|}{1-2t|x-a|}\right)$$

and hence the proof of the second inclusion follows. By l'Hôspital rule it follows that

$$\lim_{t\to 0}\frac{R}{r}=\lim_{t\to 0}\frac{1+t|x-a|}{(1-t|x-a|)(1-2t|x-a|)}=1.$$

To show that the radius r is best possible, we consider the same construction as did in the proof of Theorem 4.7. For the same choice of y_1, it is easy to verify that

$$c_{D_a}(x,y_1)=t \iff |x-y_1|=\frac{t|x-a|^2}{1+t|x-a|}.$$

Hence, we obtain

$$\tilde{j}_{D_a}(x,y_1)=\log\left(1+\frac{|x-y_1|}{|y_1-a|}\right)=\log\left(1+\frac{|x-y_1|}{|x-a|-|x-y_1|}\right)=\log\left(1+t|x-a|\right),$$

which shows that r is best possible. $\square$

It seems natural to expect that Theorem 4.10 can be extended to the case of a general domain in the following way.

Conjecture 4.11. *Let $D\subsetneq\mathbb{R}^n$, $x\in D$ and $0<t<1/d(x)$. Then the following holds:*

$$B_{\tilde{j}}(x,r)\subset B_c(x,t)\subset B_{\tilde{j}}(x,R),$$

where $r=\log(1+td(x))$ and $R=\log\left(\frac{1}{1-td(x)}\right)$. Moreover, the radii R and r are best possible and $R/r\to 1$ as $t\to 0$.

By [1, Theorem 7.56] we have

$$\tilde{j}_{\mathbb{B}^n}(x,y)\le\rho_{\mathbb{B}^n}(x,y)\le 2\tilde{j}_{\mathbb{B}^n}(x,y). \tag{4.3}$$

It immediately follows that

$$B_\rho(x,r)\subset B_{\tilde{j}}(x,r)\subset B_\rho(x,2r).$$

This leads to the following result.

Theorem 4.12. *Let $x\in\mathbb{B}^n$ and $0<t<1/(1-|x|)$. Then the following inclusion relation holds:*

$$B_\rho(x,r)\subset B_c(x,t)\subset B_\rho(x,R)$$

where $r=\log\left(1+\frac{t(1-|x|)}{1+t(1-|x|)}\right)$ and $R=\frac{2t(1-|x|)}{1-t(1-|x|)}$. Moreover, $R/r\to 2$ as $t\to 0$.

Proof. By Theorem 4.9, $B_c(x,t) \subset B_{\tilde{j}}(t(1-|x|)/(1-t(1-|x|)))$ and by (4.3), the second inclusion follows with $R = 2t(1-|x|))/(1-t(1-|x|))$. Again from (4.3) and Theorem 4.9, we have

$$B_\rho(x,r) \subset B_{\tilde{j}}(x,r) \subset B_c(x,(e^r-1)/(1-|x|)(2-e^r)).$$

Simplification yields $B_\rho(x,r) \subset B_c(x,t)$ with $r = \log(1+(t(1-|x|)/(1+t(1-|x|))))$. By l'Hôspital rule it is easy to see that

$$\lim_{t\to 0} \frac{R}{r} = 2.$$

This completes the proof of our theorem. □

Another sharp inclusion property between the $\tilde{j}_{\mathbb{B}^n}$-metric ball and $\rho_{\mathbb{B}^n}$-metric ball is derived by Klén and Vuorinen in [51]. They proved that

Lemma 4.13. *[51, Theorem 3.1] Let $x \in \mathbb{B}^n$ and $r > 0$. Then*

$$B_{\tilde{j}}(x,m) \subset B_\rho(x,r) \subset B_{\tilde{j}}(x,M)$$

where $m = \max\{m_1, m_2\}$ and $M = \log\left(1+(1+|x|)\dfrac{e^r-1}{2}\right)$;

$$m_1 = \log\left(1+(1+|x|)\sinh(r/2)\right), \quad m_2 = \log\left(1+(1-|x|)\frac{e^r-1}{2}\right).$$

Moreover, the inclusions are sharp and $M/m \to 1$ as $r \to 0$.

Using Lemma 4.13 together with Theorem 4.9 we obtain

Theorem 4.14. *Let $x \in \mathbb{B}^n$ and $t > 0$. Then the following inclusion relation holds:*

$$B_\rho(x,r) \subset B_c(x,t) \subset B_\rho(x,R)$$

where $r = \log\left(1+\dfrac{2t(1-|x|)}{(1+|x|)(1+t(1-|x|))}\right)$ and $R = \min\{R_1, R_2\}$ with

$$R_1 = 2\sinh^{-1}\Bigg(\frac{\exp\left(\dfrac{t(1-|x|)}{1-t(1-|x|)}\right)-1}{1+|x|}\Bigg) \ \text{ and } \ R_2 = \log\Bigg(1+\frac{2\exp\left(\dfrac{t(1-|x|)}{1-t(1-|x|)}\right)-1}{1-|x|}\Bigg).$$

Remark 4.15. If $R = R_1$, then Theorem 4.14 is sharper than Theorem 4.12 (since $R_1/r \to 1$ as $t \to 0$). Otherwise Theorem 4.12 is sharper than Theorem 4.14.

However, we conjecture a better estimate for radii r and R in Theorem 4.12.

Conjecture 4.16. *Let $x \in \mathbb{B}^n$ and $t > 0$. Then the following inclusion relation holds:*

$$B_\rho(x,r) \subset B_c(x,t) \subset B_\rho(x,R)$$

where

$$r = \frac{t(1-|x|)}{\sqrt{(1+|x|)(1+|x|-2t(1-|x|))}} \quad \text{and} \quad R = \frac{t(1-|x|)}{\sqrt{(1+|x|)(1+|x|+2t(1-|x|)}}.$$

Moreover, the radii r and R are sharp and $R/r \to 1$ as $t \to 0$.

In order to discuss the relation (4.1) when $d = k_D$, for a domain $D \subsetneq \mathbb{R}^n$, we recall the useful inequality [18, Lemma 2.1]

$$k_D(x,y) \geq \tilde{j}_D(x,y); \quad x,y \in D \subsetneq \mathbb{R}^n. \tag{4.4}$$

It follows immediately from (4.4) and Theorem 4.10 that

Corollary 4.17. *Let $x \in D_a$ and $t > 0$. Then the following holds:*

$$B_k(x,r) \subset B_c(x,t)$$

where $r = \log(1 + t|x-a|)$.

Conjecture 4.18. *Let $x \in D_a$ and $0 < t < 1/|x-a|$. Then the following inclusion relation holds:*

$$B_k(x,r) \subset B_c(x,t) \subset B_k(x,R)$$

where $r = \log(1+t|x-a|)$ and $R = \log\left(\dfrac{1}{1-t|x-a|}\right)$. The radii r and R are sharp and $R/r \to 1$ as $t \to 0$.

In proper subdomains of $\mathbb{R}^n$ the following inclusion relation holds in between the c_D-metric ball and the k_D-metric ball. The following lemma is useful in this setting.

Lemma 4.19. *[50, Proposition 2.2] Let $D \subsetneq \mathbb{R}^n$ be a domain and $r \in (0, \log 2)$. Then*

$$B_{\tilde{j}}(x,m) \subset B_k(x,r) \subset B_{\tilde{j}}(x,r) \subset B_k(x,M)$$

where $r = \log(2-e^r)$ and $M = \log\left(\dfrac{1}{2-e^r}\right)$. Moreover, the second inclusion is sharp and $M/m \to 1$ as $r \to 0$.

Theorem 4.20. *Let $D \subsetneq \mathbb{R}^n$ be a domain, $x \in D$, and $t \in (0, \log 2/(d(x)(1+\log 2)))$. Then*

$$B_k(x,r) \subset B_c(x,t) \subset B_k(x,R)$$

where $r = \log\left(1 + \dfrac{td(x)}{1+td(x)}\right)$ *and* $R = \log\left(\dfrac{1}{2 - \exp\Big(td(x)/(1-td(x))\Big)}\right)$*. Moreover, $R/r \to 1$ as $t \to 0$.*

Proof. By (4.4) and Theorem 4.9 we have

$$B_k(x,r) \subset B_{\tilde{j}}(x,r) \subset B_c(x,(e^r-1)/(\delta_D(x)(2-e^r)))$$

and the first inclusion follows. Again from Theorem 4.9 and Lemma 4.19 we have

$$B_c(x,t) \subset B_{\tilde{j}}\left(x, \frac{td(x)}{1-td(x)}\right) \subset B_k\left(x, \log\left(\frac{1}{2-\exp(td(x)/(1-td(x)))}\right)\right).$$

By l'Hôspital rule it follows easily that $R/r \to 1$ as $t \to 0$. Hence the proof of our theorem is complete. □

Similar to Lemma 4.1, that the $\tilde{\tau}_D$-metric balls can be written as the intersection of $\tilde{\tau}$-metric balls in punctured spaces.

Proposition 4.21. *Let $D \subsetneq \mathbb{R}^n$, $x \in D$, and $r > 0$. Then*

$$B_{\tilde{\tau}_D}(x,r) = \cap_{p\in\partial D} B_{\tilde{\tau}_{D_p}}(x,r).$$

Proof. Suppose that $y \in \cap_{p\in\partial D} B_{\tilde{\tau}_{D_p}}(x,r)$. Then $\tilde{\tau}_{D_p}(x,y) < r$ for all $p \in \partial D$. In particular,

$$\tilde{\tau}_D(x,y) = \sup_{p\in\partial D} \tilde{\tau}_{D_p}(x,y) < r.$$

So, $\cap_{p\in\partial D} B_{\tilde{\tau}_{D_p}}(x,r) \subseteq B_{\tilde{\tau}_D}(x,r)$. Conversely, suppose that $y \in B_{\tilde{\tau}_D}(x,r)$. Then

$$\tilde{\tau}_{D_p}(x,y) \le \sup_{p\in\partial D} \tilde{\tau}_{D_p}(x,y) = \tilde{\tau}_D(x,y) < r \quad \text{for all } p \in \partial D.$$

Hence, $B_{\tilde{\tau}_D}(x,r) \subseteq \cap_{p\in\partial D} B_{\tilde{\tau}_{D_p}}(x,r)$ and the proof is complete. □

Now we will discuss the metric ball inclusion property associated with the $\tilde{\tau}_D$-metric and the u_D-metric.

The following inclusion relation holds true between the u_D-metric ball and $\tilde{\tau}_D$-metric ball.

