CALCULATION OF HALL CONDUCTIVITY IN GRAVITY SETUP

M.Sc. Thesis

By

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DISCIPLINE OF PHYSICS

INDIAN INSTITUTE OF TECHNOLOGY INDORE

JUNE 2017

CALCULATION OF HALL CONDUCTIVITY IN GRAVITY SETUP

A THESIS

Submitted in partial fulfillment of the
requirements for the award of the degree
of
Master of Science

by
MRITYUNJAY KUMAR

DISCIPLINE OF PHYSICS
INDIAN INSTITUTE OF TECHNOLOGY INDORE
JUNE 2017

INDIAN INSTITUTE OF TECHNOLOGY INDORE

CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **CALCULATION OF HALL CONDUCTIVITY IN GRAVITY SETUP** in the partial fulfillment of the requirements for the award of the degree of **MASTER OF SCIENCE** and submitted in the **DISCIPLINE OF PHYSICS, Indian Institute of Technology Indore**, is an authentic record of my own work carried out during the time period from July 2015 to July 2017 under the supervision of Dr. Manavendra Mahato, Associate Professor, IIT Indore.
The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other institute.

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This is to certify that the above statement made by the candidate is correct to the best of my/our knowledge.

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MRITYUNJAY KUMAR has successfully given his/her M.Sc. Oral Examination held on **4th July,2017**.

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ACKNOWLEDGEMENTS

I would first like to thank my thesis advisor Dr. Manavendra.N.Mahato for his conntinuous support and guidence. The door to Prof. Manavendra's office was always open whenever I ran into a trouble spot or had a question about my research or writing. He consistently allowed this paper to be my own work, but steered me in the right the direction whenever he thought I needed it.

I would also like to thank my PSPC members Dr. Krushna Mavani and Dr.Sampak Samanta who were involved in the validation survey for this research project.

My sincere thanks also goes to Dr. Subhendu Rakshit for his wonderful lectures of Statistical Physics which provided us a new perception about the subject,Dr.Prashant Samantray for his great lectures of General theory of relativity which helped me in my research work,not only this he had always helped us out in clearing doubts .I would like to thank Dr. Ankhi Roy for her continuous support.

I would like to thank from core of my heart to Rahul Kumar Singh for the stimulating discussions, for the sleepless nights we were working together before deadlines, and for all the fun we have had in the last six years.

I thank my fellow lab-mates Sudhir Rode, Jaswant Singh, Sudip Naskar, Siddharth Karmakar, Sujata and Ankita Goswami. Also, I thank my friends Rakesh Prasad (IITG) and Puja Priya (Physics Gold Medalist RU).

Finally, I must express my very profound gratitude to my parents Bhushan Prasad Singh and Rekha Singh and to my family for providing me with unfailing support and continuous encouragement throughout my years of study. This accomplishment would not have been possible without them.

Mrityunjay Kumar

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Dedicated to :

My sisters Mamta Rai and Shambhawi Singh

Abstract

If we take a bunch of electrons and restrict them to move in a two dimensional plane and turn on a strong magnetic field. This simple set up provides the setting for some of the most wonderful and surprising results in physics. These phenomena are known collectively as the quantum Hall effect.

The Hall conductivity takes the fractional value. We started with building our understanding in classical and quantum Hall effect.

But our aim was to apply our knowledge and understanding to calculate conductivity in relativistic domain, for that we started with learning physics of relativity and then tried to look at some standard problems of relativity namely D3-branes solution in asymptotically flat background geometry. We took a metric for this solution, calculated its Born-Infeld action, and transforming the coordinates into Eddington Finkelstein coordinates and thus calculated longitudinal and transverse conductivity for two currents.

Gaining the boost from these calculations we took a most general AdS4 and AdS-C metric .

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Introduction

Quantum Hall physics is the study of two-dimensional (2-D) electrons in a strong perpendicular magnetic field has become an extremely important research subject during the last two and a half decades. Our main knowledge of quantum Hall systems is a system of 2D electrons in a perpendicular magnetic field , where one drives a current I through the sample and measures both the longitudinal and the transverse resistance, called Hall resistance. The name comes from observation of these surprises. A longitudinal resistance is a resistance measured between two points along the flow of current and transverse resistance is a resistance measured in perpendicular direction of the flow of current. The transport properties of 2D electrons in a strong magnetic field was revealed 50 years later after the discovery of classical Hall effect with the discovery of the integer quantum Hall effect (IQHE) by von Klitzing, Drorda and Pepper in 1980. The Hall conductivity takes quantised values $\sigma_{xy}=\frac{e^2}{2\pi\hbar}$. Initially, it was found that ν takes only integral values. The IQHE occurs at low temperatures, when the energy scale set by the temperature K_BT is significantly smaller than the Landau level spacing. It consists of a quantisation of the Hall resistance, which is no longer linear in B, as one would expect from the classical treatment presented above, but reveals plateaus at particular values of the magnetic field. Three years after the discovery of the IQHE, an even more unexpected effect was observed in a 2-D electron system of higher quality, i.e. higher mobility, the fractional quantum Hall effect (FQHE). The effect owes its name to the fact that v is an fraction contrary to the IQHE, where the number v is an integer, a Hall-resistance quantisation was discovered by Tsui, Stormer and Gossard with v = 1/3 and 2/3 [2]. Since then, a whole series of plateaux have been detected for 1/5, 2/5, 2/7 and many more. The majority of these have denominators which are odd. The graph below is showing the variation of Hall resistivity with magnetic field.

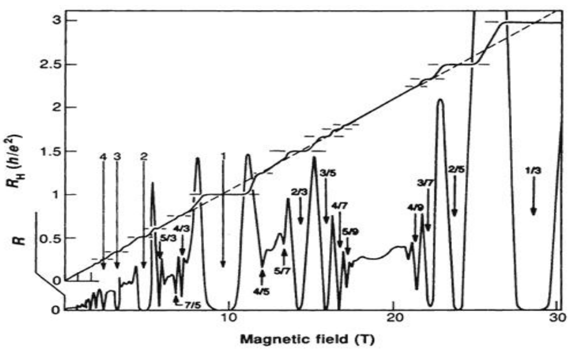

Figure 1: Hall coefficient vs magnetic field

However, the origins of the two effects are completely different, the IQHE may be understood from Landau quantisation, i.e. the kinetic-energy quantisation of independent electrons in a magnetic field whereas the FQHE[1] is due to strong electronic correlations, when a Landau level is only partially filled and the Coulomb interaction between the electrons becomes important. Here one of the most remarkable things was observed, the charged particles which are roaming on the 2-D plane carries fractional charges. These particles are neither fermions nor bosons, they are called anyons.

1.1 Landau Quantisation

The basic ingredients for the understanding of Quantum Hall Effect is Landau Quantisation i.e the kinetic-energy quantisation of a (free) charged particle in a perpendicular magnetic field. The energy levels of a particle become equally spaced, with the gap between each level proportional to the magnetic field B. These energy levels are called Landau level. The Lagrangian of a particle having mass m and charge q in a magnetic field $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$

$$L = \frac{1}{2}m\dot{\mathbf{x}}^2 - e\dot{\mathbf{x}}.\mathbf{A} \tag{1}$$

The equation of motion will remain same under gauge transformation because the Lagrangian changes by a total derivative. The canonical momentum arising from this Lagrangian will be

$$\mathbf{p} = m\dot{\mathbf{x}} - e\mathbf{A} \tag{2}$$

We can compute the Hamiltonian

$$H = \frac{1}{2m}(\mathbf{p} + e\mathbf{A})^2 \tag{3}$$

If we write the Hamiltonian in terms of mechanical momentum it looks same as in the absence of magnetic field. This shows that magnetic field does no work and so does not change the energy of the system. Since $\mathbf{x}$ and $\mathbf{p}$ are canonical i.e $\{x_i, p_j\} = \delta_{ij}$ but not gauge invariant which means that the numerical value of $\mathbf{p}$ can't have any physical meaning as it depends on the choice of the gauge. In contrast, the mechanical momenta is gauge invariant but not canonical. Poisson's bracket of mechanical momenta with itself is nonvanishing.

$$\{m\dot{x_i}, m\dot{x_j}\} = \{p_i + eA_i, p_j + eA_j\} = -e\epsilon_{ijk}B_k \tag{4}$$

Since our task is to find the energy spectrum and wavefunctions of the quantum Hamiltonian

$$\hat{H} = \frac{1}{2m}(\hat{p} + e\hat{A})^2 \tag{5}$$

Here, we have changed Hamiltonian and momentum into quantum operators. Since the particle is restricted to lie in a plane $\mathbf{x} = (x, y)$ we take the magnetic field in z-direction. In quantum formulation we will denote the mechanical momentum with $\hat{\pi}$. We write

$$m\hat{\dot{x}} = \hat{\pi} = \hat{p} + e\hat{A} \tag{6}$$

Then the commutation relation follows from Poisson bracket (4) will take the form

$$[\hat{\pi_x}, \hat{\pi_y}] = -ie\hbar B \tag{7}$$

We will derive the energy spectrum in purely algebraic method (very similar to that of Harmonic Oscillator) and has an advantage that we don't have to specify the gauge potential. Now we will introduce new variables $a = \frac{1}{\sqrt{2e\hbar B}}(\hat{\pi}_x - i\hat{\pi}_y)$ and $a^\dagger = \frac{1}{\sqrt{2e\hbar B}}(\hat{\pi}_x + i\hat{\pi}_y)$ These are the raising and lowering operators analogous to those of Harmonic oscillator. The commutaion of $\hat{\pi}$ tells us that a and $a^\dagger$ obey $[a, a^\dagger] = 1$. Writing the Hamiltonian in terms of $\hat{\pi}$ we get

$$\hat{H} = \frac{1}{2m}\hat{\pi}.\hat{\pi} = (n + \frac{1}{2})\hbar\omega$$

where $\omega = \frac{eB}{m}$ is the *cyclotron frequency.* Now it's simple to finish things off. We can construct the Hilbert space in the same way as the harmonic oscillator. We first introduce a ground state $|0\rangle$ obeying a$|0\rangle$=0 and build the rest of the Hilbert space by acting with $a^\dagger$

$$a^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle \text{ and } a\,|n\rangle = \sqrt{n}\,|n-1\rangle$$

The state $|n\rangle$ has energy

$$E_n = (n + \frac{1}{2})\hbar\omega \qquad n \in N \tag{8}$$

The energy level of a free particle is equally spaced in presence of magnetic field, with a gap proportional to magnetic field.

