ANALYTIC AND GEOMETRIC PROPERTIES OF CERTAIN CLASSES OF UNIVALENT AND p-VALENT FUNCTIONS

Ph.D. Thesis

By
Navneet Lal Sharma

DISCIPLINE OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY INDORE

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A THESIS

Submitted in partial fulfillment of the
requirements for the award of the degree

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by
NAVNEET LAL SHARMA

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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **ANALYTIC AND GEOMETRIC PROPERTIES OF CERTAIN CLASSES OF UNIVALENT AND p-VALENT FUNCTIONS** in the partial fulfillment of the requirements for the award of the degree of **DOCTOR OF PHILOSOPHY** and submitted in the **DISCIPLINE OF MATHEMATICS, Indian Institute of Technology Indore**, is an authentic record of my own work carried out during the time period from January 2011 to December 2015 under the supervision of Dr. Swadesh Kumar Sahoo, Assistant Professor, IIT Indore.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other institute.

Signature of the student with date
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This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

Signature of Thesis Supervisor with date
(**DR. SWADESH KUMAR SAHOO**)

NAVNEET LAL SHARMA has successfully given his Ph.D. Oral Examination held on

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Convener, DPGC with date

Signature of PSPC Member with date

Signature of PSPC Member with date

Signature of External Examiner with date

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DEDICATION

Dedicated to
my grand parents
Smt. Rona Devi and Sh. Govind Sharma
&
parents
Smt. Asha Devi and Sh. Gopal Sharma

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ABSTRACT

KEYWORDS: Analytic, close-to-convex, convex, spiral-like, starlike, univalent, q-close-to-convex, q-starlike, p-valent, p-valent starlike, p-valent convex, p-valent spiral-like and hypergeometric functions; area integral; coefficient inequality; convolution; Dirichlet-finite; duality technique; integral means; integral transform; subordination and q-difference operator.

This thesis deals with univalent functions as well as p-valent (or multivalent) functions defined on the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{A}$ denote the family of all normalized analytic functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in $\mathbb{D}$ and $\mathcal{S}$ denote the class of all univalent functions $f \in \mathcal{A}$.

A generalization of close-to-convex functions by means of a q-analog of the difference operator acting on analytic functions is called the q-close-to-convex functions in $\mathbb{D}$. The class of q-close-to-convex functions is denoted by $\mathcal{K}_q$. We determine the several sufficient conditions for $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ to be in $\mathcal{K}_q$, where the coefficient a_n are real, non-negative and connected with certain monotone properties. In addition, we prove the Bieberbach-de Branges Theorem for functions in the class $\mathcal{K}_q$. One of the classical problems concerns the class of analytic functions f on $\mathbb{D}$ which have finite Dirichlet integral $\Delta(1, f)$, where

$$\Delta(r, f) = \iint_{|z|<r} |f'(z)|^2 \, dxdy \quad (0 < r \le 1).$$

Computing $\Delta(r, f)$ is known as the *area problem for the function of type f*. The class $\mathcal{S}^*(A, B)$ of functions $f \in \mathcal{A}$ and satisfies the subordination condition $zf'(z)/f(z) \prec (1 + Az)/(1 + Bz)$ in $\mathbb{D}$ and for some $-1 \le B \le 0$, $A \in \mathbb{C}$ with $A \neq B$, has been studied extensively. We are mainly interested to discuss the extremal problem of determining

the value of $\max_{f\in\mathcal{S}^*(A,B)} \Delta(r, z/f)$ as a function of r. This settles the question raised by Ponnusamy and Wirths (Ann. Acad. Sci. AI. Math. 39:721-731, 2014). The class of analytic p-valent functions $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$, $p \in \mathbb{N}$ is denoted by $\mathcal{A}_p$. For $f \in \mathcal{A}_p$, let us consider the integral means

$$L(r, f, p) = \frac{r^{2p}}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta}{|f(re^{i\theta})|^2}, \quad r \in (0, 1).$$

We also focus on computing the integral means and the analog of area problems for certain subclasses of p-valent functions. We estimate the Taylor-Maclaurin coefficients of functions belonging to related p-valent functions. These estimation improve the results of Aouf [**7, 8**]. We introduce a new class, denoted by $\mathcal{P}_{a,b,c}(\beta)$, in terms of convolution $(*)$ with Gaussian hypergeometric functions ${}_2F_1(a, b; c; z)$, which is defined by

$$\mathcal{P}_{a,b,c}(\beta) = \left\{ f \in \mathcal{A} : \frac{f(z)}{z} = {}_2F_1(a, b; c; z) * p(z); \quad a \leq b < c \right\},$$

where p is an analytic function with positive real part of order β $(0 \leq \beta < 1)$ in $\mathbb{D}$ and $p(0) = 1$. Making use of duality principle, we investigate the order of starlikeness (or convexity) of the integral transform $V_\lambda(f)(z) = \int_0^1 \lambda(t) \frac{f(tz)}{t}\, dt$ over functions f in the class $\mathcal{P}_{a,b,c}(\beta)$.

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NOTATION

English Symbols

$\mathbb{D}$	unit disk $(z : \|z\| < 1)$
$\mathbb{N}$	set of natural number
$\mathcal{A}$	class of normalized analytic functions in $\mathbb{D}$
$\mathcal{A}_p$	class of normalized p-valent analytic functions in $\mathbb{D}$
$\mathbb{C}$	complex plane
$\mathcal{C}$	class of convex functions
$\mathcal{C}(\beta)$	class of convex functions of order $\beta,\ 0 \leq \beta < 1$
$\mathcal{C}_p$	class of p-valent convex functions, $p \in \mathbb{N}$
$\mathcal{C}_p(\beta)$	class of p-valent convex functions of order $\beta,\ 0 \leq \beta < p$
$\mathbb{D}_r$	disk of radius r $(z : \|z\| < r,\ 0 < r \leq 1)$
D_q	q-difference operator
${}_2F_1$	Gaussian Hypergeometric function
$f * g$	Hadamard (convolution) product of f and g
$f \prec g$	f is subordinate to g
$\mathcal{K}$	class of close-to-convex functions
$\mathcal{K}_q$	class of q-close-to-convex functions
$L_1(r, f, p)$	integral means for $f \in \mathcal{A}_p$
$\mathcal{S}$	class of univalent functions
$\mathcal{S}_\alpha$	class of α-spiral-like functions
$\mathcal{S}_\alpha(\beta)$	class of α-spiral-like functions of order $\beta,\ 0 \leq \beta < 1$
$\mathcal{S}_{\alpha,p}$	class of p-valent α-spiral-like functions, $\|\alpha\| < 1$
$\mathcal{S}_{\alpha,p}(\beta)$	class of p-valent α-spiral-like functions of order $\beta,\ 0 \leq \beta < p,\ \|\alpha\| < 1$

$\mathcal{S}^*$	class of starlike functions
$\mathcal{S}^*(A,B)$	the Janowski class, $-1 \leq B < A \leq 1$
$\mathcal{S}_p^*$	class of p-valent starlike functions
$\mathcal{S}^*(\beta)$	class of starlike functions of order $\beta,\ 0 \leq \beta < 1$
$\mathcal{S}_p^*(\beta)$	class of p-valent starlike functions of order $\beta,\ 0 \leq \beta < p$
$\mathcal{S}_q^*$	class of q-starlike functions
$V_\lambda(f)$	integral transform

Greek Symbols

$\Delta(r,f)$	area of the image of $\mathbb{D}_r$ under analytic function f

CHAPTER 1

INTRODUCTION

This thesis consists seven chapters. The first chapter is introductory in nature and provides basic definitions, background ideas and pre-requisites for the remaining chapters. Final chapter concludes with important remarks and some open problems. The thesis is endued with solutions to a number of problems. For example, we consider the following problems:

- We find conditions on the coefficients of power series of certain analytic functions in the unit disk which ensure that they generate functions in a q-analog to the well-known close-to-convex family. In addition, we discuss a method to compute the Bieberbach conjecture problem for functions in this family.
- We discuss area integral problems for certain classes of univalent and p-valent functions.
- We study the coefficient problems for certain p-valent functions.
- We also characterize starlikeness and convexity of certain integral transforms.

1.1. Univalent Functions

Let $\mathbb{C}$ be the *complex plane* and $\mathbb{D} := \{z : |z| < 1\}$ be the *open unit disk* in $\mathbb{C}$. A function f that is analytic in a domain $D \subset \mathbb{C}$ is said to be *univalent* in D, if $f(z_1) \neq f(z_2)$ for all $z_1, z_2 \in D$ with $z_1 \neq z_2$, i.e. it assumes no value more than once in D. By our definition, univalent mappings are also called conformal mappings. The question arises in the study of univalent mappings, whether an arbitrary simply connected domain can be mapped onto a disk. The Riemann Mapping Theorem [**22**, pp. 11] answers this question. It resolves that *the study of univalent functions on a simply connected domain can be* *confined to the study of these functions onto $\mathbb{D}$.* If g is univalent in $\mathbb{D}$ and has a Maclaurin series $g(z) = b_0 + \sum_{n=1}^{\infty} b_n z^n$ which is convergent in $\mathbb{D}$, then $f(z) = (g(z) - b_0))/b_1$ has the following form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

where $a_n = b_n/b_1$, is also univalent in $\mathbb{D}$ and vise-versa. Geometrically, this amounts to translating, shrinking or expanding the image domain $g(\mathbb{D})$, and possibly rotating $g(\mathbb{D})$. We say that the function f has the *normalized form* (1.1) and other normalizations are also possible. We denote the family of all normalized analytic functions in $\mathbb{D}$ of the form (1.1) by $\mathcal{A}$ and the class of univalent functions $f \in \mathcal{A}$ by $\mathcal{S}$. The *Koebe function*

$$k(z) := \frac{z}{(1-z)^2} = z + \sum_{n=2}^{\infty} n z^n \tag{1.2}$$

play an extremal role for the class $\mathcal{S}$ and for numerous subclasses of $\mathcal{S}$. The Koebe function $k(z)$ and its rotation $e^{-i\theta}k(ze^{i\theta})$ maps $\mathbb{D}$ onto the entire complex plane except the slit along the negative real axis from $-\infty$ to $-1/4$. The theory of univalent functions was initiated by Koebe [**46**] in 1907. In 1916, on the basis of the estimate $|a_2| \leq 2$ for $f \in \mathcal{S}$, Bieberbach [**14**] conjectured that $|a_n| \leq n$ for $n \geq 2$. The equality holds if and only if $f(z) = z(1 - ze^{i\theta})^{-2}$. Initially this conjecture was proved for $n = 3, 4, 5$ and 6. A number of several techniques and new methods were also invented to obtain partial solution of this conjecture. It was finally settled for the whole class $\mathcal{S}$ by de Branges [**15**] in 1985 and now it is known as de Branges's Theorem. Detailed account of the work can be found in the books by Duren [**22**], Goluzin [**28**], Graham and Kohr [**35**], Hayman [**40**] and Pommerenke [**65**]. The long gap between the formulation of the Bieberbach conjecture and its proof by de Branges encouraged researchers to introduce its legality on certain subclasses of $\mathcal{S}$. These classes include the class of starlike, convex, close-to-convex and spiral-like functions.

Let D be a set in $\mathbb{C}$. A set D is called *starlike* with respect to a point $w_0 \in D$ if the line segment joining w_0 to an arbitrary point $w \in D$ lies entirely in D. If a function f maps $\mathbb{D}$ onto a starlike domain with respect to w_0, then we say that f is *starlike with respect to* w_0. In the special case that $w_0 = 0$, we say that f is *starlike with respect to the* *origin* (or simply *starlike*). Analytically, a function $f \in \mathcal{A}$ is characterized to be starlike in $\mathbb{D}$ if and only if

$$\text{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad z \in \mathbb{D}.$$

Also we say that D is *convex domain* if the line segment joining any two points of D lies entirely in D. A function $f \in \mathcal{A}$ is called *convex* in $\mathbb{D}$ if the image domain $f(\mathbb{D})$ is convex. Analytically, a function $f \in \mathcal{A}$ is characterized to be convex in $\mathbb{D}$ if and only if

$$\text{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \quad z \in \mathbb{D}.$$

Analytical characterization of starlikeness and convexity were introduced by Nevanlinna [57] in 1921 and E. Study [92] in 1913, respectively. The classes of starlike and convex functions are denoted by $\mathcal{S}^*$ and $\mathcal{C}$, respectively. For instance, the Koebe function and $z/(1-z)$ belong to $\mathcal{S}^*$ and $\mathcal{C}$, respectively. It can be noted that "$f \in \mathcal{C} \Leftrightarrow zf' \in \mathcal{S}^*$". This was first discovered by Alexander [1] in 1915 and is popularly known as Alexander's Theorem. The classes $\mathcal{S}^*(\beta) = \{f \in \mathcal{A} : \text{ Re}\,(zf'(z)/f(z)) > \beta, 0 \leq \beta < 1\}$ and $\mathcal{C}^*(\beta) = \{f \in \mathcal{A} : \text{ Re}\,(1 + zf''(z)/f'(z)) > \beta\}$ are the classes of starlike functions of order β and convex functions of order β, respectively. Obviously, $\mathcal{S}^*(0) = \mathcal{S}^*$ and $\mathcal{C}(0) = \mathcal{C}$. Another important relation "$f \in \mathcal{C} \Rightarrow f \in \mathcal{S}^*(1/2)$" is one of the earliest result due to Marx [53] and Ströhhacker [91]. We also discuss two more subclasses of $\mathcal{S}$ that are generalizations of the class of starlike functions and have geometric characterizations.

A function $f \in \mathcal{A}$ is said to be *close-to-convex* if there exists a real number $\theta \in (-\pi/2, \pi/2)$ and $g \in \mathcal{C}$ such that

$$\text{Re}\left\{e^{i\theta}\frac{f'(z)}{g'(z)}\right\} > 0, \quad z \in \mathbb{D},$$

equivalently, by Alexander's Theorem, we get

$$\text{Re}\left\{e^{i\theta}\frac{zf'(z)}{h(z)}\right\} > 0, \quad z \in \mathbb{D},$$

for $h \in \mathcal{S}^*$. We denote the class of close-to-convex functions by $\mathcal{K}$ and it was considered by Kaplan [45] in 1952. It is well-known that

$$\mathcal{C} \subset \mathcal{S}^* \subset \mathcal{K} \subset \mathcal{S}.$$

Geometrically, a function $f \in \mathcal{S}$ is said to be close-to-convex if and only if the complement of the image domain $f(\mathbb{D})$ is the union of a family of non-intersecting half-lines or rays (expect that the origin of one ray may lie on another ray). Such a domain is clearly simply connected. For instance, see the books [**22**, **35**, **65**].

A domain D $(0 \in D)$ is said to be *α-spiral-like* with $|\alpha| < \pi/2$ if for all non-zero point $w_0 \in D$, the arc of the α-spiral joining w_0 to the origin lies entirely in D. Such a domain D is simply connected. A function $f \in \mathcal{S}$ is said to be *α-spiral-like* if $f(\mathbb{D})$ is α-spiral-like. Analytically, an α-spiral-like function f is characterized by the relation

$$\mathrm{Re}\left\{e^{i\alpha}\frac{zf'(z)}{f(z)}\right\} > 0, \quad |z| < 1 \text{ and } |\alpha| < \pi/2. \tag{1.3}$$

The class of α-spiral-like functions is denoted by $\mathcal{S}_\alpha$ and it was introduced by Špaček [**90**] in 1933. It is easy to see that the function $k_\alpha(z) = z(1-z)^{-2e^{i\alpha}\cos\alpha}$ belongs to the class $\mathcal{S}_\alpha$. This leads to the observation that the class $\mathcal{S}_\alpha$ is neither included in the class $\mathcal{K}$ nor includes the class $\mathcal{S}_\alpha$. It is obvious that 0-spiral-like functions are starlike functions.

The Bieberbach conjecture was initially proved for certain subclasses of univalent functions. Here is a partial list of them.

- In 1921, Nevanlinna [**57**] proved that "if $f \in \mathcal{S}^*$ and has the form (1.1), then $|a_n| \le n$ for all $n \ge 2$. Equality holds for all n unless f is a rotation of the Koebe function."
- In 1917, Löwner [**50**] proved that "if $f \in \mathcal{C}$ and has the form (1.1), then $|a_n| \le 1$ for all $n \ge 2$. Equality holds for all n unless f is a rotation of $l(z) = z/(1-z)$."
- In 1955, Reade [**74**] proved that "if $f \in \mathcal{K}$ and has the form (1.1), then $|a_n| \le n$ for $n = 2, 3, \ldots$. Equality holds for $n \ge 2$ unless f is a rotation of the Koebe function."

One of the problems discussed in this thesis links between geometric function theory and q-theory.

For $f \in \mathcal{A}$, the *q-difference operator*, denoted by $D_q f$, is defined by

$$(1.4) \qquad (D_q f)(z) = \frac{f(z) - f(qz)}{z(1-q)}, \quad z \in \mathbb{D} \setminus \{0\}, \quad (D_q f)(0) = f'(0),$$

where $q \in (0,1)$. It is evident that, when $q \to 1^-$, $D_q f \to f'$. Note that the q-difference operator plays an important role in special functions and quantum physics (see for instance [4, 23, 24, 48, 89]). The q-theory in geometric function theory was introduced by Ismail et al. in 1990. They introduced and studied a q-analog of starlike functions via the q-difference operator in the same paper. A similar generalization of close-to-convex functions by means of $D_q f$ is defined as follows: A function $f \in \mathcal{A}$ is said to be *q-close-to-convex function* for $q \in (0,1)$, if there exists a $g \in \mathcal{S}^*$ such that

$$(1.5) \qquad \left| \frac{z}{g(z)}(D_q f)(z) - \frac{1}{1-q} \right| \le \frac{1}{1-q}, \quad \text{for } z \in \mathbb{D}.$$

The class of q-close-to-convex functions is denoted by $\mathcal{K}_q$ and it has been studied in [72]. When $q \to 1^-$, the class $\mathcal{K}_q$ coincide with the class $\mathcal{K}$. Very less progress was made between the q-theory and geometric function theory previously. It is therefore worth to study more problems in this direction. We study two types of problems. Firstly, we obtain conditions on the coefficients of functions $f(z) = z + \sum_{n=2}^{\infty} A_n z^n$ analytic in $\mathbb{D}$ which ensure that they generate functions in the q-close-to-convex family. As a result we find certain dilogarithm functions that are contained in $\mathcal{K}_q$ family. Secondly, we also study the Bieberbach conjecture problem for coefficients of q-close-to-convex functions. This produces several power series of analytic functions convergent to basic hypergeometric functions.

The second type of problem discussed in this thesis is area problem, namely, computing the area of an image domain under an analytic function in $\mathbb{D}$. The *area* of the image of the subdisk $\mathbb{D}_r := \{z \in \mathbb{C} : |z| < r\}$ under an analytic function f in $\mathbb{D}$ is denoted by $\Delta(r, f)$, $0 \le r < 1$. Thus, we have

$$(1.6) \qquad \Delta(r, f) = \iint_{\mathbb{D}_r} |f'(z)|^2 \, dxdy.$$

Computing this area is known as the *area problem for functions of type f*. We set $\mathbb{D} := \mathbb{D}_1$. We know that in one-variable calculus, the definite integral $\int_a^b f(x)\, dx$ for a real function f can be interpreted as the *area* under the curve f over $[a, b]$. In the same way, the integral $\int\int_D f(x,y)\, dA$ of $f(x,y)$ can be interpreted as the *volume* enclosed by a surface $z = f(x,y)$ over a region D. This observation motivates us to study the area of the image domain under analytic functions in $\mathbb{D}$. The functions f/z and z/f both are non-vanishing analytic functions in $\mathbb{D}$. One can easily obtain the area problem for functions of type f and f/z when f is the class of univalent functions by using de Branges's Theorem. But it is not easy to solve for functions of type z/f. Yamashita [98] conjectured that

$$\max_{f\in\mathcal{C}} \Delta\left(r, \frac{z}{f}\right) = \pi r^2$$

for each r, $0 < r \leq 1$, and the maximum is attained only by the rotations of $l(z) = z/(1-z)$. This extremal problem is also called the *maximal area integral problem for functions of type* z/f when f ranges over an analytic family. In 2013, this conjecture was settled by Obradović et al. in [58]. We aim to prolong the discussion of Yamashita's extremal problem for related classes of analytic functions. In the following paragraph, we define some basic definitions and notations that are used in the sequel.

Most of the analytic characterizations/definitions of functions considered in this thesis use the notion of subordination principle. For two analytic functions f and g in $\mathbb{D}$, we say that f is *subordinate* to g if

$$f(z) = g(w(z)), \quad |z| < 1,$$

for some analytic function w in $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$. We express this symbolically by $f \prec g$. Geometrically, if g is univalent in $\mathbb{D}$, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$. For instance, one can easily see that $1/(1+z) \prec (1+z)/(1-z)$, $z \in \mathbb{D}$. For the theory of subordination, we refer to the textbooks [**22, 56, 65**].

The class of starlike functions are generalized in a number of ways. We consider the following: A normalized analytic function f is said to belong to the class $\mathcal{S}^*(A,B)$, if it satisfies the subordination relation

$$\mathcal{S}^*(A,B) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}, \quad z \in \mathbb{D} \right\}, \tag{1.7}$$

where $A \in \mathbb{C}, -1 \leq B \leq 0$ and $A \neq B$. In particular, $\mathcal{S}^*(1-2\beta, -1) = \mathcal{S}^*(\beta)$.

In this thesis, we discuss an *open problem* (i.e. to find the maximum value of $\Delta(r, z/f)$ when $f \in \mathcal{S}^*(A, B)$) which was posed by Ponnusamy and Wirths in [71] (see also [59]).

Next problem of this thesis deals with starlikeness and convexity of certain integral transforms. The study of integral transforms has also been an important problem in the field of geometric function theory. We consider a general *integral transform* for functions $f \in \mathcal{A}$,

$$V_\lambda(f)(z) := \int_0^1 \lambda(t) \frac{f(tz)}{t}\, dt, \tag{1.8}$$

where $\lambda : [0, 1] \to \mathbb{R}$ is a non-negative function with $\int_0^1 \lambda(t)dt = 1$. This generalized integral transform was studied by Fournier and Ruscheweyh in [25]. For some special cases of $\lambda(t)$, the integral transform $V_\lambda(f)$ reduces to various well-known integral transforms such as the Alexander transform, the Libera transform, and the Bernardi transform.

Let f and g be analytic functions in $\mathbb{D}$ and have the series representations

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ and } g(z) = \sum_{n=0}^{\infty} b_n z^n,$$

respectively. The *convolution* (or Hadamard product) of f and g is given by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n, \qquad \text{for } z \in \mathbb{D}.$$

Note that $f * g$ is also analytic in $\mathbb{D}$ and obviously,convolution

$$f(z) = f(z) * \frac{z}{1-z} \qquad \text{and} \qquad zf'(z) = f(z) * \frac{z}{(1-z)^2}.$$

The concept of duality also plays a central role in the study of problems in convolution theory. Let $\mathcal{H}$ denote the class of analytic functions in $\mathbb{D}$. Consider a subclass $\mathcal{A}_0$ of $\mathcal{H}$ defined by

$$\mathcal{A}_0 = \left\{ f \in \mathcal{H} : \ f(z) = \frac{h(z)}{z}, \quad h \in \mathcal{A} \right\}.$$

For $\mathcal{V} \subset \mathcal{A}_0$, we call the set

$$\mathcal{V}^* = \{ f \in \mathcal{A}_0 : \ (f * g)(z) \neq 0, \quad \text{for all } g \in \mathcal{V} \text{ and } z \in \mathbb{D} \}$$

dual of $\mathcal{V}$. Note that every dual set is complete (and closed). The set $\mathcal{V}^{**} = (\mathcal{V}^*)^*$ is called the second dual of $\mathcal{V}$.

Duality principle [82, Theorem 1.1] has many applications to classes of functions those are defined by properties like bounded real part, convexity, starlikeness, close-to-convexity, univalence etc. One can find basic results of duality theory for convolutions and their numerous applications in the monograph by Ruscheweyh [82] and his article [81]. Silverman et al. in [84] have established new necessary and sufficient conditions in terms of convolution for some known subclasses of analytic functions. These conditions are very useful to solve problems associated with several subclasses of univalent functions and integral transforms. Some of the characterizations of starlike and convex functions in terms of convolutions are stated as follows:

Theorem 1.1. [84, Theorem 2] *A function $f \in \mathcal{S}^*(\beta)$ for $|z| < R \leq 1$ if and only if $[f * h_\beta(z)]/z \neq 0$ for $|z| < R$. Here*

$$h_\beta(z) = z\left(1 + \frac{\rho + 2\beta - 1}{2 - 2\beta} z\right) \frac{1}{(1-z)^2}, \quad 0 \leq \beta < 1, \ |\rho| = 1. \tag{1.9}$$

Theorem 1.2. [84, Theorem 1] *A function $f \in \mathcal{C}(\beta)$ for $|z| < R \leq 1$ if and only if $[f * h_{1\beta}(z)]/z \neq 0$ for $|z| < R$. Here*

$$h_{1\beta}(z) = z\left(1 + \frac{\rho + \beta}{1 - \beta} z\right) \frac{1}{(1-z)^3}, \quad 0 \leq \beta < 1, \ |\rho| = 1.$$

We note that Theorems 1.1 and 1.2 are used to prove our main results in Chapter 6 in terms of the Gaussian hypergeometric functions defined by

$${}_2F_1(a, b; c; z) = 1 + \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n, \quad |z| < 1,$$

where $a, b, c \in \mathbb{C}$ and $c \neq \mathbb{Z}^- = \{0, -1, -2, \cdots\}$. Note that the function ${}_2F_1(a, b; c; z)$ is analytic in $\mathbb{D}$. The Pochhammer symbol $(a)_n$ is defined in terms of the Gamma function Γ, by

$$(a)_0 = 1, \quad (a)_n = a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}.$$

In 1882, Gauss established the following useful relation connected with the Euler gamma function

$${}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} < \infty,$$

for Re(c − a − b) > 0. Similarly, the function ${}_0F_1(a;z)$ is defined as

$$ {}_0F_1(a;z) = \sum_{n=0}^{\infty} \frac{1}{(a)_n} \frac{z^n}{n!}, \quad |z| < 1. $$

We recall the useful derivative formula

$$ {}_2F_1'(a,b;c;z) = \frac{d}{dz} {}_2F_1(a,b;c;z) = \frac{ab}{c} {}_2F_1(a+1,b+1;c+1;z). \tag{1.10} $$

For $f \in \mathcal{A}$, the convolution relation $zF(1,b;c;z) * f(z)$ has the integral formula (e.g. see [11]):

$$ z{}_2F_1(1,b;c;z) * f(z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \frac{f(tz)}{t} dt. $$

which is nothing but the integral transform $V_\lambda(f)$ with the non-negative weight function $\lambda(t) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} t^{b-1}(1-t)^{c-b-1}$. For basic information about Gaussian hypergeometric functions, we refer to the textbooks [5, 73].

To characterize starlikeness and convexity of integral transforms $V_\lambda(f)$ for certain analytic functions f, the duality technique introduced by Fournier and Ruscheweyh [25, 82], plays a crucial role in geometric function theory. For $\gamma \in [0,1]$ and $\beta < 1$, we consider the class

$$ \mathcal{P}_\gamma(\beta) = \left\{ f \in \mathcal{A} : \mathrm{Re}\left((1-\gamma)\frac{f(z)}{z} + \gamma f'(z) \right) > \beta \right\}. \tag{1.11} $$

With the help of the duality theory of convolution, Balasubramanian et al. found conditions so that the integral transform $V_\lambda(f)$ carries $\mathcal{P}_\gamma(\beta)$ into $\mathcal{S}^*(\mu)$ and $\mathcal{C}(\mu)$ $(0 \le \mu \le 1/2)$ in [9] and [10], respectively. Initially, the starlikeness and convexity of $V_\lambda(f)$ for $f \in \mathcal{P}_1(\beta)$ were investigated by Fournier and Ruscheweyh [25] in 1994 & Ali and Singh [2] in 1995, respectively. After that, many contributions have been made on different accounts in [3, 20, 47, 68]. These contributions motivate us to check whether we can bring the concept of hypergeometric function in connection to geometric function theory in characterizing starlikeness and convexity of $V_\lambda(f)$ for functions defined by convolution with the well-known Gaussian hypergeometric functions. The answer is yes. Indeed, we can generalize the class $\mathcal{P}_\gamma(\beta)$ in the following form:

$$ \mathcal{P}_{a,b,c}(\beta) := \left\{ f \in \mathcal{A} : \frac{f(z)}{z} = {}_2F_1(a,b;c;z) * p(z); \quad a \le b < c,\ 0 \le \beta < 1 \right\}, \tag{1.12} $$

where p is an analytic function with positive real part of order β in $\mathbb{D}$ and $p(0) = 1$. We are interested to know whether $V_\lambda(f)$ is starlike of order μ or convex of order μ for functions $f \in \mathcal{P}_{a,b,c}(\beta)$, $\mu \in [0, 1/2]$. For the range $\mu \in (1/2, 1)$, the problem remains open.

1.2. p-valent Functions

The theory of p-valent functions is much more than just a generalization of the theory of univalent functions. If T is a theorem about the set $\mathcal{S}$, the extension to p-valent function for $p \geq 2$, may be completely trivial, or extremely difficult, or perhaps false. In this section, we drive the concept of starlike and convex to the case of p-valent functions and also discuss about *Goodman's conjecture* [**30**] for certain p-valent functions.

Let p be a natural number. A function

$$f(z) = \sum_{n=1}^{\infty} a_n z^n \tag{1.13}$$

is said to be *p-valent* in $\mathbb{D}$ if it is analytic and assumes no value more than p times in $\mathbb{D}$ and there is some w such that $f(z) = w$ has exactly p solutions in $\mathbb{D}$, when roots are counted in accordance with their multiplicities. Geometrically, a function f is said to be *p-valent* (or *multivalent*) in $\mathbb{D}$ if the conditions

$$f(z_1) = f(z_2) = \cdots = f(z_{p+1}), \quad z_1, z_2, \ldots, z_{p+1} \in \mathbb{D}$$

imply that $z_r = z_s$ for some pair such that $r \neq s$, and if there is some w such that the equation $f(z) = w$ has p roots (counted in accordance with their multiplicities) in $\mathbb{D}$. For example, $f(z) = z^2$ is a 2-valent function in $\mathbb{D}$.

In 1950, Goodman [**31**] has studied the classes of p-valent starlike and convex functions. A p-valent function f of the form (1.13) is said to be *p-valent starlike* in $\mathbb{D}$, if it is analytic and if there exists a $\rho > 0$ such that for $\rho < |z| < 1$

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0$$

and

$$\int_0^{2\pi} \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} d\theta = 2p\pi,$$

for $z = re^{i\theta}$. The class of p-valent starlike functions is denoted by $\mathcal{S}_p^*$.

A p-valent function f of the form (1.13) is said to be *p-valent convex* in $\mathbb{D}$, if it is analytic and if there exists a $\rho > 0$ such that for $\rho < |z| < 1$

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0$$

and

$$\int_0^{2\pi} \mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} d\theta = 2p\pi,$$

for $z = re^{i\theta}$. The class of p-valent convex functions is denoted by $\mathcal{C}_p$. There is a close relationship between $\mathcal{S}_p^*$ and $\mathcal{C}_p$ in the same way as Alexander theorem, namely,

$$f \in \mathcal{C}_p \Longleftrightarrow \frac{zf'}{p} \in \mathcal{S}_p^*.$$

For $p = 1$, the classes $\mathcal{S}_p^*$ and $\mathcal{C}_p$ are the well-known classes of starlike and convex, respectively.

Bieberbach's conjecture was not studied for the class of p-valent functions until 1948. The initiative was first taken by Goodman. In [**30**], Goodman made a conjecture that if f is analytic p-valent and has the form (1.13), then

$$|a_n| \leq \sum_{k=1}^{p} \frac{2k(p+n)\,!}{(p+k)\,!(p-k)\,!(n-p-1)\,!(n^2-k^2)}|a_k|, \tag{1.14}$$

for $n > p$. For $p = 2$ and $n = 3$, this gives the conjecture that

$$|a_3| \leq 5|a_1| + 4|a_2|. \tag{1.15}$$

Inequality (1.14) reduces to the well-known Bieberbach conjecture when $p = 1$.

In [**31**], Goodman showed that (1.15) is valid for $f \in \mathcal{S}_2^*$ which has the form (1.13) with all the real coefficients a_n, and this bound is sharp for all pairs $|a_1|, |a_2|$, not both zero. In 1951, the full conjecture was proven by Goodman and Robertson [**32**] for functions f to be in $\mathcal{S}_p^*$ and $\mathcal{C}_p$, provided that all its coefficients are real, and sharpness for all p, $n > p$ and by Robertson [76] in 1953, when $a_1 = a_2 = \cdots = a_{p-2} = 0$ and the remaining coefficients being complex.