Corollary 4.22. *Let $D \subsetneq \mathbb{R}^n$ be any arbitrary domain and $x \in D$. Then*

$$B_{u_D}(x, r) \subseteq B_{\tilde{\tau}_D}(x, t),$$

where $r = 2t$ and the inclusion is sharp.

Proof. Suppose that $y \in B_{u_D}(x, 2t)$. Then $u_D(x, y) < 2t$. Now it follows from Theorem 1.4 that $\tilde{\tau}_D(x, y) < t$ and hence the proof is complete. For the sharpness, consider the domain D_0 and $x \in D_0$. Choose the point $y = \partial B_{u_{D_0}}(x, 2t) \cap \partial B(0, |x|)$. Now, $u_{D_0}(x, y) = 2t$ implies $\tilde{\tau}_{D_0}(x, y) = t$. This proves our corollary. □

As immediate consequence of Theorem 3.4 is the following inclusion relation.

Corollary 4.23. *Let $x \in D \subsetneq \mathbb{R}^n$ and $t > 0$. Then we have*

$$B_{u_D}(x, r) \subseteq B_{\tilde{\tau}_D}(x, t) \subseteq B_{u_D}(x, R),$$

where $r = 2t$ and $R = 4t$. The radii r and R are best possible.

Proof. Let $y \in B_{u_D}(x, r)$, $r = 2t$ and $R = 4t$. Then by Theorem 3.4 we have $\tilde{\tau}_D(x, y) < t$. So, $B_{u_D}(x, r) \subseteq B_{\tilde{\tau}_D}(x, t)$. Conversely, if $y \in B_{\tilde{\tau}_D}(x, t)$, then also by Theorem 3.4 we have $y \in B_{u_D}(x, R)$. So, $B_{\tilde{\tau}_D}(x, t) \subseteq B_{u_D}(x, R)$ and hence the inclusion follows. Next, we need to prove the sharpness part.

First we consider the domain $D = D_0$ and $x \in D_0$. Now choose $y \in B(0, |x|) \cap \partial B_{\tilde{\tau}_{D_0}}(x, t)$. Then

$$\tilde{\tau}_{D_0}(x, y) = t = \log\left(1 + \frac{|x - y|}{|x|}\right) = \frac{u_{D_0}(x, y)}{2},$$

which proves the sharpness of the first inclusion. Secondly, consider $D = \mathbb{B}^n$ and let $x \in \mathbb{B}^n$ be arbitrary. Choose $y \in \mathbb{B}^n$ such that x and y lie on a diameter of $\mathbb{B}^n$ with 0 lying in-between and $|y| \le |x|$.

$$u_{\mathbb{B}^n}(x, y) = 2\log\left(\frac{|x - y| + 1 - |y|}{\sqrt{(1 - |x|)(1 - |y|)}}\right) \text{ and } \tilde{\tau}_{\mathbb{B}^n}(x, y) = \log\left(1 + \frac{|x - y|}{\sqrt{(1 - |x|)(1 + |y|)}}\right).$$

It follows that

$$\begin{aligned}
\lim_{x \to e_1} \frac{u_{\mathbb{B}^n}(x, y)}{4\tilde{\tau}_{\mathbb{B}^n}(x, y)} &= \lim_{x \to e_1} \frac{(|x - y| + 1 - |y|)(1 - |x|)(1 + |y|)}{\sqrt{(1 - |x|)(1 - |y|)}(|x - y| + \sqrt{(1 - |x|)(1 + |y|)})^2} \\
&= \begin{cases} 1 & \text{if } y = -x \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Hence we conclude that for each $x \in \mathbb{B}^n$ with $|x| \to 1$ and $t > 0$, there exist $y = -x$ such that $y \in \partial B_{\tilde{\tau}_D}(x,t)$ and $u_D(x,y) = 4t$. This proves the sharpness of the second inclusion relation and hence the proof is complete. $\square$

Now, we establish the inclusion relation between the $\tilde{j}_D$-metric ball and the $\tilde{\tau}_D$-metric ball.

Theorem 4.24. *Let $D \subsetneq \mathbb{R}^n$ and $x \in D$ and $t > 0$. Then the following inclusion property holds true:*

$$B_{\tilde{j}_D}(x,r) \subseteq B_{\tilde{\tau}_D}(x,t) \subseteq B_{\tilde{j}_D}(x,R).$$

Here $r = t$ and $R = 2t$. The radii r and R are best possible.

Proof. The proof follows from (3.5). To show the radius r is best possible, consider the domain D_0 and $x \in D_0$. Choose $y \in \partial B(0,|x|) \cap \partial B_{\tilde{\tau}_{D_0}}(x,t)$. Now clearly, $\tilde{j}_{D_0}(x,y) = \tilde{\tau}_{D_0}(x,y) = t$. To show R is the best possible, consider the domain $D = \mathbb{B}^n$. With the similar argument given in the proof of Corollary 4.23 for the second inclusion relation, we can show that for each $x \in \mathbb{B}^n$ with $|x| \to 1$ and $t > 0$, there exist $y = -x$ such that $y \in \partial B_{\tilde{\tau}_{\mathbb{B}^n}}(x,t)$ and $\tilde{j}_{\mathbb{B}^n}(x,y) = 2t$. This completes the proof. $\square$

As an immediate consequence of Lemma 3.5 we have the following inclusion relation.

Corollary 4.25. *Let $D \subsetneq \mathbb{R}^n$, $x \in D$, and $t > 0$. Then*

$$B_{u_D}(x,r) \subseteq B_{\tilde{j}_D}(x,t) \subset B_{u_D}(x,R),$$

where $r = t$ and $R = 4t$. The radius r is best possible.

Proof. Proof follows from Lemma 3.5. $\square$

The following inclusion relation holds true.

Corollary 4.26. *Let $x \in D \subsetneq \mathbb{R}^n$ and $t > 0$. Then*

$$B_{\delta_D}(x,r) \subseteq B_{\tilde{\tau}_D}(x,t) \subseteq B_{\delta_D}(x,R),$$

where $r = t$ and $R = 4t$.

Proof. Proof follows from Theorem 3.8. $\square$

Corollary 3.9 leads to the following inclusion relation.

Corollary 4.27. *Let $x \in D \subsetneq \mathbb{R}^n$ and $t > 0$. Then*

$$B_{\delta_D}(x, r) \subseteq B_{u_D}(x, t) \subseteq B_{\delta_D}(x, R),$$

where $r = t/4$ and $R = 2t$.

Theorem 3.11 leads to the following inclusion relation.

Corollary 4.28. *Let $x \in D \subsetneq \mathbb{R}^n$ and $t > 0$. Then*

$$B_{\tilde{\tau}_D}(x, t) \subseteq B_{s_D}(x, R),$$

where $R = t/\log 3$. The inclusion is sharp.

Proof. It follows from Theorem 3.11 that for $\tilde{\tau}_D(x, y) < t$, $s_D(x, y) < t/\log 3$. Hence, $B_{\tilde{\tau}_D}(x, t) \subseteq B_{s_D}(x, R)$ with $R = t/\log 3$.

To prove the sharpness part, consider the domain $D = D_0$ and $t = \log 3$. Then we have $R = 1$ and

$$\tilde{\tau}_{D_0}(x, y) = \log 3 \iff |x - y| = 2\sqrt{|x||y|}$$

and hence

$$s_{D_0}(x, y) = \frac{2\sqrt{|x||y|}}{|x| + |y|}.$$

To show $s_{D_0}(x, y) = 1$, we need to choose points x and y such that $|x| = |y|$. This implies x and y are co-linear. i.e. $y = -x$. From the definition of the $\tilde{\tau}_D$-metric it is clear that the point $-x$ lies on the sphere $\partial B_{\tilde{\tau}_{D_0}}(x, \log 3)$. Now, for any $x \in D_0$, choose $y \in \partial B_{\tilde{\tau}_{D_0}}(x, \log 3) \cap L$, where L is the line passing through 0 and x with $|x| = |y|$. Then

$$\tilde{\tau}_{D_0}(x, y) = \log 3 \iff s_{D_0}(x, y) = 1.$$

Hence, the proof is complete. $\square$

Lemma 3.15 yields the following inclusion property.

Corollary 4.29. *Let $D \subsetneq \mathbb{R}^n$ and $x \in D$ and $t > 0$. Then the following inclusion property holds true:*

$$B_{\eta_D}(x, r) \subseteq B_{\tilde{\tau}_D}(x, t) \subseteq B_{\eta_D}(x, R).$$

Here $r = \log(e^t - 2)$ and $R = 2t$. The radii r and R are best possible.

Proof. Suppose that $y \in B_{\tilde{\tau}_D}(x,t)$. Then $\tilde{\tau}_D(x,y) < t$. From the left hand side inequality of Theorem 3.15, we have $\eta_D(x,y) < 2t(= R)$. On the other hand, if $\eta_D(x,y) < \log(e^t - 2)(= r)$, then from the right hand side inequality of Theorem 3.15, we have $\tilde{\tau}_D(x,y) < t$. With the similar argument given in the proof of the sharpness part of the second inclusion relation in Corollary 4.23, we can show that the radius R is the best possible in the punctured space D_0 and $t = \log 3$. □

Lemma 3.16 leads to the following inclusion relation.

Corollary 4.30. *Let $x \in D \subsetneq \mathbb{R}^n$ and $t > 0$. Then*

$$B_{\eta_D}(x,r) \subseteq B_{u_D}(x,t) \subseteq B_{\eta_D}(x,R),$$

where $r = \log(e^{t/4} - 2)$ and $R = t$.

CHAPTER 5

MAPPING PROPERTIES

This chapter deals with the distortion results associated with the c_D-metric, the $\tilde{\tau}_D$-metric, and the u_D-metric under certain classes of mappings, namely, the Möbius class and the quasiconformal class. Recall that the c_D-metric and the $\tilde{\tau}_D$-metric are not Möbius invariant. Hence it is natural to study the quasi-invariance (distortion) properties of these metrics under Möbius transformations. The study of quasi-invariance property under quasiconformal mappings hence natural. The study on the quasi-invariance property for some well-known hyperbolic-type metrics are listed below:

- The k_D-metric and the $\tilde{j}_D$-metric are quasi-invariant with the constant 2 under Möbius transformations of $\mathbb{R}^n$ [17, Theorem 4]. However, the constant 2 is improved to $1+|a|, a \in \mathbb{B}^n$, under Möbius transformations of $\mathbb{B}^n$ onto itself [52, Theorem 1.4,Conjecture 2.3].
- The s_D-metric is quasi-invariant with the constant $(1+|a|)/(1-|a|)$ under Möbius transformations of $\mathbb{B}^n$ onto itself [11, Theorem 3.31].
- The v_D-metric is quasi-invariant with the constant 2 under Möbius transformations of $\mathbb{B}^n$ onto itself [48, Theorem 1.2].