1.2 Symmetric Gauge

After understanding the basics of Landau levels, we are going to do it all again. To find the wavefunction of corresponding energy eigen states we need to specify a gauge potential **A**. We choose our gauge to be spherically symmetric called symmetric gauge, given as

$$\mathbf{A} = -\frac{1}{2}\mathbf{r} \times \mathbf{B} = -\frac{yB}{2}\hat{\mathbf{x}} + \frac{xB}{2}\hat{\mathbf{y}} \tag{9}$$

This choice of gauge breaks the translational symmetry in x and y directions but preserves the rotational symmetry. Thus, angular momentum will be conserved. In Landau quantisation we provided a simple algebraic derivation of the energy spectrum of a particle in a magnetic field. But we didn't provide an algebraic derivation of the degeneracies of these Landau levels. Here we rectify this. As we will see, this derivation only really works in the symmetric gauge. Recalling the mechanical momenta $\hat{\pi} = \hat{p} + e\hat{A}$ and the raising and lowering operators a and $a^\dagger$. In terms of these raising and lowering operators the energy spectrum takes the value

$$E = (n + \frac{1}{2})\hbar\omega$$

To find the wave function and the degeneracy, we define another kind of momentum

$$\hat{\tilde{\pi}} = \hat{p} - e\hat{A} \tag{10}$$

This differs from the mechanical momenta by a minus sign. The commutation relation of this new momenta differ from (7) by a minus sign.

$$[\hat{\tilde{\pi}}_x, \hat{\tilde{\pi}}_y] = ie\hbar B \tag{11}$$

However, the commutator of $\hat{\pi}$ and $\hat{\tilde{\pi}}$ will looks like

$[\hat{\pi}_x, \hat{\tilde{\pi}}_x]$=$2ie\hbar\frac{\partial A_x}{\partial x}$, $[\hat{\pi}_y, \hat{\tilde{\pi}}_y]$=$2ie\hbar\frac{\partial A_y}{\partial y}$, $[\hat{\pi}_x, \hat{\tilde{\pi}}_y]$=$[\hat{\pi}_y, \hat{\tilde{\pi}}_x]$=$ie\hbar(\frac{\partial A_x}{\partial x} - \frac{\partial A_y}{\partial y}$

It means we cannot diagonalise tha Hamiltonian and $\hat{\tilde{\pi}}$ but for symmetric gauge all these above commutators vanish and we have

$$[\pi_i, \tilde{\pi}_j] = 0$$

Now we define second kind of raising and lowering operators,

$$b = \frac{1}{\sqrt{2e\hbar B}}(\tilde{\pi}_x + i\tilde{\pi}_y) \quad and \quad b^\dagger = \frac{1}{\sqrt{2e\hbar B}}(\tilde{\pi}_x - i\tilde{\pi}_y)$$

It is the second pair of creation operators that provide the degeneracy of the Landau levels. We define the ground state $|0,0\rangle$ to be annihilated by both lowering operators, so that a$|0,0\rangle$=b$|0,0\rangle$=0. Then the general state in the Hilbert space is $|n,m\rangle$ defined by

$$|n,m\rangle = \frac{a^{\dagger n} b^{\dagger m}}{\sqrt{n!m!}}|0,0\rangle$$

The energy of this state is given by the usual Landau level expression; it depends on n but not on m. Let us now construct the wavefunctions in symmetric gauge. We are going to focus on the lowest Landau level, n=0. The lowest Landau level is annhilated by a, means a$|0,m\rangle = 0$. Since a is given as

$$\begin{aligned} a &= \frac{1}{\sqrt{2e\hbar B}}(\pi_x - i\pi_y) \\ &= \frac{1}{\sqrt{2e\hbar B}}(p_x - ip_y + e(A_x - iA_y)) \\ &= \frac{1}{\sqrt{2e\hbar B}}(-i\hbar\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) + \frac{eB}{2}(-y - ix)) \end{aligned}$$

At this stage, it will be easy and useful to work in complex coordinates plane. We introduce

$$z = x - iy \quad and \quad \bar{z} = x + iy$$

Here, we have define z in opposite manner to how we normally define because we want our wavefunctions to be holomorphic rather then anti-holomorphic. Let us introduce the holomorphic and anti-holomorphic derivatives

$$\partial = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \quad and \quad \bar{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

In terms of holomorphic coordinates a takes the simple form

$$a = -i\sqrt{2}\left(l_B\bar{\partial} + \frac{z}{4l_B}\right)$$

and correspondingly

$$a^\dagger = -i\sqrt{2}\left(l_B\partial - \frac{\bar{z}}{4l_B}\right)$$

where $l_B = \sqrt{\frac{\hbar}{eB}}$. The lowest Landau level wavefunctions $\psi_{LLL}(z\bar{z})$ are then those which are annhilated by this differential operator. Operating a on $\psi_{LLL}(z,\bar{z})$ will yield

$$-i\sqrt{2}\left(l_B\bar{\partial} + \frac{z}{4l_B}\right) psi_{LLL}(z,\bar{z}) = 0$$

$$\psi_{LLL}(z,\bar{z}) = f(z)e^{\frac{-|z|^2}{4l_B^2}}$$

where f(z) is an arbitrary holomorphic function. We can construct the specific states $|0,m\rangle$ in the lowest landau level by writing b and $b^\dagger$ as differential operators as below

$$b = -i\sqrt{2}\left(l_B\partial + \frac{\bar{z}}{4l_B}\right) \quad and \quad b^\dagger = -i\sqrt{2}\left(l_B\bar{\partial} - \frac{z}{4l_B}\right)$$

The lowest Landau level state $\psi_{LLL,m=0}$ is annhilated by both a and b. This is a unique state given as

$$\psi_{LLL,m=0} \sim e^{-|z|^2/4l_B^2}$$

Now, we can construct the higher states by acting $b^\dagger$ on $\psi_{LLL,m=0}$. Each time we operate $b^\dagger$, we get a factor of $z/2l_b$. Thus, the general wavefunction can be written in terms of analytic function.

$$\psi_{LLL,m} \sim \left(\frac{z}{l_B}\right)^m e^{-|z|^2/4l_B^2} \tag{12}$$

Since, symmetric gauge has rotational symmetry meaning angular momentum is a good quantum number.

$$J = \hbar(x\bar{\partial} - \bar{z}\partial) \tag{13}$$

Operating angular momentum on $\psi_{LLL,m}$ will yield

$$J\psi_{LLL,m} = m\psi_{LLL,m}$$

In symmetric gauge, the profiles of the wavefunctions (12) form concentric rings around the origin. The higher the angular momentum m, the further out is the ring.

1.3 Berry Phase

Berry phase, is a phase difference acquired over the course of a cycle, when a system is subjected to cyclic adiabatic processes, which results from the geometrical properties of the parameter space of the Hamiltonian. We will describe the Berry phase arising for a general Hamiltonian which we write as

$$H = H(x^a, \lambda)$$

where x^a are the degrees of freedom of the system and λ^i are the parameters of the hamiltonian. We pick some values for the parameters λ and place the system in a specific energy eigenstate $|\psi\rangle$ which, for simplicity, we will take to be the ground state. Now we very slowly vary the parameters, the Hamiltonian changes and so the ground state, it is $|\psi(\lambda(t))\rangle$. Accordiing to adiabatic theorem, if we vary λ and end up at its starting values, then it means we trace a closed path in parameter space. Since we started with ground state, we also end up in the ground state with an extra phase factor. The new state will looks like

$$|\psi\rangle = e^{i\gamma} |\psi\rangle$$

Time-dependent Schodinger equation is given as

$$i\hbar \frac{\partial |\psi\rangle}{\partial t} = H(\lambda(t)) |\psi\rangle \tag{14}$$

The adiabatic theorem means that the ground state $|\psi(t)\rangle$ obeying (14) can be written as

$$|\psi(t)\rangle = U(t) |n(\lambda(t))\rangle \tag{15}$$

where U(t) is some time dependent phase. If we have $|n(\lambda(t = 0))\rangle = |\psi(t = 0)\rangle$, it means $U(t = 0) = 1$. Taking the derivative of equation (15) and overlapping with $\langle\psi|$, we will get

$$\langle\psi| |\dot{\psi}\rangle = \dot{U}U^* + \langle n| |\dot{n}\rangle = 0$$

The above expression will give rise to time dependence phase U,

$$\dot{U}U^* = -\langle n| |\dot{n}\rangle = -\langle n| \frac{\partial}{\partial\lambda^i} |n\rangle \dot{\lambda}^i \tag{16}$$

$$\dot{U}U^* = -iA_i\dot{\lambda}^i$$

where, $A_i = -i \langle n| \frac{\partial}{\partial\lambda^i} |n\rangle$ is Berry Connection. The above equation can also be written as,

$$\dot{U} = -iA_i\dot{\lambda}^i U$$

On solving the above differential equation we will get

$$U(t) = exp\left(-i \int A_i(\lambda)\dot{\lambda}^i dt\right)$$

Our goal is to compute the phase U(t) after we have taken a closed path C in parameter space. This is simply

$$e^{i\gamma} = exp\left(-i \int A_i(\lambda)\dot{\lambda}^i dt\right) \tag{17}$$

This is the Berry phase. It doesn't depend on the time taken to change the parameters but depend on the path taken through parameter space. The Berry connection is very similar to the gauge potentials in electromagnetism. Following the idea of electromagnetism we can extract the information from Berry connection (like the field strength) and is called Curvature of the connection. Thus, the Berry Phase can be written as

$$e^{i\gamma} = exp\left(-i \int A_i(\lambda)\dot{\lambda}^i dt\right) = exp\left(-i \int_s F_{ij} ds^{ij}\right) \tag{18}$$

where F_{ij} is known as Curvature of the connection, defined as

$$F_{ij}(\lambda) = \frac{\partial A_i}{\partial \lambda^j} - \frac{\partial A_j}{\partial \lambda^i}$$

and S is a two-dimensional surface in the parameter space bounded by the path C.

1.4 Aharonov-Bohm Effect

The Aharonov–Bohm effect is a quantum mechanical phenomenon in which an electrically charged particle is affected by an electromagnetic potential (V,A), despite being confined to a region in which both the magnetic field B and electric field E are zero. Consider a set-up like the solenoid where the magnetic field is localised to some region of space (outside the solenoid magnetic field is zero). We consider a particle which sits outside the solenoid. We restrict the particle to lie in a small box. There can be some interesting physics going on inside the box, we will capture this by including a potential V($\mathbf{x}$) in the Hamiltonian and, in order to trap the particle, we take this potential to be infinite outside the box. If we place the centre of the box at $\mathbf{x} - \mathbf{X}$, then the Hamiltonian of the system can be written as,

$$H = \frac{1}{2m}(i\hbar\boldsymbol{\nabla} + e\mathbf{A})^2 + V(\mathbf{x} - \mathbf{X}) \tag{19}$$

We start by placing the centre of the box at position $\mathbf{x} - \boldsymbol{X_0}$ where we will take the gauge potential to be zero i.e $\mathbf{A}(\boldsymbol{X_0})$. We will take the ground state to be $\psi(\mathbf{x} - \boldsymbol{X_0})$. Now, we will move the box in some path in space.In doing so, the gauge potential experienced by the particle changes. We can check the Schrodinger equation for the Hamiltonian(19) can be solved by the state

$$\psi(\mathbf{x} - \mathbf{X}) = \exp\left(-\frac{ie}{\hbar}\int_{(\mathbf{x}-\mathbf{X_0})}^{\mathbf{x}-\mathbf{X}} \mathbf{A}.d\mathbf{x}\right)\psi(\mathbf{x} - \boldsymbol{X_0}) \tag{20}$$

Now we take the box in a closed path C, the wavefunction comes back to,

$$\psi(\mathbf{x} - \boldsymbol{X_0}) \to e^{i\gamma}\psi(\mathbf{x} - \boldsymbol{X_0}) \quad \text{where} \quad e^{i\gamma} = \exp\left(-\frac{ie}{\hbar}\oint_c \mathbf{A}.d\mathbf{x}\right) \tag{21}$$

Comparing this to our general expression for the Berry phase, we see that in this particular context the Berry connection is actually identified with the electromagnetic potential, given as

$$\frac{c}{\hbar}\mathbf{A}(\mathbf{x} - \mathbf{X})$$

The electron has charge -e but, in what follows, we'll have need to talk about particles with different charges. In general, if a particle of charge q goes around a region containing flux Φ, it will pick up an Aharonov-Bohm phase

$$e^{\frac{iq\Phi}{\hbar}}$$

This simple fact will play an important role in our discussion of the fractional quantum Hall effect.