Consider the function f of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p}, \quad p \in \mathbb{N} \tag{1.16}$$

with p zeros at origin. Let $\mathcal{A}_p$ denote the class of all functions f of the form (1.16) analytic and p-valent in $\mathbb{D}$. We note that $\mathcal{A}_1 =: \mathcal{S}$. For the class $\mathcal{A}_p$, Hayman in [39] has showed that $|a_{p+1}| \le 2p$ and Jenkins in [44] has showed $|a_{p+2}| \le p(2p+1)$. Both of these results are consistent with (1.14). Both inequalities for p-valent functions are the analog of the coefficient bounds $|a_2| \le 2$ and $|a_3| \le 3$ for univalent functions. The problem of estimating the Taylor-Maclaurin coefficient bounds for many other classes of p-valent functions has attracted a number of researchers. For example, in 1974, Goluzina [29] introduced and studied the class of p-valent starlike functions of order β (i.e. functions $f \in \mathcal{A}_p$ satisfying $\mathrm{Re}(zf'(z)/f(z)) > \beta$, $0 \le \beta < p$) and obtained the coefficient estimates for $f \in \mathcal{A}_p$ such that

$$|a_{n+p}| \le \frac{\Gamma(n+2p-2\beta)}{n\,!\,\Gamma(2p-2\beta)}, \quad n \ge 1,$$

and equality holds only for $\mathcal{K}_p(\beta) = z^p/(1-z)^{2(p-\beta)}$ (see also [64]). Goodman's conjecture for many other classes of p-valent functions have been studied by various authors, see for instance [51, 94]. These ideas motivate us to study the Taylor-Maclaurin coefficients estimates for functions belonging to related p-valent functions having some geometric properties.

In this thesis, we consider the function classes $\mathcal{S}_p^*(A,B,\beta), \mathcal{F}_p(\alpha,\beta,\lambda)$ and $\mathcal{C}_p(b,\lambda)$ which are defined by subordination as follows: for $p \in \mathbb{N}$,

(1.17)
$$\mathcal{S}_p^*(A,B,\beta) = \left\{ f \in \mathcal{A}_p : \frac{zf'(z)}{pf(z)} \prec \frac{1 + \big[B + (A-B)(1-\beta/p)\big]z}{1+Bz}, \quad 1 \le B < A \le 1 \right\},$$

(1.18)
$$\mathcal{F}_p(\alpha,\beta,\lambda) = \left\{ f \in \mathcal{A}_p : e^{i\alpha}\frac{zf'(z)}{pf(z)} \prec \left(\frac{1+(1-2\beta/p)\lambda z}{1-\lambda z} \right) \cos\alpha + i\sin\alpha, \quad |\alpha| < \pi/2 \right\},$$

and

$$\mathcal{C}_p(b,\lambda)=\left\{f\in\mathcal{A}_p:\ \frac{1}{p}\left(1+\frac{zf''(z)}{f'(z)}\right)\prec\frac{1+(2b-1)\lambda z}{1-\lambda z},\quad 0\neq b\in\mathbb{C}\right\},\tag{1.19}$$

where $z\in\mathbb{D}, 0\leq\beta<p$ and $0<\lambda\leq 1$. The above classes have been introduced and studied by Aouf in [7] and [8]. Furthermore, we estimate the Taylor-Maclaurin coefficients $|a_n|$, $n\geq p+1$, for functions belonging to these classes, which improve the results of Aouf [7, 8].

The integral means and Dirichlet integral in p-valent theory are defined as follows. For $r\in(0,1]$, consider a function $f\in\mathcal{A}_p$ which has the *integral means*

$$L_1(r,f,p):=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{r^{2p}}{|f(re^{i\theta})|^2}d\theta,\quad z=re^{i\theta}\in\mathbb{D},\tag{1.20}$$

and the *Dirichlet integral*

$$\Delta(r,f):=\iint_{\mathbb{D}_r}|f'(z)|^2\,dxdy=\pi pr^{2p}+\pi\sum_{n=1}^{\infty}(n+p)|a_{n+p}|^2r^{2(n+p)},\quad z=x+iy.\tag{1.21}$$

Computing the integral $L_1(r,f,p)$ is known as *the integral means problem.* The classical integral means and area problems have not been studied in p-valent setting, which is one of our objectives in this thesis.

We consider a generalization of the class $\mathcal{S}^*(A,B)$ for p-valent functions, written as $\mathcal{S}_p^*(A,B)$, defined by

$$\mathcal{S}_p^*(A,B):=\left\{f\in\mathcal{A}_p:\quad\frac{zf'(z)}{pf(z)}\prec\frac{1+Az}{1+Bz},\quad z\in\mathbb{D}\right\},\tag{1.22}$$

where $A\in\mathbb{C}$, $-1\leq B\leq 0$, $A\neq B$ and $p\in\mathbb{N}$.

One of our aims is to study the integral means and Yamashita conjecture for the class $\mathcal{S}_p^*(A,B)$ and for other related class of p-valent functions.

1.3. Organization of Thesis

In **Chapter 2**, we concentrate on problems where the coefficients A_n of functions $f(z) = z + \sum_{n=2}^{\infty} A_n z^n \in \mathcal{K}_q$ are real, non-negative and connected with certain monotone properties. For instance, we obtain the following theorems:

Theorem 1.3. *Let $\{A_n\}$ be a sequence of real numbers such that $B_n = A_n(1-q^n)/(1-q)$ for all $n \geq 1$, where*

$$1 \geq B_2 \geq \cdots \geq B_n \geq \cdots \geq 0 \text{ or } 1 \leq B_2 \leq \cdots \leq B_n \leq \cdots \leq 2.$$

Then $f(z) = z + \sum_{n=2}^{\infty} A_n z^n \in \mathcal{K}_q$ with $g(z) = z/(1-z)$.

By Theorem 1.3, one can ascertain that the function $(1-q)Li_2(z;q) \in \mathcal{K}_q$, where $Li_2(z;q)$ is the quantum dilogarithm function.

Theorem 1.4. *Let $f(z) = z + \sum_{n=2}^{\infty} A_n z^n \in \mathcal{K}_q$ and suppose that*

$$\sum_{n=1}^{\infty} |B_n - B_{n-1}| \leq 1, \quad B_n = \frac{A_{n+1}(1-q^{n+1})}{1-q} - \frac{A_n(1-q^n)}{1-q}.$$

Then $f \in \mathcal{K}_q$ with $g(z) = z/(1-z)^2$.

By Theorem 1.4, we immediately have the following result:

Theorem 1.5. *Let $\{A_n\}$ be a sequence of real numbers such that $A_0 = 0 = A_1 - 1$ and B_n defined in Theorem 1.4. Suppose that*

$$1 \geq B_1 \geq \cdots \geq B_n \geq \cdots \geq 0 \text{ or } 1 \leq B_1 \leq \cdots \leq B_n \leq \cdots \leq 2.$$

Then $f(z) = z + \sum_{n=2}^{\infty} A_n z^n \in \mathcal{K}_q$ with $g(z) = z/(1-z)^2$.

Letting $q \to 1^-$ in Theorem 1.3 and 1.5, one can obtain couple of results of Alexander [**1**] and MacGregor (see [**52**, Theorems 1, 3 and 5]).

Secondly, we analyze the Bieberbach-de Branges theorem for functions belong to $\mathcal{K}_q$.

Theorem 1.6 (Bieberbach-de Branges theorem for $\mathcal{K}_q$)**.** *If $f \in \mathcal{K}_q$, then*

$$|a_n| \leq \frac{1-q}{1-q^n}\left[n + \frac{n(n-1)}{2}(1+q)\right] \quad \text{for all } n \geq 2.$$

It is noticed that the inequality in Theorem 1.6 is not sharp. However, when $q \to 1$, this leads to the sharp inequality concerning the well-known Beiberbach problem for close-to-convex functions.

In **Chapter 3**, a general problem on the Yamashita conjecture for the class $\mathcal{S}^*(A,B)$ is considered which was suggested in [**71**]. We settle the problem in complete generality for the full class $\mathcal{S}^*(A,B)$ and the main result is stated in the following form.

Theorem 1.7. *Let $f \in \mathcal{S}^*(A,B)$ for $-1 \leq B < 0$ with $A \neq B$ and z/f be a non-vanishing analytic function in $\mathbb{D}$. Then we have*

$$\max_{f \in \mathcal{S}^*(A,B)} \Delta\left(r, \frac{z}{f}\right) = \pi|\overline{A} - B|^2 r^2 {}_2F_1(A/B, \overline{A}/B; 2; B^2r^2), \quad 0 < r \leq 1.$$

The maximum is attained by the rotations of $k_{A,B}(z) = z(1+Bz)^{(A/B)-1}$.

In **Chapter 4**, we find coefficient bounds for functions $f \in \mathcal{S}_p^*(A,B,\beta)$ and other related classes of p-valent functions. These bounds improve the results of Aouf [**7, 8**]. For example, we get

Theorem 1.8. *Let $-1 \leq B < A \leq 1$, $0 \leq \beta < p$ and $p \in \mathbb{N}$. If $f \in \mathcal{S}_p^*(A,B,\beta)$ is in the form (1.17), then we have*

$$|a_{p+1}| \leq (A-B)(p-\beta); \tag{1.23}$$

for $A(p-\beta) - B(p-\beta-1) \leq 1$ (or $A(p-\beta) - B(n-\beta-1) \leq (n-p-1)$), $n \geq p+2$,

$$|a_n| \leq \frac{(A-B)(p-\beta)}{n-p}; \tag{1.24}$$

and for $A(p-\beta) - B(n-\beta-1) > (n-p-1)$, $n \geq p+2$,

$$|a_n| \leq \prod_{j=1}^{n-p} \frac{(A(p-\beta) - B(p-\beta+j-1))}{j}. \tag{1.25}$$

The inequalities (1.23), (1.24) and (1.25) are sharp.

In **Chapter 5**, we determine the integral means and Yamashita's extremal problem for the class $\mathcal{S}_p^*(A,B)$ and for other related class of p-valent functions. Indeed, we obtain

Theorem 1.9. *Let $A \in \mathbb{C}, -1 \le B \le 0, A \neq B$ and $p \in \mathbb{N}$. If $f \in \mathcal{S}_p^*(A,B)$ and z^p/f is a non-vanishing analytic function in $\mathbb{D}$. Then, for $0 < r \le 1$, we have*

$$L_1(r,f,p) := r^{2p} I_1(r,f,p) \le \begin{cases} {}_2F_1\big(\phi p, \overline{\phi} p; 1; B^2\big) & \text{if } B \neq 0; \\ \sum_{n=0}^{\infty} (p|A|)^{2n}/(n\,!)^2 & \text{if } B = 0, \end{cases}$$

where $\phi = (A/B) - 1$. The equality attains for the function $k_{A,B,p}$ defined by

$$k_{A,B,p}(z) = \begin{cases} z^p(1+Bz)^{((A/B)-1)p} & \text{if } B \neq 0, \\ z^p e^{Apz} & \text{if } B = 0 \end{cases}. \tag{1.26}$$

Theorem 1.10. *Let $f \in \mathcal{S}_p^*(A,B)$, for $A \in \mathbb{C}, -1 \le B < 0$, $p \in \mathbb{N}$ with $A \neq B$ and z^p/f be a non-vanishing analytic function in $\mathbb{D}$, then*

$$\max_{f \in \mathcal{S}_p^*(A,B)} \Delta\left(r, \frac{z^p}{f}\right) = \pi |\overline{A} - B|^2 p^2 r^2 {}_2F_1\big(\phi p + 1, \overline{\phi} p + 1; 2; B^2 r^2\big), \quad r \in (0,1].$$

The maximum is attained by the rotations of $k_{A,B,p}, B \neq 0$ defined in (1.26).

In **Chapter 6**, we are interested to find conditions such that the integral transform $V_\lambda(f)$ is starlike of order μ (or convex of order μ) over functions $f \in \mathcal{P}_{a,b,c}(\beta)$, $\mu \in [0, 1/2]$.

To discuss the starlikeness (or convexity) of $V_\lambda(f)$, for $f \in \mathcal{P}_{a,b,c}(\beta)$, we use the following notations:

$$\Lambda_{a,b,c}(t) = \int_t^1 \frac{\lambda(u)}{u^{a+2b-c}}\, du. \tag{1.27}$$

We let $g^{\mu}_{a,b,c}(t)$, the solution of the initial value problem

$$(a+2b-c)(1+g^{\mu}_{a,b,c}(t)) + t\frac{d}{dt} g^{\mu}_{a,b,c}(t) = 2(a+2b-c)\frac{1-\mu(1+t)}{(1-\mu)(1+t)^2} \tag{1.28}$$

with $g^{\mu}_{a,b,c}(0)=1$ and let

$$L^{\mu}_{\Lambda_{a,b,c}}(h_\mu)=\inf_{z\in\mathbb{D}}\int_0^1 t^{a+2b-c-1}\Lambda_{a,b,c}(t)\left[\mathrm{Re}\left(\frac{h_\mu(tz)}{tz}\right.\right.$$
$$\left.\left.*\left(\frac{abz}{c}F(a+1,b+1;c+1;z)+(a+2b-c)F(a,b;c;z)\right)\right)\right.$$
$$\left.-(a+2b-c)\frac{1-\mu(1+t)}{(1-\mu)(1+t)^2}\right]dt>0,\quad z\in\mathbb{D},$$

where h_μ has the form (1.9). We set

$$\Lambda_{b,c}:=\Lambda_{1,b,c},\ g^{\mu}_{b,c}:=g^{\mu}_{1,b,c}\text{ and }L^{\mu}_{\Lambda_{b,c}}:=L^{\mu}_{\Lambda_{1,b,c}}. \tag{1.29}$$

Concerning this, our main result is as follows:

Theorem 1.11. *For $0\le\beta<1$, let $f\in\mathcal{P}_{a,b,c}(\beta)$ with $a\le b<c$. Suppose that $\lambda:[0,1]\to\mathbb{R}$ is a non-negative weight function so that $\int_0^1\lambda(t)dt=1$ and $\Lambda_{a,b,c}$ is defined by (1.27) with the assumption that $\lim_{t\to 0^+}t^{a+2b-c}\Lambda_{a,b,c}(t)=0$. Assume that the quantity β is related by*

$$\frac{\beta}{1-\beta}=-\int_0^1\lambda(t)g^{\mu}_{a,b,c}(t)dt, \tag{1.30}$$

where $g^{\mu}_{a,b,c}$ satisfies (1.28). Then $V_\lambda(f)\in S^*(\mu)$ $(0\le\mu\le 1/2)$ if and only if $L^{\mu}_{\Lambda_{a,b,c}}(h_\mu)\ge 0$.

We also discuss the convexity of the integral transform, $V_\lambda(f)$, of $f\in\mathcal{P}_{a,b,c}(\beta)$ in Chapter 6 of the thesis.

CHAPTER 2

ON A GENERALIZATION OF CLOSE-TO-CONVEX FUNCTIONS

In this chapter, our main aim is to study the q-theory in connection with geometric function theory. In Section 2.1, we discuss about a generalization of the class of close-to-convex functions in terms of the q-difference operator D_q. We obtain several sufficient conditions for $f \in \mathcal{A}$ to be in $\mathcal{K}_q$ in Section 2.2. While Section 2.3 gives the Bieberbach-de Branges Theorem for $f \in \mathcal{K}_q$ and also presents some special cases.

The results of this chapter appeared in:
Sahoo S.K., Sharma N.L. (2015), *On a generalization of close-to-convex functions*, Ann. Polon. Math., **113**(1), 93–108.

2.1. The class $\mathcal{K}_q$

The q-difference operator, defined by (1.4), helps us to generalize the class of starlike functions $\mathcal{S}^*$ analytically. A q-analog of the class of starlike functions was introduced in [42] by means of the q-difference operator $D_q f$ acting on $f \in \mathcal{A}$. We denote the class of functions in this generalized family by $\mathcal{S}_q^*$. For the sake of convenience, we also use functions in $\mathcal{S}_q^*$ as *q-starlike functions*. It is defined as follows: A function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{S}_q^*$ if

$$\left| \frac{z}{f(z)}(D_q f)(z) - \frac{1}{1-q} \right| \leq \frac{1}{1-q}, \quad z \in \mathbb{D}.$$

Clearly, when $q \to 1^-$, the class $\mathcal{S}_q^*$ will coincide with $\mathcal{S}^*$.

A similar form of q-analog of close-to-convex functions was expected and it is defined in Chapter 1 by (1.5). In [72], the authors have investigated some basic properties of functions that are in $\mathcal{K}_q$. Some of these results are recalled in this chapter in order to exhibit their interesting consequences. As $(D_q f)(z) \to f'(z)$ as $q \to 1^-$, we observe in the limiting sense, the closed disk $|w - (1-q)^{-1}| \leq (1-q)^{-1}$ becomes the right half-plane $\mathrm{Re}\,(zf'(z)/g(z)) > 0$, and hence the class $\mathcal{K}_q$ clearly reduces to $\mathcal{K}$. In this chapter, we refer to the functions in $\mathcal{K}_q$ as *q-close-to-convex functions*. For the sake of convenience, we use the notation $\mathcal{S}_q^*$ instead of the notation PS_q used in [42], and $\mathcal{K}_q$ instead of PK_q used in [72]. It is easy to see that $\mathcal{S}_q^* \subset \mathcal{K}_q$ for all $q \in (0,1)$. Clearly, one can easily see from the above discussion that

$$\bigcap_{0<q<1} \mathcal{K}_q \subset \mathcal{K} \subset \mathcal{S}.$$

Our main aim in this chapter is to consider the following two ideas.

The first idea has its origin in the work of Friedman [26]. He proved that there are only nine functions in the class $\mathcal{S}$ whose coefficients are rational integers. They are

$$z, \quad \frac{z}{1 \pm z}, \quad \frac{z}{1 \pm z^2}, \quad \frac{z}{(1 \pm z)^2} \quad \frac{z}{1 \pm z + z^2}.$$

It is easy to see that these functions map the unit disk $\mathbb{D}$ onto starlike domains. Using the idea of MacGregor [52], we derive some sufficient conditions for functions to be in $\mathcal{K}_q$ whose coefficients are connected with certain monotonicity properties. These sufficient conditions help us to examine functions of dilogarithm types [48, 99] which are in the $\mathcal{K}_q$ family. Certain special functions, which are in the starlike and close-to-convex family, have been investigated in [38, 54, 55, 67, 70, 83, 85].

The second idea deals with the famous Bieberbach conjecture problem in univalent function theory [15, 22]. A necessary and sufficient condition for a function f to be in S_q^* is obtained in [42] by means of an integral representation of the function zf'/f which yields the maximum moduli of the coefficients of f. Using this condition, the Bieberbach problem for q-starlike functions has been solved in the following form.

Theorem A. [**42**, Theorem 1.18] *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ belongs to the class $\mathcal{S}_q^*$, then $|a_n| \leq c_n$, with equality holds for all n if and only if f is a rotation of*

$$k_q(z) := z \exp\left[\sum_{n=1}^{\infty} \frac{-2\ln q}{1-q^n} z^n\right] = z + \sum_{n=2}^{\infty} c_n z^n, \quad z \in \mathbb{D}.$$

Note that the function k_q plays the role of the Koebe function as defined by (1.2). By differentiating once the above expression for k_q and equating the coefficients of z^{n-1} on both sides, we get a recurrence relation for the c_n:

$$c_2 = \frac{-2\ln q}{1-q}$$

or

$$(n-1)c_n = \frac{-2\ln q}{1-q^{n-1}}(n-1) + \sum_{k=2}^{n-1} \frac{-2\ln q}{1-q^{k-1}} c_{n+1-k}(k-1), \quad n \geq 3.$$

It can be easily verified that Theorem A turns into the famous conjecture of Bieberbach (known as the Bieberbach-de Branges Theorem) for the class $\mathcal{S}^*$ if $q \to 1^-$. Comparing with the Bieberbach-de Branges Theorem for close-to-convex functions, one would expect that Theorem A also holds for q-close-to-convex functions. However, this remains an open problem. Indeed, in this chapter, we obtain an optimal coefficient bound for q-close-to-convex functions leading to the Bieberbach-de Branges theorem for close-to-convex functions when $q \to 1^-$. Finally, we collect a few consequences of the Bieberbach-de Branges Theorem for $\mathcal{K}_q$ involving the nine starlike functions considered above.

2.2. Conditions for $f(z) = z + \sum_{n=2}^{\infty} A_n z^n$ to be in $\mathcal{K}_q$

In this section, we obtain several sufficient conditions for $f(z) = z + \sum_{n=2}^{\infty} A_n z^n$ to be in $\mathcal{K}_q$, where the coefficients A_n are real, non-negative and connected with certain monotone properties. Similar investigations for the class of close-to-convex functions are conducted in [**1, 52**] (see references there in for initial contributions of Fejér, Szegö and Robertson in this direction).

Rewriting the representation of f, we get

$$f(z) = \sum_{n=0}^{\infty} A_n z^n \quad (A_0 = 0,\ A_1 = 1). \tag{2.1}$$

If f is of the form (2.1), then a simple computation yields

$$(D_q f)(z) = 1 + \sum_{n=2}^{\infty} \frac{A_n(1-q^n)}{1-q}\, z^{n-1} \tag{2.2}$$

for all $z \in \mathbb{D}$. With this, we now collect a number of sufficient conditions for functions to be in $\mathcal{K}_q$.

Lemma 2.1. [**72**, Lemma 1.1(1)] *Let f be of the form* (2.1)*. Suppose that $\sum_{n=1}^{\infty} |B_{n+1} - B_n| \leq 1$, with $B_n = A_n(1-q^n)/(1-q)$. Then $f \in \mathcal{K}_q$ with $g(z) = z/(1-z)$.*

As a consequence of Lemma 2.1, we obtain Theorem 1.3 (see Chapter 1).

Proof of Theorem 1.3. We know that

$$\sum_{n=1}^{\infty} |B_{n+1} - B_n| = \lim_{k\to\infty} \sum_{n=1}^{k} |B_{n+1} - B_n|.$$

If $1 \geq B_2 \geq \cdots \geq B_n \geq \cdots \geq 0$, we see that

$$\lim_{k\to\infty} \sum_{n=1}^{k} |B_{n+1} - B_n| = \lim_{k\to\infty} (B_1 - B_{k+1}) \leq B_1 = 1.$$

Similarly, if $1 \leq B_2 \leq \cdots \leq B_n \leq \cdots \leq 2$, then we get the bound $\sum_{n=1}^{\infty} |B_{n+1} - B_n| \leq 1$. Thus, by Lemma 2.1, we obtain the assertion of our theorem. $\square$

Example. *The quantum dilogarithm* function is defined by

$$Li_2(z;q) = \sum_{n=1}^{\infty} \frac{z^n}{n(1-q^n)}, \quad |z| < 1,\ 0 < q < 1.$$

Note that this function is studied by Kirillov [**48**] (see also [**99**, p.28]) and is a q-deformation of the ordinary *dilogarithm function* [**48**] defined by $Li_2(z) = \sum_{n=1}^{\infty}(z^n/n^2)$, $|z| < 1$, in the sense that

$$\lim_{\epsilon\to 0} \epsilon Li_2(z; e^{-\epsilon}) = Li_2(z).$$

By Theorem 1.3, one can verify that the function $(1-q)Li_2(z;q) \in \mathcal{K}_q$.

Proof of Theorem 1.4. Starting with $|B_n|$, we see that

$$|B_n| = \left|\sum_{k=1}^{n}(B_k - B_{k-1}) + 1\right| \leq \sum_{k=1}^{\infty}|B_k - B_{k-1}| + 1 \leq 2.$$

Hence, for all $n \geq 2$, we have

$$\left|\frac{A_n(1-q^n)}{1-q} - \frac{A_{n-1}(1-q^{n-1})}{1-q}\right| \leq 2.$$

Now, by the triangle inequality, we see that

$$\begin{aligned}\left|\frac{A_n(1-q^n)}{1-q}\right| &= \left|\frac{A_n(1-q^n)}{1-q} - \frac{A_{n-1}(1-q^{n-1})}{1-q} + \frac{A_{n-1}(1-q^{n-1})}{1-q}\right. \\ &\qquad \left. - \frac{A_{n-2}(1-q^{n-2})}{1-q} + \cdots + \frac{A_2(1-q^2)}{1-q} - 1 + 1\right| \\ &\leq 2(n-1) + 1 = 2n - 1\end{aligned}$$

and so $|A_n| \leq (2n-1)/(1+q+\cdots+q^{n-1})$. By applying the root test, one can see that the radius of convergence of $\sum_{n=0}^{\infty} A_n z^n$ is not less than unity. Therefore, $f \in \mathcal{A}$.

Since f is of the form (2.1), by using (2.2) we compute

$$\begin{aligned}&(1-z)^2(D_q f)(z) \\ &\quad = 1 + \frac{A_2(1-q^2)}{1-q}z - 2z \\ &\qquad + \sum_{n=3}^{\infty}\left[\frac{A_n(1-q^n)}{1-q} - \frac{2A_{n-1}(1-q^{n-1})}{1-q} + \frac{A_{n-2}(1-q^{n-2})}{1-q}\right]z^{n-1}.\end{aligned}$$

By the definition of B_n as given in the hypothesis, we have

$$(1-z)^2(D_q f)(z) = 1 + (B_1 - 1)z + \sum_{n=3}^{\infty}(B_{n-1} - B_{n-2})z^{n-1}.$$

Hence,

$$\begin{aligned}\frac{1}{1-q} - \left|(1-z)^2(D_q f)(z) - \frac{1}{1-q}\right| &= \frac{1}{1-q} - \left|1 + (B_1 - 1)z\right. \\ &\qquad \left. + \sum_{n=3}^{\infty}(B_{n-1} - B_{n-2})z^{n-1} - \frac{1}{1-q}\right| \\ &\geq 1 - |B_1 - 1| - \sum_{n=3}^{\infty}|B_{n-1} - B_{n-2}| \geq 0\end{aligned}$$

if $\sum_{n=2}^{\infty}|B_{n-1}-B_{n-2}|\leq 1$. This proves the assertion of our theorem. $\square$

Theorem 2.2. *Let f be defined by $f(z)=z+\sum_{n=2}^{\infty}A_{2n-1}z^{2n-1}$ and suppose that*

$$\sum_{n=1}^{\infty}|B_{2n-1}-B_{2n+1}|\leq 1,\quad B_n=\frac{A_n(1-q^n)}{1-q}.$$

Then $f\in\mathcal{K}_q$ with $g(z)=z/(1-z^2)$.

Proof. First of all we shall prove that $f(z)=z+\sum_{n=2}^{\infty}A_{2n-1}z^{2n-1}\in\mathcal{A}$. For this, we estimate

$$|B_{2n+1}|=\left|\sum_{k=1}^{n}(B_{2k-1}-B_{2k+1})-1\right|\leq 2$$

so that $|A_n|\leq 2/(1+q+\cdots q^{n-1})$. By applying the root test, one can see that the radius of convergence of the series expansion of f is not less than 1. Therefore, $f\in\mathcal{A}$.

Since $f(z)=z+\sum_{n=2}^{\infty}A_{2n-1}z^{2n-1}$, by (1.4) we get

$$(1-z^2)(D_qf)(z)=1-\sum_{n=1}^{\infty}\left[\frac{A_{2n-1}(1-q^{2n-1})}{1-q}-\frac{A_{2n+1}(1-q^{2n+1})}{1-q}\right]z^{2n}.$$

Note that $B_n=A_n(1-q^n)/(1-q)$. So, we have

$$\frac{1}{1-q}-\left|(1-z^2)(D_qf)(z)-\frac{1}{1-q}\right|\geq 1-\sum_{n=1}^{\infty}|B_{2n-1}-B_{2n+1}|\geq 0,$$

whenever $\sum_{n=1}^{\infty}|B_{2n-1}-B_{2n+1}|\leq 1$. This proves our conclusion. $\square$

By Theorem 2.2, we immediately have the following result which generalizes a result of MacGregor (see [**52**, Theorem 2]).

Theorem 2.3. *Let $\{A_n\}$ be a sequence of real numbers and set $B_n=A_n(1-q^n)/(1-q)$ for all $n\geq 1$. Suppose that*

$$1\geq B_3\geq B_5\geq\cdots\geq B_{2n-1}\geq\cdots\geq 0$$

or

$$1\leq B_3\leq B_5\leq\cdots\leq B_{2n-1}\leq\cdots\leq 2.$$

Then $f(z)=z+\sum_{n=2}^{\infty}A_{2n-1}z^{2n-1}\in\mathcal{K}_q$ with $g(z)=z/(1-z^2)$.

Lemma 2.4. [72, Lemma 1.1(4)] *Let f be defined by* (2.1) *and suppose that*
$$\sum_{n=2}^{\infty} |B_n - B_{n-2}| \leq 1, \quad B_n = \frac{A_n(1-q^n)}{(1-q)}.$$
Then $f \in \mathcal{K}_q$ with $g(z) = z/(1-z^2)$.

Lemma 2.4 leads to the following sufficient conditions for functions to be in $\mathcal{K}_q$.

Theorem 2.5. *Let $\{A_n\}$ be a sequence of real numbers such that $A_1 = 1$ and set*
$$B_n = \frac{A_n(1-q^n)}{1-q}$$
for all $n \geq 1$. Suppose that
$$1 \geq B_1 + B_2 \geq \cdots \geq B_{n-1} + B_n \geq \cdots \geq 0$$
or
$$1 \leq B_1 + B_2 \leq \cdots \leq B_{n-1} + B_n \leq \cdots \leq 2.$$
Then $f(z) = z + \sum_{n=2}^{\infty} A_n z^n \in \mathcal{K}_q$ with $g(z) = z/(1-z^2)$.

Proof. We know that
$$\sum_{n=2}^{\infty} |B_n - B_{n-2}| = \lim_{k\to\infty} \sum_{n=2}^{k} |B_n - B_{n-2}|.$$
If $1 \geq B_1 + B_2 \geq \cdots \geq B_{n-1} + B_n \geq \cdots \geq 0$, we see that
$$\lim_{k\to\infty} \sum_{n=2}^{k} |B_n - B_{n-2}| = \lim_{k\to\infty} (1 - B_{k-1} - B_k) \leq 1 + 0 = 1.$$
Similarly, if $1 \leq B_1 + B_2 \leq \cdots \leq B_{n-1} + B_n \leq \cdots \leq 2$, then we get $\sum_{n=2}^{\infty} |B_n - B_{n-2}| \leq 1$. Thus, by Theorem 2.4, the proof is complete. □

As a consequence of Theorem 2.5, one can obtain the following new criteria for functions to be in the close-to-convex family.

Theorem 2.6. *Let $\{a_n\}$ be a sequence of real numbers such that $a_1 = 1$ and set $b_n = na_n$ for all $n \geq 1$. Suppose that*
$$1 \geq b_1 + b_2 \geq \cdots \geq b_{n-1} + b_n \geq \cdots \geq 0$$

or

$$1 \leq b_1 + b_2 \leq \cdots \leq b_{n-1} + b_n \leq \cdots \leq 2.$$

Then $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *is close-to-convex with* $g(z) = z/(1-z^2)$.

Lemma 2.7. [72, Lemma 1.1(2)] *Let* f *be defined by* (2.1) *and suppose that*

$$\sum_{n=1}^{\infty} |B_{n-1} - B_n + B_{n+1}| \leq 1, \quad B_n = \frac{A_n(1-q^n)}{1-q}.$$

Then $f \in \mathcal{K}_q$ *with* $g(z) = z/(1-z+z^2)$.

Lemma 2.7 yields the following sufficient condition.

Theorem 2.8. *Let* $\{A_n\}$ *be a sequence of real numbers such that* $A_1 = 1$ *and set* $B_n = A_n(1-q^n)/(1-q)$ *for all* $n \geq 1$. *Suppose that either*

$$\begin{aligned} 0 \geq B_2 - B_1 \geq B_3 \geq B_2 + B_4 \geq B_2 + B_3 + B_5 \geq \cdots \\ \geq B_2 + B_3 + B_4 + \cdots + B_{n-1} + B_{n+1} \geq -1 \end{aligned} \tag{2.3}$$

or

$$\begin{aligned} 0 \leq B_2 - B_1 \leq B_3 \leq B_2 + B_4 \leq B_2 + B_3 + B_5 \leq \cdots \\ \leq B_2 + B_3 + B_4 + \cdots + B_{n-1} + B_{n+1} \leq 1. \end{aligned} \tag{2.4}$$

Then $f(z) = z + \sum_{n=2}^{\infty} A_n z^n \in \mathcal{K}_q$ *with* $g(z) = z/(1-z+z^2)$.

Proof. We know that

$$\sum_{n=1}^{\infty} |B_{n-1} - B_n + B_{n+1}| = \lim_{k\to\infty} \sum_{n=1}^{k} |B_{n-1} - B_n + B_{n+1}|.$$

If (2.3) holds, on the one hand we see that

$$\lim_{k\to\infty} \sum_{n=1}^{k} |B_{n-1} - B_n + B_{n+1}| = \lim_{k\to\infty} -(B_2 + B_3 + B_4 + \cdots + B_{k-1} + B_{k+1}) \leq 1.$$

On the other hand, if (2.4) holds, then similarly $\sum_{n=1}^{\infty} |B_{n-1} - B_n + B_{n+1}| \leq 1$. Thus, by Theorem 2.7, the proof is complete. □

As a consequence of Theorem 2.8, one can obtain the following new criteria for functions to be in the close-to-convex family.

Theorem 2.9. *Let $\{a_n\}$ be a sequence of real numbers such that $a_1 = 1$ and set $b_n = na_n$ for all $n \geq 1$. Suppose that*

$$0 \geq b_2 - b_1 \geq b_3 \geq b_2 + b_4 \geq b_2 + b_3 + b_5 \geq \cdots$$
$$\geq b_2 + b_3 + b_4 + \cdots + b_{n-1} + b_{n+1} \geq -1$$

or

$$0 \leq b_2 - b_1 \leq b_3 \leq b_2 + b_4 \leq b_2 + b_3 + b_5 \leq \cdots$$
$$\leq b_2 + b_3 + b_4 + \cdots + b_{n-1} + b_{n+1} \leq 1.$$

Then $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is in the close-to-convex family with $g(z) = z/(1 - z + z^2)$.

