STATISTICAL ANALYSIS OF COMPLEX NETWORK

M.Sc. Thesis

By

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DISCIPLINE OF PHYSICS

INDIAN INSTITUTE OF TECHNOLOGY INDORE

JUNE 2017

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A THESIS

Submitted in partial fulfilment of the requirements for the award of the degree
of
Master of Science

by
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DISCIPLINE OF PHYSICS
INDIAN INSTITUTE OF TECHNOLOGY INDORE
JULY 2017

INDIAN INSTITUTE OF TECHNOLOGY INDORE

CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled STATISTICAL ANALYSIS OF COMPLEX NETWORK in the partial fulfilment of the requirements for the award of the degree of **MASTER OF SCIENCE** and submitted in the **DISCIPLINE OF PHYSICS, Indian Institute of Technology Indore**, is an authentic record of my own work carried out during the time period from July 2016 to June 2017 under the supervision of Dr. Sarika Jalan Associate Professor, Discipline of Physics IIT INDORE.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other institute.

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This is to certify that the above statement made by the candidate is correct to the best of my/our knowledge.

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Acknowledgements

It is a great feeling for me to acknowledge my thanks to Dr. Sarika Jalan who gave me opportunity to work with her and helped me to complete my project. I am thankfull to her for trusting me and giving such a interesting topic "Analysis of second largest eigen vaue". I have learnt so many new thing while doing this project and got a good research exposer. It is great honour to work under her supervision. I am also thankfull to my all the faculty members of physics department for their support and giving us wonderfull environment.

It was not possible for me to achieve my M.Sc degree from a institute like IIT, without the support of my family. I am very much thankfull to my parents and sister for their constant support in my life.

I am thankfull to the all the members of complex system lab. I also want to express my gratitude towards my friends.

Abstract

Spectra of networks adjacency matrices are known to be fingerprint of the underlying complex systems. Most of the works on the network spectra have revolved around analysis of largest eigenvalue. In this work, we analyse the second largest eigen value (λ_2) of the networks adjacency matrices. We find that λ_2 may follow an entirely different behaviour than that of the largest eigen value. While, the largest eigenvalue contains information of the largest degree of the underlying network, thereby, exhibiting an increasing behvaiour throughout with an increase in the average connectivity of a network, λ_2 emulates this behaviour until a critical connectivity, after which it exhibits a continuous decrease. The behaviour of λ_2 provides an insight to degree of freedom available to the nodes to form connected pairs. More interesting is the behvaiour of λ_2 fluctuations, in the presence of inhibitory couplings. Inspired by the presence as well as importance of inhibitory couplings in many complex systems, including brain and ecological systems, we introduce such couplings in networks and investigate its impact on λ_2. For this case, while average behaviour of the largest and the second largest eigenvalues are exactly same for various value of inhibitory probability, the fluctuation statistics for the exhibit a very different behavior indicating importance of second largest eigenvalue in getting information of underlying system.

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Chapter 1

Introduction

1.1 Structure properties of complex networks

A network is a graphical representation of many interacting units. The component of the network are called nodes and there interaction is shown by link. example of networks include social networks, biological network, technology networks etc. Some of the structural properties of complex networks are listed below

1) Average Degree

Degree of node is defined as the number of edge connected to a node. The average degree is thus defined as

$\langle k \rangle = \frac{2E}{N}$.

2) Clustering Coefficient

A common property of complex networks is that cliques form, representing circles of friends in which every person knows each other member. This property is quantified as Clustering Coefficient.

$C_i = \frac{2E_i}{N\times(N-1)}$

where E_i is the number of edges that actually exists between neighbours of node say i. k_i is number of edges connecting i node to other k_i node.

3)Betweenness Centrality

Betweenness centrality is the measure of node centrality in a network. It is the fraction of shortest path between node pairs that pass through the said node of interest.

$X_i = \sum_{i\neq j} \dfrac{n_{st}^i}{g_{st}}$

where n_{st}^i is number of path from s to t that passes through i. g_{st} is total number of path from s to t.

4) Diameter

The diameter of a network is the maximal distance between any pairs of nodes. It tell us how quickly information cab be spread in a networks and how integrated the network.

5) Degree- Degree correlation

The (dis)assortativity has emerged as an important structural measure, used for understanding (dis)likelihood in connectivity in the underlying systems assortativity. Various social networks are known to be assortative, while many of the biological and technological networks are found to be disassortative Assortativity We quantify degree-degree correlation of a network by considering the pearson(degree- degree) correlation coefficient as

r $= \frac{[N_c{}^{-1}\sum_{l=1}^{N_c} d_i^l d_j^l]-[N_c{}^{-1}\sum_{l=1}^{N_c}\frac{1}{2}(d_i^l+d_j^l)^2]}{[N_c{}^{-1}\sum_{l=1}^{N_c}\frac{1}{2}({d_i^l}^2+{d_j^l}^2)]-[N_c{}^{-1}\sum_{l=1}^{N_c}\frac{1}{2}(d_i^l+d_j^l)^2]}$

where j_i and k_i are the degree of the nodes at both the end of the i^{th} connection and M represents the total connection in the network.

if $r > 0$ the network is assortive which means nodes with similar degree tend to be connected. example:- social network.
if $r < 0$ the network is disassortative which means dissimilar degree nodes tend to connect among themselves. example :- biological networks,technological network.