Chapter 2

Fractional Quantum Hall Effect

We have come to a pretty good understanding of the Landau levels and quantum phases which are the basic ingredients of Fractional quantum hall effect. While the integer QHE(IQHE) is thought of essentially as a noninteracting electron phenomenon, the fractional QHE (FQHE)is believed to arise from a condensation of the two-dimensional (2D) electrons into a new collective state of matter as a result of interelectron interactions(Coulomb interaction between electrons).

2.1 Laughlin States

The first approach to the fractional quantum Hall effect was due to Laughlin, who described the physics at filling fractions

$$\nu = \frac{1}{m}$$

with m an odd integer. Since we are dealing with a large number of particles it not possible to diagonalise the Hamiltonian. Instead Laughlin did something very bold: he simply wrote down the answer. This was motivated by a combination of physical insight and guesswork. The Laughlin wavefunction is an ansatz for the ground state of a two-dimensional electron gas placed in a uniform background magnetic field in the presence of a uniform jellium background when the filling factor of the lowest Landau level is $\nu = 1/n$ where n is an odd positive integer. It was constructed to explain the observation of the $\nu = 1/3$ fractional quantum Hall effect, and predicted the existence of additional $\nu = 1/n$ states as well as quasiparticle excitations with fractional electric charge e/n both of which were later experimentally observed.

2.1.1 The Laughlin Wavefunctions

Two Particles

Consider two particles interacting in the lowest Landau level. We take an arbitrary central potential between them,

$$V = V(\boldsymbol{r_1} - \boldsymbol{r_2})$$

To solve these type of problems we have to deal with eigenstate of angular momentum. In Landau levels we have to work with symmetric gauge as they are the eigen functions of angular momentum. The single particle wavefunctions in the lowest Landau level take the form

$$\psi_m \sim z^m e^{-|z|^2/4l_B^2}$$

Consider, in a second step, an arbitrary two-particle wave function. This wavefunction must also be an analytic function of both positions z_1 and z_2 of the first and second particle, respectively, and may be a superposition of polynomials, such as e.g. of the basis state. Then, the two particle wavefunction take the form

$$\psi = (z_1 + z_2)^M (z_1 - z_2)^m e^{-(|z_1|^2+|z_2|^2)/4l_B^2}$$

where we have defined centre of mass coordinate $(z_1 + z_2)$ and the relative coordinate $(z_1 - z_2)$. The quantum numbers M and m are the angular momentum of centre of mass and relative angular momentum respectively.

Many-Particles

On general grounds, any lowest Landau level wavefunction must take the form,

$$\psi(z_1, z_2, ...z_n) = f(z_1, z_2, ...z_n)e^{-\Sigma_{i=1}^{N}|z_i|^2/4l_B^2} \tag{22}$$

for some analytic function f(z). Moreover,this function must be anti-symmetric under exchange of any two particle $z_i \leftrightarrow z_j$, reflecting the fact that the underlying electrons are fermions. Laughlin's proposal for the ground state wavefunction at filling fraction $\nu = 1/m$ is,

$$\psi(z_i) = \prod_{i<j}(z_i - z_j)^m e^{-\Sigma_{i=1}^{N}|z_i|^2/4l_B^2} \tag{23}$$

Clearly this is anti symmetric when m is odd integer. The pre-factor vanishes with a zero of order m whenever two electrons come together. Meanwhile, the exponential factor decreases quickly whenever the electrons get too far away from the origin. Let us first show that the wavefunction has the desired filling fraction. To do this, focus on what the wavefunction is telling us about a single particle, say z_1 . The terms that depend on z_1 in the pre-factor of the Laughlin wavefunction are

$$\prod_{i<j}(z_i - z_j)^m \sim \prod_{i=2}^{N}(z_1 - z_i)^m$$

which tells us that there are m(N-1) powers of z_1 i.e maximum angular momentum of first particle is m(N-1) and the corresponding radius is $\sqrt{2mN}l_B$ and area is $2m\pi Nl_B^2$. The number of states in full Landau level is $AB/\Phi_0 \approx mN$ which in turn gives the filling fraction $\nu = 1/m$.

2.1.2 Plasma Analogy

A compelling physical picture of Laughlin's wave function (23) has been provided by Laughlin himself, in terms of an analogy with a classical 2D one-component plasma. In the present subsection, we present the basic ideas and results of this plasma analogy, for completeness and pedagogical reasons. From basic quantum mechanics the modulus square of a quantum-mechanical wave function may be interpreted as a statistical probability distribution.

$$P[z_i] = \prod_{i,j} \frac{\mid z_i - z_j \mid^2}{l_B^2 m} e^{\Sigma_i |z_i|^2/2l_B^2} \tag{24}$$

Now, from classical statistical mechanics a probability distribution in the canonical ensemble is the Boltzmann weight, $e^{\beta H}$, of some Hamiltonian H and that the classical partition function, which encodes all relevant information, is obtained by sum over all Boltzmann weights, called Partition function. We can make this analogy sharper by writing the probability distribution (24) so it looks like a Boltzmann distribution function,

$$P[z_i] = e^{-\beta U(z_i)}$$

with

$$\beta U(z_i) = -2m \sum_{i<j} log\left(\frac{\mid z_i - z_j \mid}{l_B}\right) + \frac{1}{2l_B^2}\sum_{i=1}^{N} \mid z_i \mid^2 \tag{25}$$

If we pick the Boltzman temperature β=2/m we will have

$$U(z_i) = -m^2 \sum_{i<j} log\left(\frac{\mid z_i - z_j \mid}{l_B}\right) + \frac{m}{4l_B^2}\sum_{i=1}^{N} \mid z_i \mid^2 \tag{26}$$

Here, β is not a real temperature as it is dimensionless, to compensate this the above potential energy is also dimensionless. The first term in (26) is the Coulomb potential between two particles of charge q when both the particle and the electric field lines are restricted to lie in a two-dimensional plane. To see this, note that Poisson equation in two dimensions tells us that the electrostatic potential generated by a point charge q is

$$\nabla^2\phi = -2\pi q\delta^2(r) \quad \Rightarrow \quad \phi = -q\ log\left(\frac{r}{l_B}\right)$$

The potential energy between two particles of charge q is then $U = q\phi$, which is indeed the first term in (26). The second term in (26) describes a neutralising background of constant charge. A constant background of charge density ρ_0 would have electrostatic potential obeying

$$\nabla^2\phi = -2\pi\rho_0$$

The second term obeys

$$\nabla^2\left(\frac{\mid z \mid^2}{4l_B^2}\right) = \frac{1}{l_B^2}$$

which tells that bacground charge density is,

$$\rho_0 = -\frac{1}{2\pi l_B^2} = -\frac{eB}{2\pi\hbar} \tag{27}$$

Now, from our intution about Plasma, to minimise its energy, the plasma will want to neutralise. Each particle carries a charge $q = -m$ which means that the compensating density of particles n should be $mn = \rho_0$, or

$$n = \frac{1}{2\pi m l_B^2}$$

This argument has also told us something new. The plasma analogy tells us that the average density of particles is constant.

2.2 Quasi-Particles and Quasi-Holes

Until now, we have discussed some ground-state properties of Laughlin's wave-function. However, we have not characterised so far the nature of the excitations. There are two different sorts of excitations, known as Quasi-holes and Quasi-particles.

2.2.1 Quasi-Holes

The wavefunction describing a quasi-hole at position $\eta \in C$is

$$\psi_{hole}(z,\eta) = \prod_{i=1}^{N}(z_i - \eta)\prod_{k<l}(z_k - z_l)^m e^{-\Sigma_{i=1}^{n}|z_i|^2/4l_B^2} \tag{28}$$

The electron density now vanishes at the point η. In other words, we have created a hole in the electron fluid. More generally, we can introduce M quasi-holes in the quantum Hall fluid at positions η j with j = 1, . . . , M , with wavefunction

$$\psi_{M-hole}(z,\eta) = \prod_{j=1}^{M}\prod_{i=1}^{N}(z_i - \eta_j)\prod_{k<l}(z_k - z_l)^m e^{-\Sigma_{i=1}^{n}|z_i|^2/4l_B^2} \tag{29}$$

If we place m quasi-holes at same point η,the wavefunction will take the form,

$$\psi_{m-hole}(z,\eta) = \prod_{i=1}^{N}(z_i - \eta)^m\prod_{k<l}(z_k - z_l)^m e^{-\Sigma_{i=1}^{n}|z_i|^2/4l_B^2} \tag{30}$$

Since, η is a parameter not a dynamical variable, the above wavefunction describes deficit of an electron at position η. This means that m holes act like a deficit of a single electron, so a single quasi-hole is$1/m^{th}$ of an electron. It should therefore carry charge +e/m. We can make exactly the same argument in the context of the plasma analogy for the quasi-hole wavefunction (28). The resulting plasma potential energy has an extra term compared to (26),

$$U(z_i) = -m^2 \sum_{i<j} log\left(\frac{|\ z_i - z_j\ |}{l_B}\right) + \frac{m}{4l_B^2}\sum_{i=1}^{N} |\ z_i\ |^2 - m \sum_{i<j} log\left(\frac{|\ z_i - \eta\ |}{l_B}\right) \tag{31}$$

This extra term looks like an impurity in the plasma with charge 1. The particles in the plasma are expected to swarm around and screen this impurity. Each particle corresponds to a single electron, but has charge $q{=}{-m}$ in the plasma. The impurity carries -1/m the charge of the electron. So the effective charge that's missing is +1/m. This is the charge of the quasi-hole.

2.2.2 Quasi-Particle

There are also excitations of the quantum Hall fluid which carry charge $e^*{=}{-e/m}$, i.e. the same sign as the charge of an electron. These are quasi-particles. It is difficult to write the eigenstates of quasi-particle compared to quasi-holes. For quasi-particle wavefunction, we need to increase the electron density and hence decrease the relative angular momentum of some pair of electrons. To write the wavefunction of quasi-particle we have decrease the power of $\prod(z_i)$. This leads to a wavefunction for quasi-particle,

$$\psi_{particle}(z,\eta) = \left[\prod_{i=1}^{N}\left(2\frac{\partial}{\partial z_i} - \bar{\eta}\right)\prod_{k<l}(z_k - z_l)^m\right] e^{-\Sigma_{i=1}^{n}|z_i|^2/4l_B^2} \tag{32}$$

Here the derivatives act only on the polynomial prefactor, not on the exponential. The quasi-particle wavefunction is not quite as friendly as the quasi-hole wavefunction. For a start, the derivatives make it harder to work with and, for this reason, we will mostly derive results for quasi-holes.

2.3 Introducing Anyons

In Quantum mechanics we have been taught that particles fall into two categories, the bosons and the fermions. However if the particles are restricted to a two-dimensional plane there is a loophole to the usual argument. Consider a system of two identical particles at r_1 and r_2 described by the wavefunction $\psi(\boldsymbol{r_1}, \boldsymbol{r_2})$. Since the particles are identical, the probabilities remain same on exchanging the position of the particles i.e $\mid \psi(\boldsymbol{r_1}, \boldsymbol{r_2} \mid^2 = \mid \psi(\boldsymbol{r_2}, \boldsymbol{r_1} \mid^2$. This also means that upon exchanging the position of the particle the wavefunction differ atmost by a phase

$$\psi(\boldsymbol{r_1}, \boldsymbol{r_2}) = e^{i\pi\alpha}\psi(\boldsymbol{r_2}, \boldsymbol{r_1})$$

Now, if we further exchange these particles the phase it will pick is given by $e^{2i\pi\alpha}$. This gives the condition

$$\psi(\boldsymbol{r_1}, \boldsymbol{r_2}) = e^{2i\pi\alpha}\psi(\boldsymbol{r_1}, \boldsymbol{r_2}) \quad \Rightarrow \quad e^{2i\pi\alpha} = 1 \tag{33}$$

For fermions α=1 and for bosons α=0. The weak point is the statement that when we rotate two particles by 2π we should get back to where we came from. In 3-D space the path that the pair of particles take in spacetime can always be continuously connected to the situation where the particles don't move at all. This is the reason the resulting state should be the same as the one before the exchange. But in 2-D space this is not the case,the world lines are now wind to each other. The essence of the loophole is that, after a rotation in the two-dimensions, the wavefunction may retain a memory of the path it took through the phase. This means that may have any phase α. Particles with $\alpha \neq 0,1$ are referred to as anyons and their statistics is given by fractional statistics.