2.3. The Bieberbach-de Branges Theorem for $\mathcal{K}_q$

A necessary and sufficient condition for membership in $\mathcal{S}_q^*$ was obtained in [**42**, Theorem 1.5]: $f \in \mathcal{S}_q^*$ *if and only if* $|f(qz)/f(z)| \leq 1$ *for all* $z \in \mathbb{D}$.

A similar characterization for functions in $\mathcal{K}_q$ is

Lemma 2.10. *$f \in \mathcal{K}_q$ if and only if there exists $g \in \mathcal{S}^*$ such that*

$$\frac{|g(z) + f(qz) - f(z)|}{|g(z)|} \leq 1 \quad \textit{for all } z \in \mathbb{D}.$$

Proof. This follows directly after substituting the formula for $D_q f$ in the equation (1.5). □

Lemma 2.10 will be crucial to get coefficient bounds for series representation of functions in the class $\mathcal{K}_q$; in other words, we analyze the Bieberbach-de Branges Theorem for the class of q-close-to-convex functions. The Bieberbach conjecture for close-to-convex functions was proved by Reade [**74**] (see also [**33**] for more details). It states that *if $f \in \mathcal{K}$, then $|a_n| \leq n$ for all $n \geq 2$.*

We now prove the Bieberbach-de Branges Theorem for functions in the q-close-to-convex family which stated in Chapter 1 by Theorem 1.6.

Proof of Theorem 1.6. Since $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{K}_q$, by Lemma 2.10 there exists $w : \mathbb{D} \to \overline{\mathbb{D}}$ such that

$$g(z) + f(qz) - f(z) = w(z)g(z), \tag{2.5}$$

where $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ and $w(z) = q + \sum_{n=1}^{\infty} w_n z^n$. Clearly $w(0) = q$. By assuming $a_1 = 1 = b_1$, we have

$$\sum_{n=1}^{\infty}(b_n + a_n q^n - a_n)z^n = \sum_{n=1}^{\infty} q b_n z^n + \sum_{n=2}^{\infty}\left(\sum_{k=1}^{n-1} w_{n-k} b_k\right) z^n.$$

Equating the coefficients of z^n, for $n \geq 2$ we obtain

$$a_n(q^n - 1) = b_n(q-1) + \sum_{k=1}^{n-1} w_{n-k} b_k.$$

From the classical result [21], one can verify that $|w_n| \leq 1 - |w_0|^2 = 1 - q^2$ for all $n \geq 1$. Since $g \in S^*$, we get

$$|a_n| \leq \frac{1-q}{1-q^n}\left[n + (1+q)\sum_{k=1}^{n-1} k\right] \qquad \text{for all } n \geq 2.$$

This proves the conclusion of our theorem. $\square$

It is easy to see, by the usual ratio test, that the series

$$z + \sum_{n=2}^{\infty} \frac{1-q}{1-q^n}\left[n + \frac{n(n-1)}{2}(1+q)\right] z^n \tag{2.6}$$

converges for $|z| < 1$. Indeed, we can ascertain by using the convergence factor for the series $\sum_{n=1}^{\infty} z^n/(1-q^n)$ (see [89, 3.2.2.1]) that the series given by (2.6) converges to the function

$$\frac{1+q}{2} z^2 \frac{d^2\Psi(q;z)}{dz^2} + z\frac{d\Psi(q;z)}{dz},$$

where $\Psi(q;z) := z\Phi[q,q;q^2;q,z]$ represents the corresponding Heine hypergeometric function. Note that the q-hypergeometric series was developed by Heine [41] as a generalization of the well-known Gauss hypergeometric series:

$$\Phi[a,b;c;q,z] = \sum_{n=0}^{\infty} \frac{(a;q)_n(b;q)_n}{(c;q)_n(q;q)_n} z^n, \quad |q|<1,\ 1\neq cq^n,\ |z|<1,$$

where the q-shifted factorial $(a;q)_n$ is defined by

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}) \ \text{ and } \ (a;q)_0 = 1.$$

This is also known as the basic hypergeometric series and its convergence function is known as the basic hypergeometric function. We refer to [89] for these notation. For history of q-series related calculus and their applications, we recommend [23].

Due to Friedman 's result mentioned in the Introduction, we now study the special cases of Theorem 1.6 with respect to the nine functions having integer coefficients. However, in this situation, it is enough to consider the identity function and four other functions which contain factors $1-z$ instead of $1\pm z$ in the denominator. In particular, Theorem 1.6 reduces to the following corollaries below. We provide proofs of last two as they involve variations in the exponents, whereas the first three corollaries follow directly after making the corresponding substitution for the starlike functions g.

Corollary 2.11. *If $f\in\mathcal{K}_q$ with the Koebe function $g(z)=z/(1-z)^2$, then for all $n\geq 2$ we have*

$$|a_n| \leq \frac{1-q}{1-q^n}\left[n+(1+q)\,\frac{n(n-1)}{2}\right].$$

If $f\in\mathcal{K}$ with $g(z)=z$, then for all $n\geq 2$ it is well-known that $|a_n|\leq 2/n$. As a generalization, we have the following:

Corollary 2.12. *If $f\in\mathcal{K}_q$ with $g(z)=z$, then for all $n\geq 2$ we have $|a_n|\leq (1-q^2)/(1-q^n)$.*

Here we note that the series $z+\sum_{n=2}^{\infty}(1-q^2)/(1-q^n)z^n$ converges to the Heine hypergeometric function $(z+qz)\Phi[q,q;q^2;q,z]-qz = z+z^2\Phi[q^2,q^2;q^3;q^2,z]$, as follows from [89, 3.2.2, pp. 91].

If $f \in \mathcal{K}$ with $g(z) = z/(1-z)$, then for all $n \geq 2$ it is known that $|a_n| \leq (2n-1)/n$. We find the following analogous result:

Corollary 2.13. *If $f \in \mathcal{K}_q$ with $g(z) = z/(1-z)$, then for all $n \geq 2$ we have*

$$|a_n| \leq \frac{1-q}{1-q^n}\,[n + q(n-1)].$$

One can similarly verify that the series $z + \sum_{n=2}^{\infty} \frac{1-q}{1-q^n}[n + q(n-1)]$ converges to the function $z(1+q)\frac{d}{dz}\Psi(q;z) - q\Psi(q;z)$, where $\Psi(q;z) := z\Phi[q,q;q^2;q,z]$ represents the corresponding Heine hypergeometric function.

If $f \in \mathcal{K}$ with $g(z) = z/(1-z^2)$, then for all $m \geq 1$ it is known that

$$|a_n| \leq \begin{cases} 1, & \text{if } n = 2m-1, \\ 1, & \text{if } n = 2m. \end{cases}$$

As a generalization, we now state the following corollary along with an outline of its proof:

Corollary 2.14. *If $f \in \mathcal{K}_q$ with $g(z) = z/(1-z^2)$, then for all $m \geq 1$ we have*

$$|a_n| \leq \begin{cases} \dfrac{1-q}{1-q^n}\left(\dfrac{n}{2}(1+q) + \dfrac{1}{2}(1-q)\right), & \textit{if } n = 2m-1, \\ \left(\dfrac{1-q^2}{1-q^n}\right)\dfrac{n}{2}, & \textit{if } n = 2m. \end{cases}$$

Proof. Since $g(z) = z/(1-z^2) = \sum_{n=1}^{\infty} z^{2n-1}$, by (2.5) we get

$$\sum_{n=1}^{\infty}(q^n-1)a_n z^n = (q-1)\sum_{n=1}^{\infty} z^{2n-1} + \left(\sum_{n=1}^{\infty} z^{2n-1}\right)\left(\sum_{n=1}^{\infty} w_n z^n\right).$$

This is equivalent to

$$\begin{aligned} \sum_{n=1}^{\infty}(q^n-1)a_n z^n =& (q-1)\sum_{n=1}^{\infty} z^{2n-1} + \sum_{n=2}^{\infty}\left(\sum_{k=1}^{n-1} w_{2k}\right) z^{2n-1} \\ &+ \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} w_{2k-1}\right) z^{2n}. \end{aligned} \tag{2.7}$$

In order to prove the required optimal bound for $|a_n|$, we equate the coefficients of z^{2n-1} and z^{2n} separately.

In (2.7), first we equate the coefficients of z^{2n-1}, for $n \geq 2$, to get

$$(q^{2n-1}-1)a_{2n-1} = (q-1) + \sum_{k=1}^{n-1} w_{2k}.$$

Since $|w_k| \leq (1-q^2)$ for all $k \geq 1$ and $q \in (0,1)$, we have

$$|a_{2n-1}| \leq \frac{1-q}{(1-q^{2n-1})}\left(-q+(1+q)n\right).$$

Secondly, by equating the coefficients of z^{2n}, for $n \geq 1$, we obtain

$$(q^{2n}-1)a_{2n} = \sum_{k=1}^{n} w_{2k-1},$$

and similarly we get the bound

$$|a_{2n}| \leq \frac{1-q}{(1-q^{2n})}(1+q)n.$$

Thus, we obtain the required optimal bound for $|a_n|$. □

If $f \in \mathcal{K}$ with $g(z) = z/(1-z+z^2)$, then for all $n \geq 2$ it is known that

$$|a_n| \leq \begin{cases} \dfrac{4n+1}{3n}, & \text{if } n = 3m-1, \\[2mm] \dfrac{4}{3}, & \text{if } n = 3m, \\[2mm] \dfrac{4n-1}{3n}, & \text{if } n = 3m+1. \end{cases}$$

As a generalization, we have the following:

Corollary 2.15. *If $f \in \mathcal{K}_q$ with $g(z) = z/(1-z+z^2)$, then for all $m \geq 1$ we have*

$$|a_n| \leq \begin{cases} \dfrac{1-q}{1-q^n}\left(\dfrac{1}{3}(2-q)+\dfrac{2n}{3}(1+q)\right), & \textit{if } \; n = 3m-1, \\[2mm] \dfrac{1-q^2}{1-q^n}\dfrac{2n}{3}, & \textit{if } \; n = 3m, \\[2mm] \dfrac{1-q}{1-q^n}\left(\dfrac{2n}{3}(1+q)+\dfrac{1}{3}(1-2q)\right), & \textit{if } \; n = 3m+1. \end{cases}$$

Proof. By rewriting the function $g(z) = z/(1-z+z^2)$, we obtain

$$g(z) = \frac{z(1+z)}{1+z^3} = \sum_{n=1}^{\infty}(-1)^{n-1}z^{3n-2} + \sum_{n=1}^{\infty}(-1)^{n-1}z^{3n-1}.$$

Then simplifying (2.5), we get

$$
\begin{aligned}
&\sum_{n=1}^{\infty}(q^n-1)a_n z^n \\
&\quad= \quad (q-1)\left(\sum_{n=1}^{\infty}(-1)^{n-1}z^{3n-2}+\sum_{n=1}^{\infty}(-1)^{n-1}z^{3n-1}\right)\\
&\quad\quad + \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}(-1)^{n-k}w_{3k-2}\right)z^{3n-1}+\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}(-1)^{n-k}w_{3k-1}\right)z^{3n}\\
&\quad\quad + \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}(-1)^{n-k}w_{3k}\right)z^{3n+1}+\sum_{n=2}^{\infty}\left(\sum_{k=1}^{n-1}(-1)^{n-k}w_{3k}\right)z^{3n-1}\\
&\quad\quad + \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}(-1)^{n-k}w_{3k-2}\right)z^{3n}+\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}(-1)^{n-k-1}w_{3k-1}\right)z^{3n+1}.
\end{aligned}
\tag{2.8}
$$

First equating the coefficients of z^{3n-1}, for $n \geq 2$, in (2.8), we get

$$(q^{3n-1}-1)a_{3n-1} = (-1)^{n-k}(q-1)+\sum_{k=1}^{n}(-1)^{n-k}w_{3k-2}+\sum_{k=1}^{n}(-1)^{n-k}w_{3k}.$$

Since $|w_k| \leq (1-q^2)$ for all $k \geq 1$ and $q \in (0,1)$, we have

$$|a_{3n-1}| \leq \frac{1-q}{(1-q^{3n-1})}\left(-q+2(1+q)n\right).$$

Next, for all $n \geq 1$, we equate the coefficients of z^{3n} and z^{3n+1} in (2.8), and obtain

$$|a_{3n}| \leq \frac{2(1-q)}{(1-q^{3n})}(1+q)n \quad \text{and} \quad |a_{3n+1}| \leq \frac{(1-q)}{(1-q^{3n+1})}(1+2(1+q)n).$$

Thus, the assertion of our corollary follows. □

Remark. By making use of [**42**, Theorem 1.5], one can also obtain the Bieberbach-de Branges Theorem for $\mathcal{S}_q^*$, as shown below. This also yields a Bieberbach-de Branges type theorem for $\mathcal{S}^*$. However, it differs from Theorem A.

2.4. Appendix

In this section, we verify that a similar technique used in the previous section yields a form of the Bieberbach-de Branges Theorem for $\mathcal{S}_q^*$. This leads to a result of Bieberbach-de Branges type (different from Theorem A !) for the class $\mathcal{S}^*$, when $q \to 1^-$, as well.

Theorem 2.16 (The Bieberbach-de Branges Theorem for $\mathcal{S}_q^*$)**.** *If $f \in \mathcal{S}_q^*$, then for all $n \geq 2$ we have*

$$|a_n| \leq \left(\frac{1-q^2}{q-q^n}\right) \prod_{k=2}^{n-1} \left(1 + \frac{1-q^2}{q-q^k}\right). \tag{2.9}$$

Proof. We know that $f \in \mathcal{S}_q^*$ if and only if

$$|f(qz)/f(z)| \leq 1 \quad \text{for all } z \in \mathbb{D}.$$

Then there exists $w : \mathbb{D} \to \overline{\mathbb{D}}$ such that

$$\frac{f(qz)}{f(z)} = w(z), \quad \text{i.e. } f(qz) = w(z)f(z) \text{ for all } z \in \mathbb{D}.$$

Clearly, $w(0) = q$. In terms of series expansion, we get (with $a_1 = 1$ and $w_0 = q$)

$$\sum_{n=1}^{\infty} a_n q^n z^n = \left(\sum_{n=0}^{\infty} w_n z^n\right)\left(\sum_{n=1}^{\infty} a_n z^n\right) =: \sum_{n=1}^{\infty} c_n z^n,$$

where $c_n := \sum_{k=1}^{n} w_{n-k} a_k = q a_n + \sum_{k=1}^{n-1} w_{n-k} a_k$. Comparing the coefficients of z^n $(n \geq 2)$, we get

$$a_n(q^n - q) = \sum_{k=1}^{n-1} w_{n-k} a_k, \quad \text{for } n \geq 2.$$

Since $|w_n| \leq 1 - |w_0|^2 = 1 - q^2$ for all $n \geq 1$, we obtain

$$|a_n| \leq \frac{1-q^2}{q-q^n} \sum_{k=1}^{n-1} |a_k| \quad \text{for each } n \geq 2.$$

Thus for $n = 2$, one has $|a_2| \leq (1-q^2)/(q-q^2)$, and for $n \geq 3$, we apply a similar technique to estimate $|a_{n-1}|$ and get

$$|a_n| \leq \frac{1-q^2}{q-q^n}\left(1 + \frac{1-q^2}{q-q^{n-1}}\right) \sum_{k=1}^{n-2} |a_k|.$$

Iteratively, we conclude that

$$|a_n| \leq \frac{1-q^2}{q-q^n}\left(1+\frac{1-q^2}{q-q^{n-1}}\right)\left(1+\frac{1-q^2}{q-q^{n-2}}\right)\cdots\left(1+\frac{1-q^2}{q-q^2}\right)$$

for all $n \geq 3$. This completes the proof. □

Remark. One can easily verify that the right hand side of (2.9) approaches n as $q \to 1^-$, which will lead to the Bieberbach-de Branges Theorem for starlike functions [**22**, Theorem 2.14].

We also find that the ratio test easily provides the convergence of the series $z + \sum_{n=2}^{\infty} A_n z^n$ in the subdisk $|z| < q/(q+1-q^2)$, where

$$A_n = \left(\frac{1-q^2}{q-q^n}\right)\prod_{k=1}^{n-2}\left(1+\frac{1-q^2}{q-q^{k+1}}\right).$$

With this, we end this chapter here.

CHAPTER 3

MAXIMAL AREA INTEGRAL PROBLEM FOR UNIVALENT FUNCTIONS

In this chapter we settled an open problem on the Yamashita conjecture for the class $\mathcal{S}^*(A, B)$ that was suggested by Ponnusamy and Wriths in [**71**]. In Section 3.1, we discuss the definitions and brief introduction on area integral problem. The main results and some of their consequences are presented in Section 3.2. In Section 3.3, we present useful lemmas with which we prove our results. We conclude the proofs of our main results of this chapter in Section 3.4. Finally, Section 3.5 deals with a number of open problems.

Results of this chapter published in the articles:

Ponnusamy S., Sahoo S.K., Sharma N.L. (2016), *Maximal area integral problem for certain class of univalent analytic functions*, Mediterr. J. Math., **13**, 607–623. Published online February 12, 2015.

and

Sahoo S.K., Sharma N.L. (2015), *On maximal area integral problem for analytic functions in the starlike family*, J. Class. Anal., **6**(1), 73–84.

3.1. Preliminaries on Area Problems

In Chapter 1, we presented the definition of the class $\mathcal{S}^*(A, B)$ of $f \in \mathcal{A}$ (see (1.7)). It is easy to see that the function

$$k_{A,B}(z) := \begin{cases} ze^{Az} & \text{for } B = 0 \\ z(1+Bz)^{(A/B)-1} & \text{for } B \neq 0 \end{cases} \tag{3.1}$$

belongs to the family $\mathcal{S}^*(A,B)$ and acts the role of extremal function for this family.

If $A = e^{-i\alpha}(e^{-i\alpha} - 2\beta\cos\alpha)$ with $\beta < 1$ and $B = -1$, then $\mathcal{S}^*(A,B)$ reduces to the class $\mathcal{S}_\alpha(\beta)$ of functions f (called *α-spiral-like of order β*) satisfying the condition

$$\mathrm{Re}\left(e^{i\alpha}\frac{zf'(z)}{f(z)}\right) > \beta\cos\alpha, \quad z \in \mathbb{D},$$

and recall that each function in $\mathcal{S}_\alpha(\beta)$ is univalent in $\mathbb{D}$ if $\beta \in [0,1)$ and $\alpha \in (-\pi/2, \pi/2)$ (see [**49**]). This class was introduced and studied by Libera [**49**] in 1967. Clearly, $\mathcal{S}_\alpha(\beta) \subset \mathcal{S}_\alpha := \mathcal{S}_\alpha(0)$ whenever $0 \le \beta < 1$. Functions in $\mathcal{S}_\alpha$ are called *α-spiral-like*. The class $\mathcal{S}_\alpha$ is defined by (1.3). The set $\mathcal{S}p = \cup\{\mathcal{S}_\alpha(0) : \alpha \in (-\pi/2, \pi/2)\}$ is referred as the class of *spiral-like* functions. As remarked in [**49**], spiral-like functions have been used to obtain important counter-examples in geometric function theory (see also [**22**, p. 72 and Theorem 8.11]).

The class $\mathcal{S}^*(A,B)$ with the restriction $-1 \le B < A \le 1$ was introduced and studied by Janowski [**43**] in 1973. The values of zf'/f lie inside the close disk in the right half plane with center $(1-ABr^2)/(1-B^2r^2)$ and radius $(A-B)r/(1-B^2r^2)$ for $|z| = r < 1$ and so, the class $\mathcal{S}^*(A,B)$ becomes a subclass of $\mathcal{S}^*$ whenever $-1 \le B < A \le 1$. In Table 3.1, we listed the certain basic subclasses of the class $\mathcal{S}^*$ that are studied for various choices of the pair (A,B). Set for an abbreviation $q(z) := zf'(z)/f(z)$ and $t(z) := 1 + zf''(z)/f'(z)$.

A general form of the definition of $\mathcal{T}(\lambda,\beta) := \mathcal{S}^*((1-2\beta)\lambda, -\lambda)$, $\lambda \in (0,1]$ is earlier introduced by Aouf (see [**8**, Definition 2]). We see that if $\beta = 0$, then the class $\mathcal{T}(\lambda,\beta)$ turns into the Padmanabhan class $\mathcal{T}(\lambda)$ [**62**] (see Table 3.1). It is evident that $\mathcal{T}(\lambda) \subset \mathcal{S}^*(\beta)$ for all $\lambda, \beta \in (0,1)$ with $\beta \le (1-\lambda)/(1+\lambda)$. Also, the function $g(z) = z/(1-z) \in \mathcal{S}^*(1/2)$ guarantees that this inclusion is proper. From (3.1), one can also verify that

$$k_{(1-2\beta)\lambda,-\lambda}(z) = \frac{z}{(1-\lambda z)^{2(1-\beta)}} =: k_{\lambda,\beta}(z) \text{ and } k_{\lambda,0}(z) =: k_\lambda(z). \tag{3.2}$$

Consequently, $k_{1,\beta}(z) =: k_\beta(z)$ and $k_{1,0}(z) =: k(z)$. The functions $k_{\lambda,\beta}$ & k_λ play the role of extremal functions for $\mathcal{T}(\lambda,\beta)$ & $\mathcal{T}(\lambda)$, respectively. Also, one notes that

$$\mathcal{T}(\lambda,\beta) \subset \mathcal{T}(\lambda) \subset \mathcal{S}^*(\beta) \subset \mathcal{S}^*;\ \mathcal{T}(1,\beta) =: \mathcal{S}^*(\beta) \text{ and } \mathcal{T}(1) =: \mathcal{S}^*.$$

Year	Authors	Classes	Conditions	Subordination form
1921	Nevanlinna [57]	$\mathcal{S}^*(1,-1) =: \mathcal{S}^*$	$\operatorname{Re} q(z) > 0$	$q(z) \prec \dfrac{1+z}{1-z}$
1936	Robertson [77]	$\mathcal{S}^*(1-2\beta,-1)$ $=: \mathcal{S}^*(\beta)$, $\beta \in [0,1)$	$\operatorname{Re} q(z) > \beta$	$q(z) \prec \dfrac{1+(1-2\beta)z}{1-z}$
1968	Singh [86]	$\mathcal{S}^*(1,0)$	$\lvert q(z)-1\rvert < 1$	$q(z) \prec 1+z$
1968	Padmanabhan [62]	$\mathcal{S}^*(\lambda,-\lambda) =: \mathcal{T}(\lambda)$ for $\lambda \in (0,1]$	$\left\lvert\dfrac{q(z)-1}{q(z)+1}\right\rvert < \lambda$	$q(z) \prec \dfrac{1+\lambda z}{1-\lambda z}$
1974	Singh and Singh [87]	$\mathcal{S}^*\left(1,\frac{1-\alpha}{\alpha}\right)$ for $\alpha \geq \frac{1}{2}$	$\lvert q(z)-\alpha\rvert < \alpha$	$p(z) \prec \dfrac{1+z}{1+\frac{1-\alpha}{\alpha}z}$
1988	Aouf [8]	$\mathcal{S}^*((1-2\beta)\lambda,-\lambda)$ $=: \mathcal{T}(\lambda,\beta)$ for $\lambda \in (0,1]$, $\beta \in [0,1)$	$\left\lvert\dfrac{q(z)-1}{q(z)+1-2\beta}\right\rvert < \lambda$	$q(z) \prec \dfrac{1+(1-2\beta)\lambda z}{1-\lambda z}$

TABLE 3.1. Analytic and subordination forms of $\mathcal{S}^*(A,B)$ for different A and B

The interest to study area problems comes from computing areas of certain regions in the complex plane. In general it is a difficult problem to find area of an arbitrary region. However, our problem finds exact area formula of regions that are images of $\mathbb{D}$ under certain functions. The classical Parseval-Gutzmer formula for a function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ analytic in $\overline{\mathbb{D}}_r$ states that

$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2\, d\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}.$$

With the help of this formula, one can get the area $\Delta(r,f)$ of the form (1.6) (see Section 1.3) in terms of the coefficients of f, $f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$. Thus,

$$\Delta(r,f) = \iint_{\mathbb{D}_r} |f'(z)|^2\, dxdy = \pi \sum_{n=1}^{\infty} n|a_n|^2 r^{2n}, \tag{3.3}$$

which is called the Dirichlet integral of f. Computing this area is known as the *area problem for the functions of type f*. Thus, a function has a finite Dirichlet integral exactly when its image has finite area (counting multiplicities). All polynomials and, more generally, all functions $f \in \mathcal{A}$ for which f' is bounded on $\mathbb{D}$ are Dirichlet finite.

In 1990, Yamashita [98] discussed the extremal problems

$$A(r) = \max_{f\in\mathcal{N}} \Delta\left(r, \frac{z}{f}\right), \quad r \in (0, 1], \tag{3.4}$$

where $\mathcal{N}$ represents some geometrically motivated subclass of $\mathcal{S}$. This extremal problem also called the maximal area integral problem for functions of type z/f when f ranges over an analytic family. In [98], Yamashita conjectured that $A(r) = \pi r^2$, if $\mathcal{N}$ is the class of convex functions. In 2013, this conjecture was settled by Obradović et al. in [58]. In fact the conjecture has been solved for a wider class of functions (the class of starlike functions of order β, $0 \le \beta < 1$), which includes the class of convex functions (see also [59, 71]).

A general problem on the Yamashita conjecture for the class $\mathcal{S}^*(A, B)$ was suggested in [71, p. 725] (see also [59]). In this chapter, we solve the problem in complete generality for the full class $\mathcal{S}^*(A, B)$ and the main results are stated in Section 3.2. Partially, the authors established the maximum area problem for the functions of type z/f when $f \in \mathcal{T}(\lambda, \beta)$.

3.2. Main Theorems

We now state our main results and their proofs will be given in Section 3.4.

Theorem 3.1. *Let $f \in \mathcal{S}^*(A, 0)$ for $0 < |A| \le 1$. Then we have*

$$\max_{f\in\mathcal{S}^*(A,0)} \Delta\left(r, \frac{z}{f}\right) = E_{A,0}(r) \quad \text{for } 0 < r \le 1,$$

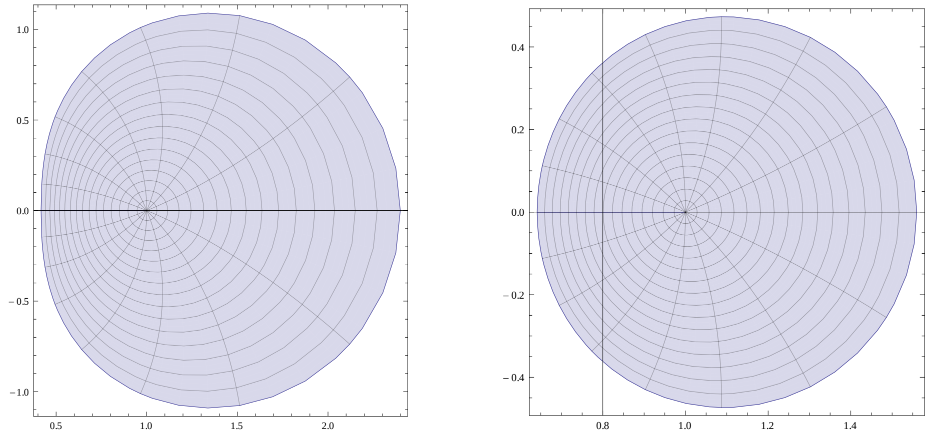
where $E_{A,0}(r) = \pi|A|^2 r^2 {}_0F_1(2; |A|^2 r^2)$. The maximum is attained by the rotations of the function $k_{A,0}(z) = ze^{Az}$.

If $A = 1$ in Theorem 3.1, then we get

Corollary 3.2. *Let $f \in \mathcal{S}^*(1, 0)$. Then we have*

$$\max_{f\in\mathcal{S}^*(1,0)} \Delta\left(r, \frac{z}{f}\right) = \pi r^2 {}_0F_1(2; r^2) \quad \text{for } 0 < r \le 1,$$

where the maximum is attained by the rotations of the function $k_{1,0}(z) = ze^{z}$.



The image domain $g_{5/6,0}(\mathbb{D})$ The image domain $g_{1/6,0}(\mathbb{D})$

FIGURE 3.1. Images of the unit disk under $g_{5/6,0}$ and $g_{1/6,0}$.

Our main theorem for $B \neq 0$ presented in Chapter 1 by Theorem 1.7.

Note that Theorem 3.1 and Theorem 1.7 generalize the results proved in [**58, 98**]. To see the bounds for the Dirichlet finite function, we write

$$E_{A,0}(1) = \pi|A|^2 \sum_{n=0}^{\infty} \frac{1}{(1)_n(2)_n}|A|^{2n}$$

and

$$E_{A,B}(1) = \pi|\overline{A} - B|^2 \sum_{n=0}^{\infty} \frac{(A/B)_n(\overline{A}/B)_n}{(1)_n(2)_n} B^{2n}.$$

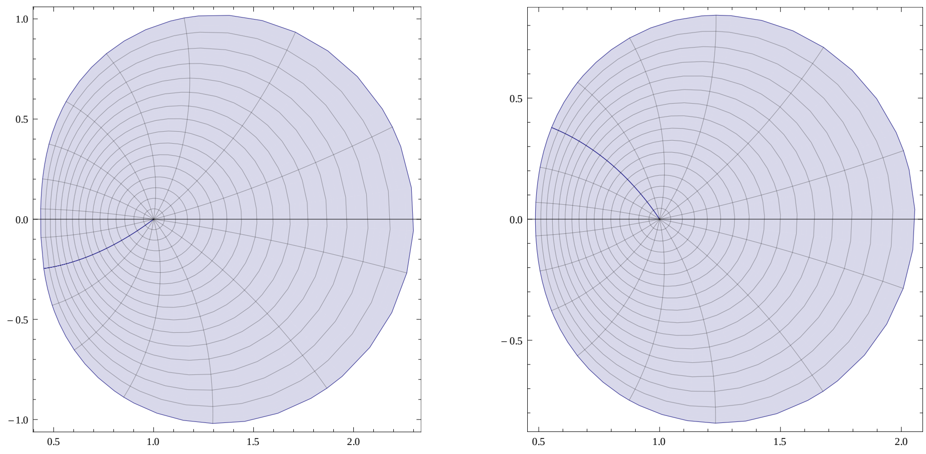
For certain values of A and B, the images of the unit disk under the extremal functions

$$g_{A,0}(z) := z/k_{A,0}(z) = e^{-Az} \quad \text{and} \quad g_{A,B}(z) := z/k_{A,B}(z) = (1+Bz)^{1-A/B},$$

and numerical values of $E_{A,0}(1)$ and $E_{A,B}(1)$ are described in Figures 3.1–3.4 and in Table 3.2, respectively. We remind the reader that for $B = -1$, $E_{A,B}(1)$ is finite only if $2 > \operatorname{Re}((A + \overline{A})/B)$, i.e. if $\operatorname{Re} A > -1$.

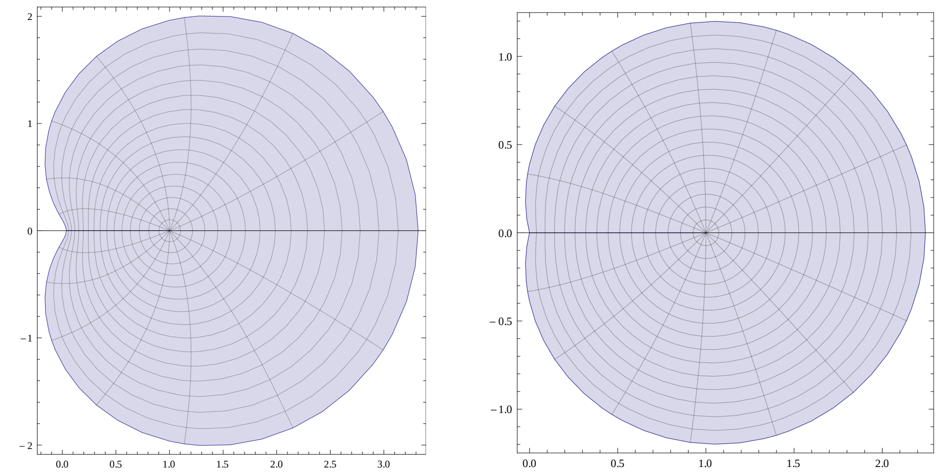
We now state certain consequences of Theorem 1.7 for the several classes introduced by several authors (refer Table 3.1). It is a simple exercise to see that Möbius transformation $w = \phi(z)$ defined by

$$w = \phi(z) = \frac{1 + Az}{1 + Bz}$$



The image domain $g_{2/3+i/2,0}(\mathbb{D})$ The image domain $g_{(2-3i)/5,0}(\mathbb{D})$

FIGURE 3.2. Images of the unit disk under $g_{2/3+i/2,0}$ and $g_{(2-3i)/5,0}$.