1.2 Model Networks

1.2.1 ER random network

ER random network is coined by Erdos and Renyi. We can generate a random network by using ER model, where we give two parameter for the construction of the network. The probability that a given node is connected to other nodes is p and total number of nodes equal to N. Some of the properties of ER networks are listed below.

1) The average degree of the node is given by $\langle k \rangle$ = p*N(N-1).
2) The degree distribution follow the binomial distribution.
3) The average path length is log N.
4) The clustering coefficient (CC) of random networks is given by $\frac{\langle k \rangle}{N}$.

1.2.2 Scalefree networks

Albert Barabasi purposed the scalefree networks. It is constructed on the algorithm of preferential attachment. Scalefree networks are constructed using BA preferential model method [18] in which each node prefers to connect with higher degree nodes. Thus node having higher degree has more chance of getting connected to upcoming nodes leading to 'rich get richer effect' The scalefree networks has many application to provide robustness to the system against the random attack. Most of the social networks are scalefree. Some of the various properties of scalefree networks are

1) The scalefree networks posses power law degree distribution.
2) Scalefree networks has lower value of diameter as compared to the random networks.
3) The clustering coefficient of scalefree networks is proportional to $N^{-}0.75$.

1.2.3 Small world networks

The smallworld networks was coined by Watts and Strogatz. Most of the social networks have a small diameter and high clustering coefficient as compared to random networks. Smallworld networks carry both these properties. Smallworld network is generated by starting with regular lattice of N nodes. We then start rewiring the network with probabiity p_r and slowly increases the value p_r such that we get a network which has high clustering and low diameter. It is build in concept of six degree of separation . The average path length between any pairs of node is given by L = log N.

1.3 Spectra properties of Complex Networks

1.3.1 Degeneracy at zero eigenvalue

A network can be represented by its adjacency matrix

$$A_{\mathrm{ij}} = \begin{cases} 1 & \text{if } i \sim j \\ 0 & \text{otherwise} \end{cases} \tag{1.1}$$

The eigen values of adjacency matrix are $\lambda 1$,$\lambda 2$,$\lambda 3$ and so on. The associated eigenvector v1,v2,v3.....vN satisfy the eigen vector equation Av(i)= λv(i). If the rank of the matrix is r then there will be N-r zero eigen values where N is total number of node in the network and hence dimension of the adjacency matrix. The factor responsible for the lowering the rank of the adjacency matrix are following.

(1) when two rows have exactly same entries it is termed as complete duplication.

R1 = R2 (1)

(2) when two or more rows (column) added together equals to some other rows(columns) then we call it as partial duplicates.

R1+R2 $\rightarrow R3 + R4 + R5$ (2)

(3) An isolated node will not be connected to any of the node and thus all the entry in its row will be zero thus lowering the rank of the matrix.

Next we will discuss the kind of topology which will generate zero degeneracy [1]. There are two cases 1) complete duplication 2) partial duplication. For example consider a network of 5 nodes as shown Fig1.1a. Node1 and Node2 both are connected to same neighour Node3, Node4, Node5. So Node1 and Node2 are said to be duplicates nodes contributing to one zero eigen values. If there are two duplicates nodes then this will lead to two zero eigen values and so on.

Further we consider the second case which lead to zero degeneracy. The kind of topology which can contribute to condition 2 can be seen from Fig1.1b. In Fig1.1b , we have considered a network of 6 nodes. The node2 is connected with two Node4 and Node5 while node 3 is connected to only one node 6. The sum of neighours of node2 and node3 i.e node4 ,node5 ,node6 are also connected to node 1 as shown in Fig1.1b.

i.e $N2 + N3 \to N1$

This lead to partial duplicates and contribute to one zero eigen value. If we carefully observe the diagram we can deduce that Node1 and Node2 are connected to same neighour Node4 and Node5. Thus they are duplicates nodes and contributing to one more zero eigen value. The number of duplicates (complete or partial) equals to the number of zero eigen values. We can find the nodes which are contributing toward zero degeneracy from the non-zero entry of the corresponding eigen vector.

1.3.2 Degeneracy at -1 eigenvalue

We try to find out the information which we can get from -1 degeneracy[2]. First we will try to focus on the occurrence of -1 degeneracy and the characteristics structure which can lead to -1 degeneracy. To study the -1 degeneracy we will do a transformation on Adjacency matrix of the network. We are adding Identity matrix to the adjacency matrix i.e A + I . We make a