2.4 Fractional Statistics

We will now compute the quantum statistics of quasi-holes in the $\nu = 1/m$ Laughlin state. During calculations we will see a more pleasant explanation of fractional charge of quasi-holes. Both calculations will involved Berry phase that arises as quasi-hole move. We consider a state of M quasi-holes which we denote as $|\eta_1, ..., \eta_M\rangle$. The corresponding wavefunction in comples plane will be

$$\psi_{M-holes}(z, \bar{z}) = \langle z, \bar{z}|\eta_1, ..., \eta_M\rangle$$

$$\psi_{M-hole}(z, \eta) = \prod_{j=1}^{M}\prod_{i=1}^{N}(z_i - \eta_j)\prod_{k<l}(z_k - z_l)^m e^{-\Sigma_{i=1}^{n}|z_i|^2/4l_B^2}$$

Whenever we calculate Berry phase we deal with normalized states. Let us call the state $|\psi\rangle$, defined by,

$$|\psi\rangle = \frac{1}{\sqrt{Z}}\,|\eta_1, ...\eta_M\rangle \tag{34}$$

where the normalisation factor is defined as

$$Z = \langle \eta_1, ...\eta_M | \eta_1, ...\eta_M \rangle$$

which is equal to

$$Z = \int \prod exp\left(\sum_{i,j} log \mid z_i - \eta_j \mid^2 m \sum_{k,l} log \mid z_k - z_l \mid^2 - \frac{1}{2l_B^2} \sum_i \mid z_i \mid^2 \right) \tag{35}$$

This is the object which plays the role of the partition function in the plasma analogy, now in the presence of impurities localised at η_i. The holomorphic Berry connection is

$$\begin{aligned} A_\eta &= -i \langle \psi | \frac{\partial}{\partial \eta} | \psi \rangle \\ &= \frac{i}{2Z} \frac{\partial Z}{\partial \eta} - \frac{i}{Z} \langle \eta | \frac{\partial}{\partial \eta} | \eta \rangle \end{aligned} \tag{36}$$

Since, $|\eta\rangle$ is holomorphic and $\langle \eta|$ is anti-holomorphic, we have

$$\frac{\partial Z}{\partial \eta} = \frac{\partial}{\partial \eta} \langle \eta | \eta \rangle = \langle \eta | \frac{\partial}{\partial \eta} | \eta \rangle$$

Thus, holomorphic Berry connection (36) will take the form

$$A_\eta(\eta, \bar{\eta}) = -\frac{i}{2} \frac{\partial log Z}{\partial \eta} \tag{37}$$

Meanwhile, the anti-holomorphic Berry connectiaon can be written as

$$A_{\bar{\eta}}(\eta, \bar{\eta}) = -i \langle \psi | \frac{\partial}{\partial \bar{\eta}} | \psi \rangle = \frac{i}{2} \frac{\partial log Z}{\partial \eta} \tag{38}$$

Now, in order to calculate the Berry connection we have to take the derivative of partition function (35). This is very difficult to do so, to ease the calculations we will use the basic idea of plasma physics. In the plasma analogy, the presence of the hole acts like a charged impurity. In the presence of such an impurity, the key physics is called screening. This is the phenomenon in which the mobile charges with positions z_i rearrange themselves to cluster around the impurity so that its effects cannot be noticed when we are suitably far away, mathematically, the electric potential due to the impurity is modified by an exponential fall-off $e^{-r/\lambda}$ where λ is called the Debye screening length. During screening phenomenon, the impurities are effectively hidden at distances much greater than λ. This means that the free energy of the plasma is independent of the positions of the impurities. However, there are two ingredients missing: the first is the energy cost between the impurities and the constant background charge, the second is the Coulomb energy between the different impurities. The correct potential energy for the plasma with M impurities should therefore be

$$\begin{aligned} U(z_k, \eta_i) = -m^2 \sum_{k<l} log \left(\frac{\mid z_k - z_l \mid}{l_B} \right) + \frac{m}{4l_B^2} \sum_{k=1}^{N} \mid z_k \mid^2 - m \sum_{k,i} log \left(\frac{\mid z_k - \eta_i \mid}{l_B} \right) \\ + \frac{1}{4l_B^2} \sum_{i=1}^{M} \mid \eta_i \mid^2 - \sum_{i<j} log \left(\frac{\mid \eta_i - \eta_j \mid}{l_B} \right) \end{aligned} \tag{39}$$

The corrected plasma partition function is then

$$Z_{new} = \int \prod d^2 z_i e^{-\beta U} = exp\left(-\frac{1}{m}\sum_{i<j} log \mid \eta_i - \eta_j \mid^2 + \frac{1}{2ml_B^2}\sum_i \mid \eta_i \mid^2\right) Z \tag{40}$$

as long as the distance between the impurities are greater then the Debye length, the above expression (Z_{new}) must be independent of η_i. This means we have

$$Z = C \;\; exp\left(\frac{1}{m}\sum_{i<j} log \mid \eta_i - \eta_j \mid^2 - \frac{1}{2ml_B^2}\sum_i \mid \eta_i \mid^2\right) \tag{41}$$

for some constant C which does not depend on η_i. Taking log on both sides will yield

$$log \;\; Z = log \;\; C + \left(\frac{1}{m}\sum_{i<j} log \mid \eta_i - \eta_j \mid^2 - \frac{1}{2ml_B^2}\sum_i \mid \eta_i \mid^2\right) \tag{42}$$

Plugging the above expression into expression for Berry connections (36) and (37), we will get

$$A_{\eta_i} = -\frac{i}{2m}\sum_{j\neq i}\frac{1}{\eta_i - \eta_j} + \frac{i\bar{\eta}_i}{4ml_B^2} \tag{43}$$

and

$$A_{\bar{\eta}_i} = \frac{i}{2m}\sum_{j\neq i}\frac{1}{\bar{\eta}_i - \bar{\eta}_j} - \frac{i\eta_i}{4ml_B^2} \tag{44}$$

where we stress that these expressions only hold as long as the quasi-holes do not get too close to each other where the approximation of complete screening breaks down.

Computing fractional charges

Calculation of charge of anyons will be based on Aharonov-Bohm phase and Berry phase. The idea is very simple, we will pick one quasi-hole and move it on a closed loop such that the path doesnot encloses any other anyons. This

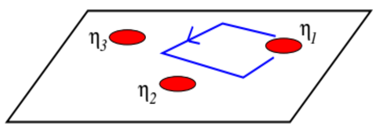

Figure 2: Path taken to compute fractional charge of quasi-hole

means only the second term of Berry connection will contribute. Thus, it will take the form

$$A_\eta = \frac{i\bar{\eta}}{4ml_B^2} \quad and \quad A_{\bar{\eta}} = -\frac{i\eta}{4ml_B^2}$$

After traversing the path C, the quasi-hole will return with a phase shift of $e^{i\gamma}$, given by the Berry phase

$$e^{i\gamma} = \exp\left(i \oint_C A_\eta d\eta + A_{\bar{\eta}} d\bar{\eta}\right) \tag{45}$$

For simplicity let us take the loop to be a circle of radius r i.e $\eta{=}re^{i\theta}$ and $\bar{\eta}{=}re^{-i\theta}$ On putting the value of η and $\bar{\eta}$ into the expression of Berry phase will yeild

$$e^{i\gamma} = \exp\left(\frac{\pi r^2}{m l_B^2}\right)$$

where $l_B = \sqrt{\frac{\hbar}{eB}}$. Comparing both sides of the above expression will give the Berry phase

$$\gamma = \frac{e\Phi}{m\hbar} \tag{46}$$

where Φ is the total magnetic flux enclosed by the path C. But there is a nice interpretation of this result, it is simply the Aharonov-Bohm phase picked up by the particle. As described in section (1.4), a particle of charge e^* will pick up a phase.

$$\gamma = \frac{e^*\Phi}{\hbar} \tag{47}$$

Comparing equation (46) and (47) will give

$$e^* = \frac{e}{m}$$

where m is an odd integer.

Chapter 3

Introduction to D-Branes

A $Dp-$ brane is an extended object with p spatial dimensions. The letter D in Dp -brane stands for Dirichlet. In the presence of a D-brane, the endpoints of open strings must lie on the brane, this requirement imposes a number of Dirichlet boundary conditions on the motion of the open string end points. A $D0$-brane is a single point, a $D1$-brane is a line (sometimes called a D-string) and a $D2$-brane is a plane.

3.1 Equations of motion of Relativistic Strings

In this section we will derive equation of motion for relativistic strings by varying string action[2] and we will assume the standard notation of string theory i.e $X^\mu = X^\mu(\tau, \sigma)$,where τ and σ are two parameters. Let us start by writing string action as

$$S = \int_{\tau_i}^{\tau_f} d\tau L = \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma \mathscr{L}(\dot{X}^\mu, X^{\mu'}) \tag{48}$$

where $\mathscr{L}$ is given by

$$\mathscr{L} = -\frac{T_0}{c}\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2} \tag{49}$$

To obtain equation of motion we need to set variation of action to zero. The variation can be written as

$$\delta S = \int_{\tau_i}^{\tau_f} d\tau \int_0^{\tau_f} d\sigma \left[\frac{\partial \mathscr{L}}{\partial \dot{X}^\mu}\delta \dot{X}^\mu + \frac{\partial \mathscr{L}}{\partial X^{\mu'}}\delta X^{\mu'}\right] \tag{50}$$

Here $X^{\mu'}$ and $\dot{X}^\mu$ are differentiation of X with respect to σ and τ respectively. Denoting the quantities $\frac{\partial \mathscr{L}}{\partial \dot{X}^\mu}$=$P^\tau_\mu$ and $\frac{\partial \mathscr{L}}{\partial X^{\mu'}}$=$P^\sigma_\mu$. The exact value of above denoted quantities can be written as

$$P^\tau_\mu = -\frac{T_0}{c}\frac{(\dot{X}\cdot X')X'_\mu - (X')^2\dot{X}_\mu}{\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}} \tag{51}$$

$$P^\sigma_\mu = -\frac{T_0}{c}\frac{(\dot{X}\cdot X')\dot{X}_\mu - (\dot{X})^2 X'_\mu}{\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}} \tag{52}$$

Thus the variation in action can be written as

$$\delta S = \int_{\tau_i}^{\tau_f} d\tau \int_0^{\tau_f} d\sigma \left[\frac{\partial(\delta X^\mu P^\tau_\mu)}{\partial \tau} + \frac{\partial(\delta X^\mu P^\sigma_\mu)}{\partial \sigma} - \delta X^\mu\right] \tag{53}$$