The study of quasi-invariance properties under quasiconformal mappings for the k_D-metric (and j_D-metric), $\tilde{j}_D$-metric, and the δ_D-metric are available respectively in [17], [30], and [59].

We begin this chapter with the quasi-invariance property of the c_D-metric under Möbius transformations of the unit ball onto itself followed by the quasi-invariance of the c_D-metric under Möbius transformations of a punctured ball onto another punctured ball.

5.1. Distortion properties under Möbius transformations

In this section we study distortion properties of the Cassinian metric $c_{\mathbb{B}^n}$ of the unit ball $\mathbb{B}^n$ under Möbius transformations of $\mathbb{B}^n$. Recall the statement of Theorem 1.9.

Theorem 5.1. *(see also, Theorem 1.9) Let ϕ be a Möbius transformation with $\phi(\mathbb{B}^n) = \mathbb{B}^n$. Then*
$$\frac{1-|\phi(0)|}{1+|\phi(0)|}c_{\mathbb{B}^n}(x,y) \le c_{\mathbb{B}^n}\big(\phi(x),\phi(y)\big) \le \frac{1+|\phi(0)|}{1-|\phi(0)|}c_{\mathbb{B}^n}(x,y)$$
for all $x, y \in \mathbb{B}^n$. The equalities in both sides can be attained.

Proof. Let ϕ be a Möbius transformation with $\phi(\mathbb{B}^n) = \mathbb{B}^n$ and put $a = \phi(0)$. If $a = 0$, then ϕ is an orthogonal matrix, i.e., $|\phi(x)| = |x|$ for each $x \in \mathbb{B}^n$. In particular, ϕ preserves the Cassinian metric. That is,

$$c_{\mathbb{B}^n}\big(\phi(x),\phi(y)\big) = c_{\mathbb{B}^n}(x,y) \qquad \text{for all} \quad x,y \in \mathbb{B}^n. \tag{5.1}$$

Suppose now that $a \neq 0$. Let σ be the inversion in the sphere $\mathbb{S}^{n-1}(a^\star, r)$, where
$$a^\star = \frac{a}{|a|^2} \qquad \text{and} \qquad r = \sqrt{|a^\star|^2-1} = \frac{\sqrt{1-|a|^2}}{|a|}.$$
Note that the sphere $\mathbb{S}^{n-1}(a^\star, r)$ is orthogonal to $\partial\mathbb{B}^n$ and that $\sigma(a) = 0$. In particular, σ is a Möbius transformation with $\sigma(\mathbb{B}^n) = \mathbb{B}^n$ and $\sigma(a) = 0$. Recall that

$$\sigma(x) = a^\star + \Big(\frac{r}{|x-a^\star|}\Big)^2\big(x-a^\star\big). \tag{5.2}$$

Then $\sigma \circ \phi$ is an orthogonal matrix (see, for example, [4, Theorem 3.5.1(i)]). In particular,

$$\Big|\sigma\big(\phi(x)\big) - \sigma\big(\phi(y)\big)\Big| = |x-y|. \tag{5.3}$$

We will need the following property of σ (see, for example, [4, p. 26]):

$$|\sigma(x)-\sigma(y)| = \frac{r^2|x-y|}{|x-a^\star||y-a^\star|}. \tag{5.4}$$

It follows from (5.3) and (5.4) that
$$|x-y| = \Big|\sigma\big(\phi(x)\big) - \sigma\big(\phi(y)\big)\Big| = \frac{r^2|\phi(x)-\phi(y)|}{|\phi(x)-a^\star||\phi(y)-a^\star|} = \frac{(|a^\star|^2-1)|\phi(x)-\phi(y)|}{|\phi(x)-a^\star||\phi(y)-a^\star|},$$
or equivalently,
$$|\phi(x)-\phi(y)| = \frac{|\phi(x)-a^\star||\phi(y)-a^\star|}{|a^\star|^2-1}|x-y|.$$
In particular, for all $x, y \in \mathbb{B}^n$ and $\eta \in \partial\mathbb{B}^n$ we have

$$\frac{|\phi(x)-\phi(y)|}{|\phi(x)-\phi(\eta)||\phi(y)-\phi(\eta)|} = \frac{|x-y|}{|x-\eta||y-\eta|}\cdot\frac{|a^\star|^2-1}{|\phi(\eta)-a^\star|^2}. \tag{5.5}$$

Note that since $\phi(\eta) \in \partial\mathbb{B}^n$ and $|a^\star| > 1$, we have
$$|a^\star|-1 \le |\phi(\eta)-a^\star| \le |a^\star|+1$$

and hence

$$\frac{1-|a|}{1+|a|}=\frac{|a^\star|-1}{|a^\star|+1}\le\frac{|a^\star|^2-1}{|\phi(\eta)-a^\star|^2}\le\frac{|a^\star|+1}{|a^\star|-1}=\frac{1+|a|}{1-|a|}. \tag{5.6}$$

Now given $x,y\in\mathbb{B}^n$, there exist $\eta_1\in\partial\mathbb{B}^n$ and $\eta_2\in\partial\mathbb{B}^n$ such that

$$c_{\mathbb{B}^n}\big(\phi(x),\phi(y)\big)=\frac{|\phi(x)-\phi(y)|}{|\phi(x)-\phi(\eta_1)||\phi(y)-\phi(\eta_1)|}\quad\text{and}\quad c_{\mathbb{B}^n}(x,y)=\frac{|x-y|}{|x-\eta_2||y-\eta_2|}.$$

Using (5.5) and (5.6) we obtain

$$c_{\mathbb{B}^n}\big(\phi(x),\phi(y)\big)=\frac{|x-y|}{|x-\eta_1||y-\eta_1|}\cdot\frac{|a^\star|^2-1}{|\phi(\eta_1)-a^\star|^2}\le\frac{1+|a|}{1-|a|}c_{\mathbb{B}^n}(x,y)$$

and

$$c_{\mathbb{B}^n}(x,y)=\frac{|\phi(x)-\phi(y)|}{|\phi(x)-\phi(\eta_2)||\phi(y)-\phi(\eta_2)|}\cdot\frac{|\phi(\eta_2)-a^\star|^2}{|a^\star|^2-1}\le\frac{1+|a|}{1-|a|}c_{\mathbb{B}^n}\big(\phi(x),\phi(y)\big).$$

The constant $(1+|a|)/(1-|a|)$ can be attained for all Möbius transformations ϕ with $\phi(\mathbb{B}^n)=\mathbb{B}^n$ and $a=\phi(0)$. To see this, it suffices to consider the map σ with $\sigma(a)=0$ for $a\in[0,e_1]\setminus\{0,e_1\}$ with $e_1=(1,0,\cdots,0)\in\mathbb{R}^n$ since $c_{\mathbb{B}^n}$ is invariant under orthogonal transformations. Choose $x=0$ and $y=te_1$, $-1<t<0$. Then we have that

$$\sigma(x)=a\qquad\text{and}\qquad\sigma(y)=\frac{|a|-t}{1-|a|t}e_1\in[a,e_1]\setminus\{a,e_1\}.$$

It is easy to see by the formula (2.1) that

$$c_{\mathbb{B}^n}(x,y)=c_{\mathbb{B}^n}(0,te_1)=-\frac{t}{1+t}.$$

Furthermore, it follows from [37, Example 3.9(B)] that

$$c_{\mathbb{B}^n}(re_1,se_1)=\frac{s-r}{(1-r)(1-s)},\quad 0\le r<s<1.$$

This gives

$$c_{\mathbb{B}^n}(\sigma(x),\sigma(y))=c_{\mathbb{B}^n}\left(a,\frac{|a|-t}{1-|a|t}e_1\right)=\frac{\dfrac{|a|-t}{1-|a|t}-|a|}{(1-|a|)\left(1-\dfrac{|a|-t}{1-|a|t}\right)}=-\frac{t}{1+t}\frac{1+|a|}{1-|a|}.$$

Therefore, we get

$$\frac{c_{\mathbb{B}^n}(\sigma(x),\sigma(y))}{c_{\mathbb{B}^n}(x,y)}=\frac{1+|a|}{1-|a|}.$$

Thus the proof is complete. □

We now prove that the same distortion constant holds true under a Möbius mapping of a punctured ball onto another punctured ball for the Cassinian metric.

Theorem 5.2. *(see also, Theorem* 1.10*) Let $a \in \mathbb{B}^n$ and $f : \mathbb{B}^n \setminus \{0\} \to \mathbb{B}^n \setminus \{a\}$ be a Möbius transformation with $f(0) = a$. Then for $x, y \in \mathbb{B}^n \setminus \{0\}$ we have*

$$\frac{1-|a|}{1+|a|} c_{\mathbb{B}^n\setminus\{0\}}(x,y) \le c_{\mathbb{B}^n\setminus\{a\}}(f(x),f(y)) \le \frac{1+|a|}{1-|a|} c_{\mathbb{B}^n\setminus\{0\}}(x,y).$$

The equalities in both sides can be attained.