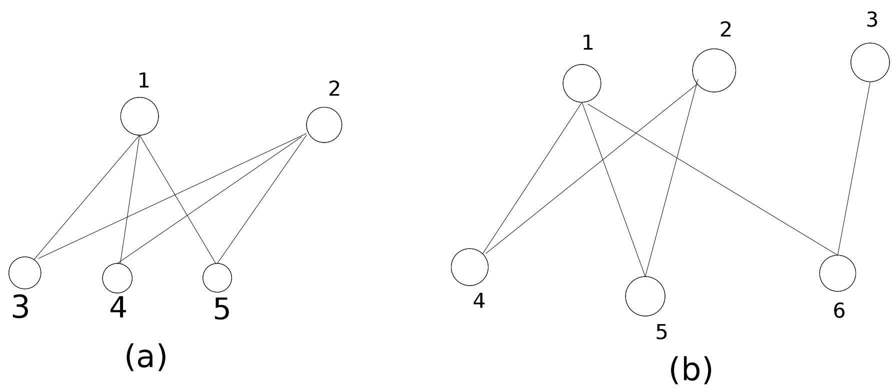

Figure 1.1: Zero degeneracy

Schematic diagram representing (a) complete node duplication and (b) partial node duplication

change of variables in the characteristic polynomial such as χ_{A+I} $(\lambda) = \chi_{A+I}$ (μ). By this way, μ is an eigenvalue of A if and only if λ is an eigenvalue of A + I. Thus -1 eigen value of adjacency matrix A can be considered as zero eigen value of transformed matrix A + I. Now we will follow the same procedure as we did for zero degeneracy of Adjacency matrix discussed in previous section. The number of zero eigen values of transformed matrix (A+I) will be equal to N-r where N is the dimension of the matrix and r is the rank of the matrix. Again there will be three condition which can lead to it.

(1) when two rows have exactly same entries and are equal.

R1 = R2 (1)

(2) when two or more rows (column) added together equals to some other rows(columns).

R1+R2 $\rightarrow R3 + R4 + R5$ (2)

(3) An isolated node will not be connected to any of the node and thus all the entry in its row will be zero thus lowering the rank of the matrix.

For the A+I matrix the condition 3 can never meet.It is difficult to define a typical structure corresponding to condition 2 [ref loic manuscript]. however we can illustrate it with the graph shown in Fig1.2

R1+R4 $\rightarrow R2 + R5$.

Here degeneracy of -1 in A+I matrix and corresponding eigen vector give us idea about the structure of the networks. The structure which can give rise to -1 degeneracy can be explained by considering a subset of n node forming complete graph K which is connected to the another set S of different nodes forming K*S network contributing to -1 eigen value with multiplicity n-1.

When we find out the eigen vector, most of the entries of such eigen-vector are zero. It turns out that the non-null entries reveals nodes which contribute to decreasing the rank of a matrix.

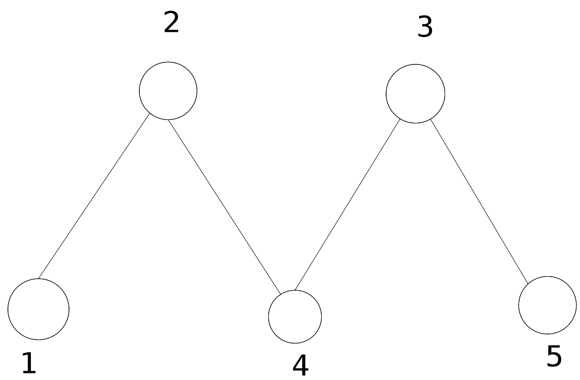

Figure 1.2: Degeneracy at -1

Schematic diagram for which the condition (2) is verified in A + I.

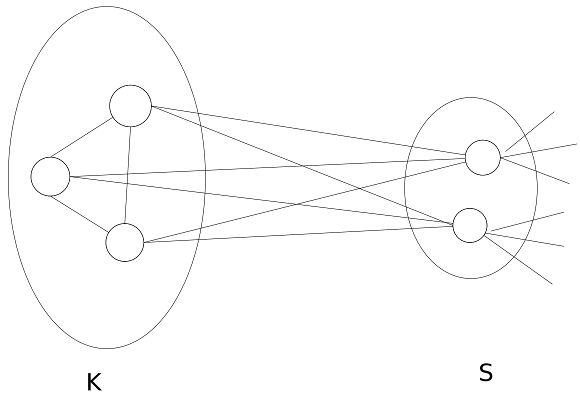

Figure 1.3: Structure giving rise to -1 degeneracy

Here k a complete subgraph of 3 node connected to set S having 2 node giving rise to two -1 eigen value.

Chapter 2

Analysis of second largest eigen value

2.1 Introduction

With an accelerated development in field of network science, various communities belonging different branches of science, economics, biology, sociology witness a growing application of networks in their respective fields. Since, the resolution of seven bridges of Königsberg problem[3], the theory of networks which deals with th study of complex interacting units represented as graphs, has demonstrated remarkable applications in various real-world networks[4]. From epidemic threshold of a disease outbreak[5] to determining stability of a systems[6], the analysis of the spectra of network plays a pivotal role in bridging dynamical and structural properties of the network. Among various aspects of spectral analysis, the second largest eigenvalue of network matrix (λ_2) has many theoretical and practical applications [7]. Various expansion (and concentration) properties of graphs in related with the second largest eigenvalue[8]. These properties in turn, is related to many applications in diverse fields including mathematics and computer science. The second largest eigenvalue is also known to denote algebraic connectivity of certain classes of graphs[9]. The expansion properties found its application in computation complexity theory[10], determining robustness of computer networks[11], graph pebbling, parallel sorting algorithms and theory of error-correcting codes[14]. It can also be used in dynamics like markov chain. It is found that graph with small λ_2 is termed as rapidly mixing for reversible markov chain[15].