The first term on the right-hand side, being a full derivative in τ, will contribute terms proportional to $\delta X^\mu(\tau_i, \sigma)$ and $\delta X^\mu(\tau_f, \sigma)$. Since the flow of τ implies the flow of time, we can imagine specifying the initial and final states of the string, and we restrict ourselves to variations for which $\delta X^\mu(\tau_i,\sigma) = \delta X^\mu(\tau_f,\sigma) = 0$ Now, our variation in action becomes

$$\delta S = \int_{\tau_i}^{\tau_f} d\tau [\delta X^\mu P^\tau_\mu]_0^{\sigma_1} - \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma \, \delta X^\mu \left(\frac{\partial P^\tau_\mu}{\partial \tau} + \frac{\partial P^\sigma_\mu}{\partial \sigma} \right) \tag{54}$$

The first term on the right hand side of equation (54) has to do with string end points. It is the collection of two terms for each value of μ. Since μ can take value 0 to d, so it represents a (d+1)dimensional space, so we need 2(d+1) boundary conditions. Suppose σ_* denotes the σ coordinate of end point of an string. Thus, σ can take two value, either 0 or 1. As we know there are two natural boundary conditions, the first is Dirichlet boundary condition in which the end point of string remain fix throughout the motion.

$$\frac{\partial X^\mu(\tau,\sigma_*)}{\partial \tau} = 0 \quad , \mu \neq 0$$

So above equation implies the μ coordinate in selected string end point is fixed in time. If the string end point is fixed, the variation are set to vanish i.e $\delta X^\mu(\tau,\sigma_*) = 0$. The second possible boundary condition is

$$P^\sigma_\mu(\tau,\sigma_*) = 0$$

This condition is called free end point condition (also called Neumann Boundary condition for non-relativistic string) because it does not impose any condition on the variation $\delta X^\mu(\tau,\sigma_*)$ of the string coordinate at the end point. The end point is free to do whatever is needed to get the variation of the action to vanish. From either of these two conditions, first term on the right hand side of (54) vanishes. The second term of the right hand side of (54) in order to satisfy the condition of extremum action must vanishes for all variations δX^μ of the motion. So we get

$$\frac{\partial P^\tau_\mu}{\partial \tau} + \frac{\partial P^\sigma_\mu}{\partial \sigma} = 0 \tag{55}$$

This is the equation of motion of relativistic string, open or closed.

Chapter 4

My Work

4.1 Gravity Dual of a Quantum Hall Effect

We have considered N_7D7-branes intersecting with N_3D3-branes over 2+1 dimensions according to

	0	1	2	3	4	5	6	7	8	9
$D3:$	×	×	×	×						
$D7:$	×	×	×		×	×	×	×	×	

Figure 3: Brane intersection

where N_3 and N_7 are the numbers of D3 and D7 branes. We have also considered N_3>>N_7, thus we will use D7 as probe branes. As per the figure (3) the D3-branes are located at $x^9 = 0$ and D7-branes at $x^9 = L$ and the directions $x^3, x^4, ...x^8$ are noncompact. The background geometry is the asymptotically flat, and the D3-brane solution[3] is given as

$$ds^2 = f^{-\frac{1}{2}}(-hdt^2 + d\vec{x}^2) + f^{\frac{1}{2}}(h^{-1}dr^2 + r^2 d\Omega_5^2) \tag{56}$$

with

$$f = 1 + \frac{R^4}{r^4}$$
$$h = 1 - \frac{{r_+}^4}{r^4}$$
$$d{\Omega_5}^2 = d\psi^2 = \sin^2\psi d{\Omega_4}^2 \tag{57}$$

Our probe spans x^0, x^1 and x^2, wraps the S^4 and lies on the curve

$$r = r(\psi)$$

Thus, induced worldvolume metric and is given by,

$$d{s_8}^2 = f^{\frac{-1}{2}}(-hdt^2 + dx^i dx^i) + f^{\frac{1}{2}}(h^{-1}r'^2 + r^2)d\psi^2 + f^{\frac{1}{2}}r^2\sin^2\psi d{\Omega_4}^2 \quad \text{with } i = 1, 2. \tag{58}$$

Our Born-Infield action becomes

$$S_{D7} = -N_7T_7\int d^8\zeta\sqrt{-g_8} = -N_7T_7\mathrm{Vol}(S^4)\int d^3x d\psi f^{\frac{1}{2}} r^4 \sqrt{r^2h + r'^2}\sin^4\psi \tag{59}$$

The equations of motions will be

$$r'' + 4r'(1 + \frac{r'^2}{B})\cot\psi - \frac{1}{B}\frac{dB}{dr}r'^2 - \frac{1}{2}\frac{dB}{dr} - \frac{(B + r'^2)}{A}\frac{dA}{dr} = 0 \tag{60}$$

where

$$A = r^4 f^{\frac{1}{2}}, \ B = r^2 h$$

To compute the conductivities on the gravity side we use the AdS/CFT correspondence. AdS/CFT is a conjectured relationship between two kinds of physical theories. On one side are anti-de Sitter spaces (AdS) which are used in theories of quantum gravity, formulated in terms of string theory. On the other side of the correspondence are conformal field theories (CFT) which are quantum field theories. Infact, since it is easy to do so, we will generalize the computation to include a nonzero magnetic field and electric field. Near the horizon limit of geometry, the metric of the probe becomes,

$$ds_8^2 = \frac{r^2}{R^2}(-hdt^2 + dx^i dx^i) + \frac{R^2}{r^2}(h^{-1} + r^2\psi'^2)dr^2 + R^2 \sin^2 d\Omega_4^2 \tag{61}$$

Here, we parametrize our probe

$$\psi = \psi(r)$$

Now to ease our calculations we will work in Eddington -Finkelstein coordinates because it extend smoothly accross future horizon. Defining the new coordinate

$$\tilde{r} = \int_{r_+}^{r} dx\sqrt{1 + hx^2\psi'2}$$

and the advance coordinates in usual manner, the metric will take the form(see appendix)

$$ds^2 = 2dvd\tilde{r} - Udv^2 + \frac{R^2}{r^2}dx^i dx^i + R^2 \sin^2\psi d\Omega_4^2 \tag{62}$$

with

$$U = \frac{r^2 h}{R^2}$$

The action of the probe is given by Born-Infeld action plus Chern-Simons term. After integrating over the S^4 the Born-Infeld contribution to the probe action becomes

$$S_{BI} = -\tau \int d^4x \sin^4\psi\sqrt{-det(g_4 + 2\pi\alpha' F)} \tag{63}$$

with

$$\tau = \frac{1}{3(\pi)^2}\frac{1}{(2\pi\alpha')^2}N_3 N_7$$

The Chern-Simons term can be written as

$$S_{cs} = \frac{1}{4\pi}\int df(r)\Lambda A\Lambda F \tag{64}$$

The function $f(\tilde{r})$ can be determined by the choice of embedding of the blackhole. We are looking for solutions of gauge field equation of motion subject to non-zero electric and magnetic fields at the boundary. Thus the field are

$$A_{\tilde{r}} = 0, \tag{65}$$

$$A_v = a_v(\tilde{r}) + E^i x^i, \tag{66}$$

$$A_j = \frac{1}{2}B\epsilon_{ij}x^i + a_j(\tilde{r}). \tag{67}$$

The gauge field strengths can be calculated as

$$F = dA$$

$$F = a_v^{'}\; dr\Lambda dv + E^i dx^i \Lambda dv + \frac{1}{2} B\epsilon_{ij} dx^i \Lambda dx^j + a_j^{'} dx^j \Lambda dr \tag{68}$$

Working out the field strengths and plugging into equation (8) gives the Born-Infield action

$$\begin{aligned} S_{BI} = -\tau \int d^4x \sin^4\psi [(B^2 + \frac{r^4}{R^4})(1 - (\partial_{\tilde{r}} a_v)^2 + U\frac{r^2}{R^2}(\partial_{\tilde{r}} a_i)^2 \\ + 2\frac{r^2}{R^2} E^i \partial_{\tilde{r}} a_i + 2B\partial_{\tilde{r}} a_v \epsilon_{ij} E^i \partial_{\tilde{r}} a_j + (\epsilon_{ij} E^i \partial_{\tilde{r}} a_j)^2]^{\frac{1}{2}} \end{aligned} \tag{69}$$

and the Chern-Simons action is given by (see appendix)

$$S_{CS} = \frac{1}{2\pi} \int d^4x \partial\tilde{r} f(\tilde{r})[B a_v(\tilde{r} + \epsilon_{ij} a_i(\tilde{r}) E_j \tag{70}$$

We now define the $\tilde{r}$ dependent charges and currents

$$\rho(\tilde{r}) = \frac{\partial L}{\partial(\partial_{\tilde{r}} a_v)} \quad and \quad j^i(\tilde{r}) = \frac{\partial L}{\partial(\partial_{\tilde{r}} a_i)} \tag{71}$$

Since the Chern-Simons term does not contribute to these expressions because there are no radial derivatives of the gauge field components. Thus we can calculate the charges and currents by using the $a_{i,v}$ equations of motion.

$$\partial_{\tilde{r}} \rho(\tilde{r}) = \frac{B}{2\pi} \partial_{\tilde{r}} f(\tilde{r}),\ \partial_{\tilde{r}} j_i(\tilde{r}) = \frac{1}{2\pi} \partial_{\tilde{r}} f(\tilde{r}) \epsilon_{ij} E_j \tag{72}$$

Thus, relative to the horizon ($\tilde{r} = \tilde{r}_+$) these functions measured at the infinity are shifted by an amount proportional to the Chern-Simons coefficient (k).

$$\rho = \rho(\tilde{r}_+) + \frac{kB}{2\pi},\ j_i = j_i(\tilde{r}_+) + \frac{1}{2\pi} \epsilon_{ij} E_j \tag{73}$$

where

$$k = \int_{\tilde{r}_+}^{\infty} d\tilde{r} \partial_{\tilde{r}} f(\tilde{r})$$

We will work in perturbatively in the electric field E^i. Our Lagrangian is given by

$$\begin{aligned} L = -\tau \sin^4\psi [(B^2 + \frac{r^4}{R^4})(1 - (\partial_{\tilde{r}} a_i)^2 + U\frac{r^2}{R^2}(\partial_{\tilde{r}} a_i)^2 \\ - 2\frac{r^2}{R^2} E^i \partial_{\tilde{r}} a_i + 2B\partial_{\tilde{r}} a_v \epsilon_{ij} E^i \partial_{\tilde{r}} a_j + (\epsilon_{ij} E^i \partial_{\tilde{r}} a_j)^2]^{\frac{1}{2}} \end{aligned} \tag{74}$$

For $E^i = 0$ and $a^i = 0$

$$\rho(\tilde{r}) = \frac{\partial L}{\partial(\partial_{\tilde{r}} a_v)} = \frac{\tau(B^2 + \frac{r^4}{R^4}) \sin^4\psi (\partial_{\tilde{r}} a_v)}{[(B^2 + \frac{r^2}{R^2})(1 - (\partial_{\tilde{r}} a_v)^2]^{\frac{1}{2}}} \tag{75}$$

By inverting we get the solution

$$\partial_{\tilde{r}} a_v = \frac{\rho(\tilde{r}}{\sqrt{\rho(\tilde{r})^2 + \tau^2 (B^2 + \frac{r^4}{R^4}) \sin^8 \psi}} \tag{76}$$

Now to calculate conductivity, we will go to first order in electric field E^i. It is most convenient to write the result at horizon where $U = 0$ and $\psi = \psi_0$

$$j^i(\tilde{r}_+) = \frac{\sqrt{\rho(\tilde{r}_+)^2 + \tau^2 (B^2 + \frac{r_+^4}{R^4}) sin^8 \psi_0}}{(B^2 + \frac{r_+^4}{R^4})} \left(\frac{{r_+}^2}{R^2} E^i + B(\partial_{\tilde{r}} a_v) \epsilon_{ij} E^i \right) \tag{77}$$