The image domain $g_{5/6,-4/5}(\mathbb{D})$ The image domain $g_{1/6,-1}(\mathbb{D})$

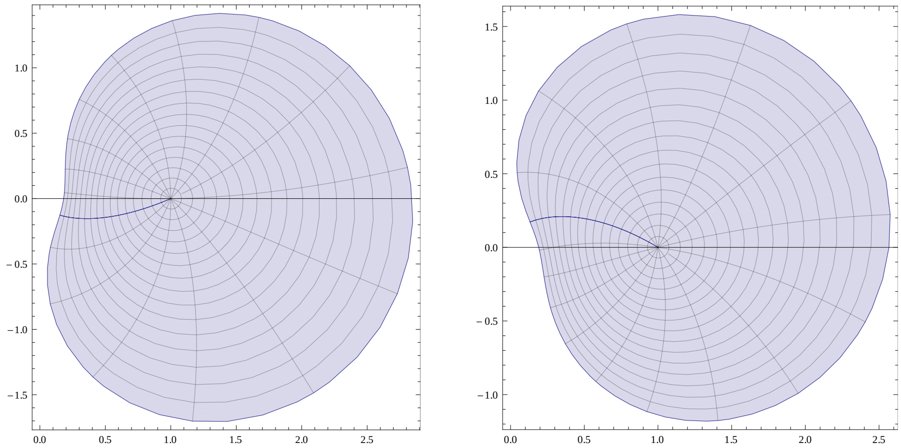
FIGURE 3.3. Images of the unit disk under $g_{5/6,-4/5}$ and $g_{1/6,-1}$.

maps the unit disk $\mathbb{D}$ onto the half-plane

$$\operatorname{Re}\left((1+\overline{A})w\right) > \frac{1-|A|^2}{2}$$

whenever $B=-1$ and $A \neq -1$. In particular, if $A = e^{i\alpha}(e^{i\alpha} - 2\beta\cos\alpha)$ $(\beta < 1)$, then as remarked in the introduction, the last condition reduces to

$$\operatorname{Re}\left(e^{-i\alpha}w\right) > \beta\cos\alpha.$$



The image domain $g_{\frac{2}{3}+\frac{i}{2},\frac{-1}{2}}(\mathbb{D})$ The image domain $g_{\frac{(2-3i)}{5},\frac{-3}{5}}(\mathbb{D})$

FIGURE 3.4. Images of the unit disk under $g_{2/3+i/2,-1/2}$ and $g_{(2-3i)/5,-3/5}$.

A	**Approximate Values of** $E_{A,0}(1)$	B	**Approximate Values of** $E_{A,B}(1)$
$5/6$	3.03211	$-4/5$	11.2917
$1/6$	0.0884841	-1	4.34607
$2/3+i/2$	3.03211	$-1/2$	6.90284
$(2-3i)/5$	2.09682	$-3/5$	5.4645

TABLE 3.2. Approximate values of $E_{A,0}(1)$ and $E_{A,B}(1)$

If $-1 < B \le 0$ and $A \neq B$, then ϕ maps $\mathbb{D}$ onto the disk

$$\left| w - \frac{1-\overline{A}B}{1-B^2} \right| < \frac{|A-B|}{1-B^2}.$$

This observation helps us to formulate important special cases.

If $A = (1-2\beta)\lambda$ and $B = -\lambda$ in Theorem 1.7, then we get

Corollary 3.3. *Let $f \in \mathcal{T}(\lambda,\beta) := \mathcal{S}^*((1-2\beta)\lambda, -\lambda)$ for $0 < \lambda \le 1$ and $0 \le \beta < 1$. Then we have*

$$\max_{f\in\mathcal{T}(\lambda,\beta)} \Delta\left(r, \frac{z}{f}\right) = 4\pi\lambda^2(1-\beta)^2 r^2 {}_2F_1(2\beta-1, 2\beta-1; 2; \lambda^2 r^2), \quad |z| < r$$

for all r, $0 < r \le 1$. The maximum is attained by the rotations of $k_{\lambda,\beta}$ as defined by (3.2).

The case $\beta = 0$ of Corollary 3.3 (i.e. $A = \lambda$ and $B = -\lambda$ of Theorem 1.7) gives

Example 3.4. If $f \in \mathcal{T}(\lambda) := \mathcal{S}^*(\lambda, -\lambda)$ for some $0 < \lambda \leq 1$, then one has

$$\max_{f \in \mathcal{T}(\lambda)} \Delta\left(r, \frac{z}{f}\right) = 2\pi\lambda^2 r^2 (2 + \lambda^2 r^2) \text{ for all } 0 < r \leq 1.$$

The maximum is attained by the rotation of k_λ as defined by (3.2).

If we choose $\lambda = 1$ in Corollary 3.3, we get

Corollary 3.5. ([**58**, Theorem 3]) *Let $f \in \mathcal{S}^*(\beta) := \mathcal{T}(1, \beta)$ for some $0 \leq \beta < 1$. Then we have*

$$\max_{f \in \mathcal{S}^*(\beta)} \Delta\left(r, \frac{z}{f}\right) = 4\pi(1-\beta)^2 r^2 {}_2F_1(2\beta - 1, 2\beta - 1; 2; r^2) \quad \textit{for } 0 < r \leq 1,$$

where the maximum is attained by the rotations of k_β as defined by (3.2).

We remark that when $A = 1$ and $B = -1$, Theorem 1.7 turns into [**58**, Theorem A]. If we choose $A = e^{-i\alpha}(e^{-i\alpha} - 2\beta \cos\alpha)$ and $B = -1$ in Theorem 1.7, then we get Yamashita's extremal problem for the class $\mathcal{S}_\alpha(\beta)$ (see [**71**, Theorem 3]). If we take $A = 1$ and $B = (1-\alpha)/\alpha$, $\alpha \geq 1/2$, then Theorem 1.7 yields

Corollary 3.6. *If $\alpha \geq 1/2$ and $f \in \mathcal{S}^*(1, (1-\alpha)/\alpha)$, then we have*

$$\max_{f \in \mathcal{S}^*(1,(1-\alpha)/\alpha)} \Delta\left(r, \frac{z}{f}\right) = \pi\left(2 - \frac{1}{\alpha}\right)^2 r^2 {}_2F_1\left(\frac{\alpha}{1-\alpha}, \frac{\alpha}{1-\alpha}; 2; \left(\frac{\alpha - 1}{\alpha}\right)^2 r^2\right),$$

for $0 < r \leq 1$, where the maximum is attained by the rotations of the function $k_{1,(1-\alpha)/\alpha}$ as defined by (3.1).

If $A = (b^2 - a^2 + a)/b$ and $B = (1-a)/b$ with $a + b \geq 1$, $a \in [b, 1+b]$, then, as a consequence of Theorem 1.7, we obtain the following result for functions in the class introduced by Silverman (see Table 3.1 in Chapter 1 for the reference).

Corollary 3.7. *Let $f \in \mathcal{S}^*((b^2-a^2+a)/b, ((1-a)/b))$. Then we have*

$$\max_{f\in\mathcal{S}^*((b^2-a^2+a)/b,((1-a)/b))} \Delta\left(r, \frac{z}{f}\right) = \pi s_1^2 r^2 {}_2F_1\left(s_2, s_2; 2; \left(\frac{1-a}{b}\right)^2 r^2\right)$$

for $0 < r \leq 1$, where

$$s_1 = (b^2-a^2+2a-1)/b \quad \textit{and} \quad s_2 = (b^2-a^2+a)/(1-a);$$

$a+b \geq 1$, $a \in [b, 1+b]$. *The maximum is attained by the rotations of the function $k_{(b^2-a^2+a)/b,(1-a)/b)}$ as defined by (3.1).*

In Section 3.3, we present useful lemmas which are the main tools to prove our main theorems.

3.3. Preparatory Results

If $f \in \mathcal{A}$ such that z/f is non-vanishing in $\mathbb{D}$ (eg. the non-vanishing condition is ensured whenever $f \in \mathcal{S}$), then

$$\frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n, \ z \in \mathbb{D}. \tag{3.5}$$

We first present a necessary coefficient condition for a function f of the form (3.5) to be in $\mathcal{S}^*(A, B)$.

Lemma 3.8. *Let $f \in \mathcal{S}^*(A, B)$ for $-1 \leq B \leq 0$ and $A \neq B$ and f be of the form (3.5). Then*

$$\sum_{k=1}^{\infty} \left(k^2 - |B-A-kB|^2\right) |b_k|^2 \leq |A-B|^2$$

holds.

Proof. Denoted by $g(z) := z/f(z)$, $f \in \mathcal{S}^*(A, B)$. Then g has the form (3.5) and satisfies the relation

$$\frac{zg'(z)}{g(z)} = 1 - \frac{zf'(z)}{f(z)} \prec 1 - \frac{1+Az}{1+Bz} = \frac{(B-A)z}{1+Bz}, \quad z \in \mathbb{D}.$$

Then by the definition of subordination, there exists an analytic function $w : \mathbb{D} \to \overline{\mathbb{D}}$ with $w(0) = 1$ such that

$$\frac{zg'(z)}{g(z)} = \frac{(B-A)zw(z)}{1+Bzw(z)}, \quad z \in \mathbb{D}.$$

Writing this in series form, we get

$$\sum_{k=1}^{\infty} kb_k z^{k-1} = \left((B-A) + \sum_{k=1}^{\infty} (B-A-kB) b_k z^k \right) w(z);$$

or equivalently

$$\sum_{k=1}^{n} kb_k z^{k-1} + \sum_{k=n+1}^{\infty} c_k z^{k-1} = \left((B-A) + \sum_{k=1}^{n-1} (B-A-kB) b_k z^k \right) w(z)$$

for certain coefficients c_k. Since $|w(z)| < 1$ in $\mathbb{D}$, by Parseval-Gutzmer formula (see also Clunie's method [18] and [19, 79, 80]), we obtain

$$\sum_{k=1}^{n} k^2 |b_k|^2 r^{2k-2} \leq |A-B|^2 + \sum_{k=1}^{n-1} |B-A-kB|^2 |b_k|^2 r^{2k},$$

or equivalently,

$$\sum_{k=1}^{n} k^2 |b_k|^2 r^{2k-2} - \sum_{k=1}^{n-1} |B-A-kB|^2 |b_k|^2 r^{2k} \leq |A-B|^2. \tag{3.6}$$

If we take $r = 1$ and allow $n \to \infty$, then we obtain the desired inequality

$$\sum_{k=1}^{\infty} \left(k^2 - |B-A-kB|^2 \right) |b_k|^2 \leq |A-B|^2.$$

This completes the proof of our lemma. $\square$

Lemma 3.9. *Let $0 < |A| \leq 1$ and $f \in \mathcal{S}^*(A, 0)$. For $|z| < r$, suppose that*

$$\frac{z}{f(z)} = 1 + \sum_{k=1}^{\infty} b_k z^k \ \textit{and} \ e^{-Az} = 1 + \sum_{k=1}^{\infty} c_k z^k, \quad r \in (0, 1].$$

Then

$$\sum_{k=1}^{N} k|b_k|^2 r^{2k} \leq \sum_{k=1}^{N} k|c_k|^2 r^{2k} \tag{3.7}$$

holds for each $N \in \mathbb{N}$.

Proof. Clearly, it suffices to prove the lemma for $0 < A \leq 1$. From Lemma 3.8, using the equation (3.6) for $B = 0$, and then multiplying the resulting equation by r^2 on both sides shows that

$$\sum_{k=1}^{n-1}(k^2 - A^2r^2)|b_k|^2r^{2k} + n^2|b_n|^2r^{2n} \leq A^2r^2. \tag{3.8}$$

The function e^{-Az} clearly shows that the equality, when $n \to \infty$, in (3.8) attains for $b_k = c_k$.

Step-I: Cramer's Rule.

We consider the inequalities corresponding to (3.8) for $n = 1, \dots, N$ and multiply the nth coefficient by a factor $\lambda_{n,N}$. These factors are chosen in such a way that the addition of the left sides of the modified inequalities results the left side of (3.7). For the calculation of the factors $\lambda_{n,N}$ we get the following system of linear equations

$$k = k^2\lambda_{k,N} + \sum_{n=k+1}^{N} \lambda_{n,N}(k^2 - A^2r^2),\ k = 1, \dots, N. \tag{3.9}$$

Since the matrix of this system is an upper triangular matrix with positive integers as diagonal elements, the solution of this system is uniquely determined. Cramer's rule allows us to write the solution of the system (3.9) in the form

$$\lambda_{n,N} = \frac{((n-1)!)^2}{(N!)^2}\,\mathrm{Det}\,A_{n,N},$$

where $A_{n,N}$ is the $(N - n + 1) \times (N - n + 1)$ matrix is constructed as follows:

$$A_{n,N} = \begin{bmatrix} n & n^2 - A^2r^2 & \cdots & n^2 - A^2r^2 \\ n+1 & (n+1)^2 & \cdots & (n+1)^2 - A^2r^2 \\ \vdots & \vdots & \vdots & \vdots \\ N & 0 & \cdots & N^2 \end{bmatrix}.$$

Determinants of these matrices can be obtained by expanding according to Laplace's rule with respect to the last row, wherein the first coefficient is N and the last one is N^2. The rest of the entries are zeros. This expansion and a mathematical induction results in the following formula. If $k \le N-1$, then

$$\lambda_{k,N} = \lambda_{k,N-1} - \frac{1}{N}\left(1 - \frac{A^2r^2}{k^2}\right)\prod_{m=k+1}^{N-1}\left(\frac{A^2r^2}{m^2}\right).$$

For fixed $k \in \mathbb{N}$ and $N \ge k$, we see that the sequence $\{\lambda_{k,N}\}$ is strictly non-increasing, i.e. $\lambda_{k,N} - \lambda_{k,N-1} < 0$ with

$$\lambda_k := \lim_{N\to\infty} \lambda_{k,N} = \frac{1}{k} - \left(1 - \frac{A^2r^2}{k^2}\right)\sum_{n=k+1}^{\infty}\frac{1}{n}\prod_{m=k+1}^{n-1}\left(\frac{A^2r^2}{m^2}\right).$$

To prove that $\lambda_{k,N} > 0$ for all $N \in \mathbb{N}, 1 \le k \le N$, it is adequate to show that $\lambda_k \ge 0$ for $k \in \mathbb{N}$. This will be completed in Step II. But before that we want to remark that the proof of the said inequality is sufficient for the proof of the theorem, since we remarked for (3.8), equality holds for $b_k = c_k$.

Step-II: Positivity of the Multipliers.

In this step again has to show

$$\sum_{n=k+1}^{\infty}\frac{1}{n}\prod_{m=k+1}^{n-1}\left(\frac{A^2r^2}{m^2}\right) \le \frac{1}{k\left(1 - \frac{A^2r^2}{k^2}\right)} = \frac{1}{k}\sum_{n=0}^{\infty}\left(\frac{A^2r^2}{k^2}\right)^n,$$

which is indeed easy to prove. The proof of our lemma is complete. □

Lemma 3.10. *Let $-1 \le B < 0$, $A \ne B$ and $f \in \mathcal{S}^*(A,B)$. For $|z| < r$, suppose that*

$$\frac{z}{f(z)} = 1 + \sum_{k=1}^{\infty} b_k z^k \text{ and } (1 - Bz)^{1-\frac{A}{B}} = 1 + \sum_{k=1}^{\infty} c_k z^k, \quad r \in (0,1].$$

Then for $N \in \mathbb{N}$, the inequality

$$\sum_{k=1}^{N} k|b_k|^2 r^{2k} \le \sum_{k=1}^{N} k|c_k|^2 r^{2k} \tag{3.10}$$

is recognized.

Proof. As in the proof of Lemma 3.8, we can rewrite (3.6) in the form

$$\sum_{k=1}^{n-1}\left(k^2 - |k-\phi|^2B^2r^2\right)|b_k|^2r^{2k} + n^2|b_n|^2r^{2n} \le B^2|\phi|^2r^2, \tag{3.11}$$

where $\phi := 1 - A/B$. The function $(1 - Bz)^{1-A/B}$ clearly shows that the equality, when $n \to \infty$, in (3.11) attains for $b_k = c_k$.

Rest of the proof is divided into two steps.

Step-I: Cramer's Rule.

We consider the inequalities corresponding to (3.11) for $n = 1, \ldots, N$ and multiply the nth coefficient by a factor $\lambda_{n,N}$. These factors are chosen in such a way that the addition of the left sides of the modified inequalities results the left side of (3.10). For the calculation of the factors $\lambda_{n,N}$ we get the following system of linear equations

$$k = k^2\lambda_{k,N} + \sum_{n=k+1}^{N} \lambda_{n,N}\left(k^2 - |k-\phi|^2B^2r^2\right),\ k = 1, \ldots, N. \tag{3.12}$$

Since the matrix of this system is an upper triangular matrix with positive integers as diagonal elements, the solution of this system is uniquely determined. Cramer's rule allows us to write the solution of the system (3.12) in the form

$$\lambda_{n,N} = \frac{((n-1)!)^2}{(N!)^2}\,\mathrm{Det}\,A_{n,N},$$

where $A_{n,N}$ is the $(N-n+1)\times(N-n+1)$ matrix is constructed as follows:

$$A_{n,N} = \begin{bmatrix} n & n^2 - |n-\phi|^2B^2r^2 & \cdots & n^2 - |n-\phi|^2B^2r^2 \\ n+1 & (n+1)^2 & \cdots & (n+1)^2 - |n+1-\phi|^2B^2r^2 \\ \vdots & \vdots & \vdots & \vdots \\ N & 0 & \cdots & N^2 \end{bmatrix}.$$

Determinants of these matrices can be found by expanding according to Laplace's rule with respect to the last row, wherein the first coefficient is N and the last one is N^2. The rest of the entries are zeros. This expansion and a mathematical induction results in the following formula. If $k \le N-1$, then

$$\lambda_{k,N} = \lambda_{k,N-1} - \frac{1}{N}\left(1 - \left|1-\frac{\phi}{k}\right|^2 B^2r^2\right)\prod_{m=k+1}^{N-1}\left|1-\frac{\phi}{m}\right|^2 B^2r^2.$$

Set as an abbreviation $U_k = 1 - |1-\phi/k|^2\,B^2r^2$, we get

$$\lambda_{k,N} = \lambda_{k,N-1} - \frac{1}{N}U_k\prod_{m=k+1}^{N-1}(1-U_m). \tag{3.13}$$

Note that U_k in (3.13) may be positive as well as negative for all $k \in \mathbb{N}$. We investigate this by including here a table (see Table 3.3).

k	A	B	r	U_k
1	$2+i$	all	0.5	-5.25
1	$1+i$	all	0.4	0.2
2	$-2+i$	-1	0.5	0.375
2	$-2+i$	-1	0.8	-0.6

TABLE 3.3. Signs of the constant U_k

Case (i): Suppose that U_k is negative.

From the relation (3.13), we see that for fixed $k \in \mathbb{N}, k \leq N-1$, the sequence $\{\lambda_{k,N}\}$ is strictly non-decreasing, i.e.

$$\lambda_{k,N} - \lambda_{k,N-1} > 0$$

so that

$$\lambda_{k,N} > \lambda_{k,N-1} > \cdots > \lambda_{k,k} = 1/k > 0,$$

and thus $\lambda_k \geq 0$ when $N \to \infty$ as required.

Case (ii): Suppose that U_k is positive.

For fixed $k \in \mathbb{N},\ N \geq k$, the sequence $\{\lambda_{k,N}\}$ is strictly non-increasing, i.e. $\lambda_{k,N} - \lambda_{k,N-1} < 0$ with

$$\lambda_k := \lim_{N\to\infty} \lambda_{k,N} = \frac{1}{k} - U_k \sum_{n=k+1}^{\infty} \frac{1}{n} \prod_{m=k+1}^{n-1} (1-U_m). \tag{3.14}$$

For all $N \in \mathbb{N}, 1 \leq k \leq N$, to prove that $\lambda_{k,N} > 0$, it is sufficient to prove $\lambda_k \geq 0$ for $k \in \mathbb{N}$. This will be completed in Step II. But before that we want to annotate that the proof of the said inequality is sufficient for the proof of the theorem, as we noted in the beginning of the proof, equality is received for $b_k = c_k$.

Step-II: Positivity of the Multipliers.

Let for an abbreviation

$$S_k = \sum_{n=k+1}^{\infty} \frac{1}{n} \prod_{m=k+1}^{n-1} (1-U_m), \quad k \in \mathbb{N}.$$

We now prove that

$$S_k \leq \frac{1}{kU_k}.$$

From the relation (3.14), we get

$$\lambda_k = \frac{1}{k} - S_k + (1 - U_k)S_k.$$

Again set for an abbreviation

$$T_k = \frac{1}{k} + (1 - U_k)S_k.$$

It is enough to prove that

$$T_k \leq \frac{1}{kU_k}. \tag{3.15}$$

To prove (3.15) we use the inequality

$$\frac{1}{nU_n} > \frac{1}{(n+1)U_{n+1}} \tag{3.16}$$

and the identity

$$\frac{1}{nU_n} = \frac{1}{n} + \frac{1 - U_n}{nU_n} \tag{3.17}$$

which are valid for each $n \in \mathbb{N}$. Repeated application of (3.16) and (3.17) for $n = k, k+1, \ldots, P$ results the inequality

$$\frac{1}{kU_k} > \sum_{n=k}^{P} \frac{1}{n} \prod_{m=k}^{n-1} (1 - U_m) + \frac{\prod_{m=k}^{P}(1 - U_m)}{PU_P} =: S_{k,P} + R_{k,P}, \text{ for } k \leq P.$$

Since $R_{k,P} > 0$, taking the limit as $P \to \infty$ we obtain

$$\frac{1}{kU_k} \geq \lim_{P \to \infty} S_{k,P} = \sum_{n=k}^{\infty} \frac{1}{n} \prod_{m=k}^{n-1} (1 - U_m),$$

and we complete the inequality (3.15). This completes the proof of Lemma 3.10. □

3.4. Proofs of the Main Results

Proof of Theorem 3.1. Let $f \in \mathcal{S}^*(A,0)$. By the definition of the class $\mathcal{S}^*(A,0)$, it suffices to assume that $0 < A \leq 1$ and

$$\frac{zf'(z)}{f(z)} \prec 1 + Az = \frac{zk'_{A,0}(z)}{k_{A,0}(z)}, \quad z \in \mathbb{D}.$$

By the subordination principle, we obtain that $z/f(z) \prec e^{-Az}$ which in terms of the Taylor coefficients may be written as

$$1 + \sum_{n=1}^{\infty} b_n z^n \prec e^{-Az} = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad c_n = (-1)^n \frac{A^n}{n!}.$$

By Lemma 3.9, we have

$$\sum_{n=1}^{N} n|b_n|^2 r^{2n} \leq \sum_{n=1}^{N} n|c_n|^2 r^{2n}, \ N \in \mathbb{N}, \ r \in (0,1],$$

which implies that

$$\Delta\left(r, \frac{z}{f}\right) = \pi \sum_{n=1}^{\infty} n|b_n|^2 r^{2n} \leq \Delta\left(r, \frac{z}{k_{A,0}}\right) = \pi \sum_{n=1}^{\infty} n|c_n|^2 r^{2n}.$$

We claim that

$$\pi \sum_{n=1}^{\infty} n|c_n|^2 r^{2n} = E_{A,0}(r),$$

where $E_{A,0}(r) = \pi A^2 r^2 {}_0F_1(2; A^2 r^2)$ with $0 < A \leq 1$. To prove the claim, we observe that

$$\begin{aligned}
\pi^{-1}\Delta\left(r, \frac{z}{k_{A,0}}\right) &= \sum_{n=1}^{\infty} n \frac{A^{2n}}{(n!)^2} r^{2n} \\
&= A^2 r^2 \sum_{n=0}^{\infty} \frac{1}{(2)_n (1)_n} A^{2n} r^{2n} \\
&= A^2 r^2 {}_0F_1(2; A^2 r^2) \\
&=: \pi^{-1} E_{A,0}(r)
\end{aligned}$$

and thus,

$$\Delta\left(r, \frac{z}{f}\right) \leq \Delta\left(r, \frac{z}{k_{A,0}}\right) = E_{A,0}(r).$$

The equality case is obvious from $z/k_{A,0}(z) = e^{-Az}$. The proof of the theorem is complete.

□

Proof of Theorem 1.7. Suppose $f \in \mathcal{S}^*(A,B)$, $-1 \le B < 0$ and $A \neq B$. Then by setting $g(z) = z/f(z)$, we write

$$\frac{zg'(z)}{g(z)} = 1 - \frac{zf'(z)}{f(z)} \prec 1 - \frac{1+Az}{1+Bz} = \frac{(B-A)z}{1+Bz}, \quad z \in \mathbb{D}.$$

By a well-known subordination result, we get

$$g(z) = \frac{z}{f(z)} \prec (1+Bz)^{1-\frac{A}{B}} = \frac{z}{k_{A,B}(z)},$$

where $k_{A,B}$ is defined by (3.1). If

$$\frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n \text{ and } \frac{z}{k_{A,B}(z)} = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad |z| < r,$$

then Lemma 3.10 gives that

$$\sum_{n=1}^{N} n|b_n|^2 r^{2n} \le \sum_{n=1}^{N} n|c_n|^2 r^{2n}$$

for each $N \in \mathbb{N}$ and $r \in (0,1]$. Allowing $N \to \infty$, we obtain

$$\Delta\left(r, \frac{z}{f}\right) = \sum_{n=1}^{\infty} n|b_n|^2 r^{2n} \le \Delta\left(r, \frac{z}{k_{A,B}}\right) = \sum_{n=1}^{\infty} n|c_n|^2 r^{2n}.$$

Clearly,

$$c_n = (-1)^n \frac{(\zeta)_n}{(1)_n} B^n \text{ with } \zeta = (A/B) - 1.$$

Now, applying the area formula (3.3) for the function $z/k_{A,B}$, we obtain

$$\begin{aligned}
\pi^{-1}\Delta\left(r, \frac{z}{k_{A,B}}\right) &= \sum_{n=1}^{\infty} n|c_n|^2 r^{2n}, \quad |z| < r \\
&= \sum_{n=1}^{\infty} n \frac{(\zeta)_n (\overline{\zeta})_n}{(1)_n (1)_n} B^{2n} r^{2n} \\
&= |\zeta|^2 B^2 r^2 \sum_{n=0}^{\infty} \frac{(\zeta+1)_n (\overline{\zeta}+1)_n}{(2)_n (1)_n} B^{2n} r^{2n} \\
&= |\overline{A} - B|^2 r^2 {}_2F_1(A/B, \overline{A}/B; 2; B^2 r^2) \\
&=: \pi^{-1} E_{A,B}(r),
\end{aligned}$$

and the proof of Theorem 1.7 is complete. ☐

We end this chapter with the following remarks.

3.5. Concluding Remarks

It would be interesting to solve analog of Yamashita's extremal problem (3.4) for many interesting geometric subclasses of functions from $\mathcal{S}$. For example, determining the analog of Theorems 3.1 and 1.7 when zf' belongs to the class $\mathcal{S}^*(A,B)$ (see also [**59, 71**]) and also for functions f in the Bazilevič class [**13**] or for functions convex in some direction, would be interesting to study.

CHAPTER 4

COEFFICIENT ESTIMATES FOR p-VALENT FUNCTIONS

This chapter concerns with the Taylor-Maclaurin coefficient estimates for some classes of p-valent functions. This problem was initially studied by Aouf in [**7, 8**]. The proof given by Aouf is found to be partially erroneous and a correct proof of it is derived in this chapter. Section 4.1 provides an introduction. We study a sharp correct form of the coefficient bounds for the classes $\mathcal{S}_p^*(A,B,\beta), \mathcal{F}_p(\alpha,\beta,\lambda)$ and $\mathcal{C}_p(b,\lambda)$ and their proofs in Section 4.2.

The results of this chapter will appear in:
Sahoo S.K., Sharma N.L., *A note on a class of p-valent starlike functions of order beta*, Siberian Math. J., pages 6, Accepted.

and

Sharma N.L., *A note on coefficient estimates for some classes of p-valent functions*, Ukrainian Math. J., pages 16, Accepted.

4.1. Introduction

We refer to Chapter 1 for related definitions and notations used in this chapter. In Chapter 1, we consider the class $\mathcal{S}_p^*(A,B,\beta)$ defined by the equation (1.17) and for special values of A,B,β and p, the class $\mathcal{S}_p^*(A,B,\beta)$ reduces to the following classes:

$$\mathcal{S}_p^*(1,-1,\beta) =: \mathcal{S}_p^*(\beta),\ \mathcal{S}_p^*(1,-1,0) =: \mathcal{S}_p^*,\ \mathcal{S}_1^*(1,-1,\beta) =: \mathcal{S}^*(\beta),\ \mathcal{S}_1^*(1,-1,0) =: \mathcal{S}^*,$$
$$\text{and } \mathcal{S}_1^*(A,B,0) =: \mathcal{S}^*(A,B) \quad \text{with} \quad -1 \le B < A \le 1.$$

Note that $\mathcal{S}_p^*(\beta)$ is the class of p-valent starlike functions of order β and it was introduced by Goluzina [29], and $\mathcal{S}_p^*$ is the usual class of p-valent starlike functions.

In [62], Padmanabhan introduced the class of starlike functions of order λ $(0 < \lambda \leq 1)$ defined as

Definition 4.1. A function $f \in \mathcal{A}$ is said to be in $\mathcal{T}(\lambda)$, if

$$\left| \left(\frac{zf'(z)}{f(z)} - 1 \right) \Big/ \left(\frac{zf'(z)}{f(z)} + 1 \right) \right| < \lambda,$$

equivalently,

$$\frac{zf'(z)}{f(z)} \prec \frac{1+\lambda z}{1-\lambda z} \quad \text{or} \quad \frac{zf'(z)}{f(z)} \prec \frac{1-\lambda z}{1+\lambda z},$$

for all $z \in \mathbb{D}$ and $0 < \lambda \leq 1$.

A function $f \in \mathcal{A}_p$ is said to be *p-valent α-spiral-like function of order β* in $\mathbb{D}$, if it is analytic and if there exists a $\rho > 0$ such that for $\rho < |z| < 1$

$$\mathrm{Re}\left\{ e^{i\alpha} \frac{zf'(z)}{f(z)} \right\} > \beta \cos\alpha$$

and

$$\int_0^{2\pi} \mathrm{Re}\left\{ e^{i\alpha} \frac{zf'(z)}{f(z)} \right\} d\theta = 2p\pi,$$

for $z = re^{i\theta}$. The class of p-valent α-spiral-like of order β is denoted by $\mathcal{S}_{\alpha,p}(\beta)$. In [63], Patil and Thakare introduced the class $\mathcal{S}_{\alpha,p}(\beta)$.

Two subclasses $\mathcal{F}_p(\alpha, \beta, \lambda)$ and $\mathcal{C}_p(b, \lambda)$ of p-valent functions in $\mathbb{D}$ were acquainted by Aouf in [8] which are defined as follows:

Definition 4.2. A function $f \in \mathcal{A}_p$ is said to belong to the class $\mathcal{F}_p(\alpha, \beta, \lambda)$, if it satisfies the following condition

$$\left| \frac{H(f(z)) - 1}{H(f(z)) + 1} \right| < \lambda, \quad z \in \mathbb{D},$$

where

$$H(f(z)) = \frac{e^{i\alpha} \frac{zf'(z)}{f(z)} - \beta\cos\alpha - ip\sin\alpha}{(p-\beta)\cos\alpha}.$$

By subordination property, equivalently, it can be written in the form (1.18) in Chapter 1.

Definition 4.3. Let b be a non-zero complex number. For $0 < \lambda \leq 1$ and $p \in \mathbb{N}$, let $\mathcal{C}_p(b, \lambda)$ denote the class of functions $f \in \mathcal{A}_p$ satisfying the relation

$$\left| \frac{H(f(z)) - 1}{H(f(z)) + 1} \right| < \lambda \quad \text{for } z \in \mathbb{D},$$

where

$$H(f(z)) = 1 + \frac{1}{pb} \left(1 + \frac{zf''(z)}{f'(z)} - p \right).$$

By subordination relation, it can be written in the form (1.19) in Chapter 1.

We note that a number of subclasses have been studied by several authors and the subclasses can be obtain by putting for different values of $p, \alpha, \beta, \lambda$ and b. We list some of them here.

1. $\mathcal{F}_p(0, 0, 1) =: \mathcal{S}_p^*$ and $\mathcal{F}_p(0, \beta, 1) =: \mathcal{S}_p^*(\beta)$. $\mathcal{C}_p(1, 1) =: \mathcal{C}_p$ and $\mathcal{C}_p((1 - \beta/p), 1) =: \mathcal{C}_p(\beta)$, $0 \leq \beta < p$, is the class of p-valent convex functions of order β (i.e. $zg'/p \in \mathcal{S}_p^*(\beta)$).
2. $\mathcal{F}_p(\alpha, 0, 1) =: \mathcal{S}_{\alpha,p}$ and $\mathcal{F}_p(\alpha, \beta, 1) =: \mathcal{S}_{\alpha,p}(\beta)$ are the classes of p-valent α-spiral-like functions and p-valent α-spiral-like functions of order β, respectively.
3. $\mathcal{C}_p(e^{-i\alpha} \cos \alpha, 1)$ and $\mathcal{C}_p(e^{-i\alpha}(1 - \beta/p) \cos \alpha, 1)$ for $|\alpha| < \pi/2$, are the classes of p-valent functions g for which zg'/p are p-valent α-spiral-like functions and p-valent α-spiral-like functions of order β, respectively.
4. The class $\mathcal{F}_1(\alpha, \beta, \lambda) =: \mathcal{F}(\alpha, \beta, \lambda)$ was inquired by Gopalakrishna and Umarani [**34**].
5. $\mathcal{C}_p(b, 1)$ is the class of p-valent functions $g \in \mathcal{A}_p$ and it satisfies

$$\operatorname{Re} \left\{ p + \frac{1}{b} \left(1 + \frac{zg''(z)}{g'(z)} - p \right) \right\} > 0 \quad \text{for } z \in \mathbb{D}.$$

   This class was considered by Aouf in [**6**].
6. $\mathcal{F}_1(0, 0, 1) =: \mathcal{S}^*$, $\mathcal{F}_1(0, \beta, 1) =: \mathcal{S}^*(\beta)$, $\mathcal{C}_1(1, 1) =: \mathcal{C}$. The class $\mathcal{C}_1(1 - \beta, 1) =: \mathcal{C}(\beta)$ is the class of convex functions of order β for $0 \leq \beta < 1$ and this class was introduced by Robertson [**77**]. $\mathcal{F}_1(0, 0, \lambda) =: \mathcal{T}(\lambda)$ (see Definition 4.1), and $\mathcal{C}_1(1, \lambda) =: \mathcal{C}(\lambda)$ is the class of functions g for which $zg' \in \mathcal{T}(\lambda)$.
7. $\mathcal{F}_1(\alpha, 0, 1) =: \mathcal{S}_\alpha$ and $\mathcal{C}_1(e^{-i\alpha} \cos \alpha, 1)$, $|\alpha| < \pi/2$, respectively define the classes of α-spiral-like functions and the class of functions g for which $zg' \in \mathcal{S}_\alpha$ was introduced

by Robertson [78]; $\mathcal{F}_1(\alpha, \beta, 1) =: \mathcal{S}_\alpha(\beta)$ and $\mathcal{C}_1\big(e^{-i\alpha}(1-\beta)\cos\alpha, 1\big) =: \mathcal{C}_\alpha(\beta)$ for $|\alpha| < \pi/2$, are the classes of α-spiral-like functions of order β and functions g for which zg' is α-spiral-like of order β introduced by Chichra [17] and Sizuk [88].