Furthermore, inhibition which introduces directness in between interacting units of a graph, plays a pivotal role in various real-world networks including ecological systems[16] and brain networks. The interaction between neurons often studied using a synaptic graph with includes inhibitive properties of neurons[17]. The ecological network is constructed keeping the predator-prey nature of various species acting as nodes and thus includes inhibition from top predators of the food chain. In this report, we investigate the statistics of the λ_2 of network adjacency matrix which includes inhibition to mimic properties of real world networks under GEV framework. We consider inhibitory and excitatory couplings in between the interactions of individual units and study the statistical properties of the λ_2.

2.2 Model:

We conduct the analysis for various model network, particularly Erdös-Renýi (ER) random, scalefree and regular networks. ER random networks are constructed using ER model where every pair of nodes is connected with a probability p [4]. Scalefree networks are constructed using BA preferential model method [18] in which each node prefers to connect with higher degree nodes. The degree distribution of scalefree networks thus created follow a power law. In regular networks every node has exactly the same degree and each node is connected to its nearest neighbors.

We can represent a network by its adjacency matrix. The adjacency matrix A of a network has entries 1 or 0 depending upon whether i and j nodes are connected or not. The diagonal entries of A are zero depicting no self connection.

$$A_{\mathrm{ij}} = \begin{cases} 1 & \text{if } i \sim j \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

The eigen values of adjacency matrix of a complex network are refereed as spectra of networks. The spectra of a network provides lot of information about the structural properties of networks as well as dynamical behaviour of interacting nodes on these networks. For example degeneracy of zero and one eigenvalue provide clue to the structural symmetries in the networks [19]. The largest eigenvalue, in addition to capture the information of the largest degree, is related with the synchronization phenomena of diffusively coupled dynamical units on the network [20]. Eigenvalues of the adjaceny matrix can be written as $\lambda_1 > \lambda_2 > \lambda_3....\lambda_N$.

The inhibitory node in the network is introduced as follow. An introduction of the inhibitor coupling with probability p_{in} leads to -1 entries in the corresponding rows of the adjacency matrices and consequently symmetrical property of the matrix is lost [21] leading to complex eigenvalues. For this case, the first and the second largest eigenvalue denote the first and the second largest real part of the eigenvalues, respectively. We will analyze the mean as well as fluctuation behaviour of the second largest eigenvalue around its mean. Particularly, we will study the role of inhibition on the behaviour of λ_2 and will compare the behaviour with that of the largest eigen value λ_1.

Further, the GEV statistics has been successfully applied to many real-word systems including stock markets, natural disasters, galaxy distributions as a model for extreme events. The GEV statistics of independent, identically distributed random variable can be characterized entirely in terms of three universal probability distribution functions (PDF) namely Weibull, Gumbel and Fr'echet depending on the tail of density function being a power law, faster than the power law and bounded or unbounded, respectively. The probability density function for these three distributions can be written as [22].

$$\rho(x) = \begin{cases} \frac{1}{\sigma}\big[1+\big(\xi\frac{(x-\mu)}{\sigma}\big)\big]^{-1-\frac{1}{\xi}} \exp\big[-\big(1+\big(\xi\frac{(x-\mu)}{\sigma}\big)\big)^{-\frac{1}{\xi}}\big] \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } \xi \neq 0 \\ \frac{1}{\sigma}\exp\big(-\frac{x-\mu}{\sigma}\big)\exp\big[-\exp\big(-\frac{x-\mu}{\sigma}\big)\big] \quad \text{if } \xi = 0. \end{cases} \tag{2.2}$$

where μ, σ, ξ represent location parameter, scale parameter and shape parameter, respectively. The underlying statistics can be determined by the value of the shape parameter as follows; $\xi > 0$ (Fréchet Statistics), $\xi = 0$ (Gumbel statistics) and $\xi < 0$ (Weibull statistics).

Chapter 3

Result and Discussion

The largest eigen value λ_1 of a network exhibits an increases with the average degree. From the Gershgorin circle theorem, the eigen values of a square matrix a_{ij} lie on atleast one of Gershgorin disks [23] whose radius is defined as

$R_i = \sum_{i \neq j} a_{ij}$ with centre lying a_{ii}

$$. \tag{3.1}$$

If the disks are disjoint from each other than they will contain exactly one eigen value. Since the concentric circles are disjoint so each cirle will contain exactly one eigen value.

For a adjacency matrix corresponding to a simple graph, devioding of self connection, diagonal entries are zero. Consequently, every eigenvalue will lie within a circle having center at zero and radius being the degree of a node.

Thus λ_1 and λ_2 both will have upper bound of the highest and the second highest degree of the network, i.e., k_{max} and k_{max2}, respectively. Further, λ_1 has lower bound given by the average degree of the network [24]. Since λ_1 has upper bound of k_{max} and lower bound of $\langle k \rangle$ and for a regular graph the largest degree k_{max} and the average degree are equal, i.e $\langle k \rangle = k_{max}$ it is very clear that $\lambda_1 = \langle k \rangle$.