We write

$$j^i = \sigma^L E^i + \sigma^T \epsilon_{ij} E^j \tag{78}$$

Comparing the above two equations and restoring the factors of $2\pi\alpha^{'}$ we get,

$$\sigma^L = \frac{\sqrt{(2\pi\alpha'\rho)^2/(2\pi\alpha')^4\tau^2 + \left((2\pi\alpha' B)^2 + \frac{r_+^4}{R^4}\right) \sin^8 \psi_0}}{\left((2\pi\alpha')B^2 + \frac{r_+^4}{R^4}\right)} \frac{r_+^2}{R^2} \tag{79}$$

and

$$\sigma^T = \frac{(2\pi\alpha'\rho)(2\pi\alpha' B)}{\left((2\pi\alpha' B)^2 + \frac{r_+^4}{R^4}\right)} + \frac{k}{2\pi} \tag{80}$$

4.2 Conductivity in AdS_4 metric

Keeping our physical background same as the previous one we will now calculate the conductivity for an another 10-dimensional metric[4].

$$ds_{10}^2 = e^{2A} ds_{AdS_7}^2 + ds_{M_3}^2 \tag{81}$$

where

$$ds_{M_3}^2 = dr^2 + \frac{1}{16} e^{2A} (1 - x^2)[d\theta^2 + sin^2 \phi]$$

and

$$ds_{AdS_7}^2 = -\frac{q^2}{\alpha^2} dt^2 + \frac{\alpha^2}{q^2} dq^2 + \frac{q^2}{\alpha^2} dx^i dx^i + d\psi^2 + \sinh^2 d\xi^2 + \sinh^2 \psi \ \sinh^2 \xi \, d\eta^2$$
$$with \quad i = 1, 2$$

Combining above equations our original metric will take the form,

$$\begin{aligned} ds^2 = e^{2A} \left[-\frac{q^2}{\alpha^2} dt^2 + \frac{\alpha^2}{q^2} dq^2 + \frac{q^2}{\alpha^2} dx^i dx^i \right] + e^{2A} [d\psi^2 + \sinh^2 \psi \, d\xi^2 \\ + \sinh^2 \psi \ \sinh^2 \xi \, d\eta^2] + \left[dr^2 + \frac{1}{16} e^{2A} (1 - x^2)[d\theta^2 + \sinh^2 \phi \right] \end{aligned} \tag{82}$$

where A and x depends on r. This metric spans x^0,x^1 and x^2,wraps a three dimensional sphere $S^3(\psi, \xi, \eta)$ and a two dimensional sphere $S^2(\theta, \phi)$. It also lies on a curve $q = q(\psi)$. The internal space M_3 is an S^2 -fibration over an interval, whose coordinate we call r. For a constant r (because our probe does not extend in that direction), the induced world volume metric takes the form

$$ds_8^2 = e^{2A}\left[-\frac{q^2}{\alpha^2}dt^2 + \left(\frac{\alpha^2}{q^2}q'^2 + 1\right)d\psi^2 + \frac{q^2}{\alpha^2}dx^i dx^i\right] + e^{2A}[\sinh^2\psi\, d\xi^2 + \sinh^2\psi\ \sinh^2\xi\, d\eta^2] + \left[\frac{1}{16}e^{2A}(1-x^2)(d\theta^2 + \sinh^2\phi)\right] \quad (83)$$

In absence of gauge fields, the action is given by Born-Infeld action

$$S_{D_7} = -N_7 T_7 \int d^8\xi \sqrt{-g_8}$$

Plugging the value of determinant of worldvolume metric into Born-Infeld action will yield

$$S_{D_7} = -N_7 T_7 \,\mathrm{Vol}(S_1^2)\,\mathrm{Vol}(S_2^2) e^{8A}\left[\frac{1}{16}e^{2A}(1-x^2)\right]\int d^3x\ d\psi \frac{q^2}{\alpha^2}\sqrt{\frac{\alpha^2}{q^2}q'^2 + 1}\ \sinh^2\psi \quad (84)$$

The equation of motion will be

$$q'' + 2q'\coth\psi\left(1 + \frac{q'^2\alpha^2}{q^2}\right) + \frac{\alpha^2 q'^4}{q^3} - \alpha^4 q'^4 + \alpha^2 q'^2 q^2 - 2q^4 = 0$$

To compute the conductivities on the gravity side we parametrize our metric via ψ=$\psi(q)$, then the metric is given as

$$ds_8^2 = e^{2A}\left[-\frac{q^2}{\alpha^2}dt^2 + \left(\frac{\alpha^2}{q^2} + \psi'^2\right)dq^2 + \frac{q^2}{\alpha^2}dx^i dx^i\right] + e^{2A}[\sinh^2\psi\, d\xi^2 + \sinh^2\psi\ \sinh^2\xi\, d\eta^2] + \left[\frac{1}{16}e^{2A}(1-x^2)(d\theta^2 + \sinh^2\phi)\right] \quad (85)$$

Here, we parametrize our probe via ψ=$\psi(q)$.

$$ds_8^2 = e^{2A}\left[-\frac{q^2}{\alpha^2}dt^2 + \frac{1}{q^2}(\alpha^2 + \psi'^2 q^2)dq^2 + \frac{q^2}{\alpha^2}dx^i dx^i\right] + e^{2A}[\sinh^2\psi\, d\xi^2 + \sinh^2\psi\ \sinh^2\xi\, d\eta^2] + \left[\frac{1}{16}e^{2A}(1-x^2)(d\theta^2 + \sinh^2\phi)\right] \quad (86)$$

It is easier to work in Eddington-Finkelstein coordinates as they extend smoothly across the horizon. We will now define a new coordinate $d\tilde{q}$ given as

$$d\tilde{q} = \int dq\sqrt{\alpha^2 + \psi'^2 q^2} \quad (87)$$

We define the advance Eddington-Finkelstein coordinates

$$v = t + q^*$$
$$dt = dv - dq^*$$

Putting the value of dt in the above metric will yield

$$ds_8^2 = \left[-e^{2A}\frac{q^2}{\alpha^2}dv^2 + 2dvd\tilde{q} + e^{2A}\frac{q^2}{\alpha^2}dx^idx^i\right] + e^{2A}\sinh^2\psi\, d\xi^2$$
$$+e^{2A}\sinh^2\psi\ \sinh^2\xi\, d\eta^2 + \left[\frac{1}{16}e^{2A}(1-x^2)(d\theta^2+\sinh^2\phi)\right] \quad (88)$$

where we have also defined

$$\frac{e^{2A}q^2}{\alpha^2}dq^* = d\tilde{q} \quad (89)$$

Defining

$$\frac{e^{2A}q^2}{\alpha^2} = U$$

our metric takes the form

$$ds_8^2 = -Udv^2 + 2dvd\tilde{q} + Udx^idx^i + e^{2A}\sinh^2\psi[\,d\xi^2 +$$
$$\sinh^2\xi\, d\eta^2] + \left[\frac{1}{16}e^{2A}(1-x^2)(d\theta^2+\sinh^2\phi)\right] \quad (90)$$

The action of the probe is given by Born-Infeld action plus Chern-Simons term. After integrating over the S^4 the Born-Infeld contribution to the probe action becomes

$$S_{BI} = -\tau\int d^4x\sinh^2\psi\sqrt{-det(g_4+2\pi\alpha{\cdot}F)} \quad (91)$$

The Chern-Simons term can be written as

$$S_{cs} = \frac{1}{4\pi}\int df(\tilde{q})\Lambda A\Lambda F \quad (92)$$

The function $f(\tilde{q})$ can be determined by the choice of embedding. We are looking for solutions gauge field equation of motion subject to the non-zero electric and magnetic fields at the boundary. Thus the field are

$$A_{\tilde{q}} = 0, \quad (93)$$
$$A_v = a_v(\tilde{q}) + E^ix^i, \quad (94)$$
$$A_j = \frac{1}{2}B\epsilon_{ij}x^i + a_j(\tilde{q}). \quad (95)$$

The gauge field strengths can be calculated as

$$F = dA$$

$$F = a_v^{'}\ dq\Lambda dv + E^idx^i\Lambda dv + \frac{1}{2}B\epsilon_{ij}dx^i\Lambda dx^j + a_j^{'}dx^j\Lambda dq \quad (96)$$

where prime is derivative with respect $\tilde{q}$.
The term inside the squareroot of Born-Infeld action can be calculated as

$$g_4 + F = \begin{bmatrix} 0 & (1+a_v^{'}) & a_1^{'} & a_2^{'} \\ (1-a_v^{'}) & -U & -E^1 & -E^2 \\ a_1^{'} & E^1 & U & B \\ a_2^{'} & E^2 & -B & U \end{bmatrix} \quad (97)$$

Further solving for the determinant of the above matrix ,we will get

$$-det(g_4+F) = (B^2+U^2)(1-a_v^{'2})+U^2(a_i^{'})^2+-2UE^i a_i^{'}+2Ba_v^{'}\epsilon_{ij}E^i a_j^{'}-(\epsilon_{ij}E^i a_j^{'})^2 \tag{98}$$

Thus, the Born-Infeld equation takes the form

$$S_{BI} = -\tau \int d^4x \sinh^2\psi[(B^2+U^2)(1-a_v^{'2}) + U^2(a_i^{'})^2 + -2UE^i a_i^{'} \\ +2Ba_v^{'}\epsilon_{ij}E^i a_j^{'} - (\epsilon_{ij}E^i a_j^{'})^2]^{\frac{1}{2}} \tag{99}$$

and the Chern-Simons action is given by

$$S_{CS} = \frac{1}{2\pi}\int d^4x \partial_{\tilde{q}} f(\tilde{q})[Ba_v(\tilde{q}) + \epsilon_{ij}a_i(\tilde{q})E_j] \tag{100}$$

We now define the $\tilde{q}$ dependent charges and currents

$$\rho(\tilde{q}) = \frac{\partial L}{\partial(\partial_{\tilde{q}}a_v)} \quad \text{and} \quad j^i(\tilde{q}) = \frac{\partial L}{\partial(\partial_{\tilde{q}}a_i)} \tag{101}$$

Since the Chern-Simons term doesnot contribute to these expressions because there are no radial derivatives of the gauge field components. Thus, we can calculate the charges and currents by using the $a_{i,v}$ equations of motion.

$$\partial_{\tilde{q}}\rho(\tilde{q}) = \frac{B}{2\pi}\partial_{\tilde{q}}f(\tilde{q}) \quad , \quad \partial_{\tilde{q}}j_i(\tilde{q}) = \frac{1}{2\pi}\partial_{\tilde{q}}f(\tilde{q})\epsilon_{ij}E_j \tag{102}$$

Thus, relative to the horizon ($\tilde{q} = \tilde{q}_+$) these functions measured at the infinity are shifted by an amount proportional to the Chern-Simons coefficient(k).

$$\rho = \rho(\tilde{q}_+) + \frac{kB}{2\pi} \quad \text{and} \quad j_i = j_i(\tilde{q}_+) + \frac{1}{2\pi}\epsilon_{ij}E_j \tag{103}$$

where

$$k = \int_{\tilde{q}_+}^{\infty} d\tilde{q}\partial_{\tilde{q}}f(\tilde{q})$$

Our Lagrangian is given by

$$L = -\tau\sinh^2\psi[(B^2+U^2)(1-a_v^{'2}) + U^2(a_i^{'})^2 + -2UE^i a_i^{'} \\ +2Ba_v^{'}\epsilon_{ij}E^i a_j^{'} - (\epsilon_{ij}E^i a_j^{'})^2]^{\frac{1}{2}} \tag{104}$$