Proof. The case $a = 0$ is already proved in Theorem 1.9. Now, assume that $a \neq 0$ and let σ be the inversion in the sphere $\mathbb{S}^{n-1}(a^\star, r)$ as defined in (5.2). Note that the sphere $\mathbb{S}^{n-1}(a^\star, r)$ is orthogonal to $\mathbb{S}^{n-1}$ and that $\sigma(a) = 0$. In particular, σ is a Möbius map with $\sigma(\mathbb{B}^n \setminus \{a\}) = \mathbb{B}^n \setminus \{0\}$. Then $\sigma \circ f$ is an orthogonal matrix (see, for example, [4, Theorem 3.5.1(i)]). Now,

$$c_{\mathbb{B}^n\setminus\{0\}}(x,y) = \frac{|x-y|}{\min\{|x||y|, \inf_{z\in\partial\mathbb{B}^n} |x-z||z-y|\}}$$

and

$$c_{\mathbb{B}^n\setminus\{a\}}(f(x),f(y)) = \frac{|f(x)-f(y)|}{\min\{|f(x)-a||a-f(y)|, \inf_{w\in\partial\mathbb{B}^n} |f(x)-w||w-f(y)|\}}.$$

Denote by $P = \min\{|f(x)-a||a-f(y)|, \inf_{w\in\partial\mathbb{B}^n} |f(x)-w||w-f(y)|\}$. Now we have two choices for P. The choice of $P = \inf_{w\in\partial\mathbb{B}^n} |f(x)-w||w-f(y)|$ is already proved in Theorem 1.9. We only need to consider the other choice for P. Let $P = |f(x)-a||a-f(y)|$. It follows from (5.3) and (5.4) that

$$|f(x)-a| = \frac{|f(x)-a^\star||a-a^\star|}{(|a^\star|^2-1)}|x| \quad \text{and} \quad |a-f(y)| = \frac{|f(y)-a^\star||a-a^\star|}{(|a^\star|^2-1)}|y|.$$

Now,

$$c_{\mathbb{B}^n\setminus\{a\}}(f(x),f(y)) = \frac{|x-y|}{|x||y|}.\frac{|a^\star|^2-1}{|a-a^\star|^2} \le \frac{1+|a|}{1-|a|} c_{\mathbb{B}^n\setminus\{0\}}(x,y).$$

To prove the sharpness, consider the map σ defined by (5.2). For $0 < s < t < 1$, choose the points $x = -te_1$ and $y = -se_1$ in such a way that

$$c_{\mathbb{B}^n\setminus\{0\}}(x,y) = \frac{t-s}{(1-t)(1-s)} \quad \text{and} \quad c_{\mathbb{B}^n\setminus\{a\}}(\sigma(x),\sigma(y)) = \frac{|\sigma(x)-\sigma(y)|}{(1-|\sigma(x)|)(1-|\sigma(y)|)}.$$

The image points of x and y under σ is given by

$$\sigma(x) = \frac{|a|+t}{1+|a|t}e_1, \quad \sigma(y) = \frac{|a|+s}{1+|a|s}e_1 \in [a, e_1] \setminus \{a, e_1\}.$$

Now, the Cassinian distance between $\sigma(x)$ and $\sigma(y)$ is

$$\begin{aligned} c_{\mathbb{B}^n\setminus\{a\}}(\sigma(x),\sigma(y)) &= \frac{|\sigma(x)-\sigma(y)|}{(1-|\sigma(x)|)(1-|\sigma(y)|)} = \frac{\left|\dfrac{|a|+t}{1+|a|t}-\dfrac{|a|+s}{1+|a|s}\right|}{\left(1-\left|\dfrac{|a|+t}{1+|a|t}\right|\right)\left(1-\left|\dfrac{|a|+s}{1+|a|s}\right|\right)} \\ &= \frac{t-s}{(1-t)(1-s)}.\frac{1+|a|}{1-|a|} = \frac{1+|a|}{1-|a|}c_{\mathbb{B}^n\setminus\{0\}}(x,y). \end{aligned}$$

The lower bound can be seen by considering the inverse of σ and hence the conclusion follows. $\square$

Remark 5.3. It is clear from Theorems 1.9 and 1.10 that the distortion constant $(1+|a|)/(1-|a|)\to\infty$ as $|a|\to 1$. Hence, we conclude that there does not exist any finite distortion constant for the c_D-metric under Möbius transformations of $\mathbb{R}^n$. However, if we replace Möbius mappings by bi-Lipschitz mappings of $\mathbb{R}^n$, then we can guarantee that distortion constant exists. Indeed, if $f:\mathbb{R}^n\to\mathbb{R}^n$ is an L-bilipschitz mapping, that is

$$|x-y|/L \le |f(x)-f(y)| \le L|x-y| \tag{5.7}$$

for all $x,y\in\mathbb{R}^n$, which maps $D\subsetneq\mathbb{R}^n$ onto $D'\subsetneq\mathbb{R}^n$, then

$$\frac{1}{L^3}c_D(x_1,x_2)\le c_{D'}(f(x_1),f(x_2))\le L^3 c_D(x_1,x_2)$$

for all $x_1,x_2\in D$.

Similar to the distortion property of the $c_{\mathbb{B}^n}$-metric, the $\tilde{\tau}_{\mathbb{B}^n}$-metric distorts under Möbius transformations of $\mathbb{B}^n$. The distortion of $\tilde{\tau}_{\mathbb{B}^n}$ is proved by Ibragimov in [40]. He proved that

Lemma 5.4. *[40, Theorem 7.1] If ϕ is a Möbius transformation with $\phi(\mathbb{B}^n)=\mathbb{B}^n$, then for all $x,y\in\mathbb{B}^n$ we have*

$$\tilde{\tau}_{\mathbb{B}^n}(x,y)-k\le\tilde{\tau}_{\mathbb{B}^n}\big(\phi(x),\phi(y)\big)\le\tilde{\tau}_{\mathbb{B}^n}(x,y)+k,$$

where $k=\log((1+|\phi(0)|)/(1-|\phi(0)|))$.

Using the well-known Bernoulli's inequality

$$\log(1+ax)\le a\log(1+x),\quad a\ge 1, x>0 \tag{5.8}$$

in the proof of [40, Theorem 7.1], we can reformulate Lemma 5.4 in the following way.

Corollary 5.5. *If ϕ is a Möbius transformation with $\phi(\mathbb{B}^n) = \mathbb{B}^n$, then for all $x, y \in \mathbb{B}^n$ we have*

$$\frac{1-|\phi(0)|}{1+|\phi(0)|}\tilde{\tau}_{\mathbb{B}^n}(x,y) \le \tilde{\tau}_{\mathbb{B}^n}\big(\phi(x),\phi(y)\big) \le \frac{1+|\phi(0)|}{1-|\phi(0)|}\tilde{\tau}_{\mathbb{B}^n}(x,y).$$

We next prove the quasi-invariance of the $\tilde{\tau}_D$-metric under the Möbius transformation of a punctured ball onto another punctured ball stated in Theorem 1.11. First we recall the statement of Theorem 1.11.

Theorem 5.6. *(see also, Theorem* 1.11*) Let $a \in \mathbb{B}^n$ and $f : \mathbb{B}^n \setminus \{0\} \to \mathbb{B}^n \setminus \{a\}$ be a Möbius transformation with $f(0) = a$. Then for $x, y \in \mathbb{B}^n \setminus \{0\}$ we have*

$$\frac{1-|a|}{1+|a|}\tilde{\tau}_{\mathbb{B}^n\setminus\{0\}}(x,y) \le \tilde{\tau}_{\mathbb{B}^n\setminus\{a\}}(f(x),f(y)) \le \frac{1+|a|}{1-|a|}\tilde{\tau}_{\mathbb{B}^n\setminus\{0\}}(x,y).$$

Proof. The proof technique is similar to that of the proof technique of Theorem 1.10. Denote by

$$Q = \min\{\sqrt{|f(x)-a||f(y)-a|}, \inf_{p\in\partial\mathbb{B}^n}\sqrt{|f(x)-p||f(y)-p|}\}.$$

It follows from the definition that

$$\tilde{\tau}_{\mathbb{B}^n\setminus\{a\}}(f(x),f(y)) = \log\left(1+\frac{|f(x)-f(y)|}{Q}\right)$$

and

$$\tilde{\tau}_{\mathbb{B}^n\setminus\{0\}}(x,y) = \log\left(1+\frac{|x-y|}{\min\{\sqrt{|x||y|}, \inf_{z\in\partial\mathbb{B}^n}\sqrt{|x-z||y-z|}\}}\right).$$

The case $Q = \inf_{p\in\partial\mathbb{B}^n}\sqrt{|f(x)-p||f(y)-p|}$ is already treated by Ibragimov in [40].We need to consider the other choice only. That is $Q = \sqrt{|f(x)-a||f(y)-a|}$.

From (5.5), it is clear that

$$|f(x)-a| = \frac{|f(x)-a^\star||a-a^\star|}{|a^\star|^2-1}|x| \quad \text{and } |f(y)-a| = \frac{|f(y)-a^\star||a-a^\star|}{|a^\star|^2-1}|y|.$$

Now,

$$\begin{aligned}
\tilde{\tau}_{\mathbb{B}^n\setminus\{a\}}(f(x),f(y)) &= \log\left(1+\frac{|f(x)-f(y)|}{\sqrt{|f(x)-a||f(y)-a|}}\right) \le \log\left(1+\frac{1+|a|}{1-|a|}\frac{|x-y|}{\sqrt{|x||y|}}\right) \\
&\le \log\left(\frac{1+|a|}{1-|a|}\left(1+\frac{|x-y|}{\sqrt{|x||y|}}\right)\right) \le \tilde{\tau}_{\mathbb{B}^n\setminus\{0\}}(x,y) + \log\frac{1+|a|}{1-|a|}.
\end{aligned}$$

The lower bound follows from the inverse mapping. This completes the proof of our theorem. □

Similar to Corollary 5.5, using Bernoulli's inequality (5.8), we can reformulate Theorem 1.11 in the following way.

Theorem 5.7. *Let $a \in \mathbb{B}^n$ and $f : \mathbb{B}^n \setminus \{0\} \to \mathbb{B}^n \setminus \{a\}$ be a Möbius map with $f(0) = a$. Then for $x, y \in \mathbb{B}^n \setminus \{0\}$ we have*

$$\frac{1-|a|}{1+|a|}\tilde{\tau}_{\mathbb{B}^n\setminus\{0\}}(x,y) \le \tilde{\tau}_{\mathbb{B}^n\setminus\{a\}}(f(x),f(y)) \le \frac{1+|a|}{1-|a|}\tilde{\tau}_{\mathbb{B}^n\setminus\{0\}}(x,y).$$

Remark 5.8. Corollary 5.5 and Theorem 1.11 guarantee that there does not exist a finite distortion constant for the $\tilde{\tau}_D$-metric under Möbius transformations of $\mathbb{R}^n$.

5.2. Quasi-invariance properties under quasiconformal mappings

Given domains D and D' in $\mathbb{R}^n$, $n \ge 2$, let $f : D \to D'$ be a homeomorphism. For $x \in D, r \in (0, d(x))$, and $1 \le K < \infty$, let

$$L(x,r) = \max\{|f(x) - f(y)| : |x-y| = r\},$$

and

$$l(x,r) = \min\{|f(x) - f(y)| : |x-y| = r\}.$$

We say that f is in the class $\mathcal{F}(D, D'; K)$ if, for each point $x \in D$,

$$H(f,x) = \limsup_{r\to 0} \frac{L(x,r)}{l(x,r)} \le K < \infty.$$

It was shown by Heinonen and Koskela in 1995 that, surprisingly, the quantity $H(f,x)$ can be replaced by

$$\liminf_{r\to 0} \frac{L(x,r)}{l(x,r)}$$

in the definition of quasiconformality (see [32, Theorem 1.4]). It is clear that we can regard the class $\mathcal{F}(D, D'; 1)$ consisting of the conformal mappings of D onto D'. However, it seems best to use Väisälä's definition of K-quasiconformality, see [63], to keep our results compatible with the standard terminology. For more on quasiconformal theory, we refer to [1, 10, 63, 67].