8. $\mathcal{C}_1(b, 1) =: \mathcal{C}(b)$ is the class of functions $g \in \mathcal{A}$ and it satisfies

$$\operatorname{Re}\left(1 + \frac{1}{b}\frac{zg''(z)}{g'(z)}\right) > 0 \quad \text{for } z \in \mathbb{D}.$$

   This class was introduced by Wiatrowski [97] and studied in [60, 61].

Aouf estimated the coefficient bounds for the functions from the classes $\mathcal{S}_p^*(A, B, \beta)$; $\mathcal{F}_p(\alpha, \beta, \lambda)$ and $\mathcal{C}_p(b, \lambda)$ in [7, 8], receptively, in which the proof is found to be incorrect. In this chapter, we present their correct proofs.

4.2. Coefficient Estimates

The following Lemma is obtained by Goel and Mehrok:

Lemma 4.4. [27, Theorem 1] *Let $-1 \le B < A \le 1$ and $f \in \mathcal{S}^*(A, B)$. Then*

$$|a_2| \le A - B; \tag{4.1}$$

for $A - 2B \le 1$, $n \ge 3$,

$$|a_n| \le \frac{A - B}{n - 1}; \tag{4.2}$$

and for $A - (n-1)B > (n-2)$, $n \ge 3$,

$$|a_n| \le \frac{1}{(n-1)\,!}\prod_{j=2}^{n}(A - (j-1)B). \tag{4.3}$$

The equality signs in (4.1) and (4.2) are attained for the rotation of the functions

$$k_{n,A,B}(z) = \begin{cases} z(1 + Bz^{n-1})^{(A-B)/(n-1)B}, & \text{if } B \neq 0; \\ z\ \exp\left(\frac{\mathrm{Az^{n-1}}}{\mathrm{n-1}}\right), & \text{if } B = 0, \end{cases},$$

and in (4.3) equality is attained for the rotation of the functions

$$k_{A,B}(z) = \begin{cases} z(1 + Bz)^{(A-B)/B}, & \text{if } B \neq 0; \\ ze^{Az}, & \text{if } B = 0, \end{cases}.$$

However, a p-valent analog of Lemma 4.4 was wrongly proven by Aouf in the following form:

Theorem A. [7, Theorem 3] *Let $-1 \leq B < A \leq 1$ and $p \in \mathbb{N}$. If $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \in \mathcal{S}_p^*(A, B, \beta)$, then*

$$|a_n| \leq \prod_{j=0}^{n-p-1} \frac{|(B-A)(p-\beta)+Bj|}{j+1},$$

for $n \geq p+1$, and these bounds are sharp for all admissible A, B, β and for each n.

The following theorems were mistakenly proven by Aouf in [8].

Theorem B. [8, Theorem 2] *Let $0 < \lambda \leq 1$, $0 \leq \beta < p$, $p \in \mathbb{N}$ and $|\alpha| < \pi/2$. If $f(z) \in \mathcal{F}_p(\alpha, \beta, \lambda)$ and has the form (1.16), then*

$$|a_n| \leq \prod_{j=0}^{n-p-1} \frac{\lambda \big| j + 2(p-\beta)e^{-i\alpha}\cos\alpha \big|}{j+1},$$

for $n \geq p+1$, and these bounds are sharp for all admissible α, β, λ and for each n.

Theorem C. [8, Theorem 3] *Let $0 < \lambda \leq 1$, $p \in \mathbb{N}$ and $b \neq 0$ be any complex number. If $f(z) \in \mathcal{C}_p(b, \lambda)$ and has the form (1.16), then*

$$|a_n| \leq \prod_{j=0}^{n-p-1} \frac{\lambda |j + 2bp|}{j+1},$$

for $n \geq p+1$, and these bounds are sharp for all admissible α, β, λ and for each n.

In Chapter 1, we provided the correct form (Theorem 1.8) of the coefficients bounds for the function $f \in \mathcal{S}_p^*(A, B, \beta)$ as stated in Theorem A.

Proof of Theorem 1.8. Let $f \in \mathcal{S}_p^*(A, B, \beta)$. By the relation (1.17), we can guarantee an analytic function $\phi : \mathbb{D} \to \overline{\mathbb{D}}$ with $\phi(0) = 0$ such that

$$\frac{zf'(z)}{pf(z)} = \frac{1 + \big[B + (A-B)(1-\beta/p)\big]\phi(z)}{1 + B\phi(z)},$$

i.e.

$$zf'(z) - pf(z) = \big[(pB + (A-B)(p-\beta))f(z) - Bzf'(z)\big]\phi(z).$$

Substituting the series expansion (1.16), of f, and cancelling the factor z^p on both sides, we obtain

$$\sum_{k=1}^{\infty} k a_{p+k} z^k = \left((A-B)(p-\beta) - \sum_{k=1}^{\infty} \big[B(p+k) + (-pB + (B-A)(p-\beta))\big] a_{p+k} z^k\right) \phi(z).$$

Rewriting it, we get

$$\sum_{k=1}^{\infty} k a_{p+k} z^k = \left((A-B)(p-\beta) + \sum_{k=1}^{\infty} \big[A(p-\beta) - B(k+p-\beta)\big] a_{p+k} z^k\right) \phi(z).$$

By Clunie's method [**18**] (for instance see [**80, 79**]) for $n \in \mathbb{N}$, we observe that

$$\sum_{k=1}^{n} k^2 |a_{p+k}|^2 \le (A-B)^2(p-\beta)^2 + \sum_{k=1}^{n-1} \big[A(p-\beta) - B(k+p-\beta)\big]^2 |a_{p+k}|^2.$$

Simplification of the above inequality leads to

$$|a_{p+n}|^2 \le \frac{1}{n^2}\left((A-B)^2(p-\beta)^2 + \sum_{k=1}^{n-1}\left(\big[A(p-\beta) - B(k+p-\beta)\big]^2 - k^2\right)|a_{p+k}|^2\right)$$

or

$$|a_{p+n}|^2 \le \frac{1}{n^2}\left((A-B)^2(p-\beta)^2 + \sum_{k=2}^{n}\left(\big[A(p-\beta) - B(k+p-\beta-1)\big]^2 - (k-1)^2\right)|a_{p+k-1}|^2\right).$$

Above inequality can be rewritten by replacing $p+n$ by n as

$$|a_n|^2 \le \frac{1}{(n-p)^2}\left((A-B)^2(p-\beta)^2 + \sum_{k=2}^{n-p}\left(\big[A(p-\beta) - B(k+p-\beta-1)\big]^2 - (k-1)^2\right)|a_{p+k-1}|^2\right) \tag{4.4}$$

for $n \ge p+1$.

Note that the terms under the summation in the right hand side of (4.4) may be positive as well as negative. We investigate it by including here a table (see Table 4.1) for values of $W := \big(A(p-\beta) - B(k+p-\beta-1)\big)^2 - (k-1)^2$ for various choices of A, B, k, β and p. So, we can not apply direct mathematical induction in (4.4) to establish the required bounds for $|a_n|$. Therefore, we are considering different cases for this.

k	p	A	B	β	W
2	1	0.8	0.5	0	-0.96
2	1	-0.5	-0.8	0	0.21
3	2	0.5	0.4	0.5	-3.5775
3	2	-0.1	-0.7	0.5	1.29

TABLE 4.1. Signs of the constant W

(*This is the place where the incorrectness of Aouf's proof is found!*)

First, for $n = p + 1$, we easily see that (4.4) reduces to

$$|a_{p+1}| \leq (A - B)(p - \beta),$$

which establishes (1.23).

Secondly, $A(p-\beta)-B(p-\beta-1) \leq 1$ if and only if $A(p-\beta)-B(n-\beta-1) \leq (n-p-1)$ for $n \geq p+2$. Since all the terms under the summation in (4.4) are non-positive, it reduces to

$$|a_n| \leq \frac{(A - B)(p - \beta)}{n - p},$$

for $A(p - \beta) - B(p - \beta + 1) \leq 1$, $n \geq p + 2$. This proves (1.24). The equality holds in (1.23) and (1.24) for the rotation of the functions

$$k_{n,p,A,B,\beta}(z) = \begin{cases} z^p\left(1 + Bz^{n-1}\right)^{(A-B)(p-\beta)/(n-1)B}, & B \neq 0; \\ z^p \exp\left(\frac{\mathrm{A(p-\beta)z^{n-1}}}{\mathrm{n-1}}\right), & B = 0, \end{cases}.$$

Finally, let us prove (1.25) when $A(p-\beta) - B(n-\beta-1) > (n-p-1)$, $n \geq p+2$. We see that all the terms under the summation in (4.4) are positive. We prove the inequality by the usual mathematical induction. Fix n, $n \geq p + 2$ and suppose that (1.25) holds for

$k = 3, 4, \ldots, n-p$. Then from (4.4), we find

(4.5)

$$|a_n|^2 \le \frac{1}{(n-p)^2}\left((A-B)^2(p-\beta)^2 + \sum_{k=2}^{n-p}\left(\big[A(p-\beta) - B(k+p-\beta-1)\big]^2 - (k-1)^2\right)\right.$$
$$\left.\prod_{j=1}^{k-1}\frac{\big[A(p-\beta)-B(p-\beta+j-1)\big]^2}{j^2}\right).$$

It is now enough to show that the square of the right hand side of (1.25) is equal to the right hand side of (4.5), that is

(4.6)

$$\prod_{j=1}^{m-p}\frac{\big[A(p-\beta)-B(p-\beta+j-1)\big]^2}{j^2} = \frac{1}{(m-p)^2}\left((A-B)^2(p-\beta)^2\right.$$
$$\left.+\sum_{k=2}^{m-p}\left(\big[A(p-\beta)-B(k+p-\beta-1)\big]^2-(k-1)^2\right)\prod_{j=1}^{k-1}\frac{\big[A(p-\beta)-B(p-\beta+j-1)\big]^2}{j^2}\right),$$

for $A(p-\beta) - B(m-\beta-1) > (m-p-1)$, $m \ge p+2$. We also use the induction principle to prove (4.6).

The equation (4.6) is recognized for $m = p+2$. Suppose that (4.6) is true for all m, $p+2 < m \le n-p$. Then from (4.5), we obtain

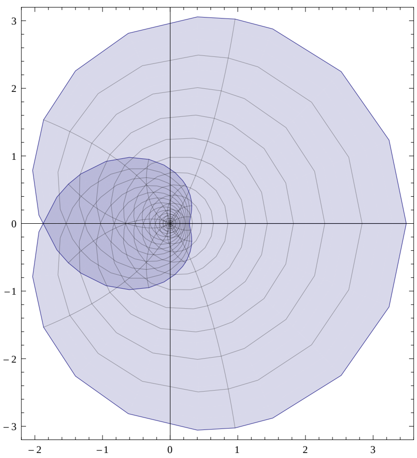
$$|a_n|^2 \le \frac{1}{(n-p)^2}\left((A-B)^2(p-\beta)^2 + \sum_{k=2}^{n-p-1}\left(\big[A(p-\beta)-B(k+p-\beta-1)\big]^2-(k-1)^2\right)\right.$$
$$\times\prod_{j=1}^{k-1}\frac{\big[A(p-\beta)-B(p-\beta+j-1)\big]^2}{j^2} + \left(\big[A(p-\beta)-B(n-\beta-1)\big]^2\right.$$
$$\left.\left.-(n-p-1)^2\right)\times\prod_{j=1}^{n-p-1}\frac{\big[A(p-\beta)-B(p-\beta+j-1)\big]^2}{j^2}\right).$$

Using the induction hypothesis, for $m = n-1$, we get

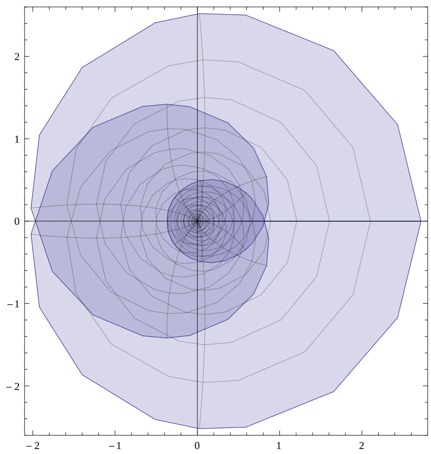
$$|a_n|^2 \le \frac{1}{(n-p)^2}\left((n-p-1)^2\prod_{j=1}^{n-p-1}\frac{\big[A(p-\beta)-B(p-\beta+j-1)\big]^2}{j^2}\right.$$
$$\left.+\left(\big[A(p-\beta)-B(n-\beta-1)\big]^2-(n-p-1)^2\right)\prod_{j=1}^{n-p-1}\frac{\big[A(p-\beta)-B(p-\beta+j-1)\big]^2}{j^2}\right).$$

h_n	p	A	B	β
h_1	2	5/6	0	0.5
h_2	3	1/2	0	1
h_3	2	5/6	-1	0.5
h_4	3	1/2	-0.6	1

TABLE 4.2. The extremal function h_n



The image domain $h_1(\mathbb{D})$



The image domain $h_2(\mathbb{D})$

FIGURE 4.1. Images of the unit disk under h_1 and h_2.

Hence

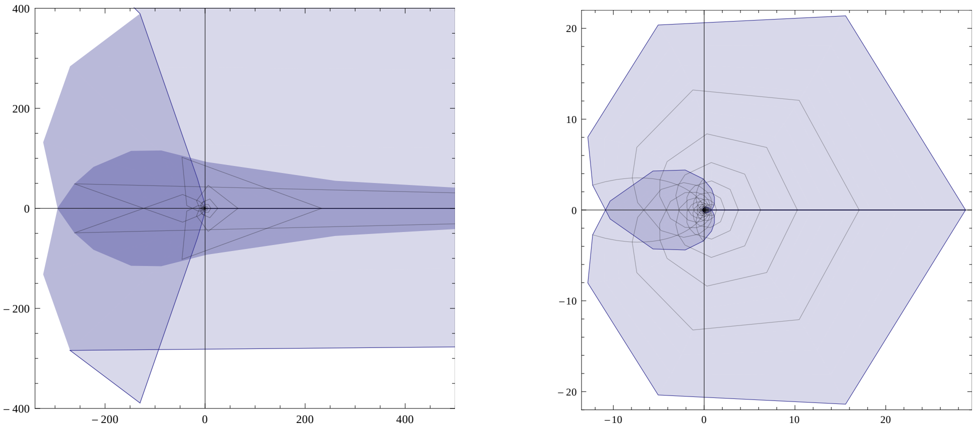
$$|a_n| \leq \prod_{j=1}^{n-p} \frac{\left[A(p-\beta) - B(p-\beta+j-1)\right]}{j}.$$

It is easy to prove that the bounds are sharp for the rotation of the functions

$$k_{p,A,B,\beta}(z) = \begin{cases} z^p\left(1+Bz\right)^{(A-B)(p-\beta)/B}, & B \neq 0; \\ z^p e^{A(p-\beta)z}, & B = 0, \end{cases}.$$

This completes the proof of Theorem 1.3. We remark that, choosing $p = 1$ and $\beta = 0$ in Theorem 1.3, it turns into Lemma 4.4.

For different values of p, A, B and β (see Table 4.2), the images of the unit disk under the extremal functions $h_n := k_{p,A,B,\beta}$ are described in Figures 4.1 & 4.2.



The image domain $h_3(\mathbb{D})$ The image domain $h_4(\mathbb{D})$

FIGURE 4.2. Images of the unit disk under h_3 and h_4.

Secondly, we now give the correct form of the coefficients bounds for $f \in \mathcal{F}_p(\alpha, \beta, \lambda)$ as stated in Theorem B and its proof.

Theorem 4.5. *Let $0 < \lambda \leq 1$, $0 \leq \beta < p$, $p \in \mathbb{N}$ and $|\alpha| < \pi/2$. If $f \in \mathcal{F}_p(\alpha, \beta, \lambda)$ is in the form (1.16), then we have*

$$|a_{p+1}| \leq 2\lambda(p-\beta)\cos\alpha; \tag{4.7}$$

for $\lambda^2\big(2p - 2\beta + (n-p-1)\big)^2 \leq (n-p-1)^2\big(\sec^2\alpha - \lambda^2\tan^2\alpha\big)$,

$$|a_n| \leq \frac{2\lambda(p-\beta)}{n-p}\cos\alpha, \quad n \geq p+2; \tag{4.8}$$

and for $\lambda^2\big(2p - 2\beta + (n-p-1)\big)^2 > (n-p-1)^2\big(\sec^2\alpha - \lambda^2\tan^2\alpha\big)$,

$$|a_n| \leq \prod_{j=1}^{n-p} \frac{\lambda\big|2(p-\beta)e^{-i\alpha}\cos\alpha + j - 1\big|}{j}, \quad n \geq p+2. \tag{4.9}$$

The equality signs in (4.7), (4.8) and (4.9) are attained.

Proof. Let $f \in \mathcal{F}_p(\alpha, \beta, \lambda)$. It follows from (1.18) that

$$e^{i\alpha}\frac{zf'(z)}{pf(z)} = \left(\frac{1 + (1 - (2\beta)/p)\lambda\phi(z)}{1 - \lambda\phi(z)}\right)\cos\alpha + i\sin\alpha,$$

for some analytic function ϕ in $\mathbb{D}$ with $\phi(0) = 0$ and $|\phi(z)| < 1$. We divide the expansion by $\cos\alpha$ on both sides and get

$$e^{i\alpha}\sec\alpha z f'(z) - (p + ip\tan\alpha)f(z) = \lambda\Big(e^{i\alpha}\sec\alpha z f'(z) + (p - 2\beta - ip\tan\alpha)f(z)\Big)\phi(z).$$

Substituting this in the series expansion (1.16), of f, we find that

$$\sum_{k=0}^{\infty}\Big(e^{i\alpha}(k+p)\sec\alpha - p - ip\tan\alpha\Big)a_{k+p}z^{k+p}$$
$$= \lambda\left(\sum_{k=0}^{\infty}\Big(e^{i\alpha}(k+p)\sec\alpha + p - 2\beta - i\tan\alpha\Big)a_{k+p}z^{k+p}\right)\phi(z),$$

where $a_p = 1$ and $\phi(z) = \sum_{k=0}^{\infty} w_{k+p}z^{k+p}$. Rewriting it, we obtain

$$\sum_{k=0}^{m}\Big(e^{i\alpha}(k+p)\sec\alpha - p - ip\tan\alpha\Big)a_{k+p}z^{k+p} + \sum_{k=m+1}^{\infty} C_k z^{k+p}$$
$$= \lambda\left(\sum_{k=0}^{m-1}\Big(e^{i\alpha}(k+p)\sec\alpha + p - 2\beta - i\tan\alpha\Big)a_{k+p}z^{k+p}\right)\phi(z)$$

for certain coefficients C_k. Since $|\phi(z)| < 1$ in $\mathbb{D}$, then by Parseval-Gutzmer formula (see also Clunie's method [**18**] and [**80, 79**]), we get

$$\sum_{k=0}^{m}\Big|e^{i\alpha}(k+p)\sec\alpha - p - ip\tan\alpha\Big|^2|a_{k+p}|^2 r^{2p+2k} + \sum_{k=m+1}^{\infty}|C_k|^2 r^{2p+2k}$$
$$\leq \lambda^2\left(\sum_{k=0}^{m-1}\Big|e^{i\alpha}(k+p)\sec\alpha + p - 2\beta - i\tan\alpha\Big|^2|a_{k+p}|^2 r^{2p+2k}\right).$$

Letting $r \to 1$, the above inequality can be written as

$$\Big|e^{i\alpha}(m+p)\sec\alpha - p - ip\tan\alpha\Big|^2|a_{m+p}|^2 \leq \sum_{k=0}^{m-1}\left(\lambda^2\Big|e^{i\alpha}(k+p)\sec\alpha + p - 2\beta - i\tan\alpha\Big|^2\right.$$
$$\left. - \Big|e^{i\alpha}(k+p)\sec\alpha - p - ip\tan\alpha\Big|^2\right)|a_{k+p}|^2.$$

Simplification of the above inequality leads

$$m^2\sec^2\alpha|a_{m+p}|^2 \leq \sum_{k=0}^{m-1}\Big(\lambda^2(k+2p-2\beta)^2 - k^2(\sec^2\alpha - \lambda^2\tan^2\alpha)\Big)|a_{k+p}|^2$$

k	p	α	β	λ	T
all	1	all	all	1	positive
2	1	$\pm\pi/4$	0.9	0.9	-0.0236
3	2	$\pm\pi/3$	1	0.6	-5.92
3	2	$\pm\pi/3$	1	0.8	1.92

TABLE 4.3. Signs of the constant T

(*This is the place where the incorrectness of Aouf's proof is found!*)

or

$$|a_{m+p}|^2 \leq \frac{\cos^2\alpha}{m^2}\left(4\lambda^2(p-\beta)^2 + \sum_{k=2}^{m}\Big(\lambda^2(k-1+2p-2\beta)^2 - (k-1)^2(\sec^2\alpha - \lambda^2\tan^2\alpha)\Big)\right)|a_{k+p-1}|^2.$$

Above inequality can be rewritten by replacing $m+p$ by n as

(4.10)
$$|a_n|^2 \leq \frac{\cos^2\alpha}{(n-p)^2}\left(4\lambda^2(p-\beta)^2 + \sum_{k=2}^{n-p}\Big(\lambda^2(k-1+2p-2\beta)^2 - (k-1)^2(\sec^2\alpha - \lambda^2\tan^2\alpha)\Big)\right)|a_{k+p-1}|^2, \quad \text{for } n \geq p+1.$$

Note that the terms under the summation in the right hand side of (4.10) may be positive as well as negative. We verify it by including here a table (see Table 4.3) for values of

$$T := \lambda^2(k-1+2p-2\beta)^2 - (k-1)^2(\sec^2\alpha - \lambda^2\tan^2\alpha)$$

for various choices for k, p, α, β and λ. So, we can not apply direct principle of mathematical induction in (4.10) to establish the desired bounds for $|a_n|$. Therefore, we are considering different cases for this.

First, for $n = p+1$, we readily see that (4.10) reduces to

$$|a_{p+1}| \leq 2\lambda(p-\beta)\cos\alpha,$$

which is equivalent to (4.7).

Secondly, $\lambda^2\big(2p-2\beta+(n-p-1)\big)^2 \le (n-p-1)^2\big(\sec^2\alpha-\lambda^2\tan^2\alpha\big)$ for $n \ge p+2$. Since all the terms under the summation in (4.10) are negative, we get

$$|a_n| \le \frac{2\lambda(p-\beta)}{n-p}\cos\alpha.$$

This gives the bound for $|a_n|$ as asserted in (4.8). The equality holds in (4.7) and (4.8) for the rotation of the functions

$$k_{n,p,\alpha,\beta,\lambda,}(z) = \frac{z^p}{(1+\lambda z^{n-1})^{\zeta_n}}.$$

Here $\zeta_n := 2(p-\beta)e^{-i\alpha}\cos\alpha/(n-1)$.

Finally, we consider the case $\lambda^2\big(2p-2\beta+(n-p-1)\big)^2 > (n-p-1)^2\big(\sec^2\alpha-\lambda^2\tan^2\alpha\big)$ for $n \ge p+2$ and obtain bound for $|a_n|$ as stated in (4.9). We see that all the terms under the summation in (4.10) are non-negative. We prove the inequality by the usual induction principle. Fix n, $n \ge p+2$ and suppose that (4.9) holds for $k = 3, 4, \ldots, n-p$. Then by (4.10), we obtain

(4.11)
$$|a_n|^2 \le \frac{\cos^2\alpha}{(n-p)^2}\left(4\lambda^2(p-\beta)^2 + \sum_{k=2}^{n-p}\Big(\lambda^2(2p-2\beta+k-1)^2 - (k-1)^2(\sec^2\alpha-\lambda^2\tan^2\alpha)\Big)\right)\prod_{j=1}^{k-1}\frac{\lambda^2\left|2(p-\beta)e^{-i\alpha}\cos\alpha+j-1\right|^2}{j^2}.$$

It is now sufficient to prove that the square of the right hand side of (4.9) is equal to the right hand side of (4.11), that is

(4.12)
$$\prod_{j=1}^{m-p}\frac{\lambda^2\left|2(p-\beta)e^{-i\alpha}\cos\alpha+j-1\right|^2}{j^2} = \frac{\cos^2\alpha}{(m-p)^2}\left(4\lambda^2(p-\beta)^2 + \sum_{k=2}^{m-p}\Big(\lambda^2(2p-2\beta+k-1)^2 - (k-1)^2(\sec^2\alpha-\lambda^2\tan^2\alpha)\Big)\right)\prod_{j=1}^{k-1}\frac{\lambda^2\left|2(p-\beta)e^{-i\alpha}\cos\alpha+j-1\right|^2}{j^2},$$

when $\lambda^2\big(2p-2\beta+(m-p-1)\big)^2 > (m-p-1)^2\big(\sec^2\alpha-\lambda^2\tan^2\alpha\big)$ for $m \geq p+2$.

The equation (4.12) is valid for $m = p+2$. Suppose that (4.12) is true for all m, $p+2 < m \leq n-p$. Then by (4.11), we obtain

$$\begin{aligned}
|a_n|^2 \leq \frac{\cos^2\alpha}{(n-p)^2}\Bigg\{ & 4\lambda^2(p-\beta)^2 + \sum_{k=2}^{n-p-1}\Big(\lambda^2(2p-2\beta+k-1)^2 \\
& - (k-1)^2(\sec^2\alpha-\lambda^2\tan^2\alpha)\Big)\prod_{j=1}^{k-1}\frac{\lambda^2\big|2(p-\beta)e^{-i\alpha}\cos\alpha+j-1\big|^2}{j^2} \\
& + \Bigg(\lambda^2(2p-2\beta+n-p-1)^2-(n-p-1)^2(\sec^2\alpha-\lambda^2\tan^2\alpha)\Bigg) \\
& \times \prod_{j=1}^{n-p-1}\frac{\lambda^2\big|2(p-\beta)e^{-i\alpha}\cos\alpha+j-1\big|^2}{j^2}\Bigg\}.
\end{aligned}$$

By induction hypothesis for $m = n-1$, we get

$$\begin{aligned}
|a_n|^2 \leq \frac{\cos^2\alpha}{(n-p)^2}\Bigg\{ & \frac{(n-p-1)^2}{\cos^2\alpha}\prod_{j=1}^{n-p-1}\frac{\lambda^2\big|2(p-\beta)e^{-i\alpha}\cos\alpha+j-1\big|^2}{j^2} \\
& + \Bigg(\lambda^2(2p-2\beta+n-p-1)^2-(n-p-1)^2(\sec^2\alpha-\lambda^2\tan^2\alpha)\Bigg) \\
& \times \prod_{j=1}^{n-p-1}\frac{\lambda^2\big|2(p-\beta)e^{-i\alpha}\cos\alpha+j-1\big|^2}{j^2}\Bigg\},
\end{aligned}$$

i.e

$$\begin{aligned}
|a_n|^2 \leq \frac{\lambda^2}{(n-p)^2} & \Bigg((2p-2\beta+n-p-1)^2\cos^2\alpha+(n-p-1)^2\sin^2\alpha\Bigg) \\
& \times \prod_{j=1}^{n-p-1}\frac{\lambda^2\big|2(p-\beta)e^{-i\alpha}\cos\alpha+j-1\big|^2}{j^2}.
\end{aligned}$$

On simplification, the above inequality leads to

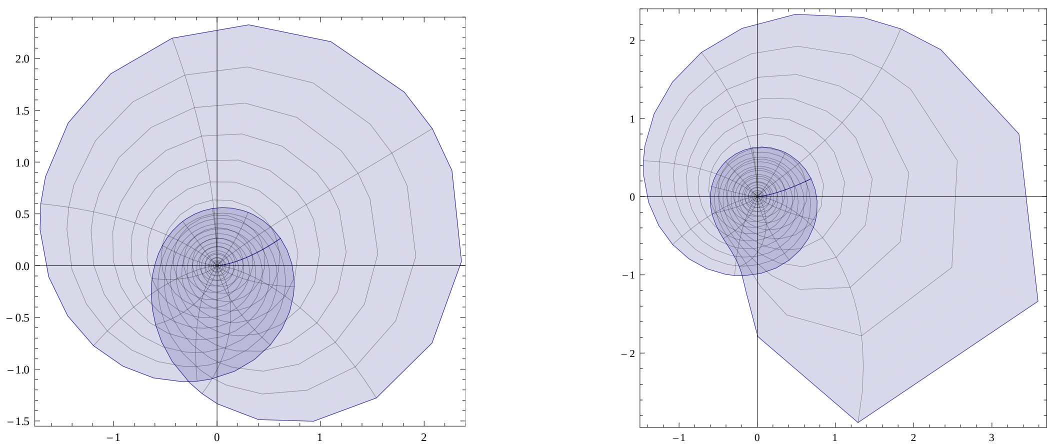
$$|a_n| \leq \prod_{j=1}^{n-p}\frac{\lambda\big|2(p-\beta)e^{-i\alpha}\cos\alpha+j-1\big|}{j}.$$

It is easy to prove that the bounds are sharp as can be seen by the rotation of the function

$$k_{p,\alpha,\beta,\lambda}(z) = \frac{z^p}{(1+\lambda z)^{\zeta}}.$$

k_n	p	α	β	λ
k_1	2	$\pi/4$	1	0.5
k_2	2	$\pi/4$	1.5	0.9
k_3	3	$-\pi/3$	2	0.8
k_4	3	$-\pi/3$	0.5	0.2

TABLE 4.4. The extremal function k_n



The image domain $k_1(\mathbb{D})$ The image domain $k_2(\mathbb{D})$

FIGURE 4.3. Images of the unit disk under k_1 and k_2.

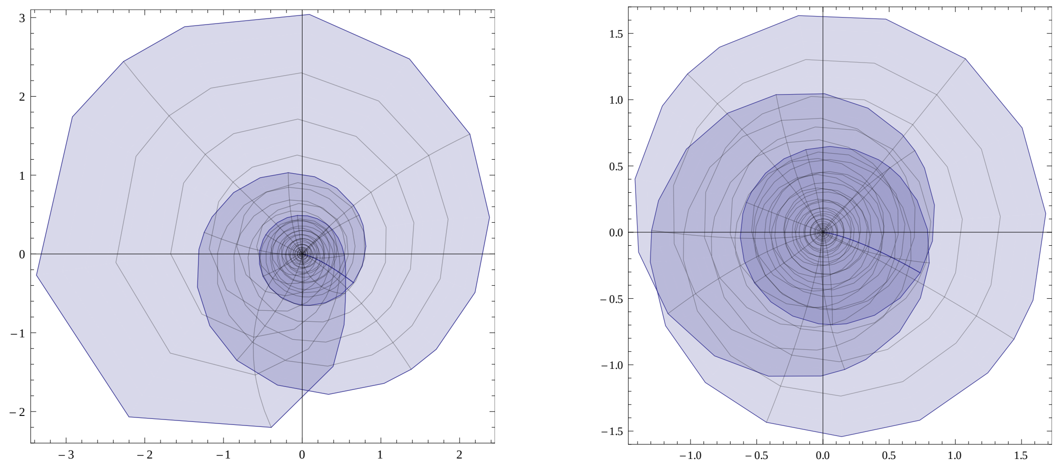
Here $\zeta := 2(p-\beta)e^{-i\alpha}\cos\alpha$. This completes the proof of Theorem 4.5. $\square$

Remark 4.6. Letting the different values of p, α, β and λ in Theorem 4.5, we obtain results which were proved in [**29**, **30**, **31**, **34**, **49**, **63**, **64**, **77**, **100**].

For different values of p, α, β and λ (see Table 4.4), the images of the unit disk under the extremal functions $k_n := k_{p,\alpha,\beta,\lambda}$ are described in Figures 4.3 & 4.4. We now give the correct form of the statement stated in Theorem C and its proof.