We will work perturbatively in the electric field E^i. For E^i=0 and a_i=0,

$$\rho(\tilde{q}) = \frac{\partial L}{\partial(\partial_{\tilde{q}}a_v)} = \frac{\tau\sinh^2\psi(B^2+U^2)a_v'}{[(B^2+U^2)(1-(a_v')^2]^{\frac{1}{2}}} \tag{105}$$

and by inverting the above equation

$$a_v' = \frac{\rho}{[\rho^2 + \tau^2(B^2+U^2)\sinh^4\psi]^{\frac{1}{2}}} \tag{106}$$

The current density can be calculated as

$$j^i(\tilde{q}) = \frac{\tau\sinh^2\psi[U^2a_i' + UE^i + Ba_v'\epsilon_{ij}E^j]}{[(B^2+U^2)(1-a_v'^2) + U^2a_i'^2 - 2UE^ia_i' + 2BE^ia_v'a_i' - \epsilon_{ij}E^ia_j']^{\frac{1}{2}}} \tag{107}$$

Now, to calculate conductivity, we have to go to first order in electric field, the current density will then take the form,

$$j^i(\tilde{q}) = \frac{\tau \sinh^2\psi [UE^i + Ba'_v \epsilon_{ij} E^j]}{\left[(B^2+U^2)\left(1 - \frac{\rho^2}{\rho^2+\tau^2(B^2+U^2)\sinh^4\psi}\right)\right]^{\frac{1}{2}}} \tag{108}$$

This can also be written as,

$$j^i(\tilde{q}) = \frac{\sqrt{\rho^2+\tau^2(B^2+U^2)\sinh^4\psi}}{(B^2+U^2)}[UE^i + B\epsilon_{ij}a'_v E^j] \tag{109}$$

Writing the current density as

$$j^i = \sigma^L E^i + \sigma^T \epsilon_{ij} E^j \tag{110}$$

Comparing equations (100) and (101) we get,

$$\sigma^L = \frac{\sqrt{\rho^2+\tau^2(B^2+U^2)\sinh^4\psi_0}}{(B^2+U^2)}U$$

$$\sigma^L = \frac{\sqrt{\rho^2+\tau^2(B^2+e^{4A}\frac{q_+^4}{\alpha^4})\sinh^4\psi_0}}{(B^2+e^{4A}\frac{q_+^4}{\alpha^4})} \quad e^{2A}\frac{q_+^2}{\alpha^2} \tag{111}$$

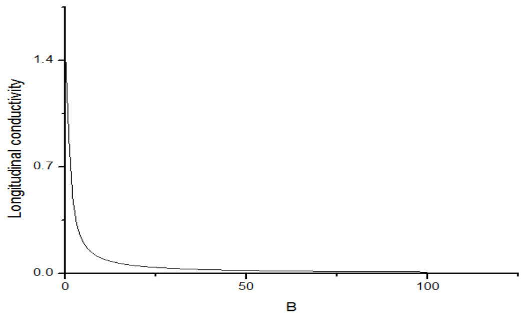

Figure 4: longitudinal Conductivity v/s magnetic field

Proceeding in similar manner, tranverse conductivity is given as

$$\sigma^T = \frac{\sqrt{\rho^2+\tau^2\left[B^2+e^{4A}\left(\frac{q_+^2}{\alpha^2}\right)^2\right]\sinh^4\psi_0}}{\left[B^2+e^{4A}\left(\frac{q_+^2}{\alpha^2}\right)^2\right]} \quad Ba'_v$$

Putting the value of a'_v we get

$$\sigma^T = \frac{\rho B}{\left(B^2+e^{4A}\frac{q_+^4}{\alpha^4}\right)} + \frac{k}{2\pi} \tag{112}$$

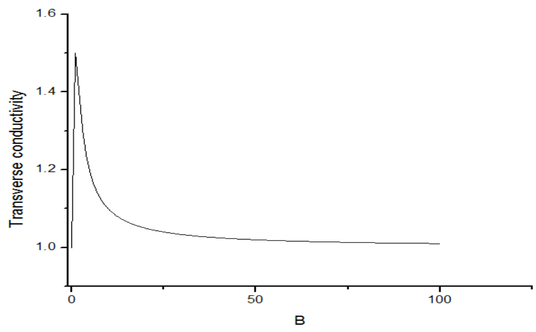

Figure 5: Tranverse Conductivity v/s magnetic field

4.3 Conductivity in AdS-C metric

We will now replace the AdS_4 metric by a general AdS-C metric[5]. The line element of a general AdS metric in C coordinates is given by

$$ds^2 = \frac{l^2}{(x-y)^2}\left[-\frac{F(y)}{1+\lambda}dt^2 + \frac{dx^2}{G(x)} + \frac{dy^2}{F(y)} + G(x)d\zeta^2\right] \tag{113}$$

In the limit $\lambda \rightarrow$ -1, we define new coordinates given as

$$(x-y) = q \Rightarrow (dx - dy) = dq$$
$$(x+y) = p \Rightarrow (dx + dy) = dp$$

Putting the above values in equation (113),the metric will take the form,

$$ds^2 = -\frac{F(q,p)}{1+\lambda}dt^2 + \left(\frac{1}{G(p,q)} + \frac{1}{F(q,p)}\right)\frac{dq^2}{4} + \left(\frac{1}{G(p,q)} + \frac{1}{F(q,p)}\right)\frac{dp^2}{4} + \left(\frac{1}{G(p,q)} - \frac{1}{F(q,p)}\right)\frac{dq\ dp}{2} + G(q,p)d\zeta^2 \tag{114}$$

Replacing AdS-4 metric by equation (114) we get D3-brane solution,

$$ds^2 = -\frac{e^{2A}l^2}{q^2}\left[\frac{F}{1+\lambda}dt^2 + \left(\frac{1}{G} + \frac{1}{F}\right)\frac{dq^2}{4} + \left(\frac{1}{G} + \frac{1}{F}\right)\frac{dp^2}{4} + \left(\frac{1}{G} - \frac{1}{F}\right)\frac{dq\ dp}{2}\right] + \frac{e^{2A}l^2}{q^2}G(q,p)d\zeta^2 + e^{2A}d\psi^2 + e^{2A}\sinh^2\psi(d\xi^2 + \sinh^2\ d\eta^2) + dr^2 + \frac{1}{16}e^{2A}(1-x^2)(d\theta^2 + \sinh^2\theta d\phi^2) \tag{115}$$

Again, taking r=constant and q=q(ψ), the above metric takes the form,

$$ds^2 = -\frac{e^{2A}l^2}{q^2}\frac{F}{1+\lambda}dt^2 + \frac{e^{2A}l^2}{4q^2}M dp^2 + \frac{e^{2A}Gl^2}{q^2}d\zeta^2 + e^{2A}\left(\frac{l^2q'^2M}{4q^2} + 1\right)d\psi^2 + \frac{e^{2A}l^2}{2q^2}N\,dp\,d\psi + e^{2A}\sinh^2\psi(d\xi^2 + \sinh^2\xi\ d\eta^2) + \frac{1}{16}e^{2A}(1-x^2)(d\theta^2 + \sinh^2\theta d\phi^2) \tag{116}$$

where

$$\left(\frac{1}{G}+\frac{1}{F}\right)=M \quad \text{and} \quad \left(\frac{1}{G}-\frac{1}{F}\right)=N$$

In the absent of gauge fields, the world volume action is given as,

$$S=-\int d^3x\,d\psi\,\sinh^2\psi\frac{1}{q^4}\left[q'^2+(G+F)q^2\right]^{\frac{1}{2}} \tag{117}$$

Thus, the lagrangian can be written as,

$$L=-\sinh^2\psi\frac{1}{q^4}\left[q'^2+(G+F)q^2\right]^{\frac{1}{2}} \tag{118}$$

$$L=-\sinh^2\psi A\left[q'^2+B\right]^{\frac{1}{2}} \tag{119}$$

where A=$\frac{1}{q^4}$ and B=$(G+F)q^2$.
Working out the equation of motion, we get

$$q"+2q'\coth\psi\left(1+\frac{q'^2}{B}\right)-\frac{1}{B}\frac{dB}{dq}q'^2-\frac{1}{2}\frac{dB}{dq}-\frac{(B+q'^2)}{A}\frac{dB}{dq}=0 \tag{120}$$

Again, to compute the conductivities on the gravity side we parametrize our metric via ψ=$\psi(q)$, then the metric is given as

$$ds^2=-\frac{e^{2A}l^2}{q^2}\frac{F}{1+\lambda}dt^2+\frac{e^{2A}l^2}{4q^2}M\,dp^2+\frac{e^{2A}Gl^2}{q^2}d\zeta^2+e^{2A}\left(\frac{Ml^2}{4q^2}+\psi'^2\right)dq^2$$
$$+\frac{e^{2A}l^2}{2q^2}N\,dq\,dp+e^{2A}\sinh^2\psi(d\xi^2+\sinh^2\xi\,d\eta^2)+\frac{1}{16}e^{2A}(1-x^2)(d\theta^2+\sinh^2\theta d\phi^2) \tag{121}$$

Following the same logic we now write our metric in Eddington-Finkelstein coordinates and defining

$$d\tilde{q}=\frac{1}{2q}\sqrt{Ml^2+4q^2\psi'^2}\,dq$$

and advance coordinates in the same manner, the metric takes the form

$$ds^2=e^{2A}\left[-U\,dv^2+2dv\,d\tilde{q}+\frac{Ml^2}{4q^2}dp^2+\frac{Gl^2}{q^2}d\zeta^2+\frac{Nl^2}{2q^2\sqrt{Ml^2+4q^2\psi'^2}}dpd\tilde{q}\right]$$
$$+e^{2A}\sinh^2\psi(d\xi^2+\sinh^2\xi d\eta^2)+\frac{1}{16}e^{2A}(1-x^2)(d\theta^2+\sinh^2\theta d\phi^2) \tag{122}$$

where

$$U=\frac{Fl^2}{q^2(1+\lambda)}$$

Again keeping the gauge fields same, the Born-Infeld action is given as

$$S=-\tau\int d^4x\sinh^4\psi\sqrt{-det(g_4+F)}$$

$$S = -\tau \int d^4x \sinh^4 \psi \sqrt{\gamma} \tag{123}$$

where we have defined γ=$-det(g_4 + F)$(see appendix). Thus, the Lagrangian will be

$$L = -\tau \sinh^2 \psi \sqrt{\gamma} \tag{124}$$

Again working in the perturbatively in electric field E^i. For E^i=0 and a'_v=0, the charge density can be written as,

$$\rho = \frac{\tau \sinh^2 \psi \, a'_v \left[WZe^{2A} + B^2\right]}{[(e^{4A} - a'^2_v)(WZe^{2A} + B^2]^{\frac{1}{2}}} \tag{125}$$

where

$$W = \frac{Ml^2}{4q^2} \quad and \quad Z = \frac{Gl^2}{q}$$

From the above equation we can write

$$a'_v = \frac{\rho e^{2A}}{\left[\rho^2 + \tau^2(B^2 + WZe^{2A}) \sinh^4 \psi\right]^{\frac{1}{2}}} \tag{126}$$