As an example of quasiconformal mapping, consider the L-bilipschitz mappings as defined in (5.7). That is, $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfying

$$\frac{1}{L}|x-y| \le |f(x) - f(y)| \le L|x-y|, \quad x, y \in D, L \ge 1.$$

Then clearly, $L(x,r) \leq L$ and $l(x,r) \geq 1/L$. Hence, $H(f,x) \leq L^2$. That is, f is L^2-quasiconformal. Moreover, one can verify that f is L^2-bilipschitz with respect to the $\tilde{\tau}_D$-metric. That is,

$$\frac{1}{L^2}\tilde{\tau}_D(x,y) \leq \tilde{\tau}_{f(D)}(f(x),f(y)) \leq L^2\tilde{\tau}_D(x,y), \quad x,y \in D.$$

Indeed,

$$\begin{aligned}
\tilde{\tau}_{f(D)}(f(x),f(y)) &= \log\left(1+\sup_{f(p)\in\partial f(D)}\frac{|f(x)-f(y)|}{\sqrt{|f(x)-f(p)||f(p)-f(y)|}}\right)\\
&\leq \log\left(1+\sup_{p\in\partial D}\frac{L^2|x-y|}{\sqrt{|x-p||p-y|}}\right)\\
&\leq L^2\log\left(1+\sup_{p\in\partial D}\frac{|x-y|}{\sqrt{|x-p||p-y|}}\right) = L^2\,\tilde{\tau}_D(x,y),
\end{aligned}$$

where the first inequality follows from the bilipschitz condition on f and the second inequality follows from the well known Bernoulli's inequality (5.8). Observe that the distortion constant L^2 is independent upon the dimension of the space. Now the question arises whether we can distort the $\tilde{\tau}_D$-metric if we replace the L-bilipschitz map by any arbitrary K-quasiconformal mapping? The answer is "yes" and the distortion constant will depend upon n (the dimension of the space) and K. In this regard, Gehring and Osgood first studied the quasi-invariance property of the k_D-metric under quasiconformal mappings of $\mathbb{R}^n$. They proved that

Lemma 5.9. *[17, Theorem 3] For $n \geq 1$ and $K \geq 1$, there exists a constant c depending only on n and K with the following property: if f is a K-quasiconformal mapping of D onto D', then*

$$k_{D'}(f(x),f(y)) \leq c\max\{k_D(x,y), k_D(x,y)^\alpha\}, \quad \alpha = K^{1/(1-n)},$$

for all $x,y \in D$.

Recall that the k_D-metric does not change by more than a factor 2 under Möbius maps of $\mathbb{R}^n$ [18, Corollary 2.5]. If we take a Möbius map of the unit ball $\mathbb{B}^n$ onto itself then we get a smaller distortion constant $1+|a|$, $a \in \mathbb{B}^n$ [52, Theorem 1.4]. However, a conformal map in the complex plane $\mathbb{C}$ leads to the sharp distortion constant 4 for the k_D-metric [52, Proposition 1.6]. The above discussion guarantees that the distortion constant in Lemma 5.9 does not tend to 1 whenever $K \to 1$ (see also the discussions given in [52, p. 313]).

A similar investigation has also been made by Seittenranta in [59, Theorem 1.2] for the δ_D-metric. He proved that for a K-quasiconformal mapping f of $\overline{\mathbb{R}^n}$, the δ_D-metric satisfies the following relation:

$$\delta_{f(D)}(f(x), f(y)) \leq c(K,n) \max\{\delta_D(x,y), \delta_D(x,y)^\alpha\},$$

where $c(K,n) = \lambda_n^{\beta-1}\beta\eta_{K,n}(1)$ and $\alpha = K^{1/(1-n)} = 1/\beta$. Here $c(K,n)$ tends to 1 as K tends to 1. The constant λ_n is the Grötzsch ring constant, with $\lambda_n \in [4, 2e^{n-1})$ and $\lambda_2 = 4$ (see [1, Ch. 12]). For the function $\eta_{K,n}$ and estimates for $\eta_{K,n}$, see [59, Theorem 5.2]. The particular case, $D = \mathbb{B}^n$, is proved in [52, Corollary 2.10]. That is

Let $f : \mathbb{B}^n \to \mathbb{B}^n$ be a K-quasiconformal mapping and $x, y \in \mathbb{B}^n$. Then

$$\rho_{\mathbb{B}^n}(f(x), f(y)) \leq c(K,n) \max\{\rho_{\mathbb{B}^n}(x,y), \rho_{\mathbb{B}^n}(x,y)^\alpha\}, \quad \alpha = K^{1/(1-n)},$$

where $c(K,n) = \lambda_n^{\beta-1}\beta\eta_{K,n}(1)$ and $\alpha = K^{1/(1-n)} = 1/\beta$.

Obtaining a distortion constant (which tend to 1 as $K \to 1$) under a K-quasiconformal mapping of $\mathbb{R}^n$ for hyperbolic-type metrics (other than the δ_D-metric) is a challenging problem in geometric function theory. Hence we are interested to study such distortion properties under quasiconformal mappings for the metrics of our interest.

Now, in the sequel, we discuss the quasi-invariance property of the $\tilde{\tau}_D$-metric followed by the quasi-invariance property of the u_D-metric under quasiconformal mappings of $\mathbb{R}^n$.

Using Lemma 5.9, the following quasi-invariance property of the distance ratio metric in arbitrary domains of $\mathbb{R}^n$ is studied in [30].

Lemma 5.10. *[30, Lemma 2.3] If f is a K-quasiconformal mapping of $\mathbb{R}^n$ which maps D onto D', then there exists a constant C depending only on n and K such that*

$$\tilde{j}_{D'}(f(x), f(y)) \leq C \max\{\tilde{j}_D(x,y), \tilde{j}_D(x,y)^\alpha\}$$

for all $x, y \in D$, where $\alpha = K^{1/(1-n)}$.

Now using [57, (3.1)] and Lemma 5.10 we have a quasi-invariance property associated with the $\tilde{\tau}_D$-metric stated in Theorem 1.12.

Theorem 5.11. *(see also, Theorem 1.12) For $n \geq 1$ and $K \geq 1$, if $f : \mathbb{R}^n \to \mathbb{R}^n$ is a K-quasiconformal mapping of $\mathbb{R}^n$ which maps $D \subsetneq \mathbb{R}^n$ onto $D' \subsetneq \mathbb{R}^n$ then there exists a constant C_1 depending only on n and K such that*

$$\tilde{\tau}_{D'}(f(x), f(y)) \leq C_1 \max\{\tilde{\tau}_D(x, y), \tilde{\tau}_D(x, y)^\alpha\}$$

for all $x, y \in D$, where $\alpha = K^{1/(1-n)}$.

Proof. For all $x, y \in D$, we have

$$\begin{aligned}\tilde{\tau}_{D'}(f(x), f(y)) \leq \tilde{j}_D(f(x), f(y)) &\leq C \max\{\tilde{j}_D(x, y), \tilde{j}_D(x, y)^\alpha\} \\ &\leq C \max\{2\tilde{\tau}_D(x, y), 2^\alpha \tilde{\tau}_D(x, y)^\alpha\} \\ &\leq C_1 \max\{\tilde{\tau}_D(x, y), \tilde{\tau}_D(x, y)^\alpha\},\end{aligned}$$

where the first and third inequality follows from [57, (3.1)], the second inequality follows from Lemma 5.10 and the constant C_1 is depending upon n and K. □

Using the bilipschitz property of the $\tilde{\tau}_D$-metric and the u_D-metric (see [57, Theorem 3.5]) and Theorem 1.12, one can obtain the quasi-invariance property of the u_D-metric in arbitrary subdomains of $\mathbb{R}^n$ as well.

Theorem 5.12. *(see also, Theorem 1.13) For $n \geq 1$ and $K \geq 1$, if $f : \mathbb{R}^n \to \mathbb{R}^n$ is a K-quasiconformal mapping of $\mathbb{R}^n$ which maps $D \subsetneq \mathbb{R}^n$ onto $D' \subsetneq \mathbb{R}^n$ then there exists a constant C_2 depending only on n and K such that*

$$u_{D'}(f(x), f(y)) \leq C_2 \max\{u_D(x, y), u_D(x, y)^\alpha\}$$

for all $x, y \in D$, where $\alpha = K^{1/(1-n)}$.

Proof. It follows from Theorem 3.4 and Theorem 1.12. □

5.3. Modulus of continuity

The definition of the modulus of continuity is described in Chapter 1. Here we will mainly be motivated by geometric function theory and therefore give a few related examples. The well-known relation $|x-y| \leq 2\tanh(\rho_{\mathbb{B}^n}(x, y)/4)$ says that $\omega(t) = 2\tanh(t/4)$ is the modulus of continuity for the identity map $id : (\mathbb{B}^n, \rho_{\mathbb{B}^n}) \to (\mathbb{B}^n, |.|)$. If $X_1 = \mathbb{B}^n = X_2$ and $f : \mathbb{B}^n \to \mathbb{B}^n$ is K-quasiconformal, then the quasiconformal counterpart of the

Schwarz lemma says that $f : (\mathbb{B}^n, \rho_{\mathbb{B}^n}) \to (\mathbb{B}^n, \rho_{\mathbb{B}^n})$ is uniformly continuous where $\rho_{\mathbb{B}^n}$ is the hyperbolic metric of $\mathbb{B}^n$. If $X_1 = \mathbb{B}^2$, $X_2 = \mathbb{R}^2 \setminus \{0,1\}$, the Schottky theorem gives, in an explicit form, a growth estimate for $|f(z)|$ in terms of $|z|$ when $f : \mathbb{B}^2 \to \mathbb{R}^2 \setminus \{0,1\}$ is an analytic function [31, p. 685, 702]. In fact, Nevanlinna's principle of the hyperbolic metric [31, p. 683] yields an estimate for the modulus of continuity of $f : (\mathbb{B}^2, \rho_{\mathbb{B}^2}) \to (X_2, d_2)$ where d_2 is the hyperbolic metric of the twice-punctured plane X_2. If q is the *chordal metric* and $f : (\mathbb{B}^2, \rho_{\mathbb{B}^2}) \to (\overline{\mathbb{R}^2}, q)$ is a meromorphic function, then f is normal (in the sense of Lehto and Virtanen [55]) if and only if it is uniformly continuous. In the context of quasiregular mappings, uniform continuity has been discussed in [66,68]. Uniform continuity of the $\tilde{j}$-metric and the quasihyperbolic metric in the unit ball has been discussed in [47].