Theorem 4.7. *Let $0 < \lambda \leq 1$, $p \in \mathbb{N}$ and $b \neq 0$ be any complex number. If $f \in \mathcal{C}_p(b, \lambda)$ is of the form (1.16), then we have*

$$|a_{p+1}| \leq \frac{2\lambda p^2 |b|}{1+p}; \tag{4.13}$$



The image domain $k_3(\mathbb{D})$ The image domain $k_4(\mathbb{D})$

FIGURE 4.4. Images of the unit disk under k_3 and k_4.

for $|2bp+n-p-1| \leq n-p-1$ (*equivalently* $|1+2bp| \leq 1$), *we get*

$$|a_n| \leq \frac{2\lambda p^2|b|}{n(n-p)}, \quad n \geq p+2; \tag{4.14}$$

and for $|2bp+n-p-1| > n-p-1$,

$$|a_n| \leq \frac{p}{n} \prod_{j=0}^{n-p-1} \frac{\lambda|j+2bp|}{j+1}, \quad n \geq p+2. \tag{4.15}$$

The equality signs in (4.13), (4.14) *and* (4.15) *are attained.*

Proof. Let $f(z) \in \mathcal{C}_p(b,\lambda)$. By the equation (1.19), we see that there is an analytic function $\phi : \mathbb{D} \to \overline{\mathbb{D}}$ with $\phi(0) = 0$ such that

$$1 + \frac{zf''(z)}{f'(z)} = \frac{p(1+(2b-1)\lambda\phi(z))}{1-\lambda\phi(z)},$$

or

$$zf''(z) - (p-1)f'(z) = -\lambda\Big((p-2bp-1)f'(z) - zf''(z)\Big)\phi(z).$$

Using the representation (1.16), we observe that

$$\sum_{k=1}^{\infty} k(k+p)a_{k+p}z^k = \lambda\Big(2p^2b + \sum_{k=1}^{\infty}(k+p)(k+2bp)a_{k+p}z^k\Big)\phi(z).$$

k	p	b	λ	V
2	1	1	0.1	-3.998
2	1	1	0.6	1.76
4	2	$3-2i$	0.2	-3.2
4	2	$3-2i$	0.3	12.8

TABLE 4.5. Signs of the constant V

(*This is the place where the in correctness of Aouf's proof is found!*)

We apply Clunie's method [18] for $m \in \mathbb{N}$ (see also [80, 79]) and obtain

$$\sum_{k=1}^{m} k^2(k+p)^2|a_{k+p}|^2 \leq \lambda^2\Big(4p^4|b|^2 + \sum_{k=1}^{m-1}(k+p)^2|k+2bp|^2|a_{k+p}|^2\Big).$$

The above inequality yields

$$|a_{m+p}|^2 \leq \frac{1}{m^2(m+p)^2}\Big(4\lambda^2p^4|b|^2 + \sum_{k=1}^{m-1}(k+p)^2\big(\lambda^2|k+2bp|^2-k^2\big)|a_{k+p}|^2\Big).$$

Replacing $m+p$ by n, we get

$$|a_n|^2 \leq \frac{1}{n^2(n-p)^2}\Big(4\lambda^2p^4|b|^2 + \sum_{k=1}^{n-p-1}(k+p)^2\big(\lambda^2|k+2bp|^2-k^2\big)|a_{k+p}|^2\Big) \tag{4.16}$$

for $n \geq p+1$.

Note that the terms under the summation in the right hand side of (4.16) may be positive as well as negative. We inspect this by including here a table (see Table 4.5) for values of

$$V := \lambda^2|k+2bp|^2 - k^2$$

for different choices of k, p, b and λ. So, we can not apply direct mathematical induction in (4.16) to prove the required coefficients bounds for $f \in \mathcal{C}_p(b,\lambda)$. Therefore, we are taking different cases for this.

First, for $n = p+1$, (4.16) reduces to

$$|a_{p+1}| \leq \frac{2\lambda p^2|b|}{1+p}.$$

This proves (4.13).

Secondly, we consider the case $|2bp+n-p-1| \leq n-p-1$ (equivalently, $|1+2bp| \leq 1$) for $n \geq p+2$. Since all the terms under the summation in (4.16) are non-positive, we get

$$|a_n| \leq \frac{2\lambda p^2|b|}{n(n-p)},$$

which establishes (4.14). The equality holds in (4.13) and (4.14) for the rotation of the functions $k_{n,p,b,\lambda} \in \mathcal{C}_p(b,\lambda)$ given by

$$k'_{n,p,b,\lambda}(z) = \frac{pz^{p-1}}{(1+\lambda z^{n-1})^{2bp/(n-1)}}.$$

Finally, we prove (4.15) when $|1+2bp| \geq |2bp+n-p-1| > n-p-1$ for $n \geq p+2$. We see that all the terms under the summation in (4.16) are positive. We prove the inequality by the mathematical induction. We consider that (4.15) holds for $k = 3, 4, \ldots, n-p$. Then from (4.16), we obtain

$$|a_n|^2 \leq \frac{1}{n^2(p-n)^2}\left(4\lambda^2 p^4|b|^2 + \sum_{k=1}^{n-p-1} p^2\left(\lambda^2|k+2bp|^2 - k^2\right)\prod_{j=0}^{k-1}\frac{\lambda^2|j+2bp|^2}{(j+1)^2}\right). \tag{4.17}$$

We now prove that the square of the right hand side of (4.15) is equal to the right hand side of (4.17), that is

$$\prod_{j=0}^{m-p-1}\frac{\lambda^2|j+2bp|^2}{(j+1)^2} = \frac{1}{(p-m)^2}\left(4\lambda^2p^2|b|^2 + \sum_{k=1}^{n-p-1}\left(\lambda^2|k+2bp|^2-k^2\right)\right.$$
$$\left.\times\prod_{j=0}^{k-1}\frac{\lambda^2|j+2bp|^2}{(j+1)^2}\right), \tag{4.18}$$

when $|2bm+p-p-1| > m-p-1$, $m \geq p+2$.

For $m = p+2$, the equation (4.18) is recognized. Suppose that (4.18) is true for all m, $p+2 < m \leq n-p$. Then from (4.17), we obtain

$$|a_n|^2 \leq \frac{1}{n^2(p-n)^2}\Bigg(4\lambda^2p^4|b|^2 + \sum_{k=1}^{n-p-2} p^2\Big(\lambda^2|k+2bp|^2 - k^2\Big)\prod_{j=0}^{k-1}\frac{\lambda^2|j+2bp|^2}{(j+1)^2}$$
$$+ p^2\Big(\lambda^2|n-p-1+2bp|^2 - (n-p-1)^2\Big)\prod_{j=0}^{n-p-2}\frac{\lambda^2|j+2bp|^2}{(j+1)^2}\Bigg).$$

Using relation (4.18) for $m = n-1$, we find that

$$|a_n|^2 \leq \frac{1}{n^2(p-n)^2}\Bigg(p^2(p-n+1)^2\prod_{j=0}^{n-p-2}\frac{\lambda^2|j+2bp|^2}{(j+1)^2}$$
$$+ p^2\Big(\lambda^2|n-p-1+2bp|^2 - (n-p-1)^2\Big)\prod_{j=0}^{n-p-2}\frac{\lambda^2|j+2bp|^2}{(j+1)^2}\Bigg).$$

It is equivalent to

$$|a_n| \leq \frac{p\lambda|j+2bp|}{n(p-n)}\prod_{j=0}^{n-p-2}\frac{\lambda|j+2bp|}{(j+1)},$$

Which establishes (4.15).

The bounds are sharp for the rotation of the function $k_{p,b,\lambda} \in \mathcal{C}_p(b,\lambda)$ which is given by

$$k'_{p,b,\lambda}(z) = \frac{pz^{p-1}}{(1+\lambda z)^{2bp}}.$$

This completes the proof of Theorem 4.5. □

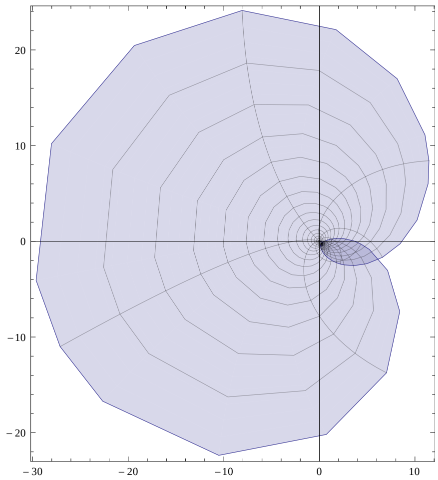
Remark 4.8. Letting the different values of p, b and λ in Theorem 4.7, we obtain results which were proved in [**6, 31, 77, 97**].

For different values of p, b and λ (see Table 4.6), the images of the unit disk under the extremal functions $g_n := k'_{p,b,\lambda}(z)$ are described in Figures 4.5 & 4.6.

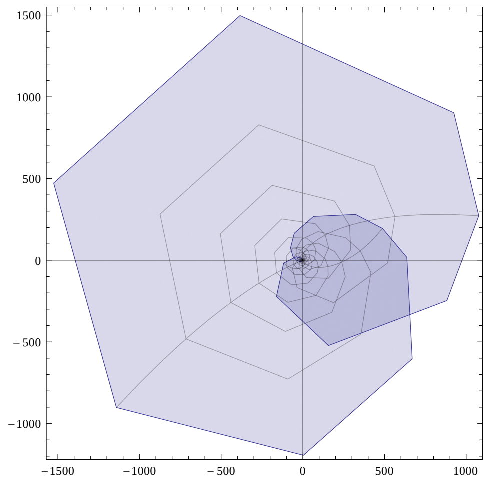
We end this chapter here.

g_n	p	b	λ
g_1	2	$1+i$	0.4
g_2	2	$2-3i$	0.4
g_3	3	$1-2i$	0.7
g_4	3	$3-2i$	0.7

TABLE 4.6. The extremal function g_n

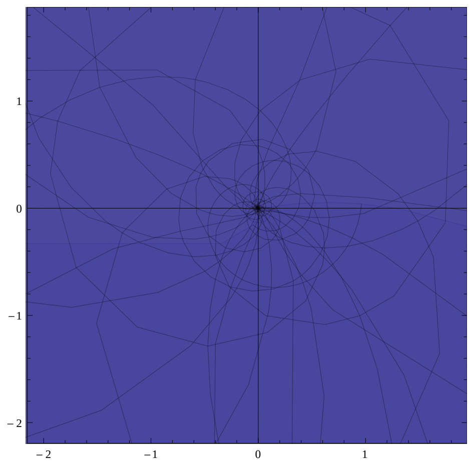


The image domain $g_1(\mathbb{D})$

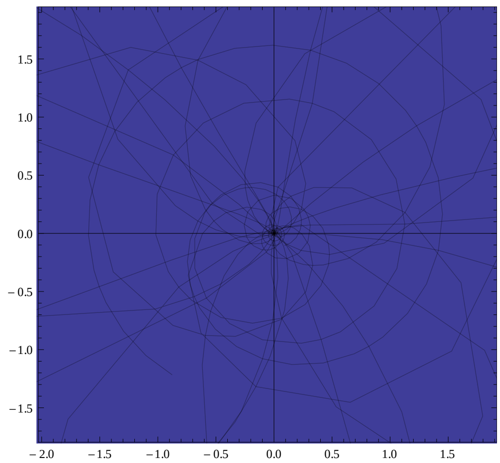


The image domain $g_2(\mathbb{D})$

FIGURE 4.5. Images of the unit disk under g_1 and g_2.



The image domain $g_3(\mathbb{D})$



The image domain $g_4(\mathbb{D})$

FIGURE 4.6. Images of the unit disk under g_3 and g_4.

CHAPTER 5

INTEGRAL MEANS AND MAXIMAL AREA INTEGRAL PROBLEMS FOR p-VALENT FUNCTIONS

This chapter is composed of two types of problems. First one is the integral means and second one is the maximal area integral problems for certain classes of p-valent functions. The structure of this chapter is as follows. Section 5.1 gives some preliminary information on the family $\mathcal{S}_p^*(A,B)$ and other basic definitions that are used in the sequel. Section 5.2 and 5.3 deal with the statements of our main results and some of their important consequences. In section 5.4, we state and prove some lemmas which are used as tools to prove our main results. We prove our main results in Section 5.5. Finally, in section 5.6, we propose some open problems.

The results of this chapter have been included in:
Sharma N.L., *Integral means and maximum area integral problems for certain family of p-valent functions*, Communicated.

5.1. Basic Information

The motivation to study p-valent functions comes from the theory of univalent functions. One of the basic problems in p-valent function theory is to see how results from univalent function theory fit analogously into the theory of p-valent functions for $p \geq 2$. Background on some of the important problems in the theory of p-valent functions, for instance, can be found in [**16, 31, 29, 63, 64, 75**]. However, in this collection, the classical integral means and area problems have not been studied in p-valent setting, which is our objective in this chapter.

Recall, we considered the class $\mathcal{S}_p^*(A,B)$ in Chapter 1 by (1.22) with the conditions $A \in \mathbb{C}, A \neq B$ and $-1 \leq B \leq 0$. The function $k_{A,B,p}$, defined by (1.26), plays the role of an extremal function for the class $\mathcal{S}_p^*(A,B)$.

Choosing $A = \lambda e^{-i\alpha}(e^{-i\alpha} - (2\beta/p)\cos\alpha)$ and $B = -\lambda$, the class $\mathcal{S}_p^*(A,B)$ reduces to the class $\mathcal{F}_p(\alpha,\beta,\lambda)$ of functions $f \in \mathcal{A}_p$, satisfying the relation

$$e^{i\alpha}\frac{zf'(z)}{pf(z)} \prec \frac{e^{i\alpha} + (e^{-i\alpha} - (2\beta/p)\cos\alpha)\,\lambda z}{1-\lambda z}, \quad z \in \mathbb{D}, \tag{5.1}$$

or the equation (5.1) is equivalent to the equation (1.18) (see also Definition 4.2). Here $0 < \lambda \leq 1$, $0 \leq \beta < p$, $p \in \mathbb{N}$ and $|\alpha| < \pi/2$. In Chapter 4, we obtained the correct forms of the coefficient bounds for functions to be in the class $\mathcal{F}_p(\alpha,\beta,\lambda)$ and other related classes of p-valent functions. If we take different values of p, α, β and λ in the class $\mathcal{F}_p(\alpha,\beta,\lambda)$, then we get certain subclasses of p-valent functions (see p. 55, Chapter 4). It is easy to see that the function $k_{p,\alpha,\beta,\lambda}$ is defined by

$$k_{p,\alpha,\beta,\lambda}(z) = \frac{z^p}{(1-\lambda z)^{\xi}}, \quad \xi = 2(p-\beta)e^{i\alpha}\cos\alpha \tag{5.2}$$

belongs to the class $\mathcal{F}_p(\alpha,\beta,\lambda)$.

We note that, by taking distinct parameters A, B and p in the class $\mathcal{S}_p^*(A,B)$, we get the following classes which were investigated and studied by several authors. We list down some of them as follows:

1. $\mathcal{S}_p^*(1-(2\beta/p),-1) =: \mathcal{S}_p^*(\beta)$, $0 \leq \beta < p$ and $\mathcal{S}_p^*(1,-1) =: \mathcal{S}_p^*$.
2. $\mathcal{S}_p^*((1-(2\beta/p))\lambda,-\lambda) =: \mathcal{T}_p(\lambda,\beta)$ (i.e. $\mathcal{F}_p(0,\beta,\lambda) =: \mathcal{T}_p(\lambda,\beta)$), the class of p-valent functions of $\mathcal{T}(\lambda,\beta)$ which is studied in [8].
3. The class $\mathcal{S}_1^*((1-2\beta)\lambda,-\lambda) =: \mathcal{T}(\lambda,\beta)$ $(0 \leq \beta < 1)$ (see Table 3.1).
4. $\mathcal{S}_1^*(\beta) =: \mathcal{S}^*(\beta)$ $(0 \leq \beta < 1)$ and $\mathcal{S}_1^*(1,-1) =: \mathcal{S}^*$.

In this chapter, we consider the functions f in $\mathcal{A}_p$ $(p \in \mathbb{N})$ such that z^p/f is non-vanishing in $\mathbb{D}$, hence it can be represented as Taylor's series of the form

$$\frac{z^p}{f(z)} = 1 + \sum_{n=1}^{\infty} b_{n+p-1} z^n, \; z \in \mathbb{D}. \tag{5.3}$$

5.2. Integral Means Problems

Suppose, $f \in \mathcal{A}_p$, let us consider the integral means

$$M(r, f, p, \lambda_1, \lambda_2) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^{\lambda_1} |f'(re^{i\theta})|^{\lambda_2} d\theta,$$

for λ_1, $\lambda_2 \in \mathbb{R}$ and $r \in (0,1)$. For the special case $\lambda_1 = -2$ and $\lambda_2 = 0$, we find the following interesting integral means such that

$$I_1(r, f, p) := M(r, f, p, -2, 0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{|f(re^{i\theta})|^2} d\theta.$$

The integral means $L_1(r, f, p) := r^{2p} I_1(r, f, p)$ defined in Chapter 1 by (1.20).

One of the motivation to study this form of the integral means comes from the following observations. The integral means $L_1(r, f, 1)$ are associated with some functionals appearing in planar fluid mechanics concerning isoperimetric problems for moving phase domains; see [**95, 96**]. Another aim to study integral means problem was to solve the Bieberbach conjecture; see [**22, 80**] and references therein. In 2002, Gromova and Vasil'ev [**36**] made a conjecture that if $f \in \mathcal{S}^*(\beta) := \mathcal{S}_1^*(1-2\beta, -1)$ for $\beta \in [0,1)$, then the estimate

$$L_1(r, f) := L_1(r, f, 1) \leq \frac{\Gamma(5 - 4\beta)}{\Gamma^2(3 - 2\beta)}$$

holds, where Γ is the classical gamma function. The estimate was proven sharp only for $\beta = 0$ and $\beta = 1/2$. This conjecture has been recently settled by Ponnusamy and Wirths in [**71**] for a more general setting by considering the family $\mathcal{S}^*(A, B) := \mathcal{S}_1^*(A, B)$ with $-1 \leq B < A \leq 1$.

In this chapter, we estimate the quantity $L_1(r, f, p)$ for $f \in \mathcal{S}_p^*(A, B)$, $A \in \mathbb{C}$, $-1 \leq B \leq 0$ and $A \neq B$ and for other related class of p-valent functions. We stated our first main result by Theorem 1.9 in Section 1.3 and its proof presented in Section 5.5. We now discuss some of their consequences.

We remark that when $p = 1$ in Theorem 1.9, then we obtain [**71**, Theorem 1]. For $A = 1 - (2\beta/p)$ and $B = -1$, Theorem 1.9 implies the following corollaries:

Corollary 5.1. *For $0 \le \beta < p$ and $p \in \mathbb{N}$, let $f \in \mathcal{S}_p^*(\beta)$. Then we have*

$$L_1(r, f, p) \le \frac{\Gamma(1 + 4(p - \beta))}{\Gamma^2(1 + 2(p - \beta))}, \quad r \in (0, 1].$$

In particular, we have

- $L_1(r, f, p) \le \Gamma(1 + 4p)/\Gamma^2(1 + 2p)$ *for* $f \in \mathcal{S}_p^*$.
- $L_1(r, f, 1) =: L_1(r, f) \le \Gamma(5 - 4\beta)/\Gamma^2(3 - 2\beta)$ *for* $f \in \mathcal{S}^*(\beta)$ *and* $L_1(r, f) \le 6$ *for* $f \in \mathcal{S}_1^*(0) =: \mathcal{S}^*$ (*see* [**71**, Corollary 1]).

All inequalities are sharp.

Moreover, choosing $B = -A$ in Theorem 1.9, we have

Corollary 5.2. *Let $f \in \mathcal{S}_p^*(A, -A)$ for $0 < A \le 1$ and $p \in \mathbb{N}$. Then we have*

$$L_1(r, f, p) \le {}_2F_1(-2p, -2p; 1; A^2), \quad 0 < r \le 1.$$

In particular, $L_1(r, f, 1) \le 1 + 4A^2 + A^4$ for $f \in \mathcal{S}_1^*(A, -A)$ (see [**71**, Corollary 2]).

If we choose $A = \lambda e^{-i\alpha}(pe^{-i\alpha} - 2\beta \cos\alpha)/p$ and $B = -\lambda$, then

$$\begin{aligned} p\phi = p\left(\frac{A}{B} - 1\right) &= -e^{-i\alpha}(pe^{-i\alpha} - 2\beta\cos\alpha) - p \\ &= -2e^{-i\alpha}(p - \beta)\cos\alpha =: -\xi. \end{aligned}$$

By Theorem 1.9, for $B \neq 0$, we get the following integral means for $f \in \mathcal{F}_p(\alpha, \beta, \lambda) := \mathcal{S}_p^*\left(\lambda e^{-i\alpha}(e^{-i\alpha} - (2\beta/p)\cos\alpha), -\lambda\right)$.

Theorem 5.3. *For $0 < \lambda \le 1$, $0 \le \beta < p$, $p \in \mathbb{N}$ and $|\alpha| < \pi/2$. Let $f \in \mathcal{F}_p(\alpha, \beta, \lambda)$ be such that z^p/f has the form (5.3), then we have*

$$L_1(r, f, p) := r^{2p} I_1(r, f, p) \le \sum_{n=0}^{\infty} \left|{}^{\xi}C_n\right|^2 \lambda^{2n},$$

where $\xi = 2(p - \beta)e^{-i\alpha}\cos\alpha$ and ${}^{\xi}C_n$ denotes the combination. The equality is attained for the functions $k_{p,\alpha,\beta,\lambda}$ as defined by (5.2).

If we let $\lambda = 1$, then Theorem 5.3 yields:

Corollary 5.4. *Let $f \in \mathcal{F}_p(\alpha, \beta, 1) =: \mathcal{S}_{\alpha,p}(\beta)$, for $0 \le \beta < p$, $p \in \mathbb{N}$ and $|\alpha| < \pi/2$. Then we have*

$$L_1(r, f, p) \le \sum_{n=0}^{\infty} \left|{}^{\xi}C_n\right|^2,$$

where $\xi = 2(p - \beta)e^{i\alpha} \cos\alpha$. The estimate is sharp. In particular, we have the following

- $L_1(r, f, p) \le \sum_{n=0}^{\infty} \left|{}^{\eta}C_n\right|^2$ *for $f \in \mathcal{F}_p(\alpha, 0, 1) =: \mathcal{S}_{\alpha,p}$ where $\eta = 2pe^{-i\alpha} \cos\alpha$.*
- $L_1(r, f, p) \le \Gamma(1 + 4p)/\Gamma^2(1 + 2p)$ *for $f \in \mathcal{F}_p(0, 0, 1) =: \mathcal{S}_p^*$.*

All inequalities are sharp.

5.3. Area Integral Problems

The area $\Delta(r, f)$ of the multi-sheeted image of $\mathbb{D}_r$, $0 < r \le 1$ under $f \in \mathcal{A}_p$ is defined by (1.21) in Chapter 1. We now state our second main results and some of their consequences.

Theorem 5.5. *Let $f \in \mathcal{S}_p^*(A)$, for $0 < |A| \le 1$ and $p \in \mathbb{N}$, be of the form (5.3). Then we have*

$$\max_{f \in \mathcal{S}_p^*(A)} \Delta\left(r, \frac{z^p}{f}\right) = \pi|A|^2 p^2 r^2 {}_0F_1(2, |A|^2 p^2 r^2) =: E_A(r, p), \tag{5.4}$$

where r, $0 < r \le 1$, and the maximum is attained by the rotations of $k_{A,p}(z) = z^p e^{Apz}$.

The case $A = 1$ simplifies to

Corollary 5.6. *If $f \in \mathcal{S}_p^*(1)$, for $p \in \mathbb{N}$. Then we have*

$$\max_{f \in \mathcal{S}_p^*(1)} \Delta\left(r, \frac{z^p}{f}\right) = \pi p^2 r^2 {}_0F_1(2, p^2 r^2), \quad r \in (0, 1].$$

The maximum is attained by the rotations of the function $k_{1,p}(z) = z^p e^{pz}$.

Our main theorem for $B \neq 0$ presented in Chapter 1 by Theorem 1.10. Moreover, Theorem 1.10, for $A = 1 - (2\beta/p)$ and $B = -1$, gives the following result.

Corollary 5.7. *If $f \in \mathcal{S}_p^*(\beta)$, for $0 \le \beta < p$ and $p \in \mathbb{N}$, then we have*

$$\max_{f\in\mathcal{S}_p^*(\beta)} \Delta\left(r, \frac{z^p}{f}\right) = 4\pi(p-\beta)^2 r^2 {}_2F_1\big((2\beta-2p+1),(2\beta-2p+1);2;r^2\big),$$

for all $r \in (0,1]$. The maximum is attained for the rotations of the function $z^p/(1-z)^{2p-2\beta}$. In particular, for $f \in \mathcal{S}_p^*(0) =: \mathcal{S}_p^*$, one has

$$\max_{f\in\mathcal{S}_p^*} \Delta\left(r, \frac{z^p}{f}\right) = 4\pi p^2 r^2 {}_2F_1\big(1-2p,1-2p;2;r^2\big), \quad r \in (0,1],$$

and the maximum is attained for the rotations of the function $z^p/(1-z)^{2p}$.

For the case $p = 1$, the above Theorems and Corollaries of Section 5.3 have been proved in Chapter 3. Choosing $A = (1-(2\beta/p))\lambda$ and $B = -\lambda$ in Theorem 1.10, then we find that

Corollary 5.8. *Let $f \in \mathcal{T}_p(\lambda,\beta)$, for $0 < \lambda \le 1$, $0 \le \beta < p$ and $p \in \mathbb{N}$. Then we have*

$$\max_{f\in\mathcal{T}_p(\lambda,\beta)} \Delta\left(r, \frac{z^p}{f}\right) = 4\pi\lambda^2(p-\beta)^2 r^2 {}_2F_1\big((2\beta-2p+1),(2\beta-2p+1);2;\lambda^2 r^2\big),$$

where r, $0 < r \le 1$, and the maximum is attained for the rotations of $z^p/(1-\lambda z)^{2p-2\beta}$. In particular, for $f \in \mathcal{T}_p(1,\beta) =: \mathcal{S}_p^*(\beta)$, we get Corollary 5.7.

We end this section with the following special results.

The case $A = \lambda e^{-i\alpha}(e^{-i\alpha} - (2\beta/p)\cos\alpha)$ and $B = -\lambda$, simplifies that

$$\begin{aligned} p\left(\frac{A}{B} - 1\right) + 1 &= -e^{-i\alpha}(pe^{-i\alpha} - 2\beta\cos\alpha) - p + 1 \\ &= 1 - 2e^{-i\alpha}(p-\beta)\cos\alpha =: 1 - \xi. \end{aligned}$$

By Theorem 1.10, we obtain Yamashita's conjecture on area maximum property for the class $\mathcal{F}_p(\alpha,\beta,\lambda)$.

Theorem 5.9. *Let λ, β, α such that $0 < \lambda \le 1$, $0 \le \beta < p$, $-\pi/2 < \alpha < \pi/2$ and $p \in \mathbb{N}$. If the function f, defined by (5.3), belongs to the class $\mathcal{F}_p(\alpha,\beta,\lambda)$, then we have*

$$\max_{f\in\mathcal{F}_p(\alpha,\beta,\lambda)} \Delta\left(r, \frac{z^p}{f}\right) = E_{\alpha,\beta,\lambda}(r,p),$$

where

$$E_{\alpha,\beta,\lambda}(r,p) = \pi r^2\lambda^2|\overline{\xi}|^2{}_2F_1(1-\xi, 1-\overline{\xi}; 2; \lambda^2r^2),$$

with $\xi = 2(p-\beta)e^{-i\alpha}\cos\alpha$. *The maximum is attained for the rotations of* $k_{p,\alpha,\beta,\lambda}$ *as defined by* (5.2).

The case $\lambda = 1$ of Theorem 5.9 gives

Corollary 5.10. *Let* $f \in \mathcal{S}_{\alpha,p}(\beta) := \mathcal{F}_p(\alpha,\beta,1)$ *be of the form* (5.3). *Then we have*

$$\max_{f\in\mathcal{S}_{\alpha,p}(\beta)} \Delta\left(r, \frac{z^p}{f}\right) = \pi r^2|\overline{\xi}|^2{}_2F_1(1-\xi, 1-\overline{\xi}; 2; r^2), \quad 0 < r \le 1.$$

The maximum is attained for the rotations of $k_{p,\alpha,\beta,1}$ *as defined by* (5.2).
In particular, for $f \in \mathcal{S}_{\alpha,p} := \mathcal{S}_{\alpha,p}(0)$, *one has*

$$\max_{f\in\mathcal{S}_{\alpha,p}} \Delta\left(r, \frac{z^p}{f}\right) = \pi r^2|\overline{\eta}|^2{}_2F_1\big(1-\eta, 1-\overline{\eta}; 2; r^2\big), \quad \eta = 2pe^{-i\alpha}\cos\alpha,$$

for all $r \in (0,1]$.

For the case $p = 1$, Corollaries 5.8 and 5.10 give the results which are obtained in Chapter 3 (see Corollary 3.3), and in [**71**, Theorem 3 and Corollary 4], respectively.

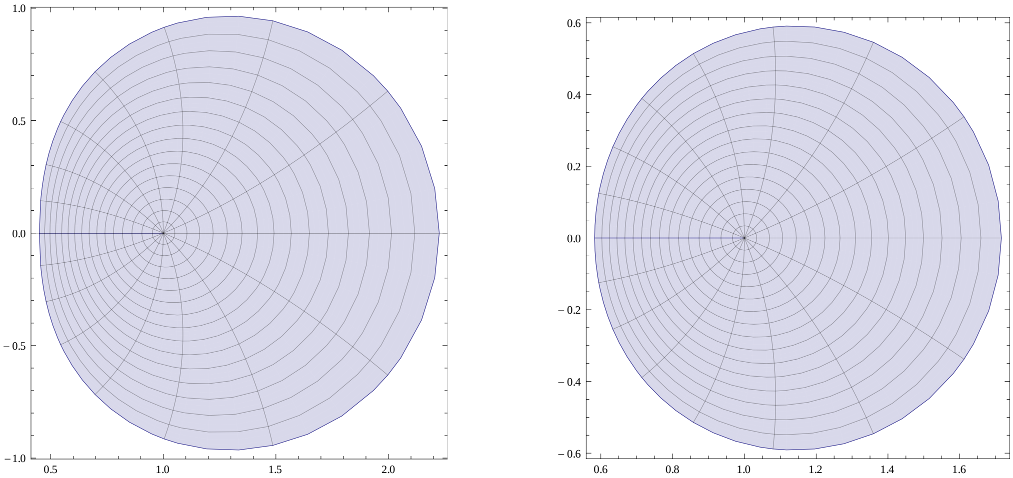
Proofs of Theorems 5.5 and 1.10 are presented in Section 5.5. To see the bounds for the Dirichlet finite function, we denote

$$E_A(1,p) = \pi p^2|A|^2\sum_{n=0}^{\infty}\frac{1}{(1)_n(2)_n}p^{2n}|A|^{2n},$$

$$E_{A,B}(1,p) = \pi p^2|\overline{A}-B|^2\sum_{n=0}^{\infty}\frac{(p\phi+1)_n(p\overline{\phi}+1)_n}{(2)_n(1)_n}B^{2n} \quad \text{and}$$

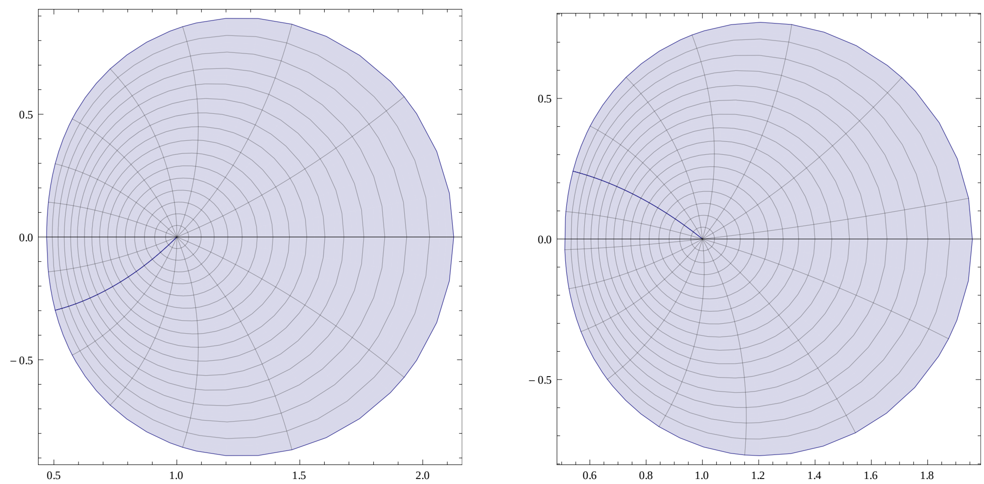
$$E_{\alpha,\beta,\lambda}(1,p) = \pi\lambda^2|\overline{\xi}|^2\sum_{n=0}^{\infty}\frac{(1-\xi)_n(1-\overline{\xi})_n}{(2)_n(1)_n}\lambda^{2n}.$$

The images of the disk $\mathbb{D}_r$ ($r \in (0,1]$) under the extremal functions $g_{A,p}(z) := z^p/k_{A,p}(z) = e^{-Apz}$, $g_{A,B,p}(z) := z^p/k_{A,B,p}(z) = (1+Bz)^{(1-A/B)p}$ and $z^p/k_{p,\alpha,\beta,\lambda}(z) =: l_{p,\alpha,\beta,\lambda}(z) = (1-\lambda z)^{\xi}$ and numerical values of $E_A(r,p)$, $E_{A,B}(r,p)$ and $E_{\alpha,\beta,\lambda}(r,p)$ are described in Figures 5.1–5.5 and Tables 5.1 & 5.2, respectively, for several values of $A, B, \alpha, \beta, \lambda, r$ and



The image domain $g_{2/5,2}(\mathbb{D})$ The image domain $g_{1/5,3}(\mathbb{D}_{0.9})$

FIGURE 5.1. Images of the disk $\mathbb{D}_r$ under $g_{2/5,2}$ and $g_{1/5,3}$.