To calculate conductivities again we have to go to first order in electric field. The charge density is given as,

$$j^1(q_+) = \frac{\partial L}{\partial a'_1} = \frac{[-2Z_+e^{4A}E^1 + (e^{2A} - a'_v)BE^2] \left[\rho^2 + \tau^2(B^2 + W_+Z_+e^{2A}) \sinh^4 \psi_0\right]^{\frac{1}{2}}}{e^{2A}[(B^2 + W_+Z_+e^{2A}]} \tag{127}$$

Writing current density in terms of conductivities

$$j^1 = \sigma_1^L E^1 + \sigma_1^T E^2$$

Comparison of the above expresions will give

$$\sigma_1^L = \frac{-2Z_+e^{4A}) \left[\rho^2 + \tau^2(B^2 + W_+Z_+e^{2A}) \sinh^4 \psi\right]^{\frac{1}{2}}}{e^{2A}[(B^2 + W_+Z_+e^{2A}]} \tag{128}$$

and the transverse conductivity is given as

$$\sigma_1^T = \frac{(e^{2A} - a'_v)B \left[\rho^2 + \tau^2(B^2 + W_+Z_+e^{2A}) \sinh^4 \psi_0\right]^{\frac{1}{2}}}{e^{2A}([B^2 + W_+Z_+e^{2A}]} \tag{129}$$

Putting the value of a'_v, the above equation can be written as,

$$\sigma_1^T = \frac{Be^{2A}[\sqrt{\rho^2 + \tau^2(B^2 + W_+Z_+e^{2A}) \sinh^4 \psi_0} - \rho}{e^{2A}[B^2 + W_+Z_+e^{2A}]} \tag{130}$$

Proceeding in same manner

$$j^2(q_+) = \frac{\partial L}{\partial a'_2} = \frac{[BE^1 + W_+e^{2A}E^2] \left[\rho^2 + \tau^2(B^2 + W_+Z_+e^{2A}) \sinh^4 \psi_0\right]^{\frac{1}{2}}}{e^{2A}[(B^2 + W_+Z_+e^{2A}]} \tag{131}$$

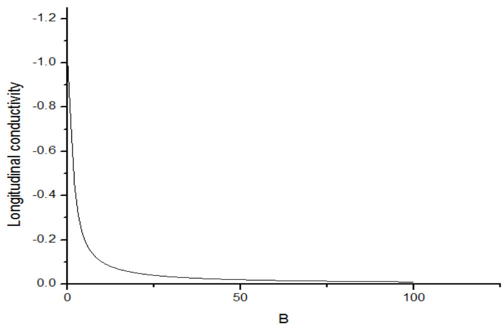

Figure 6: Longitudinal Conductivity vs magnetic field

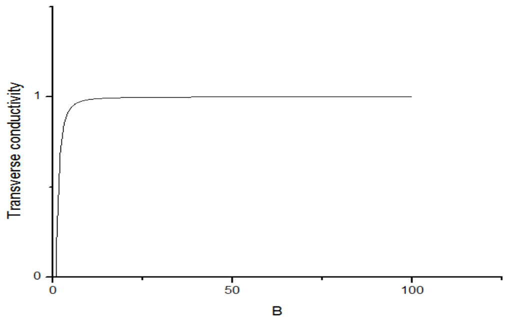

Figure 7: Transverse Conductivity vs magnetic field

Since we can write $j^2 = \sigma_2^T E^1 + \sigma_2^L E^2$, comparing the above expression with expression of j^2 will give

$$\sigma_2^L = \frac{W e^{2A} \left[\rho^2 + \tau^2 (B^2 + W_+ Z_+ e^{2A}) \sinh^4 \psi_0\right]^{\frac{1}{2}}}{e^{2A}[(B^2 + W_+ Z_+ e^{2A}]} \tag{132}$$

and

$$\sigma_2^T = \frac{B \left[\rho^2 + \tau^2 (B^2 + W_+ Z_+ e^{2A}) \sinh^4 \psi_0\right]^{\frac{1}{2}}}{e^{2A}[(B^2 + W_+ Z_+ e^{2A}]} \tag{133}$$

where

$$W_+ = \frac{Ml^2}{4q_+^2} \quad and \quad Z_+ = \frac{Gl^2}{q_+}$$

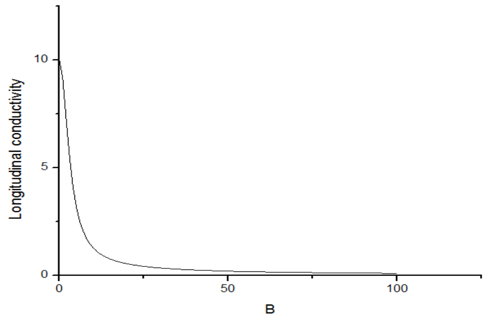

Figure 8: Longitudinal Conductivity vs magnetic field

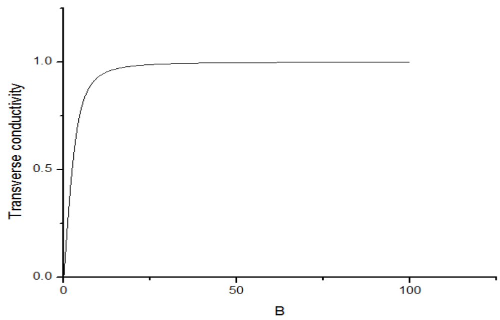

Figure 9: Transverse Conductivity vs magnetic field

We have plotted all the graphs in natural units in which B is $1\mathrm{eV}^2$ and conductivity is in 1eV.

Chapter 5

Results and Disscussion

For AdS4 metric the longitudinal conductivity falls sharply for small range of magnetic field. However it becomes constant for higher values of magnetic field (figure 4),whereas, the tranverse conductivity first increases for small values of magnetic field and as its value increases conductivity falls sharply (figure 5). At $B = 0$ tranverse conductivity takes a constant value $k/2\pi$ and has a maxima at $B = e^{2A}\frac{q^2}{\alpha^2}$.
For AdS C-metric we have two currents j^1 and j^2. Corresponding to each of them we got longitudinal and tranverse conductivity. For j^1, longitudinal conductivity takes negative value (figure 6). As we increase the value of magnetic field longitudinal conductivity increases and for large values of magnetic fields it tends to zero. But the transverse conductivity is positive and increases sharply for very small range of magnetic field(figure 7). It becomes constant for higher magnetic field. For j^2, longitudinal conductivity is very high when B tends to ver small values. As we increase magnetic field it starts falling asymtotically and becomes zero for higher vaalues of magnetic field. However, the nature of tranverse conductivity is quite different. It is zero for B=0 and increases sharply for very small range of magnetic field and finally becomes constant for high values of magnetic field.

Chapter 6

Conclusion and scope for Future Work

We have shown that how one can calculate conductivity for different systems in constant electric and magnetic field. In section (4.1) we have calculated conductivity in terms of relative motion of D3 and D7-branes. Our present theory has a weak coupling description in terms of D-branes and Minkowski space, hence has a purely integer charge spectrum. Since our metrics are anisotropic thus we get conductivity in two directions i.e longitudinal and tranverse conductivity which are not equal. For j^1 (AdS C-metric) longitudinal conductivity is negative which means reistivity along longitudinal direction is negative. In the current analysis we have used probe branes action to calculate conductivity, such probe branes does not take care the background fluctuations on the metric into account. A more sophisticated way is to incorporate them using linear fluctuation theory and use Kubo's formula to calculate conductivity. In this frame we will also able to calculate viscosity, thermal conductivity and diffusivity. There is also scope to obtain fractional case, any weak coupling limit would have to exhibit charge fractionalisation, possibly due to an orbifold geometry.

Appendix

Conversion into Eddington-Finkelstein Coordinates

$$ds_8^2 = \frac{r^2}{R^2}(-hdt^2 + dx^i dx^i) + \frac{R^2}{r^2}(h^{-1} + r^2\psi'^2 dr^2 + R^2 sin^2 d\Omega_4^2 \tag{134}$$

and

$$\tilde{r} = \int_{r_+}^{r} dx\sqrt{1 + hx^2\psi'2}$$

The Eddington-Finkelstein coordinates is defined as

$$v = t + r^*$$

$$dt = dv - dr^*$$

Putting the se values in in the line element we get

$$ds_8^2 = -\frac{hr^2}{R^2}(dv^2 + dr^{*2} - 2dv\,dr^*) + \frac{R^2}{hr^2}d\tilde{r}^2 + \frac{hr^2}{R^2}dx^i dx^i + R^2\sin^2 d\Omega_4^2 \tag{135}$$

$$ds_8^2 = -\frac{hr^2}{R^2}dv^2 - \frac{hr^2}{R^2}dr^{*2} + \frac{hr^2}{R^2}2dv\,dr^*) + \frac{R^2}{hr^2}d\tilde{r}^2 + \frac{hr^2}{R^2}dx^i dx^i + R^2\sin^2 d\Omega_4^2 \tag{136}$$

Defining $\frac{hr^2}{R^2}dr^*$=$d\tilde{r}$ and putting it in the above expression we get,

$$ds^2 = 2dvd\tilde{r} - Udv^2 + \frac{R^2}{r^2}dx^i dx^i + R^2\sin^2\psi d\Omega_4^2 \; where \; U = \frac{r^2 h}{R^2} \tag{137}$$

Chern-Simmons Action

$$S_{cs} = \frac{1}{4\pi}\int df(\tilde{r})\Lambda A\Lambda F$$

$$df(\tilde{r}) = f' d\tilde{r}$$
$$A = A_v dv + A_j dx^j$$

$$A\Lambda F = \frac{1}{2}B\epsilon_{ij}a_v' dv\Lambda dx^i\Lambda dx^j + a_v a_j dv\Lambda dx^j\Lambda dr + a_j' E^i x^i dv\Lambda dx^j\Lambda dr + \frac{1}{2}B\epsilon_{ij}x^i a_v' dx^j\Lambda dr\Lambda dv$$
$$+ a_j a_v' dx^j\Lambda dr\Lambda dv + a_j E^i dx^j\Lambda dx^i\Lambda dv$$

$$f' dr\Lambda A\Lambda F = f'(Ba_v + a_j E^i)d^4x \tag{138}$$

where

$$d^4x = dr\Lambda dv\Lambda dx^i\Lambda dx^j$$

Calculation of det($g_4 + F$) of AdS-C metric

The term in squareroot of Born-Infeld action(AdS-C)can be calculated as,

$$g_4 + F = \begin{bmatrix} 0 & (e^{2A} + a_v^{'}) & X - a_1^{'} & a_2^{'} \\ (e^{2A} - a_v^{'}) & -Ue^{2A} & -E^1 & -E^2 \\ a_1^{'} + X & E^1 & W & B \\ a_2^{'} & E^2 & -B & Z \end{bmatrix} = \gamma \tag{139}$$

where we have denoted $\frac{Nl^2}{2q^2\sqrt{Ml^2+4q^2\psi'^2}}$=X.
γ can be written as,

$$\begin{aligned}\gamma = -Ue^{2A}[Za_1'e^{2A} - ZX^2e^{2A} + 2Ba_2'X + Wa_2'^2e^{2A}] + (a_v'^2 - e^{4A})(B^2 + WZE^{2A}) + 2ZE^1a_1'e^{4A} \\ +2ZXa_v'E^1e^{2A} + a_1'a_v'^2B - a_1'BE^2e^{2A} + 2BE^2Xe^{2A} - 2E^1Ba_2' - (E^1a_2')^2 + 2E^1E^2a_2'X \\ +E^1E^2a_1'a_2' + BE^2a_v'e^{2A} + Ba_v'^2E^2 + E^1E^2a_2'a_v' + (E^2X)^2 - (E^2a_v')^2 - WE^2a_2'e^{2A}\end{aligned}$$

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