In this section, we consider a problem to find a bound, as sharp as possible, for the modulus of continuity of the identity mapping

$$id : (\mathbb{B}^n, c_{\mathbb{B}^n}) \to (\mathbb{B}^n, |\cdot|). \tag{5.9}$$

In this setting the well-known Jung's theorem is useful.

Lemma 5.13. *[7, Theorem 11.5.8] Let $D \subset \mathbb{R}^n$ be a domain with $\operatorname{diam} D < \infty$. Then there exists $z \in \mathbb{R}^n$ such that $D \subset B^n(z,r)$, where $r \le \sqrt{n/(2n+2)}\operatorname{diam} D$.*

Theorem 5.14. 1. *If x, y are on a diameter of $\mathbb{B}^n$ and $w = |x-y|\, e_1/2$, then we have*

$$c_{\mathbb{B}^n}(x,y) \ge c_{\mathbb{B}^n}(-w,w) = \frac{4|x-y|}{4-|x-y|^2} \ge |x-y|.$$

The first inequality becomes equality when $y = -x$.

2. *If $x, y \in \mathbb{B}^n$ are arbitrary and $w = |x-y|\, e_1/2$, then*

$$c_{\mathbb{B}^n}(x,y) \ge c_{\mathbb{B}^n}(-w,w) = \frac{4|x-y|}{4-|x-y|^2} \ge |x-y|.$$

where the first inequality becomes equality when $y = -x$.

Proof. Let $x, y \in \mathbb{B}^n$ with $|x| \le |y|$.

1. If x and y are lying on a diameter of $\mathbb{B}^n$. Without loss of generality, since the Cassinian metric is invariant under rotations, we assume that x and y are lying on the interval $(-1,1)$. Then it follows from the definition that

$$c_{\mathbb{B}^n}(x,y) = \frac{|x-y|}{(1+|x|)(1-|y|)}$$

and hence

$$c_{\mathbb{B}^n}(-w,w)=\frac{2|w|}{1-|w|^2}=\frac{4|x-y|}{4-|x-y|^2}.$$

Geometrically, it can easily be seen that the maximal Cassinian oval with foci at x and y will lie inside the maximal Cassinian oval with foci at $-w$ and w lying on the same diameter. Analytically, we say

$$\inf_{p\in\partial\mathbb{B}^n}|x-p||p-y|\le\inf_{p\in\partial\mathbb{B}^n}|w+p||p-w|.$$

i.e.

$$(1+|x|)(1-|y|)\le 1-\frac{|x-y|^2}{4}.$$

Hence, $c_{\mathbb{B}^n}(x,y)\ge c_{\mathbb{B}^n}(-w,w)$. Since $4-|x-y|^2\ge 4$ for all x and y, the second inequality follows.

2. Let $x,y\in\mathbb{B}^n$ be arbitrary. Choose $x',y'\in\mathbb{B}^n$ such that $|x-y|=|x'-y'|$ with $y'=-x'$. Then the maximal Cassinian oval $C(x,y)$ with foci at x and y is not larger than the maximal Cassinian oval foci at x' and y' (see Figure 5.1).

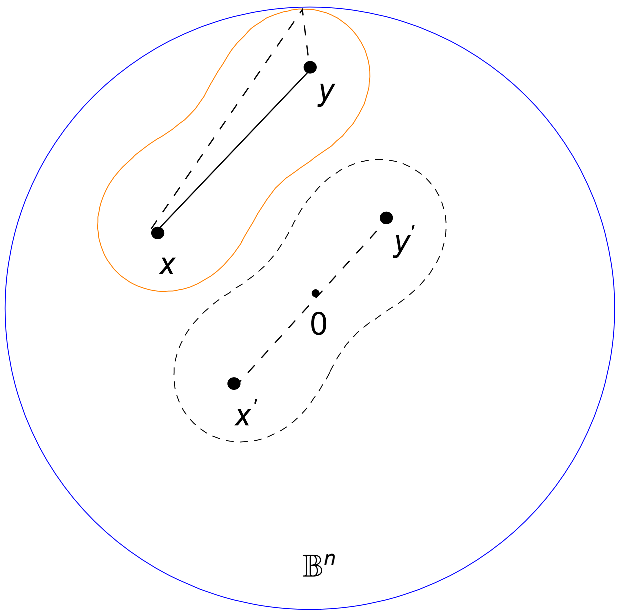

FIGURE 5.1. The maximal Cassinian oval with foci at x and y is not larger than the maximal Cassinian oval with foci at x' and y'.

Analytically,

$$\inf_{p\in\partial\mathbb{B}^n}|x-p||p-y|\le 1-\frac{|x-y|^2}{4}$$

Hence, $c_{\mathbb{B}^n}(x,y) \geq c_{\mathbb{B}^n}(x',y') = \dfrac{4|x-y|}{4-|x-y|^2}$. The second inequality follows from (1).

The proof is complete. □

Now, we obtain the modulus of continuity of the *id* map (5.9) when D is a bounded proper subdomain of $\mathbb{R}^n$.

Theorem 5.15. *(see also, Theorem* 1.14*) Let* $D \subsetneq \mathbb{R}^n$ *be a domain with* $\operatorname{diam} D < \infty$ *and* $r = \sqrt{n/(2n+2)}\operatorname{diam} D$*. Then we have*

$$c_D(x,y) \geq \frac{4|x-y|}{4-|x-y|^2} \geq \frac{|x-y|}{r}$$

for all distinct $x,y \in D$ *with equality in the first step when* $D = B^n(z,r)$ *and* $z = (x+y)/2$*. In particular, the modulus of continuity for id* : $(D, c_D) \to (D, |.|)$ *is* $\omega(t) = rt$*.*

Proof. Given that $D \subsetneq \mathbb{R}^n$ is arbitrary domain with $\operatorname{diam} D < \infty$. By Lemma 5.13, there exists $z \in \mathbb{R}^n$ with $D \subset B(z,r)$, where $r = \sqrt{n/(2n+1)}\operatorname{diam} D$. Then by the monotone property of the c_D-metric we have

$$c_D(x,y) \geq c_{B(z,r)}(x,y).$$

Without loss of generality assume that $z = 0$. Choose $u,v \in B(0,r)$ in such a way that $|u-v| = 2|u| = |x-y|$. Then by Theorem 5.14 (2), we have

$$c_D(x,y) \geq c_{B(z,r)}(x,y) \geq c_B(-u,u) = \frac{4|x-y|}{4r-|x-y|^2} \geq \frac{|x-y|}{r}.$$

This completes the proof of our theorem. □

Next we obtain the modulus of continuity for the identity map

$$id : (\mathbb{B}^n, \tilde{\tau}_{\mathbb{B}^n}) \to (\mathbb{B}^n, |.|).$$

First we obtain the modulus of continuity in the unit ball $\mathbb{B}^n$.

Theorem 5.16. 1. *If* x,y *are on a diameter of* $\mathbb{B}^n$ *and* $w = |x-y|\, e_1/2$*, then we have*

$$\tilde{\tau}_{\mathbb{B}^n}(x,y) \geq \tilde{\tau}_{\mathbb{B}^n}(-w,w) = \log\left(1 + \frac{2|x-y|}{\sqrt{4-|x-y|^2}}\right) \geq c\,|x-y|,$$

where $c\,(\approx 0.76)$ *is the solution of the equation*

$$(4-t^2)(2t+\sqrt{4-t^2})\log\left(1+\frac{2t}{\sqrt{4-t^2}}\right) - 8t = 0.$$

The first inequality becomes equality when $y = -x$.

2. *If $x, y \in \mathbb{B}^n$ are arbitrary and $w = |x-y|\, e_1/2$, then*

$$\tilde{\tau}_{\mathbb{B}^n}(x,y) \geq \tilde{\tau}_{\mathbb{B}^n}(-w,w) = \log\left(1 + \frac{2|x-y|}{\sqrt{4-|x-y|^2}}\right) \geq c\,|x-y|\,,$$

where c is the same number as stated in the first part. The first inequality becomes equality when $y = -x$.

Remark 5.17. Mathematica experiment suggests that the value of $c \approx 0.763286$ attained at $t = 1.16032$.

Proof. Let $x, y \in \mathbb{B}^n$ with $|x| \leq |y|$.

1. If x and y are lying on a diameter of $\mathbb{B}^n$, then it follows from the definition that

$$\tilde{\tau}_{\mathbb{B}^n}(x,y) = \log\left(1 + \frac{|x-y|}{\sqrt{(1+|x|)(1-|y|)}}\right)$$

and hence

$$\tilde{\tau}_{\mathbb{B}^n}(-w,w) = \log\left(1 + \frac{2|w|}{\sqrt{1-|w|^2}}\right) = \log\left(1 + \frac{2|x-y|}{\sqrt{4-|x-y|^2}}\right).$$

Geometrically, it can easily be seen that the maximal Cassinian oval with foci at x and y will lie inside the maximal Cassinian oval with foci at $-w$ and w. Analytically, we say

$$\inf_{p\in\partial\mathbb{B}^n} \sqrt{|x-p||p-y|} \leq \inf_{p\in\partial\mathbb{B}^n} \sqrt{|w+p||p-w|}.$$

i.e.

$$\sqrt{(1+|x|)(1-|y|)} \leq \sqrt{1 - \frac{|x-y|^2}{4}}.$$

Hence, $\tilde{\tau}_{\mathbb{B}^n}(x,y) \geq \tilde{\tau}_{\mathbb{B}^n}(-w,w)$. For the second inequality, we need to find the minimum of the function

$$\frac{1}{t}\log\left(1 + \frac{2t}{\sqrt{4-t^2}}\right) \quad (t = |x-y|).$$

By the derivative test, it can be seen that the minimum attains at the point $t \approx 1.16$ and the minimum value is approximately 0.76.

2. The proof of this case is similar to the proof of Theorem 5.14(2).

The proof is complete. $\square$

Now, we obtain the modulus of continuity of the id map (5.9) when D is a bounded proper subdomain of $\mathbb{R}^n$.