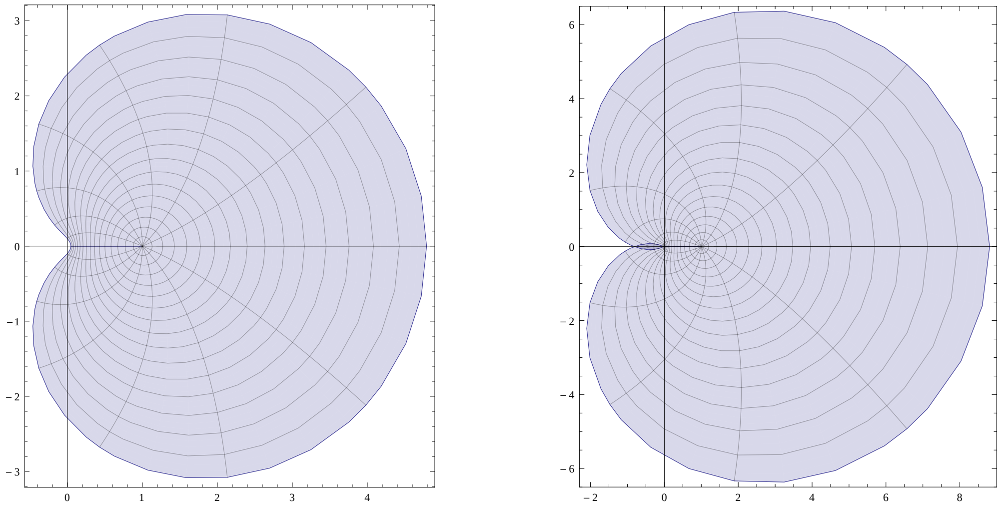


The image domain $g_{(1+i)/3,2}(\mathbb{D}_{0.8})$ The image domain $g_{2/8-i/5,3}(\mathbb{D}_{0.7})$

FIGURE 5.2. Images of the disk $\mathbb{D}_r$ under $g_{(1+i)/3,2}$ and $g_{2/8-i/5,3}$.

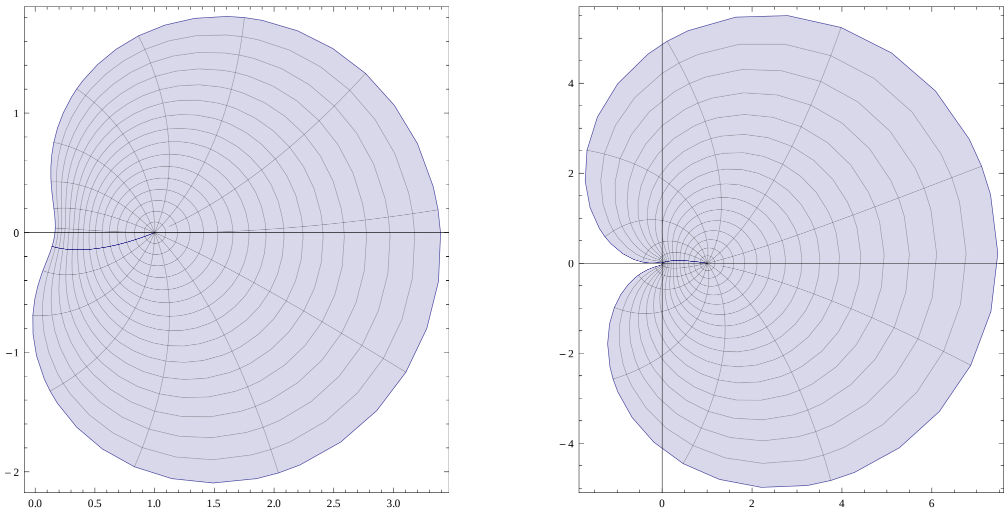
p. We remind the reader that for $B=-1$, $E_{A,B}(1,p)$ is finite only if $2>\operatorname{Re}(2+p(\phi+\overline{\phi})/B)$, i.e. if $\operatorname{Re}A>-1$.

In the next section, we present the following crucial lemmas which play important roles for the proofs of our main results.



The image domain $g_{2/5,-3/5,52}(\mathbb{D})$ The image domain $g_{1/5,-9/10,3}(\mathbb{D}_{0.9})$

FIGURE 5.3. Images of the disk $\mathbb{D}_r$ under $g_{2/5,-3/5,2}$ and $g_{1/5,-9/10,3}$.

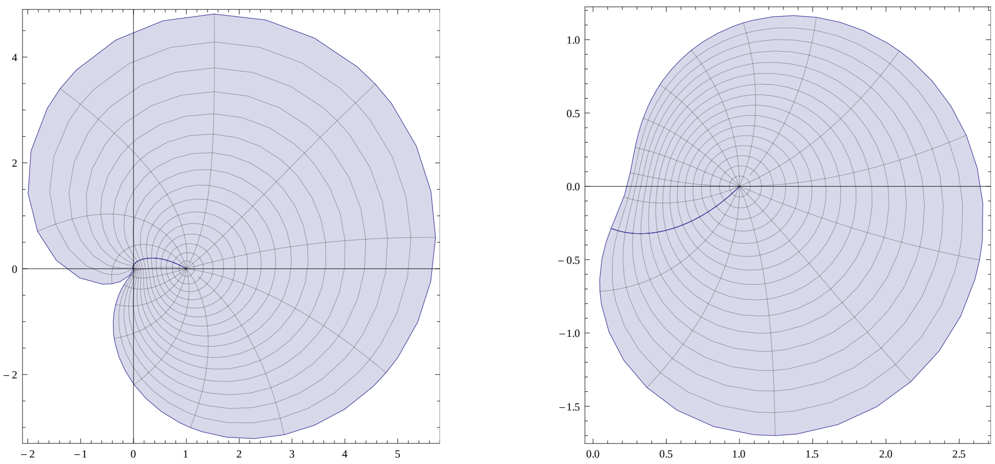


The image domain $g_{(1+i)/3,-0.5,2}(\mathbb{D}_{0.8})$ The image domain $g_{2/8-i/5,0.99,3}(\mathbb{D}_{0.7})$

FIGURE 5.4. Images of the disk $\mathbb{D}_r$ under $g_{(1+i)/3,-0.5,2}$ and $g_{2/8-i/5,0.99,3}$.

5.4. Preparatory Results

We first present a necessary coefficient condition for a function $f \in \mathcal{S}_p^*(A, B)$.



The image domain $l_{2,-\pi/6,1,0.9}(\mathbb{D})$ The image domain $l_{3,\pi/4,1.5,0.6}(\mathbb{D}_{0.9})$

FIGURE 5.5. Images of the disk $\mathbb{D}_r$ under $l_{2,-\pi/6,1,0.9}$ and $l_{3,\pi/4,1.5,0.6}$.

p	A	r	**Approximate Values of** $E_A(r,p)$	B	**Approximate Values of** $E_{A,B}(r,p)$
2	$2/5$	1	2.7264	$-3/5$	26.19994
3	$1/5$	0.9	1.05631	$-9/10$	112.473
2	$(1+i)/3$	0.8	2.34613	$-1/2$	10.5859
3	$2/8-i/5$	0.7	1.76615	$-99/100$	26.98

TABLE 5.1. Approximate values of $E_A(r,p)$ and $E_{A,B}(r,p)$

p	α	β	λ	r	**Approximate Values of** $E_{\alpha,\beta,\lambda}(r,p)$
2	$-\pi/6$	1	$9/10$	1	11.2667
3	$\pi/4$	1.5	$3/5$	0.9	7.1980

TABLE 5.2. Approximate values of $E_{\alpha,\beta,\lambda}(r,p)$

Lemma 5.11. *Let $f \in \mathcal{S}_p^*(A,B)$, for $A \in \mathbb{C}, -1 \leq B \leq 0$, $A \neq B$, $p \in \mathbb{N}$ and f be of the form* (5.3). *Then*

$$\sum_{k=1}^{\infty} \left(k^2 - |kB + (\overline{A} - B)p|^2\right) |b_{k+p-1}|^2 \leq |\overline{A} - B|^2 p^2$$

holds. Equality is attained for the function $k_{A,B,p}$ *as defined by* (1.26).

Proof. Let $f \in \mathcal{S}^*(A,B)$ and $g(z) := z^p/f(z)$. Then by subordination principle, we obtain

$$\frac{zg'(z)}{pg(z)} = \frac{(B-A)zw(z)}{1+Bzw(z)}, \quad z \in \mathbb{D},$$

where $w(0) = 1$ in $\mathbb{D}$. Substituting this in the series expansion (5.3) of g, we get

$$\sum_{k=1}^{\infty} kb_{k+p-1}z^{k-1} = -\left((A-B)p + \sum_{k=1}^{\infty}\big(kB+(A-B)p\big)b_{k+p-1}z^k\right)w(z).$$

It is equivalent to

$$\sum_{k=1}^{n} kb_{k+p-1}z^{k-1} + \sum_{k=n+1}^{\infty} c_k z^{k-1} = -\left((A-B)p + \sum_{k=1}^{n-1}\big(kB+(A-B)p\big)b_{k+p-1}z^k\right)w(z)$$

for certain coefficients c_k. By Clunie's method [**18**] (see also [**19, 79, 80**]) for $n \in \mathbb{N}$, since $|w(z)| < 1$ in $\mathbb{D}$, we find

$$\sum_{k=1}^{n} k^2|b_{k+p-1}|^2r^{2k-2} \leq |A-B|^2p^2 + \sum_{k=1}^{n-1}\big|kB+(A-B)p\big|^2|b_{k+p-1}|^2r^{2k},$$

it holds for all $r \in (0,1)$ and for all large n. It is equivalent to

$$\sum_{k=1}^{n} k^2|b_{k+p-1}|^2r^{2k-2} - \sum_{k=1}^{n-1}\big|kB+(A-B)p\big|^2|b_{k+p-1}|^2r^{2k} \leq |A-B|^2p^2. \tag{5.5}$$

If we take $r \to 1^-$ and allow $n \to \infty$, then we get the desired inequality

$$\sum_{k=1}^{\infty}\left(k^2 - \big|kB+(\overline{A}-B)p\big|^2\right)|b_k|^2 \leq |\overline{A}-B|^2p^2.$$

Equality occurs in the above inequality for the function $k_{A,B,p}$ as defined by (1.26). The proof of Lemma 5.11 is complete. □

Lemma 5.12. *Let* $0 < |A| \leq 1$ *and* $f \in \mathcal{S}_p^*(A)$. *For* $|z| < r$, *suppose that*

$$\frac{z^p}{f(z)} = 1 + \sum_{k=1}^{\infty} b_{k+p-1}z^k \text{ and } e^{-Apz} = 1 + \sum_{k=1}^{\infty} c_{k+p-1}z^k, \quad r \in (0,1].$$

Then for all $N \in \mathbb{N}$,

$$\sum_{k=1}^{N} k|b_{k+p-1}|^2r^{2k} \leq \sum_{k=1}^{N} k|c_{k+p-1}|^2r^{2k} \tag{5.6}$$

holds.

Proof. It is enough to prove the lemma for $0 < A \leq 1$. From the relation (5.5) for $B = 0$, we get

$$\sum_{k=1}^{n-1}(k^2 - A^2p^2r^2)|b_{k+p-1}|^2r^{2k-2} + n^2|b_{n+p-1}|^2r^{2n-2} \leq A^2p^2.$$

Multiplying by r^2 on both sides, we obtain

$$\sum_{k=1}^{n-1}(k^2 - A^2p^2r^2)|b_{k+p-1}|^2r^{2k} + n^2|b_{n+p-1}|^2r^{2n} \leq A^2p^2r^2. \tag{5.7}$$

Obviously, the function e^{-Apz} shows that the equality, when $n \to \infty$, in (5.7) attains with $b_{k+p-1} = c_{k+p-1}$.

We split remaining part of the proof into three following steps.

Step-I: Cramer's Rule.

We consider the inequalities corresponding to (5.7) for $n = 1, 2, \ldots, N$ and multiply the n^{th} coefficient by a factor $\lambda_{n,N}$. These factors are chosen in such a way that the addition of the left sides of the modified inequalities results the left side of (5.6) and hence from the modified inequalities, we get

$$\sum_{k=1}^{N} k|b_{k+p-1}|^2r^{2k} \leq A^2p^2r^2\lambda_{n,N}. \tag{5.8}$$

First, we shall evaluate the suitable multipliers $\lambda_{n,N}$ by Cramer's rule. Secondly, in Step-II, we will prove that these multipliers are all positive. Finally, from (5.6) and (5.8), we will prove the inequality

$$A^2p^2r^2\lambda_{n,N} \leq \sum_{k=1}^{N} k|c_{k+p-1}|^2r^{2k} \tag{5.9}$$

in Step-III. Here $c_{k+p-1} = (Ap)^k/(k!)$.

For the calculation of the factors $\lambda_{n,N}$, we get the following system of linear equations

$$k = k^2\lambda_{k,N} + \sum_{n=k+1}^{N} \lambda_{n,N}(k^2 - A^2p^2r^2), \quad k = 1, 2, \ldots, N. \tag{5.10}$$

Since the matrix of this system is an upper triangular matrix with positive integers as diagonal elements, the solution of this system is uniquely determined. Cramer's rule allows us to write the solution of the system (5.10) in the form

$$\lambda_{n,N} = \frac{((n-1)!)^2}{(N!)^2} \operatorname{Det} A_{n,N},$$

where $A_{n,N}$ is the $(N-n+1) \times (N-n+1)$ matrix constructed as follows:

$$A_{n,N} = \begin{bmatrix} n & n^2 - A^2p^2r^2 & \cdots & n^2 - A^2p^2r^2 \\ n+1 & (n+1)^2 & \cdots & (n+1)^2 - A^2p^2r^2 \\ \vdots & \vdots & \vdots & \vdots \\ N & 0 & \cdots & N^2 \end{bmatrix}.$$

Determinants of these matrices can be obtained by expanding, according to Laplace's rule with respect to the last row, wherein the first coefficient is N and the last one is N^2. The rest of the entries are zeros. This expansion and a mathematical induction lead to the following formula: if $k \leq N-1$, then

$$\lambda_{k,N} = \lambda_{k,N-1} - \frac{1}{N}\left(1 - \frac{A^2p^2r^2}{k^2}\right)\prod_{m=k+1}^{N-1}\left(\frac{A^2p^2r^2}{m^2}\right).$$

We see that the sequence $\{\lambda_{k,N}\}$ is strictly decreasing in N when $k \in \mathbb{N}$ is fixed and $N \geq k$, i.e. $\lambda_{k,N} < \lambda_{k,N-1}$ with

$$\lambda_k := \lim_{N\to\infty} \lambda_{k,N} = \frac{1}{k} - \left(1 - \frac{A^2p^2r^2}{k^2}\right)\sum_{n=k+1}^{\infty}\frac{1}{n}\prod_{m=k+1}^{n-1}\left(\frac{A^2p^2r^2}{m^2}\right). \tag{5.11}$$

To prove that $\lambda_{k,N} > 0$ for all $N \in \mathbb{N}, 1 \leq k \leq N$, it is adequate to show that $\lambda_k \geq 0$ for $k \in \mathbb{N}$. This will be completed in Step II. But before that we want to remark that the proof of the said inequality is sufficient for the proof of the theorem, since, as we remarked for (5.7), equality holds for $b_{k+p-1} = c_{k+p-1}$.

Step-II: Positivity of the Multipliers.

In this step, we show that

$$\sum_{n=k+1}^{\infty}\frac{1}{n}\prod_{m=k+1}^{n-1}\left(\frac{A^2p^2r^2}{m^2}\right) \leq \frac{1}{k\left(1 - \frac{A^2p^2r^2}{k^2}\right)} = \frac{1}{k}\sum_{n=k+1}^{\infty}\left(\frac{A^2p^2r^2}{k^2}\right)^n,$$

which is indeed easy to prove, i.e. from (5.11), $\lambda_k \geq 0$.

Step-III:

Since the sequence $\{\lambda_{n,N}\}$ is strictly decreasing in N for each fixed n, $n \le N$, i.e. $\lambda_{n,N} < \lambda_{n,n}$, so that

$$A^2p^2r^2\lambda_{n,N} < A^2p^2r^2\lambda_{n,n} = \frac{A^2p^2r^2}{n}$$
$$< A^2p^2r^2 \le \sum_{k=1}^{N} \frac{k(Ap)^{2k}}{(k!)^2} r^{2k}.$$

This means that inequality (5.9) holds. The proof of our lemma is complete. □

Lemma 5.13. *Let $f \in \mathcal{S}_p^*(A,B)$, for $A \in \mathbb{C}, -1 \le B < 0$, $A \ne B$ and $p \in \mathbb{N}$. Suppose that*

$$(1-Bz)^{(1-(A/B))p} = 1 + \sum_{k=1}^{\infty} d_{k+p-1}z^k \text{ and } \frac{z^p}{f(z)} = 1 + \sum_{k=1}^{\infty} b_{k+p-1}z^k$$

for all r, $0 < r \le 1$. Then the inequality

$$\sum_{k=1}^{N} k|b_{k+p-1}|^2 r^{2k} \le \sum_{k=1}^{N} k|d_{k+p-1}|^2 r^{2k} \tag{5.12}$$

is valid for all $N \in \mathbb{N}$.

Proof. From Lemma 5.11, using inequality (5.5), and then multiplying the resulting inequality by r^2 on both sides, we obtain

$$\sum_{k=1}^{n-1}\left(k^2 - \left|kB + (A-B)p\right|^2 r^2\right)|b_{k+p-1}|^2 r^{2k} + n^2|b_{n+p-1}|^2 r^{2n} \le |A-B|^2p^2r^2.$$

Set for an abbreviation $\phi := (A/B) - 1$, we get

$$\sum_{k=1}^{n-1}\left(k^2 - |k+p\phi|^2B^2r^2\right)|b_{k+p-1}|^2 r^{2k} + n^2|b_{n+p-1}|^2 r^{2n} \le B^2p^2r^2|\phi|^2. \tag{5.13}$$

It is apparent that in the inequality (5.13), the equality is attained for the function $(1-Bz)^{(1-(A/B))p}$ with $b_{k+p-1} = d_{k+p-1}$, when $n \to \infty$.

We split remaining part of the proof into three following steps.

Step-I: Cramer's Rule.

We consider the inequalities corresponding to (5.13) for $n = 1, 2, \dots, N$ and multiply the

nth coefficient by a factor $\lambda_{n,N}$. These factors are chosen in such a way that the addition of the left sides of the modified inequalities results the left side of (5.12) and hence from the modified inequalities, we get

$$\sum_{k=1}^{N} k|b_{k+p-1}|^2 r^{2k} \leq B^2 p^2 r^2 |\phi|^2 \lambda_{n,N}. \tag{5.14}$$

First, we shall evaluate the suitable multipliers $\lambda_{n,N}$ by Cramer's rule. Secondly, in Step II, we shall prove that these multipliers are all positive. Finally, from (5.12) and (5.14), we shall prove the inequality

$$B^2 p^2 r^2 |\phi|^2 \lambda_{n,N} \leq \sum_{k=1}^{N} k|d_{k+p-1}|^2 r^{2k}, \quad n = 1, 2, \ldots, N \tag{5.15}$$

in Step III. Here $d_{k+p-1} = B^k (p\phi)_k/(k!)$.

For the calculation of the factors $\lambda_{n,N}$, we get the following system of linear equations

$$k = k^2 \lambda_{k,N} + \left(k^2 - |k + p\phi|^2 B^2 r^2\right) \sum_{n=k+1}^{N} \lambda_{n,N}, \quad k = 1, 2, \ldots, N. \tag{5.16}$$

Since the matrix of this system is an upper triangular matrix with positive integers as diagonal elements, the solution of this system is uniquely determined. Cramer's rule allows us to write the solution of the system (5.16) in the form

$$\lambda_{n,N} = \frac{((n-1)!)^2}{(N!)^2} \operatorname{Det} A_{n,N},$$

where $A_{n,N}$ is the $(N-n+1) \times (N-n+1)$ matrix constructed as follows:

$$A_{n,N} = \begin{bmatrix} n & n^2 - |n + p\phi|^2 B^2 r^2 & \cdots & n^2 - |n + p\phi|^2 B^2 r^2 \\ n+1 & (n+1)^2 & \cdots & (n+1)^2 - |n + 1 + p\phi|^2 B^2 r^2 \\ \vdots & \vdots & \vdots & \vdots \\ N & 0 & \cdots & N^2 \end{bmatrix}.$$

Determinants of these matrices can be found by expanding according to Laplace's rule with respect to the last row, wherein the first coefficient is N and the last one is N^2. The rest of the entries are zeros. This expansion and a mathematical induction result in the

k	A	B	p	r	$U_{k,p}$
1	0.9	-0.6	2	0.4	0.0784
2	3	-0.4	2	0.8	-4.76
3	$3-i$	-0.9	2	0.2	0.8666
2	0.8	-0.7	5	0.9	-6.5350
3	0.5	-1	5	0.6	0.19
2	$2+3i$	-0.8	5	0.3	-7.5221

TABLE 5.3. Signs of the constant $U_{k,p}$

following formula. If $k \leq N-1$, then

$$\lambda_{k,N} = \lambda_{k,N-1} - \frac{1}{N}\left(1 - \left|1+\frac{p\phi}{k}\right|^2 B^2r^2\right)\prod_{m=k+1}^{N-1}\left(\left|1+\frac{p\phi}{m}\right|^2 B^2r^2\right). \tag{5.17}$$

Note that $U_{k,p} := \left(1 - |1+(p\phi/k)|^2 B^2r^2\right)$ in (5.17) may be positive as well as negative for all $k \in \mathbb{N}$. We investigate this by including here a table (see Table 5.3).

Case (i): Suppose that $U_{k,p}$ is non-positive.

From (5.17), we see that, the sequence $\{\lambda_{k,N}\}$ is strictly increasing in N for every fixed $k \in \mathbb{N}, k \leq N-1$, i.e.

$$\lambda_{k,N} - \lambda_{k,N-1} > 0$$

so that

$$\lambda_{k,N} > \lambda_{k,N-1} > \cdots > \lambda_{k,k} = 1/k > 0,$$

and thus $\lambda_k \geq 0$ when $N \to \infty$ as is required.

Case (ii): Suppose that $U_{k,p}$ is non-negative.

From (5.17), for each fixed $k \in \mathbb{N},\ N \geq k$, the sequence $\{\lambda_{k,N}\}$ is strictly decreasing in N, i.e. $\lambda_{k,N} - \lambda_{k,N-1} < 0$ with

$$\lambda_k := \lim_{N\to\infty} \lambda_{k,N} = \frac{1}{k} - \left(1 - \left|1+\frac{p\phi}{k}\right|^2 B^2r^2\right)\sum_{n=k+1}^{\infty}\frac{1}{n}\prod_{m=k+1}^{n-1}\left(\left|1+\frac{p\phi}{m}\right|^2 B^2r^2\right). \tag{5.18}$$

To show that $\lambda_{k,N} > 0$ for all $N \in \mathbb{N}, k \in [1, N]$, it is enough to show that $\lambda_k \geq 0$ for $k \in \mathbb{N}$. This will be completed in Step II. But before that, we want to note that the proof of the said inequality is adequate for the proof of the theorem, since, we observed in the beginning of the proof, equality is obtained for $b_{k+p-1} = d_{k+p-1}$.

Step-II: Positivity of the Multipliers.

Let for an abbreviation

$$S_k = \sum_{n=k+1}^{\infty} \frac{1}{n} \prod_{m=k+1}^{n-1} \left(\left| 1 + \frac{p\phi}{m} \right|^2 B^2 r^2 \right), \quad k \in \mathbb{N}.$$

We now show that

$$S_k \leq \frac{1}{k \left(1 - \left| 1 + \frac{p\phi}{k} \right|^2 B^2 r^2 \right)}.$$

From the equation (5.18), we get

$$\lambda_k = \frac{1}{k} - S_k + \left(\left| 1 + \frac{p\phi}{k} \right|^2 B^2 r^2 \right) S_k.$$

Again set for an abbreviation

$$T_k = \frac{1}{k} + \left(\left| 1 + \frac{p\phi}{k} \right|^2 B^2 r^2 \right) S_k.$$

It is enough to prove that

$$T_k \leq \frac{1}{k \left(1 - \left| 1 + \frac{p\phi}{k} \right|^2 B^2 r^2 \right)}. \tag{5.19}$$

To prove (5.19) we use the inequality

$$\frac{1}{n \left(1 - \left| 1 + \frac{p\phi}{n} \right|^2 B^2 r^2 \right)} > \frac{1}{(n+1) \left(1 - \left| 1 + \frac{p\phi}{n+1} \right|^2 B^2 r^2 \right)} \tag{5.20}$$

(this inequality follows from the fact that $n \left(1 - \left| 1 + (p\phi/n) \right|^2 B^2 r^2 \right)$ is an increasing by derivative test) and the identity

$$\frac{1}{n \left(1 - \left| 1 + \frac{p\phi}{n} \right|^2 B^2 r^2 \right)} = \frac{1}{n} + \frac{\left| 1 + \frac{p\phi}{n} \right|^2 B^2 r^2}{n \left(1 - \left| 1 + \frac{p\phi}{n} \right|^2 B^2 r^2 \right)}, \tag{5.21}$$

which are admissible for each $n \in \mathbb{N}$. Repeated application of (5.20) and (5.21) for $n = k, k+1, \ldots, Q$ results in the inequality

$$\begin{aligned}\frac{1}{k\left(1-\left|1+\frac{p\phi}{k}\right|^2 B^2r^2\right)} &> \sum_{n=k}^{Q}\frac{1}{n}\prod_{m=k}^{n-1}\left(\left|1+\frac{p\phi}{m}\right|^2 B^2r^2\right) \\ &\quad + \frac{\prod_{m=k}^{Q}\left(\left|1+\frac{p\phi}{m}\right|^2 B^2r^2\right)}{Q\left(1-\left|1+\frac{p\phi}{Q}\right|^2 B^2r^2\right)} \\ &=: S_{k,Q} + R_{k,Q}, \text{ for } k \le Q.\end{aligned}$$

Since $R_{k,Q} > 0$, allow the limit as $Q \to \infty$, we get

$$\frac{1}{k\left(1-\left|1+\frac{p\phi}{k}\right|^2 B^2r^2\right)} \ge \lim_{T\to\infty} S_{k,Q} = \sum_{n=k}^{\infty}\frac{1}{n}\prod_{m=k}^{n-1}\left(\left|1+\frac{p\phi}{m}\right|^2 B^2r^2\right) = Q_k,$$

and we complete the inequality (5.19).

Step-III:

Taking the left side of (5.15) for $N = 2, n = 1$ and using the inequality (5.17), we obtain

$$\begin{aligned}B^2p^2r^2|\phi|^2\lambda_{1,2} &= B^2p^2r^2|\phi|^2\left(\lambda_{1,1} - \frac{1}{2}\left(1-|1+p\phi|^2B^2r^2\right)\right) \\ &= \frac{B^2p^2r^2|\phi|^2}{2} + \frac{B^4p^2r^4|\phi|^2|1+p\phi|^2}{2} \quad (\text{ since } \lambda_{1,1} = 1) \\ &< B^2p^2r^2|\phi|^2 + \frac{B^4p^2r^4|\phi|^2|1+p\phi|^2}{2} = \sum_{k=1}^{2}\frac{k|(p\phi)_k|^2}{(k!)^2}(Br)^{2k}.\end{aligned}$$

Since, $d_{k+p-1} = B^k(p\phi)_k/(k!)$, then the inequality (5.15) holds for $N = 2, n = 1$.

Now, we can complete the proof by a method of induction. Therefore, if we assume that the inequality (5.15) is true for $N = m$ i.e.

(5.22) $$B^2p^2r^2|\phi|^2\lambda_{n,m} \le \sum_{k=1}^{m} k|d_{k+p-1}|^2r^{2k}, \quad n = 1, 2, \ldots, m.$$

Then for $N = m+1$, using the inequality (5.17), we deduce that

$$
\begin{aligned}
B^2p^2r^2|\phi|^2\lambda_{n,m+1} &= B^2p^2r^2|\phi|^2 \left[\lambda_{n,m} - \frac{1}{m+1}\left(1-\left|1+\frac{p\phi}{n}\right|^2 B^2r^2\right)\right. \\
&\qquad \left. \times \prod_{t=n+1}^{m}\left(\left|1+\frac{p\phi}{t}\right|^2 B^2r^2\right)\right] \\
&\leq \sum_{k=1}^{m} k|d_{k+p-1}|^2 r^{2k} - \frac{1}{m+1}\left(1-\left|1+\frac{p\phi}{n}\right|^2 B^2r^2\right) \\
&\qquad \times \prod_{t=n+1}^{m}\left(\left|1+\frac{p\phi}{t}\right|^2 B^2r^2\right) B^2p^2r^2|\phi|^2 \quad (\text{ by } (5.22)) \\
&= \sum_{k=1}^{m} k|d_{k+p-1}|^2 r^{2k} - \frac{1}{m+1}\prod_{t=n+1}^{m}\left(\left|1+\frac{p\phi}{t}\right|^2 B^2r^2\right) B^2p^2r^2|\phi|^2 \\
&\qquad + \frac{1}{m+1}\prod_{t=n}^{m}\left(\left|1+\frac{p\phi}{t}\right|^2 B^2r^2\right) B^2p^2r^2|\phi|^2 \\
&\leq \sum_{k=1}^{m} \frac{k|(p\phi)_k|^2}{(k!)^2}(Br)^{2k} + \frac{1}{m+1}\prod_{t=n}^{m}\left(\left|1+\frac{p\phi}{t}\right|^2 B^2r^2\right) B^2p^2r^2|\phi|^2.
\end{aligned}
$$

Since $d_{k+p-1} = B^k(p\phi)_k/(k!)$, the last inequality implies that

$$
B^2p^2r^2|\phi|^2\lambda_{n,m+1} < \sum_{k=1}^{m} \frac{k|(p\phi)_k|^2}{(k!)^2}(Br)^{2k} + \frac{1}{m+1}\prod_{t=1}^{m}\left(\left|1+\frac{p\phi}{t}\right|^2 B^2r^2\right) B^2p^2r^2|\phi|^2
$$

or equivalently,

$$
\begin{aligned}
B^2p^2r^2|\phi|^2\lambda_{n,m+1} &< \sum_{k=1}^{m} \frac{k|(p\phi)_k|^2}{(k!)^2}(Br)^{2k} + \frac{(m+1)(B^2r^2)^{m+1}}{(1)_{m+1}^2}\prod_{t=1}^{m}\left(\left|1+\frac{p\phi}{t}\right|^2\right)(1)_m^2 p^2|\phi|^2 \\
&= \sum_{k=1}^{m} \frac{k|(p\phi)_k|^2}{(k!)^2}(Br)^{2k} + \frac{(m+1)(B^2r^2)^{m+1}}{(1)_{m+1}^2}\prod_{t=1}^{m}\left(|t+p\phi|^2\right) p^2|\phi|^2 \\
&= \sum_{k=1}^{m} \frac{k|(p\phi)_k|^2}{(k!)^2}(Br)^{2k} + \frac{(m+1)(B^2r^2)^{m+1}}{(1)_{m+1}^2}|(p\phi)_{m+1}|^2 \\
&= \sum_{k=1}^{m+1} \frac{k|(p\phi)_k|^2}{(k!)^2}(Br)^{2k}.
\end{aligned}
$$

Hence, we obtain the desired inequality (5.15). The proof of Lemma 5.13 is complete. □

5.5. Proofs of the Main Results

Proof of Theorem 1.9. Let $f \in \mathcal{S}_p^*(A,B)$. We apply the theorem of Hallenbeck and Ruscheweyh [**37**, Theorem 2] and get

$$\frac{f(z)}{z^p} \prec \frac{1}{(1+Bz)^{(1-(A/B))p}}, \quad z \in \mathbb{D},$$

so that

$$\frac{z^p}{f(z)} \prec (1+Bz)^{(1-(A/B))p} =: \chi_{A,B,p}(z), \quad z \in \mathbb{D}, \tag{5.23}$$

where

$$\chi_{A,B,p}(z) = \frac{z^p}{k_{A,B,p}(z)} = \begin{cases} (1+Bz)^{(1-(A/B))p} & \text{if } B \neq 0 \\ e^{-Apz} & \text{if } B = 0 \end{cases}.$$

For $B \neq 0$, we rewrite the quantity $\chi_{A,B,p}(z)$ in hypergeometric notation and get

$$\chi_{A,B,p}(z) = \begin{cases} {}_2F_1(p\phi, 1; 1; -Bz) & \text{if } B \neq 0 \\ e^{-Apz} & \text{if } B = 0 \end{cases}$$

$$=: \sum_{n=0}^{\infty} d_{n+p-1} z^n \tag{5.24}$$

with $\phi = (A/B) - 1$ and

$$d_{n+p-1} = \begin{cases} \frac{(-1)^n (p\phi)_n B^n}{n!} & \text{if } B \neq 0 \\ \frac{(-1)^n (Ap)^n}{n!} & \text{if } B = 0 \end{cases}.$$

If z^p/f and $\chi_{A,B,p}(z)$ are two analytic functions such that (5.23) holds, and both the functions have the forms (5.3) and (5.24) (with $b_{p-1} = 1 = d_{p-1}$), respectively. Then by Rogosinski's result (see [**22, 80**]) we get

$$\sum_{n=0}^{k} |b_{n+p-1}|^2 r^{2n} \leq \sum_{n=0}^{k} |d_{n+p-1}|^2 r^{2n},$$

for $0 < r < 1$ and $k \in \mathbb{N}$. Thus, from (5.23) and (5.24), we obtain

$$\sum_{n=0}^{k} |b_{n+p-1}|^2 r^{2n} \leq \begin{cases} \displaystyle\sum_{n=0}^{k} \frac{(p\phi)_n (p\overline{\phi})_n}{(n!)^2} B^{2n} r^{2n} & \text{if } B \neq 0 \\ \displaystyle\sum_{n=0}^{k} \frac{1}{(n!)^2} \big(p|A|\big)^{2n} r^{2n} & \text{if } B = 0 \end{cases}.$$

If we take $r \to 1$ and allow $k \to \infty$, then we find the inequality

$$1 + \sum_{n=1}^{\infty} |b_{n+p-1}|^2 \leq \begin{cases} \displaystyle\sum_{n=0}^{\infty} \frac{(p\phi)_n (p\overline{\phi})_n}{(n!)^2} B^{2n} & \text{if } B \neq 0 \\ \displaystyle\sum_{n=0}^{\infty} \frac{1}{(n!)^2} \big(p|A|\big)^{2n} & \text{if } B = 0 \end{cases}$$

$$= \begin{cases} {}_2F_1\big(p\phi, p\overline{\phi}; 1; B^2\big) & \text{if } B \neq 0 \\ J_0(2ip|A|) & \text{if } B = 0 \end{cases},$$

where $J_0(z)$ is the Bessel function of zero order (see for the definition [93]).