For other graphs the upper and lower bound do not shed light on the exact value an eigenvalue will take. Since by increasing the average degree keeping N constant will increase the highest degree k_{max} node in the network. it is interesting to notice that for ER and SF networks as well λ_1 follows exactly the highest degree displaying an increasing trend with an increase in the average connectivity . Such a behaviour is not exhibited by the second largest eigen value λ_2 which also has upper bound of k_{max2}. λ_2 does not depict a continuous increase with an increase in average connectivity or the second largest degree of a network. If by keeping the network size fixed, we increase the average connectivity of a network, λ_2 first exhibits an increase until $\langle k \rangle \sim N/2$, which is not surprising as λ_1 also exhibits an increase and one can think of a similar trend followed by the second, however, the surprising phenomena emerges if we increase the average connectivity further, λ_2 shows a decreasing behaviour with $\langle k \rangle$. Figure 3.2(a) depicts behaviour of λ_1 and λ_2 for a regular lattice. λ_1 shows a linear function of the average degree ($\lambda_1 = \langle k \rangle$). However, the second largest eigenvalue λ_2 manifests first an increase with an increase in the average connectivity but for the higher average connectivity shows a decreasing trend. The similar behavior of first an increase and then decrease in λ_2 after $\langle k \rangle = N/2$ is depicted by ER and SF networks also. This is more surprising for these networks, as for these networks the second largest degree k_{max2} keeps on increasing as average con-

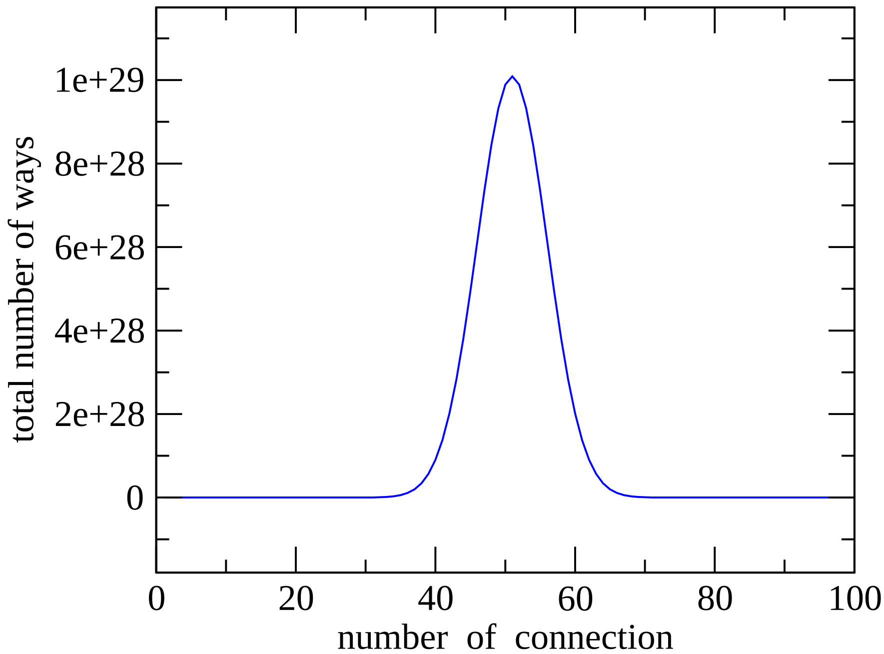

Figure 3.1: degree of freedom of connections

Number of possible ways $N\langle k\rangle$ connections can be distributed among N nodes as a function of average degree.

Intuitively, the reason behind this behaviour of λ_2 can be related with the degree of freedom associated with the distribution of connections among the nodes. For N nodes, the total number of possible connections among them is given by $N_T = {}^{N}C_2 = \frac{N(N-1)}{2}$. Next, number of ways r connections from these N_T connections can be selected is ${}^{N_T}C_2$. It is easy to see that upon varying r from 0 to N_T, the possible ways in which r pairs of the nodes can be selected from N_T pairs, avoiding self and duplicate connections, will first increase followed by a decrease. Additionally, from ${}^{N}C_r = {}^{N}C_{N-r}$, we know that it will be symmetrical in the nature. The philosophy behind this discussion is that as we increase the value of r, the possible number of choices

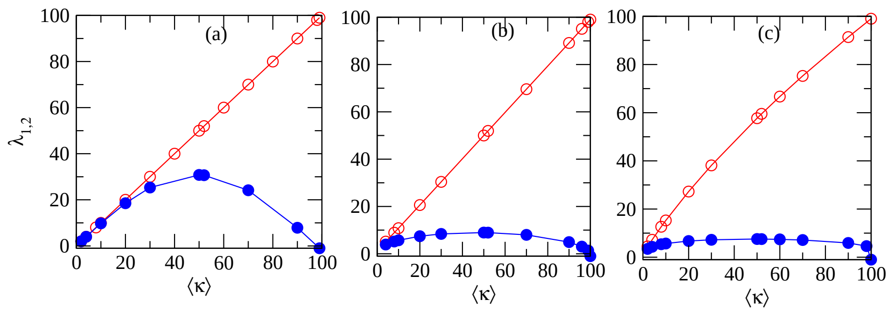

Figure 3.2: Variation of λ_1 and λ_2 with $\langle k \rangle$
The analysis has been averaged for 5000 realizations of network generation for SF and ER networks. Note that for a regular network only one realization of the graph is possible for a given network parameter. ($\bigcirc$) and ($\bullet$) are used for λ_1 and λ_2 respectively. (a) Regular network, (b) ER random and (c) SF networks.

for the connections among N nodes first increases for r being exactly being equal to the half of total number of possible connections for these nodes. For a further increase in r, the number of available choice to make connections or degree of freedom available to the network become less. Figure 3.1 illustrates that as the average connectivity increases, the possible ways of distributing connections first increases followed by a decrease exactly as average degree becomes $N/2$. Since, λ_2 emulates the similar behaviour as discussed about the degree of freedom (Figure 3.1), we can intuitively relate it with the degree of freedom.