Theorem 5.18. *(see also, Theorem 1.15) Let $D \subsetneq \mathbb{R}^n$ be a domain with* $\operatorname{diam} D < \infty$ *and $r = \sqrt{n/(2n+2)}\operatorname{diam} D$. Then we have*

$$\tilde{\tau}_D(x,y) \geq \log\left(1 + \frac{2|x-y|}{\sqrt{4r^2 - |x-y|^2}}\right) \geq c\,\frac{|x-y|}{r}$$

for all distinct $x, y \in D$ with equality in the first step when $D = B^n(z,r)$ and $z = (x+y)/2$. Here $c\,(\approx 0.76)$ is the solution of the equation

$$(4-t^2)(2t+\sqrt{4-t^2})\log\left(1+\frac{2t}{\sqrt{4-t^2}}\right) - 8t = 0.$$

In particular, the modulus of continuity for $id : (D, \tilde{\tau}_D) \to (D, |.|)$ is $\omega(t) = rt/c$.

Proof. Given that $D \subsetneq \mathbb{R}^n$ is arbitrary domain with $\operatorname{diam} D < \infty$. By Lemma 5.13, there exists $z \in \mathbb{R}^n$ with $D \subset B(z,r)$, where $r = \sqrt{n/(2n+1)}\operatorname{diam} D$. Then by the monotone property of $\tilde{\tau}_D$ we have

$$\tilde{\tau}_D(x,y) \geq \tilde{\tau}_{B(z,r)}(x,y).$$

Without loss of generality assume that $z = 0$. Choose $u, v \in B(0,r)$ in such a way that $|u-v| = 2|u| = |x-y|$. Then by Theorem 5.16 (2), we have

$$\tilde{\tau}_D(x,y) \geq \tilde{\tau}_{B(z,r)}(x,y) \geq \tilde{\tau}_B(-u,u) = \log\left(1 + \frac{2|x-y|}{\sqrt{4r^2-|x-y|^2}}\right).$$

This completes the proof of our theorem. $\square$

Now, we shall find the modulus of continuity for the following map:

$$id : (\mathbb{B}^n, u_{\mathbb{B}^n}) \to (\mathbb{B}^n, |\cdot|).$$

Theorem 5.19. 1. *If x, y are on a diameter of $\mathbb{B}^n$ and $w = |x-y|\,e_1/2$, then we have*

$$u_{\mathbb{B}^n}(x,y) \geq u_{\mathbb{B}^n}(-w,w) = 2\log\left(\frac{2+|x-y|}{2-|x-y|}\right) \geq |x-y|.$$

where the first inequality becomes equality when $y = -x$.

2. *If $x, y \in \mathbb{B}^n$ are arbitrary and $w = |x-y|\,e_1/2$, then*

$$u_{\mathbb{B}^n}(x,y) \geq u_{\mathbb{B}^n}(-w,w) = 2\log\left(\frac{2+|x-y|}{2-|x-y|}\right) \geq |x-y|.$$

where the first inequality becomes equality when $y = -x$.

Proof. Let $x, y \in \mathbb{B}^n$ with $|x| \le |y|$.

1. If $0 \in [x, y]$, then by definition

$$u_{\mathbb{B}^n}(x, y) = 2\log\left(\frac{|x-y|+1-|x|}{\sqrt{(1-|x|)(1-|y|)}}\right) \quad \text{and} \quad u_{\mathbb{B}^n}(-w, w) = 2\log\left(\frac{2+|x-y|}{2-|x-y|}\right).$$

By AM-GM inequality we have

$$\frac{1}{\sqrt{(1-|x|)(1-|y|)}} \ge \frac{2}{2-|x|-|y|}. \tag{5.10}$$

To prove our claim, it is enough to show

$$\frac{|x-y|+1-|x|}{\sqrt{(1-|x|)(1-|y|)}} \ge \frac{2+|x-y|}{2-|x-y|}.$$

Since $|x-y| = |x|+|y|$ and $|x| \le |y|$, we have

$$\frac{|x-y|+1-|x|}{\sqrt{(1-|x|)(1-|y|)}} \ge \frac{2(1+|y|)}{2-|x|-|y|} \ge \frac{2+|x|+|y|}{2-|x|-|y|} = \frac{2+|x-y|}{2-|x-y|},$$

where the first inequality follows from (5.10).

If $x \in [0, y]$, then

$$u_{\mathbb{B}^n}(x, y) = 2\log\left(\frac{1+|y|-2|x|}{\sqrt{(1-|x|)(1-|y|)}}\right) \quad \text{and} \quad u_{\mathbb{B}^n}(-w, w) = 2\log\left(\frac{2+|y|-|x|}{2-|y|+|x|}\right).$$

To show $u_{\mathbb{B}^n}(x, y) \ge u_{\mathbb{B}^n}(-w, w)$, it is enough to show

$$\frac{1+|y|-2|x|}{\sqrt{(1-|x|)(1-|y|)}} \ge \frac{2+|y|-|x|}{2-|y|+|x|}.$$

From (5.10), we have

$$\frac{1+|y|-2|x|}{\sqrt{(1-|x|)(1-|y|)}} \ge \frac{2(1+|y|-2|x|)}{2-|x|-|y|}.$$

Now our aim is to show

$$\frac{2(1+|y|-2|x|)}{2-|x|-|y|} \ge \frac{2+|y|-|x|}{2-|y|+|x|}$$

or, equivalently,

$$2|y| - 2|x| - |y|^2 + 6|x||y| - 5|x|^2 \ge 0.$$

Since, $|x| \le |y|$, we have

$$2|y| - 2|x| - |y|^2 + 6|x||y| - 5|x|^2 \ge 2|y| - 2|x| - |y|^2 + |x|^2 = (1-|x|)^2 - (1-|y|)^2 \ge 0.$$

Again, since the function

$$f(t) = 2\log\left(\frac{2+t}{2-t}\right) - t$$

is increasing in t, the conclusion follows.

2. Given that $x, y \in \mathbb{B}^n$ are arbitrary. Choose $y' \in \mathbb{B}^n$ such that $|x-y| = |x-y'|$ and $x, 0,$ and y' are co-linear. By geometry, it is clear that $|y'| \leq |y|$. Hence,

$$u_{\mathbb{B}^n}(x,y) \geq u_{\mathbb{B}^n}(x,y') \geq u_{\mathbb{B}^n}(-w,w).$$

The proof is complete. $\square$

CHAPTER 6

SUMMARY AND FUTURE DIRECTIONS

Apart from the first chapter, which gives an introduction to the research area and available literatures including preliminaries for the upcoming chapters, the second and the third chapters are dedicated to the comparison results among the c_D-metric, the $\tilde{\tau}_D$-metric and the u_D-metric with other well-known hyperbolic-type metrics. These comparisons lead to several inclusion results which are recorded in the forth chapter. The forth chapter also consists of the local starlikeness and convexity properties of the Cassinian metric balls.

In the fifth chapter, the quasi-invariance properties of the c_D-metric under Möbius transformations of the unit ball as well as under the Möbius transformations of a punctured ball onto another punctured ball are discussed. We also discuss the quasi-invariance property of the $\tilde{\tau}_D$-metric under Möbius transformations of a punctured ball onto another punctured ball. In addition, we study the quasi-invariance of the $\tilde{\tau}_D$-metric and the u_D-metric under quasiconformal mappings of $\mathbb{R}^n$. At last we focus on the modulus of continuity for the identity mappings on bounded domains equipped with the Cassinian metric onto the same domain with the Euclidean metric.

Note that several nice domains (eg. uniform domains, John domains, quasidisks,) are characterized in terms of metric inequalities. We do expect that some of the inequalities discussed in chapters two and three can be used to characterize certain domains. Also, if a metric is of interest then so its isometries and geodesics. It would be interesting to study the geodesics and the isometries of the metrics under consideration. We expect that some of the investigations would lead to new results in different areas of research in function theory.

BIBLIOGRAPHY

[1] Anderson G. D., Vamanamurthy M. K., Vuorinen M. (1997), *Conformal Invariants, Inequalities, and Quasiconformal Maps*, John Wiley & Sons, Inc., New York.

[2] Ahlfors L. V. (2010), *Conformal invariants: Topics in Geometric Function Theory*, AMS Chelsea Publishing, Providence, RI.

[3] Barbilian D. (1934-35), *Einordnung von Lobayschewskys Massenbestimmung in einer gewissen allgemeinen Metrik der Jordansche Bereiche*, Casopsis Mathematiky a Fysiky, **64**, 182–183.

[4] Beardon A. F. (1995), *Geometry of discrete groups*, Springer-Verlag, New York.

[5] Beardon A. F. (1998), *The Apollonian metric of a domain in* $\mathbb{R}^n$, In: Duren P., Heinonen J., Osgood B., and Palka B. (Eds.), *Quasiconformal mappings and analysis* (Ann Arbor, MI, 1995), pp. 91–108. Springer-Verlag, New York.

[6] Beardon A. F., Minda D. (2007), *The hyperbolic metric and geometric function theory*, In: Ponnusamy S., Sugawa T., Vuorinen M. (Eds.), Quasiconformal Mappings and their Applications, Narosa Publishing House, New Delhi, India, pp. 9–56.

[7] Berger M. (1987), *Geometry I,* Springer-Verlag, Berlin.

[8] Bonk M., Schramm O. (2000), *Embeddings in Gromov hyperbolic spaces*, Geom. Funct. Anal., **10** (2), 266–306.

[9] Borovikova M., Ibragimov Z. (2008), *Convex bodies of constant width and the Apollonian metric*, Bull. Malays. Math. Sci. Soc., **31** (2), 117–128.

[10] Caraman P. (1974), *n-dimensional quasiconformal (QCf) mappings*, Revised, enlarged and translated from the Romanian by the author. Editura Academiei Romane, Bucharest; Abacus Press, Tunbridge Wells; Haessner Publishing, Inc., Newfoundland, N.J., 553 pp.

[11] Chen J., Hariri P., Klén R., Vuorinen M. (2015), *Lipschitz conditions, Triangular ratio metric and Quasiconformal maps*, Ann. Acad. Sci. Fenn. Math., **40**, 683–709.

[12] Deza M. M., Deza E. (2014), *Encyclopedia of distances*, Third Edition, Springer-Verlag, Berlin Heidelberg.

[13] Ferrand J. (1988), *A characterization of quasiconformal mappings by the behavior of a function of three points*, In: Laine I., Rickman S., Sorvali T. (Eds.) *Proceedings of the 13th Rolf Nevalinna Colloquium* (Joensuu, 1987), pp. 110–123. Lecture Notes in Mathematics Vol. 1351, Springer-Verlag, New York.

[14] Fréchet M. (1906), *Sur quelques points du calcul fonctionnel*, Thèse, Paris, Rendiconti Circolo Mat. Palermo, **22**, 1–72.