Now, we evaluate the integral means for the function z^p/f and get

$$\begin{aligned} L_1(r, f, p) := r^{2p} I_1(r, f, p) &= \frac{1}{2\pi} \int_0^{2\pi} \frac{r^{2p}}{|f(re^{i\theta})|^2} d\theta = \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{z^p}{f(z)} \right|^2 d\theta \\ &= 1 + \sum_{n=1}^{\infty} |b_{n+p-1}|^2 r^{2n} \\ &\leq 1 + \sum_{n=1}^{\infty} |b_{n+p-1}|^2, \end{aligned}$$

which establishes the desired inequality. The result is sharp and it can be easily verified by considering the function $z^p/k_{A,B,p}$, defined in (1.26).

Proof of Theorem 5.5. Suppose $f \in \mathcal{S}_p^*(A)$, $0 < |A| \leq 1$ and $p \in \mathbb{N}$. It is enough to prove the theorem for $0 < A \leq 1$. By the definition of $\mathcal{S}_p^*(A)$, we get

$$\frac{zf'(z)}{pf(z)} \prec 1 + Az = \frac{zk'_{A,p}(z)}{pk_{A,p}(z)}, \ z \in \mathbb{D}.$$

Let $g(z) = z^p/f(z)$ be of the form (5.3). Then using the theorem of Hallenbeck and Ruscheweyh [37, Theorem 2] and subordinate property, we get

$$g(z) \prec e^{-Apz} = \frac{z^p}{k_{A,p}(z)}.$$

By rewriting the last subordination relation in power series form, we have

$$1 + \sum_{k=1}^{\infty} b_{k+p-1} z^k \prec e^{-Apz} = 1 + \sum_{k=1}^{\infty} c_{k+p-1} z^k,$$

where $c_{k+p-1} = (-1)^k (Ap)^k/(k!)$. Now, by Lemma 5.12, for $r \in (0, 1]$, we have

$$\sum_{k=1}^{N} k|b_{k+p-1}|^2 r^{2k} \le \sum_{k=1}^{N} k|c_{k+p-1}|^2 r^{2k},\ N \in \mathbb{N}.$$

If we assume $N \to \infty$, then it follows

$$\pi \sum_{k=1}^{\infty} k|b_{k+p-1}|^2 r^{2k} \le \pi \sum_{k=1}^{\infty} k|c_{k+p-1}|^2 r^{2k},$$

i.e.

$$\Delta\left(r, \frac{z^p}{f}\right) \le \Delta\left(r, \frac{z^p}{k_{A,p}}\right).$$

It is easy to simplifies that $\Delta\left(r, z^p/k_{A,p}\right) = \pi|A|^2 p^2 r^2 {}_0F_1(2, |A|^2 p^2 r^2) = E_A(r, p)$, then we get the desired identity (5.4). The maximum is attained by rotations of $k_{A,p}(z) = z^p e^{Apz}$.

The proof of our theorem is complete. $\square$

Proof of Theorem 1.10. Let $g(z) = z^p/f(z)$ be of the form (5.3). Now, by the definition of $\mathcal{S}_p^*(A, B)$, we obtain

$$\frac{zf'(z)}{pf(z)} \prec \frac{1 + Az}{1 + Bz} = \frac{zk'_{A,B,p}(z)}{pk_{A,B,p}(z)}.$$

By Hallenbeck and Ruscheweyh's result [37] and subordinate principle, we find that

$$g(z) \prec (1 + Bz)^{(1-(A/B))p} = \frac{z^p}{k_{A,B,p}(z)}.$$

Suppose, $z^p/k_{A,B,p}$ has the power series representation $1 + \sum_{k=1}^{\infty} d_{k+p-1} z^n$ with $d_{k+p-1} = (-1)^k B^k (p\phi)_k/(k!)$. Then it follows from Lemma 5.13, for $N \in \mathbb{N}$,

$$\sum_{k=1}^{N} k|b_{k+p-1}|^2 r^{2k} \le \sum_{k=1}^{N} k|d_{k+p-1}|^2 r^{2k}, \quad 0 < r \le 1,$$

which implies that

$$\pi\sum_{k=1}^{\infty}k|b_{k+p-1}|^2r^{2k}\leq\pi\sum_{k=1}^{\infty}k|d_{k+p-1}|^2r^{2p},$$

i.e.

$$\Delta\left(r,\frac{z^p}{f}\right)\leq\Delta\left(r,\frac{z^p}{k_{A,B,p}}\right).$$

By the area formula for $z^p/k_{A,B,p}$, we easily have

$$\begin{aligned}\pi^{-1}\Delta\left(r,\frac{z}{k_{A,B,p}}\right)&=\sum_{k=1}^{\infty}k|d_{k+p-1}|^2r^{2k}\\&=\sum_{k=1}^{\infty}k\frac{(p\phi)_k(p)\overline{\phi})_k}{(1)_k^2}B^{2k}r^{2k}\\&=B^2p^2r^2|\phi|^2\sum_{k=0}^{\infty}\frac{(p\phi+1)_k(p\overline{\phi}+1)_k}{(2)_k(1)_k}B^{2k}r^{2k}.\end{aligned}$$

Hence,

$$\Delta\left(r,\frac{z}{k_{A,B,p}}\right)=\pi|\overline{A}-B|^2p^2r^2{}_2F_1\left(p\phi+1,p\overline{\phi}+1;2;B^2r^2\right)=E_{A,B}(r,p),$$

and the proof of Theorem 5.3 is complete. $\square$

5.6. Concluding Remarks and Open Problem

For $-1\leq B\leq 0$ and $A\in\mathbb{C}$, $A\neq B$, define

$$\mathcal{C}_p(A,B):=\left\{f\in\mathcal{A}_p:\ \frac{1}{p}\left(1+\frac{zf''(z)}{f'(z)}\right)\prec\frac{1+Az}{1+Bz},\quad z\in\mathbb{D}\right\}.$$

The choices $A=1-(2\beta/p)$, $0\leq\beta<p$ and $B=-1$ turn the class $\mathcal{C}_p(A,B)$ into the class $\mathcal{C}_p(\beta)$, the class of p-valent convex of order β. The class $\mathcal{C}_p(0)=:\mathcal{C}_p$ is the usual class of p-valent convex functions. The results of this paper (e.g. Theorem 5.5 and 1.10) motivate the following problems for further research in this direction:

Open problem 5.25. *Discuss the maximal area integral problem for the functions of type z^p/f when $f\in\mathcal{C}_p(A,B)$, in particular when f is in $\mathcal{C}_P$ and $\mathcal{C}(\beta)$, respectively.*

With this, we end this chapter here.

CHAPTER 6

STARLIKENESS AND CONVEXITY OF INTEGRAL TRANSFORMS

The duality principle is used to determine the starlikeness (or convexity) of the integral transform $V_\lambda(f)$ defined by (1.8). Section 6.1 discusses about the class $\mathcal{P}_{a,b,c}(\beta)$. In Section 6.2 and 6.3, we investigate the necessary and sufficient conditions for $V_\lambda(f)$ to be starlike and convex.

The results in this chapter are from the article: Sahoo S.K., Sharma N.L. (2014), *Duality technique on a class of function defined by convolution with Gaussian hypergeometric functions*, J. Analysis, **113**(1), 145–155.

6.1. The class $\mathcal{P}_{a,b,c}(\beta)$

Study of starlikeness and convexity of certain *integral transforms* of analytic functions in the open unit disk plays a significant role in geometric function theory. In Chapter 1, we discussed about the integral transform $V_\lambda(f)$ for $f \in \mathcal{A}$ in (1.8). More precisely, $V_\lambda(f)$ over the class $\mathcal{P}_{a,b,c}(\beta)$ is taken, which is defined by (1.12), using the notion of convolution and expressed in the form of Gaussian hypergeometric functions. The newly considered class $\mathcal{P}_{a,b,c}(\beta)$ is a generalization of the family $\mathcal{P}_\gamma(\beta)$ defined in (1.11). In [**66**], Ponnusamy used the idea of differential subordination to discuss the starlikeness of the Bernardi integral transform of functions belonging to a family related to the family $\mathcal{P}_\gamma(\beta)$. Indeed, the family $\mathcal{P}_\gamma(\beta)$ has been considered by a number of authors in the literature to study certain problems in analytic function theory; see for instance [**12, 66, 68**] and references therein. For $f \in \mathcal{P}_1(\beta)$, the starlikeness of $V_\lambda(f)$ was first studied in [**25**]

Year	Authors	Ref.	Order of Starlikeness of V_λ	$f \in \mathcal{P}_\gamma(\beta)$
1994	Fournier and Ruscheweyh	[25]	zero	$\mathcal{P}_1(\beta)$
1997	Ponnusamy and Rønning	[68]	$\mu \in [0, 1/2]$	$\mathcal{P}_1(\beta)$
2001	Kim and Rønning	[47]	zero	$\mathcal{P}_\gamma(\beta),\ \gamma > 0$
2004	Balasubramanian, Ponnusamy, and Prabhakaran	[9]	$\mu \in [0, 1/2]$	$\mathcal{P}_\gamma(\beta),\ \gamma > 0$

TABLE 6.1. Order of Starlikeness of $V_\lambda(f)$

by Fournier and Ruscheweyh and later the case of starlikeness of order μ $(0 \leq \mu \leq \frac{1}{2})$ was investigated by Ponnusamy and Rønning [68, 69]. The case $1/2 < \mu < 1$ is yet to be settled. Significant contributions on starlikeness of the integral transform $V_\lambda(f)$ of $f \in \mathcal{P}_1(\beta)$ are listed in Table 6.1 where duality principle [82] played a crucial role in their proofs. In fact, for $f \in \mathcal{P}_\gamma(\beta)$, contributions on convexity of $V_\lambda(f)$ have also been made by several authors. We discuss this part separately in Section 6.3.

We now observe that if $f \in \mathcal{P}_\gamma(\beta)$ in (1.11) then we write

$$(1-\gamma)\frac{f(z)}{z} + \gamma f'(z) = \frac{f(z)}{z} * F\left(1, 1+\frac{1}{\gamma}; \frac{1}{\gamma}; z\right) =: k(z)$$

where $k(z) \in \mathcal{P}(\beta) := \{p(z) : p(z)$ is analytic in $\mathbb{D}$, $p(0) = 1$ and $\operatorname{Re} p(z) > \beta\}$. In view of this observation, f takes to the form

$$\frac{f(z)}{z} = F\left(1, \frac{1}{\gamma}; 1+\frac{1}{\gamma}; z\right) * k(z).$$

This representation motivates us to introduce the new three parameter family, denoted as $\mathcal{P}_{a,b,c}(\beta)$, which is defined by (1.12) in chapter 1. In particular, if we substitute $a = 1$ in the definition of $\mathcal{P}_{a,b,c}(\beta)$, it reduces to the two parameter family $\mathcal{P}_{b,c}(\beta)$ which can be defined in the form

$$\mathcal{P}_{b,c}(\beta) := \left\{ f \in \mathcal{A} : \operatorname{Re}\left(\frac{f(z)}{z} * F(1, c; b; z)\right) > \beta; \quad 1 \leq b < c,\ 0 \leq \beta < 1 \right\}.$$

This form of the definition of $\mathcal{P}_{b,c}(\beta)$ is possible due to the fact that

$$f(z) = {}_2F_1(1, r; s; z) * g(z) \iff f(z) * {}_2F_1(1, s; r; z) = g(z)$$

for $f,\ g \in \mathcal{A}$. For special values given to the parameters a, b and c, the class $\mathcal{P}_{a,b,c}(\beta)$ reduces to various well-known classes already studied in the literature. Thus, this new class may be looked up as a unifying class of all those classes of functions.

In this chapter, we are interested to find conditions such that either $V_\lambda(f) \in \mathcal{S}^*(\mu)$ or $V_\lambda(f) \in \mathcal{C}(\mu)$ for functions $f \in \mathcal{P}_{a,b,c}(\beta)$. Section 6.2 is devoted to the discussion on the starlikeness of $V_\lambda(f)$.

6.2. Starlikeness of the Integral Transform $V_\lambda(f)$

We stated our main theorem concerning the starlikeness of the integral transform in Chapter 1 by Theorem 1.11.

Proof of Theorem 1.11. Definition of $\mathcal{P}_{a,b,c}(\beta)$ and the duality principle [**82**] (see also [**9, 68**]) guarantees that

$$\frac{f(z)}{z} = {}_2F_1\,(a, b; c; z) * \left((1-\beta)\frac{1+xz}{1+yz} + \beta\right), \quad z \in \mathbb{D}. \tag{6.1}$$

By Theorem 1.1 (see Chapter 1), we know that

$$V_\lambda(f) \in S^*(\mu) \iff \frac{V_\lambda(f)(z)}{z} * \frac{h_\mu(z)}{z} \neq 0,$$

where $z \in \mathbb{D}$ and $h_\mu(z)$ is defined by (1.9). Therefore, we have

$$V_\lambda(f) \in S^*(\mu) \iff \int_0^1 \frac{\lambda(t)}{1-tz}dt * \frac{f(z)}{z} * \frac{h_\mu(z)}{z} \neq 0.$$

Since $\dfrac{\beta}{1-\beta} * {}_2F_1(a,b;c;z) = \dfrac{\beta}{1-\beta}$, (6.1) yields

$$\begin{aligned}
&\int_0^1 \frac{\lambda(t)}{1-tz}dt * \frac{f(z)}{z} * \frac{h_\mu(z)}{z} \\
&\qquad = \int_0^1 \frac{\lambda(t)}{1-tz}dt * \left({}_2F_1(a,b;c;z) * \left((1-\beta)\frac{1+xz}{1+yz} + \beta\right)\right) * \frac{h_\mu(z)}{z} \\
&\qquad = (1-\beta)\int_0^1 \frac{\lambda(t)}{1-tz}dt * \left({}_2F_1(a,b;c;z) * \frac{h_\mu(z)}{z} + \frac{\beta}{1-\beta}\right) * \frac{1+xz}{1+yz}.
\end{aligned}$$

It is equivalent to

$$\begin{aligned}
&\int_0^1 \frac{\lambda(t)}{1-tz}dt * \frac{f(z)}{z} * \frac{h_\mu(z)}{z} \\
&\qquad = (1-\beta)\int_0^1 \lambda(t)\left({}_2F_1(a,b;c;zt) * \frac{h_\mu(z)}{z} + \frac{\beta}{1-\beta}\right)dt * \frac{1+xz}{1+yz}.
\end{aligned}$$

This, together with the relation (1.30), yields

$$\begin{aligned}
\int_0^1 \frac{\lambda(t)}{1-tz}dt * \frac{f(z)}{z} * \frac{h_\mu(z)}{z} = (1-\beta)\int_0^1 \lambda(t)\Big({}_2F_1(a,b;c;zt) * \frac{h_\mu(z)}{z} \\
- g^\mu_{a,b,c}(t)\Big)dt * \frac{1+xz}{1+yz}.
\end{aligned} \tag{6.2}$$

We also know from [**82**, Theorem 1.6, p. 23] that for $f \in \mathcal{A}$

$$\frac{f(z)}{z} * \frac{1+xz}{1+yz} \neq 0 \iff \operatorname{Re}\frac{f(z)}{z} > \frac{1}{2}$$

holds. Therefore, from (6.2) and the last implication, it follows that $V_\lambda(f)$ belongs to $S^*(\mu)$ if and only if

$$\operatorname{Re}\left((1-\beta)\int_0^1 \lambda(t)\left({}_2F_1(a,b;c;zt) * \frac{h_\mu(z)}{z} - g^\mu_{a,b,c}(t)\right)dt\right) > \frac{1}{2},$$

or equivalently,

$$\operatorname{Re}\left(\int_0^1 \lambda(t)\left({}_2F_1(a,b;c;zt) * \frac{h_\mu(z)}{z} - g^\mu_{a,b,c}(t)\right)dt\right) - \frac{1}{2(1-\beta)} > 0. \tag{6.3}$$

We see that the relation (1.30) is equivalent to $\dfrac{1}{1-\beta} = \displaystyle\int_0^1 \lambda(t)(1-g^\mu_{a,b,c}(t))dt$, and so (6.3) simplyfies to

$$\operatorname{Re}\left(\int_0^1 \lambda(t)\left({}_2F_1(a,b;c;zt) * \frac{h_\mu(z)}{z} - \frac{1+g^\mu_{a,b,c}(t)}{2}\right)dt\right) > 0. \tag{6.4}$$

It is evident from (1.27) that $\Lambda_{a,b,c}(1) = 0$. In order to complete the required proof, we use the representation of $\Lambda_{a,b,c}(t)$ and rewrite (6.4) by dividing and multiplying by the factor t^{a+2b-c}, then we get

$$\begin{aligned}&\mathrm{Re}\left(\int_0^1 (-\Lambda_{a,b,c}^{'}(t))\left(\frac{h_\mu(z)}{z} * \left(t^{a+2b-c}{}_2F_1(a,b;c;zt)\right)\right.\right.\\ &\qquad\left.\left. -\frac{t^{a+2b-c}(1+g_{a,b,c}^{\mu}(t))}{2}\right)dt\right) > 0. \end{aligned}\tag{6.5}$$

Using the facts $\Lambda_{a,b,c}(1) = 0$ and $\lim_{t\to 0+} t^{a+2b-c}\Lambda_{a,b,c}(t) = 0$, and applying the integration by parts to the relation (6.5), it yields

$$\begin{aligned}&\mathrm{Re}\left(\int_0^1 t^{a+2b-c-1}\Lambda_{a,b,c}(t)\left[\frac{h_\mu(z)}{z} * \left(t\frac{d}{dt}{}_2F_1(a,b;c;tz) + (a+2b-c){}_2F_1(a,b;c;zt)\right)\right.\right.\\ &\qquad\left.\left. -\frac{1}{2}\Big((a+2b-c)(1+g_{a,b,c}^{\mu}(t)) + t\frac{d}{dt}g_{a,b,c}^{\mu}(t)\Big)\right]dt\right) > 0.\end{aligned}$$

From (1.28), we have

$$\begin{aligned}&\mathrm{Re}\left(\int_0^1 t^{a+2b-c-1}\Lambda_{a,b,c}(t)\left[\frac{h_\mu(z)}{z} * \left(t\frac{d}{dt}{}_2F_1(a,b;c;tz) + (a+2b-c){}_2F_1(a,b;c;zt)\right)\right.\right.\\ &\qquad\left.\left. -(a+2b-c)\frac{1-\mu(1+t)}{(1-\mu)(1+t)^2}\right]dt\right) > 0.\end{aligned}$$

Using (1.10), we obtain

$$\begin{aligned}&\mathrm{Re}\left(\int_0^1 t^{a+2b-c-1}\Lambda_{a,b,c}(t)\left[\frac{h_\mu(tz)}{tz} * \left(\frac{abz}{c}{}_2F_1(a+1,b+1;c+1;z)\right.\right.\right.\\ &\qquad\left.\left.\left. +(a+2b-c){}_2F_1(a,b;c;z)\right) - (a+2b-c)\frac{1-\mu(1+t)}{(1-\mu)(1+t)^2}\right]dt\right) > 0, \quad z\in\mathbb{D}.\end{aligned}$$

The assertion follows.

If we choose $a = 1$ in Theorem 1.11, then we obtain

Theorem 6.1. *For $0 \le \beta < 1$, let $f \in \mathcal{P}_{b,c}(\beta)$ with $1 \le b < c$. Suppose that $\lambda : [0,1] \to \mathbb{R}$ is a non-negative weight function so that $\int_0^1 \lambda(t)dt = 1$ and $\Lambda_{b,c}$ is defined by (1.29) with the assumption that $\lim_{t\to 0^+} t^{2b-c+1}\Lambda_{b,c}(t) = 0$. Assume that the quantity β is related by*

$$\frac{\beta}{1-\beta} = -\int_0^1 \lambda(t)g_{b,c}^{\mu}(t)dt,$$

where $g_{b,c}^{\mu}$ satisfies (1.29). Then $V_\lambda(f) \in S^*(\mu)$, $0 \le \mu \le 1/2$ if and only if $L_{\Lambda_{b,c}}^{\mu}(h_\mu) \ge 0$.

Year	**Authors**	**Ref.**	**Order of Convexity of $V_\lambda(f)$**	$f \in \mathcal{P}_\gamma(\beta)$
1995	Ali and Singh	[2]	zero	$\mathcal{P}_1(\beta)$
2002	Choi, Kim, and Saigo	[20]	zero	$\mathcal{P}_\gamma(\beta),\ \gamma > 0$
2005	Durai and Parvatham	[3]	$\mu \in [0, 1/2]$	$\mathcal{P}_1(\beta)$
2007	Balasubramanian, Ponnusamy, and Prabhakaran	[10]	$\mu \in [0, 1/2]$	$\mathcal{P}_\gamma(\beta),\ \gamma > 0$

TABLE 6.2. Order of Convexity of $V_\lambda(f)$

Remark 6.2. When $b = 1, c = 2$ and $\mu = 0$, Theorem 6.1 leads to a result of Fournier and Ruscheweyh; see [25, Theorem 2]. If we set $b = 1$ and $c = 2$ in Theorem 6.1 then we obtain [68, Theorem 2.1], the result due to Ponnusamy and Rønning. If we substitute $b = 1/\gamma,\ c = 1 + b$ and $\mu = 0$ in Theorem 6.1 then we get [47, Theorem 2.1] proved by Kim and Rønning. Putting $b = 1/\gamma$ and $c = 1 + b$ in Theorem 6.1, we easily see that our result reduces to an equivalent result of [9, Theorem 1.2] due to Balasubramanian, et al.

6.3. Convexity of the Integral Transform $V_\lambda(f)$

In this section, we find conditions so that the integral transform $V_\lambda(f)$ carrying functions from $\mathcal{P}_{a,b,c}(\beta)$ into $\mathcal{C}(\mu),\ 0 \le \mu \le 1/2$. In 1995, Ali and Singh [2] first discussed the convexity of the integral transform $V_\lambda(f)$ of functions belonging to $\mathcal{P}_1(\beta)$ with the help of the duality theory of convolution developed by Ruscheweyh in [82]. Subsequently, a number of authors investigated this problem in a more general setting. We now list down them in Table 6.2.

Our main aim in this section is to generalize the convexity result of Balasubramanian et al. [10] for functions belonging to the class $\mathcal{P}_{a,b,c}(\beta)$. The following basic notations are useful: For the function $\Lambda_{a,b,c}(t)$ stated in (1.27), we define

$$M^{\mu}_{\Lambda_{a,b,c}}(h_\mu) = \inf_{z\in\mathbb{D}} \int_0^1 t^{a+2b-c-1}\Lambda_{a,b,c}(t)\Bigg[\mathrm{Re}\left(\frac{1}{z}\frac{d}{dt}h_\mu(tz)\right.$$
$$\left.* \left(\frac{abz}{c}F(a+1,b+1;c+1;z)+(a+2b-c)_2F_1(a,b;c;z)\right)\right)$$
$$-(a+2b-c)\frac{(1-\mu)-t(1+\mu)}{(1-\mu)(1+t)^3}\Bigg]dt,$$

where a,b,c are real parameters and $h_\mu(z)$ is defined as in (1.9). We set

$$M^{\mu}_{\Lambda_{b,c}} := M^{\mu}_{\Lambda_{1,b,c}}. \tag{6.6}$$

Let $\phi^{\mu}_{a,b,c}(t)$ be the solution of the initial value problem

$$\frac{d}{dt}\left(t^{a+2b-c}\phi^{\mu}_{a,b,c}(t)\right) = (a+2b-c)\frac{t^{a+2b-c-1}\Big((1-\mu)-t(1+\mu)\Big)}{(1-\mu)(1+t)^3}, \tag{6.7}$$

with $\phi^{\mu}_{a,b,c}(0)=1$. Set $\phi^{\mu}_{b,c} := \phi^{\mu}_{1,b,c}$.

Theorem 6.3. *For $0\le\beta<1$, let $f\in\mathcal{P}_{a,b,c}(\beta)$, $a\le b<c$. Suppose that λ and $\Lambda_{a,b,c}$ satisfy as in the hypothesis of Theorem 1.11. Then $V_\lambda(f)$ is convex of order μ $(0\le\mu\le 1/2)$ if and only if $M^{\mu}_{\Lambda_{a,b,c}}(h_\mu)\ge 0$, where $\beta\in[0,1)$ is related by*

$$\frac{\beta-1/2}{1-\beta} = -\int_0^1 \lambda(t)\phi^{\mu}_{a,b,c}(t)dt, \tag{6.8}$$

and $\phi^{\mu}_{a,b,c}$ is defined as in (6.7).

Proof. Let $F(z)=V_\lambda(f)(z)$. It is well-known that F is convex of order μ if and only if zF' is starlike of order μ. Then, by Theorem 1.2 (see for instance [**82**]), we have

$$F\in\mathcal{C}(\mu) \Longleftrightarrow \frac{1}{z}(zF'(z)*h_\mu(z))\neq 0.$$

It is enough to show that

$$\begin{aligned}
0\neq \frac{1}{z}(zF'(z)*h_\mu(z)) &= \frac{1}{z}(F(z)*zh'_\mu(z))\\
&= \int_0^1 \lambda(t)\frac{f(tz)}{tz}dt * h'_\mu(z)\\
&= \int_0^1 \frac{\lambda(t)}{1-tz}dt * \frac{f(z)}{z} * h'_\mu(z). \qquad (6.9)
\end{aligned}$$

Using (6.1), the relation (6.9) holds if and only if

$$\begin{aligned}0 \neq& \int_0^1 \frac{\lambda(t)}{1-tz}dt * \left({}_2F_1(a,b;c;z) * \left((1-\beta)\frac{1+xz}{1+yz}+\beta\right)\right) * h_\mu^{'}(z)\\ =& (1-\beta)\int_0^1 \lambda(t)\left({}_2F_1(a,b;c;zt) * h_\mu^{'}(z) + \frac{\beta}{1-\beta}\right)dt * \frac{1+xz}{1+yz},\end{aligned}$$

which clearly holds if and only if

$$\mathrm{Re}\left((1-\beta)\int_0^1 \lambda(t)\left({}_2F_1(a,b;c;zt) * h_\mu^{'}(z) + \frac{\beta}{1-\beta}\right)dt\right) > \frac{1}{2},$$

or equivalently,

$$\mathrm{Re}\left(\int_0^1 \lambda(t)\left({}_2F_1(a,b;c;zt) * h_\mu^{'}(z) + \frac{\beta-1/2}{1-\beta}\right)dt\right) > 0. \tag{6.10}$$

Dividing and multiplying by the factor t^{a+2b-c} with the integrand in (6.10), the relation (6.8) leads

$$\mathrm{Re}\left(\int_0^1 (-\Lambda_{a,b,c}^{'}(t))\left(t^{a+2b-c}{}_2F_1(a,b;c;zt) * h_\mu^{'}(z) - t^{a+2b-c}\phi_{a,b,c}^{\mu}(t)\right)dt\right) > 0.$$

This on integration by parts and using $\Lambda_{a,b,c}(1) = 0, \lim_{t\to 0^+} t^{a+2b-c}\Lambda_{a,b,c}(t) = 0$ and (6.7), we obtain

$$\begin{aligned}\mathrm{Re}\Bigg(\int_0^1 t^{a+2b-c-1}\Lambda_{a,b,c}(t)\Bigg[h_\mu^{'}(z) * \left(t\frac{d}{dt}{}_2F_1(a,b;c;tz) + (a+2b-c){}_2F_1(a,b;c;zt)\right)\\ -(a+2b-c)\frac{(1-\mu)-t(1+\mu)}{(1-\mu)(1+t)^3}\Bigg]dt\Bigg) > 0.\end{aligned}$$

From (1.10), we have

$$\begin{aligned}\mathrm{Re}\Bigg(\int_0^1 t^{a+2b-c-1}\Lambda_{a,b,c}(t)\Bigg[\frac{1}{z}\frac{d}{dt}h_\mu(tz)\\ * \left(\frac{abz}{c}{}_2F_1(a+1,b+1;c+1;z) + (a+2b-c){}_2F_1(a,b;c;z)\right)\\ -(a+2b-c)\frac{(1-\mu)-t(1+\mu)}{(1-\mu)(1+t)^3}\Bigg]dt\Bigg) > 0.\end{aligned}$$

This means that $M_{\Lambda_{a,b,c}}^{\mu}(h_\mu) > 0$, which completes the proof of our theorem. □

If we set $a = 1$ in Theorem 6.3, we get

Theorem 6.4. *For $0 \leq \beta < 1$, let $f \in \mathcal{P}_{b,c}(\beta)$, $1 \leq b < c$. Suppose that λ and $\Lambda_{b,c}(t)$ satisfy as in the hypothesis of Theorem 6.1. Then $V_\lambda(f)$ is convex of order μ $(0 \leq \mu \leq 1/2)$ if and only if $M^{\mu}_{\Lambda_{b,c}}(h_\mu) \geq 0$, where $0 \leq \beta < 1$ is related by*

$$\frac{\beta - 1/2}{1-\beta} = -\int_0^1 \lambda(t)\phi^{\mu}_{b,c}(t)dt,$$

and $\phi^{\mu}_{b,c}$ and $M^{\mu}_{\Lambda_{b,c}}(h_\mu)$ are respectively defined by (6.7) and (6.6).

Remark 6.5. When $b = 1, c = 2$ and $\mu = 0$, Theorem 6.4 leads to a result of Ali and Singh; see [**2**, Theorem 1]. If we choose $b = 1/\gamma, c = 1+b$ and $\mu = 0$ in Theorem 6.4, then we obtain a result due to Choi, et al. (see [**20**, Lemma 3]). If we substitute $b = 1$ and $c = 2$ in Theorem 6.4, then we get the result [**3**, Theorem 2] due to Durai and Parvatham. If we set $b = 1/\gamma$ and $c = 1 + b$, then one can easily see that Theorem 6.4 reduces to an equivalent form of [**10**, Theorem 2.3] due to Balasubramanian, et al.

We end our discussion of this chapter here.

CHAPTER 7

CONCLUSION AND SCOPE FOR FUTURE WORK

In Chapter 1, we focus on basic definitions and properties of univalent and p-valent functions in $\mathbb{D}$ which are used in the subsequent chapters.

In the second chapter, we study the class of q-close-to-convex functions and determine several sufficient conditions for $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ to be in $\mathcal{K}_q$. In addition, we prove the Bieberbach-de Branges Theorem for functions in the class $\mathcal{K}_q$. This produces several power series of analytic functions convergent to q-hypergeometric function. Since not much work has been done in the literature about the class $\mathcal{K}_q$ and thus it is worth to deduce some new results in this topic. Further work in this field will certainly bring a strong base between q-theory and geometric function theory. It would be interesting to investigate the q-theory and its applications more in geometric function theory, in particular, to analyse the q-analog of convex functions and other related functions.

In the third and fifth chapter, we are particularly interested to solve Yamashita's conjecture on area maximum property for univalent and p-valent functions, respectively. We also discuss some integral means problem for several class of p-valent functions. It would be interesting to solve the analog of Yamashita's extremal and the integral means problems for other geometric subclasses of functions from $\mathcal{S}$ and $\mathcal{A}_p$. For example, to determine the analog of Yamashita's conjecture when zf' belongs to the class $\mathcal{S}^*(A, B)$ and also for functions f in the Bazilević class [13], and to derive Yamashita's extremal problem for p-valent convex functions.

In the fourth chapter, we present a correct form of the coefficient bounds for a function to be in certain family of p-valent functions. There are many application of Bieberbach's conjecture problem in univalent function theory. It would be interesting to find possible applications of coefficient estimates that we obtained for p-valent functions.

The duality theory for convolutions is used to investigate the starlikeness and convexity of order $\mu \in [0, 1/2]$ of the integral transform $V_\lambda(f)$ in the sixth chapter. For the order $\mu \in (1/2, 1)$, this problem is unsolved. By using Duality technique or otherwise, this problem and other related problems with their applications can be interested for researchers to investigate.

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