Comparison of λ_2 with λ_1 for various model networks is illustrated in Figure 3.3. The largest eigenvalue of SF network is higher than that of the random and the regular network due to the presence of a hub nodes having much higher degree in the scalefree network as compared to the highest degree nodes of the regular and ER random networks. As we are increasing

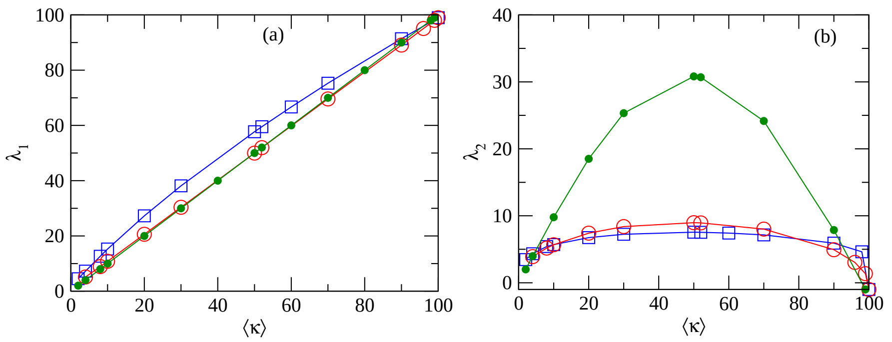

Figure 3.3: Comparison of λ_1 and λ_2

The analysis has been averaged for 5000 realizations of network generation for SF and ER networks. Note that for a regular network only one realization of the graph is possible for a given network parameter. ($\square$), ($\bigcirc$) and ($\bullet$) are used for scalefree, random and regular networks. (a) λ_1 (b) λ_2

the average connectivity highest degree node in the network increases. The increase in the largest degree provides a scaling factor to λ_1 yielding λ_1 for SF greater than that of regular and ER network at all average connectivity. Whereas for λ_2, regular networks have higher values than those of the ER random and scalefree networks at all average connectivity.

Though, the degree of freedom network connections have provide clue to the overall behaviour of λ_2 for various average degree, it does not shed light on higher value of λ_2 for 1D lattice as compared to SF and ER random networks. It is rather difficult here to prove that why degree of freedom connections have for 1D is much higher than that for the random networks and may require additional insights. Nevertheless, algebraic connectivity of networks provides a different view to λ_2 behaviour.

The second smallest eigenvalue or first nonzero eigenvalue of the Laplacian matrix of a network is refereed as the algebraic connectivity [25] α of the

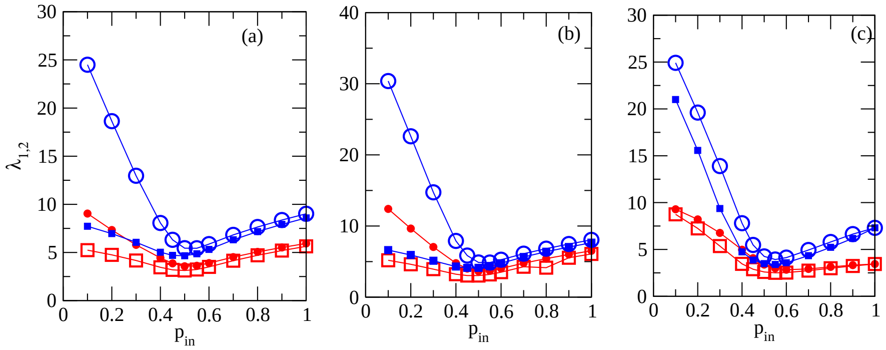

Figure 3.4: λ_1 and λ_2 with p_{in}

Variation of λ_1 (open symbol) and λ_2 (closed symbol) as a function of p_{in} for two different value of the average degree $\langle k \rangle = 10, 30$. The analysis is done for $N = 100$ and for 5000 random realizations of the networks arising due to different manner entries in a matrix change there sign with probability p_{in}. (a) ER random network (b) SF network and (c) regular network. (◯), (■) are used for λ_1 and λ_2 for $\langle k \rangle = 30$ and (•) and (□) are λ_1 and λ_2 for average degree 10.

network. The algebraic connectivity of a graph is greater than zero if and only if the graph is connected. For a regular lattice, the algebraic connectivity α is related with the second largest eigen value λ_2 [25] as $\alpha = \langle k \rangle$ - λ_2. $\Rightarrow$ $\lambda_2 = \langle k \rangle$ - α.