[15] Gehring F. W. (1999), *Characterizations of quasidisks*, Quasiconformal geometry and dynamics (Lublin, 1996), 11–41, Banach center publ., **48**, Polish Acad. Sci., Warsaw.

[16] Gehring F. W., Hag K. (2000), *The Apollonian metric and quasiconformal mappings*. pp. 143–163. *In:* Kra, I. and Maskit, B. (eds.) *In the tradition of Ahlfors and Bers* (Stony Brook, NY, 1998), *Contemp. Math.* **256**, Amer. Math. Soc., Providence, RI.

[17] Gehring F. W., Osgood B. G. (1979), *Uniform domains and the quasihyperbolic metric*, J. Analyse Math., **36**, 50–74.

[18] Gehring F. W., Palka B. P. (1976), *Quasiconformally homogeneous domains*, J. Analyse Math., **30**, 172–199.

[19] Gromov M. (1987), *Hyperbolic groups*, Essays in Group Theory, 75–263, Math. Sci. Res. Inst. Publ., **8**, Springer, New York.

[20] Hariri P., Klén R., Vuorinen M., Zhang Z. (2017), *Some remarks on the Cassinian metric*, Publ. Math. Debrecen, **90** (3-4), 269–285.

[21] Hästö P. (2002), *A new weighted metric: the relative metric. I*, J. Math. Anal. Appl., **274** (1), 38–58.

[22] Hästö P. (2003), *The Apollonian metric: uniformity and quasiconvexity,* Ann. Acad. Sci. Fenn. Math., **28** (2), 385–414.

[23] Hästö P. (2007), *Isometries of the quasihyperbolic metric*, Pacific J. Math., **230** (2), 315–326.

[24] Hästö P., Ibragimov Z. (2005), *Apollonian isometries of planar domains are Möbius mappings*, J. Geom. Anal., **15** (2), 229–237.

[25] Hästö P. (2006), *Gromov hyperbolicity of the j_G and $\tilde{j}_G$ metrics*, Proc. Amer. Math. Soc., **134** (4), 1137–1142.

[26] Hästö P., Ibragimov Z. (2007), *Apollonian isometries of regular domains are Möbius mappings*, Ann. Acad. Sci. Fenn., Ser. A I Math., **32** (1), 83–98.

[27] Hästö P., Ibragimov Z., Linden H. (2006), *Isometries of relative metrics*, Comput. Methods Funct. Theory, **6** (1), 15–28.

[28] Hästö P., Linden H. (2004), *Isometries of the half-apollonian metric*, Complex Var. Theory Appl., **49**, 405–415.

[29] Hästö P., Ibragimov Z., Minda D., Ponnusamy S., Sahoo S. K., (2007) *Isometries of some hyperbolic-type path metrics, and the hyperbolic medial axis*, In the tradition of Ahlfors-Bers, IV, Contemporary Math., **432**, 63–74.

[30] Hästö P., Klén R., Sahoo S. K., Vuorinen M., *Geometric properties of φ-uniform domains*, arXiv:1510.05847.

[31] Hayman W. K. (1989), *Subharmonic functions. Vol. 2*, London Mathematical Society Monographs, 20, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London, pp. i–xxvi and 285–875.

[32] Heinonen J., Koskela P. (1995), *Definitions of quasiconformality,* Invent. Math., **120**, 61–79.

[33] Herron D., Ibragimov Z., Minda D. (2008), *Geodesics and curvature of Möbius invariant metrics*, Rocky Mountain J. Math., **38** (3), 891–921.

[34] Hokuni S., Klén R., Li Y., Vuorinen M. (2016), *Balls in the triangular ratio metric*, In: Complex Analysis and Dynamical Systems VI. Part 2, pp. 105–123, Contemp. Math. 667, Amer. Math. Soc., Providence, RI.

[35] Ibragimov Z. (2003), *On the Apollonian metric of domains in $\overline{\mathbb{R}^n}$*, Complex Var. Theory Appl., **48** (10), 837–855.

[36] Ibragimov Z. (2003), *Conformality of the Apollonian metric*, Comput. Methods Funct. Theory, **3** (1-2), 397–411.

[37] Ibragimov Z. (2009), *The Cassinian metric of a domain in $\bar{\mathbb{R}}^n$*, Uzbek. Mat. Zh., **1**, 53–67.

[38] Ibragimov Z. (2011), *Hyperbolizing metric spaces,* Proc. Amer. Math. Soc., **139** (12), 4401–4407.

[39] Ibragimov Z. (2011), *Hyperbolizing hyperspaces,* Michigan Math. J., **60**, 215–239.

[40] Ibragimov Z. (2016), *A scale-invariant Cassinian metric*, J. Anal., **24** (1), 111–129.

[41] Ibragimov Z., Sahoo S. K., *Invariant Cassinian metrics*, Preprint.

[42] Ibragimov Z., Mohapatra M. R., Sahoo S. K., Zhang X. (2017), *Geometry of the Cassinian metric and its inner metric*, Bull. Malays. Math. Sci. Soc., **40** (1), 361–372.

[43] Klén R. (2008), *Local convexity properties of j-metric balls*, Ann. Acad. Sci. Fenn. Math., **33** (1), 281–293.

[44] Klén R. (2008), *Local convexity properties of quasihyperbolic balls in punctured space*, J. Math. Anal. Appl., **342** (1), 192–201.

[45] Klén R. (2010), *Close-to-convexity of quasihyperbolic and j-metric balls*, Ann. Acad. Sci. Fenn. Math., **35** (2), 493–501.

[46] Klén R. (2013), *Local convexity properties of balls in Apollonian and Seittenranta's metrics*, Conform. Geom. Dyn., **17**, 133–144.

[47] Klén R., Li L., Vuorinen M. (2013), *Subdomain geometry of hyperbolic type metrics*, Transactions of the Institute of Mathematics of the National Academy of Sciences of Ukraine, **10** (4-5), 190–206.

[48] Klén R., Linden H., Vuorinen M., Wang G. (2014), *The visual angle metric and Möbius transformations*, Comput. Methods Funct. Theory, **14** (2-3), 577–608.

[49] Klén R., Mohapatra M. R., Sahoo S. K. (2016), *Geometric properties of the Cassinian metric*, Math. Nachr., `DOI: 10.1002/mana.201600117`.

[50] Klén R., Vuorinen M. (2012), *Inclusion relations of hyperbolic type metric balls*, Publ. Math. Debrecen, **81** (3-4), 289–311.

[51] Klén R., Vuorinen M. (2013), *Inclusion relations of hyperbolic type metric balls II*, Publ. Math. Debrecen, **83** (1-2), 21–42.

[52] Klén R., Vuorinen M., Zhang X.-H. (2014), *Quasihyperbolic metric and Möbius transformations*, Proc. Amer. Math. Soc., **142** (1), 311–322.

[53] Kim, K., Langmeyer, N. (1998), *Harmonic measure and hyperbolic distance in John disks,* Math. Scand., **83**, 283–299.

[54] Kulkarni R., Pinkall U. (1994), *A canonical metric for Möbius structures and its applications,* Math. Z., **216**, 89–129.

[55] Lehto O., Virtanen K. I. (1973), *Quasiconformal mappings in the plane,* Springer-Verlag, New York.

[56] Menger, K. (1928), *Untersuchungen über allgemeine Metrik*, Math. Ann., **100**, 75–163.

[57] Mohapatra M. R., Sahoo S. K., *A Gromov hyperbolic metric vs the hyperbolic and other related metrics*, Preprint.

[58] Pick G. (1915), *Über eine Eigenschaft der konformen Abbildung kreisförmiger Bereiche*, Math. Ann., **77** (1), 1–6.

[59] Seittenranta P. (1999), *Möbius-invariant metrics*, Math. Proc. Cambridge Philos. Soc., **125**, 511–533.

[60] Simić S., Vuorinen M. (2015), *Lipschitz conditions and the distance ratio metric*, Filomat, **29** (9), 2137–2146.

[61] Simić S., Vuorinen M., Wang G. (2015), *Sharp Lipschitz constants for the distance ratio metric*, Math. Scand., **115**, 86–103.

[62] Tasković M. R. (2005), *Frechét's metric spaces-100th next*, Math. Morav., **9**, 69–75.

[63] Väisälä J. (1971), *Lectures on n-dimensional Quasiconformal Mappings*, Lecture Notes in Math., Vol. 229, Springer, Berlin, p. 152.

[64] Väisälä J. (1999), *The free quasiworld. Freely quasiconformal and related maps in Banach spaces*, In: Bojarski B., Lawrynowicz J., Martio O., Vuorinen M., Zajhac J. (Eds.), Quasiconformal Geometry and Dynamics (Lublin, 1996). Banach Center Publ., **48**, Polish Acad. Sci., Warsaw, 1999, 55–118.

[65] Väisälä J. (2005), *Gromov hyperbolic spaces*, Expo. Math., **23**, 187–231.

[66] Vuorinen M. (1985), *Conformal invariants and quasiregular mappings*, J. Anal. Math., **45**, 69–115.

[67] Vuorinen M. (1988), *Conformal geometry and quasiregular mappings*, Lecture note in mathematics, Vol. 1319, Springer-Verlag, Berlin.

[68] Vuorinen M. (2007), *Metrics and quasiregular mappings*, In: Ponnusamy S., Sugawa T., Vuorinen M. (Eds.), Quasiconformal Mappings and their Applications, Narosa Publishing House, New Delhi, India, pp. 291–325.

INDEX

p_D, 18

absolute ratio (Seittenranta's) metric, 7
apollonian metric, 31
arc-length parametrization, 22

Bernoulli's inequality, 55
bilipschitz mapping, 54

Cassinian isometry, 4
Cassinian metric, 4
convex, 10

distance ratio metric, 6
domain monotone property, 4

Gromov hyperbolic, 5
Gromov hyperbolic metric, 5, 23
Gromov product, 5

half-apollonian metric, 8
hyperbolic metric, 6

identity map, 13
inclusion relations, 2, 35, 39
inner Cassinian metric, 20
inversion, 52

linear dilatation, 57

Möbius transformation, 11, 51
metric, 1
metric balls, 2, 35
metric space, 1
modulus of continuity, 3, 59

quasi-invariance (distortion) property, 11, 51
quasi-isometry, 5
quasiconformal mapping, 12, 57
quasihyperbolic metric, 7

scale-invariant Cassinian metric, 4
scaling map, 4
Schwarz-Pick lemma, 5
similarity mapping, 6
starlike, 10
strictly convex, 10
strictly starlike, 10

triangular ratio metric, 8

visual angle metric, 8