Though this relation holds good only for the regular lattice, what we understand from this relation is that for a fixed average degree λ_2 is higher if the algebraic connectivity is less and vice versa. The algebraic connectivity for the various model networks such as regular, scle-free is displayed in the Table 3.4. We can see that algebraic connectivity is always increasing function of the average connectivity. But for the average degree being less than that of N/2, α of the regular network is much lower than that of scale free networks which in turn reflects a higher value for λ_2 of the regular networks

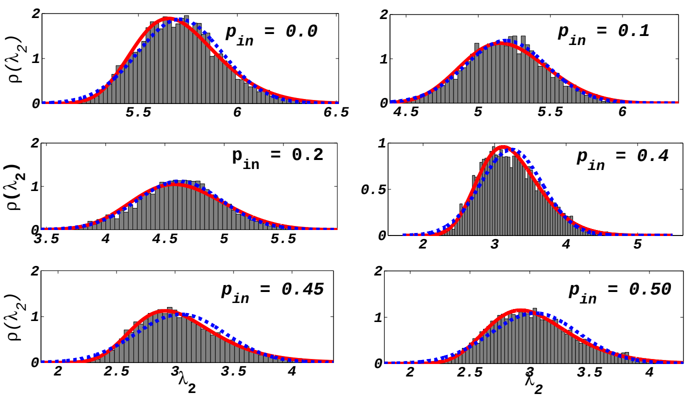

Figure 3.5: λ_2 statistics for SF network

Distribution of λ_2 for various p_{in} with average degree $\langle k \rangle = 10$ of scalefree networks. The blue and red line indicate normal and GEV distribution, respectively.

than scalefree networks. However as the average degree becomes higher than N/2, there is a sudden increase in the value of α for a regular network leading to decrease in the value of λ_2.For the scalefree networks no such change is observed in α and thus λ_2 is almost constant even for the higher average degree.

As discussed earlier that inspired by the role of inhibition in many real world systems including brain [26] and ecological we introduce inhibitory node in networks. and study the variation of second largest eigen value λ_2 for different value of p_{in}. For a fixed average degree, the average behaviour of second largest eigen value λ_2 is similar to that of largest eigen value λ_1 for introduction of inhibitory nodes. λ_1 is asymmetric about $p_{in} = 0.5$. From the Figure 3.4 , we can see that both the largest eigen value λ_1 and second the largest eigen value λ_2 are decreasing up to $p_{in} = 0.5$ and then they start increasing. Thus second largest eigen value λ_2 is more related to the average

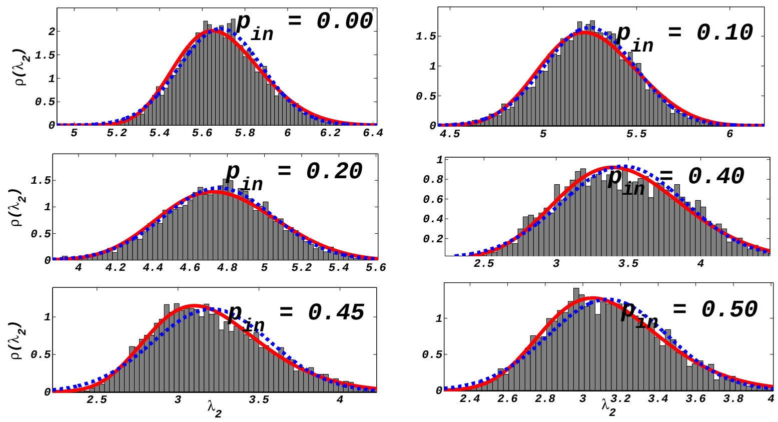

Figure 3.6: λ_2 statistics for ER network

Distribution of λ_2 for various p_{in} with average degree $\langle k \rangle = 10$ of ER random networks. The blue and red line indicate normal and GEV distribution, respectively.

degree than inhibitory nodes. Further the reason behind the behaviour of λ_2 with the inhibitory node can be explained by simple matrix algebra. Since when $p_{in} = 1$ all the 1s entries of the adjacency matrix for $p_{in} = 0$ become equal to -1. So the adjacency matrix A has changed to -A for $p_{in} = 1$. Thus all the eigen values of the adjacency matrix A will change their sign for -A. Hence, second minimum eigen value λ_{min2} of adjacency matrix for $p_{in} = 0$ becomes equal to second largest eigen value λ_2 of the adjacency matrix for $p_{in} = 1$ and since $\lambda_{min2} \neq \lambda_2$ for a adjacency matrix when $p_{in} = 0$. Thus, there will be asymmetric graph .

Next, we analyse statistics of the second largest eigen value. First we will discuss the case $p_{in} = 0$. We use kolmogorov Smirnov test [27] for the second largest eigen value statistics. As there exists only one realisation possible for the architecture of regular lattice the statistic can be drawn from the way inhibitory nodes are distributed. The statistics of λ_2 for the ER and scalefree network manifest GEV distribution. The shape parameter characterizes the statistics as the Weibull distribution of the GEV statistics. The value parameters are depicted in the Table 3.1 and Table 3.2 . As inhibitory nodes are introduced althogh the behaviour of λ_1 and λ_2 are similar, there statistics manifest different behaviour. Statistics of second largest eigen value is more dependent on the inhibitory coupling than the average degree. The largest eigen value λ_1 statistics accepts both GEV and gaussian statistics however, for λ_2, KS test accepts only GEV statistics for $p_{in} = 0$ and the value of shape parameter characterizes it as the weibull distributon. Few intermediate p_{in} values can also be modeled with the normal distribution excepts for $p_{in} =$ 0.45, 0.50, 0.55 which always shows GEV statistics for both the Scalefree and random networks. For higher average degrees, there exists a transition from the weibull to Frechet via gumbell at $p_{in} = 0.40$ as shown in Table 3.3 . Note, there exists no such transition for λ_1.

p_{in}	ξ of GEV	σ of GEV	μ of GEV	p-value of KS test for GEV	μ of Normal	σ of Normal	p-value of KS test for Normal
0.00	-0.14	0.19	5.62	0.211	5.70	0.21	0.006
0.20	-0.23	0.36	4.48	0.00019	4.62	0.35	0.309
0.40	-0.14	0.38	3.05	0.10	3.23	0.42	0.0001
0.45	-0.08	0.32	2.88	0.97	3.04	0.37	0.0010
0.50	-0.10	0.32	2.88	0.58	3.04	0.36	0.001
0.55	-0.15	0.38	3.03	0.17	3.20	0.42	0.0006

Table 3.1: KS test for SF networks

Estimated parameters of KS test for fitting GEV and normal distributions of λ_2 for different inhibitory inclusion probability (p_{in}) of SF network over 5000 population. Other parameters are network size $N = 100$ and average degree $\langle k \rangle = 10$.

p_{in}	ξ of GEV	σ of GEV	μ of GEV	p-value of KS test for GEV	μ of Normal	σ of Normal	p-value of KS test for Normal
0.0000	-0.1900	0.1800	5.6000	0.1100	5.6800	0.1900	0.0200
0.2000	-0.2400	0.2900	4.6400	0.0100	4.7500	0.2900	0.4800
0.4000	-0.2000	0.4000	3.2900	0.0026	3.4600	0.4200	0.0001
0.4500	-0.1300	0.3200	3.0500	0.2900	3.2080	0.3600	0.0000
0.5000	-0.1500	0.2900	3.0000	0.8800	3.1300	0.3100	0.0009
0.5500	0.1600	0.3200	3.0900	0.9000	3.2300	0.3400	0.0003

Table 3.2: KS test for random network

Estimated parameters of KS test for fitting GEV and normal distributions of λ_2 for different inhibitory inclusion probability (p_{in}) of random network over 5000 population. Other parameters are network size $N = 100$ and average degree $\langle k \rangle = 10$.

p_{in}	λ_1	λ_2
0.00	GEV+normal	GEV(Weibull)
0.10	Normal	GEV(Weibull)
0.20	None	Normal
0.30	None	GEV(Gumbell)
0.40	None	GEV(Frechet)
0.42	None	GEV(Frechet)
0.46	None	GEV(Frechet)

Table 3.3: KS test for higher average degree

λ_1, λ_2 statistics for isolated random networks $\langle k \rangle = 60$. The study is done for 5000 realization.

$\langle k \rangle$	α Regular	α S.F
10	0.21	3.005
30	4.68	12.52
50	19.17	22.83
70	45.84	32.15

Table 3.4: algebric connectivity with $\langle k \rangle$

Variation of algebraic connectivity for scalefree and regular networks. Total number of node is equal to 100 and average degree is varied from 10 to 70.

Chapter 4

Conclusion and Future direction

To conclude, we analysed the second largest eigen value λ_2 of the adjacency matrix of a network and report that it behave completely different from the largest eigen value λ_1. The largest eigenvalue λ_1 is more dependent on the highest degree of the networks. Larger the value of k_{max} ,higher will be the value of λ_1 but λ_2 though having upper bound of k_{max2} is independent of it. λ_1 of scalefree networks is always higher than the regular and random network since it has higher k_{max} as compare to the random and regular networks. While λ_2 is higher for the regular networks than the random and scalefree networks. Intitutively we can relate this behaviour with the degree of freedom of the network. Thus, we say that regular networks has more degree of freedom than the random and the scalefree networks although k_{max2} of scalefree networks is higher than regular networks. Additionaly, we know that λ_2 has a relation with the algebraic connectivity for a regular networks. Using this relation we have indicated that there lies a drastic change in the algebraic connectivity of the regular networks for the higher average degree which intern leads to a decrease in λ_2 of regular networks for higher average degree. There exists no such drastic change in the algebraic connectivity for scale free networks and λ_2 remains almost constant for scalefree networks. Since for a regular networks λ_1 is equal to average connectivity so we can not get much information about the network from λ_1 and thus λ_2 become more important for the regular networks than scalefree and random networks. Further as inhibitory nodes are introduces in the networks we analysed the change in behaviour of λ_2. We have illustrated that although λ_1 and λ_2 both have similar behaviour as a behaviour of inhibitory couplings of there statistics are different from each other. While λ_1 follows normal distribution for $p_{in} = 0$, λ_2 shows weibull distribution of GEV statistics. There is also phase transition of λ_2 for lower average degree but no such transition is observed for λ_2 but for the higher average degree λ_2 shows transition while statistics of λ_1 remain unchanged. Thus, what we understanding from our work is that both λ_1 and λ_2 have completely different behaviour. λ_1 is more related to the macroscopic structural property of the network while λ_2 provide mere information about detailed topology of the networks. In the future scope I would like to go more insight in the statistics of eigen values.

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