LOCALIZATION THEOREMS AND PERTURBATION ANALYSIS ON QUATERNIONIC EIGENVALUE PROBLEMS

Ph.D. Thesis

by

ISTKHAR ALI

DISCIPLINE OF MATHEMATICS

INDIAN INSTITUTE OF TECHNOLOGY INDORE

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LOCALIZATION THEOREMS AND PERTURBATION ANALYSIS ON QUATERNIONIC EIGENVALUE PROBLEMS

A THESIS

Submitted in partial fulfillment of the

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by

ISTKHAR ALI

DISCIPLINE OF MATHEMATICS

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INDIAN INSTITUTE OF TECHNOLOGY INDORE
CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **"LOCALIZATION THEOREMS AND PERTURBATION ANALYSIS ON QUATERNIONIC EIGENVALUE PROBLEMS"** in the partial fulfillment of the requirements for the award of the degree of **DOCTOR OF PHILOSOPHY** and submitted in the **DISCIPLINE OF MATHEMATICS, Indian Institute of Technology Indore,** is an authentic record of my own work carried out during the time period from January 2011 to August 2016 under the supervision of Dr. Sk. Safique Ahmad, Assistant Professor, Indian Institute of Technology Indore.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other institute.

Signature of the student with date
(ISTKHAR ALI)

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This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

Signature of Thesis Supervisor with date
(Dr. SK. SAFIQUE AHMAD)

- -

ISTKHAR ALI has successfully given his Ph.D. Oral Examination held on **November 20, 2017**.

Sign. of Chairperson (OEB) Date:	Sign. of External Examiner Date:	Sign. of Thesis Supervisor Date:
Sign. of PSPC Member #1 Date:	Sign. of PSPC Member #2 Date:	Sign. of Convener, DPGC Date:
Sign. of Head of Discipline Date:		

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List of Publications

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ABSTRACT

Quaternions are extensively used in programming video games, computer graphics, control theory and quantum physics, etc. A solution of a quaternionic linear differential equation with constant coefficients can be presented in terms of right eigenvalues as well as their corresponding eigenvectors of the associated quaternionic matrix. The study of a quaternionic linear differential equation with constant coefficients is based on finding the zeros of its corresponding quaternionic polynomial. In contrast to the complex case, the location of left eigenvalues of a quaternionic matrix plays an important role in the characterization of zeros of quaternionic polynomials. The stability of linear difference/differential equations with quaternionic matrix coefficients is based on the location of right eigenvalues of their corresponding block companion matrices.

This thesis mainly deals with localization theorems for the left and right eigenvalues of a quaternionic matrix and their applications for finding bounds/location of zeros of quaternionic polynomials. Bounds for the left and right eigenvalues of quaternionic matrix polynomials are derived. In the proposed research work we also discuss about perturbation bounds for right eigenvalues/generalized right eigenvalues of a quaternionic matrix/quaternionic matrix pencil. The entire work of this thesis is divided into seven chapters and has been briefly described below:

Chapter 1 describes preliminaries and basic facts related to the development of our theory.

In Chapter 2, inclusion regions for eigenvalues of a quaternionic matrix are derived and bounds for the zeros of quaternionic polynomials are presented. In this chapter, we study Gerschgorin, Ostrowski, and Brauer type theorems for the left and right eigenvalues of a quaternionic matrix. Thereafter a sufficient condition for the stability of a continuous-time quaternionic system is given.

Chapter 3 presents inclusion regions of zeros of quaternionic polynomials.

Chapter 4 discusses basic properties of regular quaternionic matrix pencils, localization theorems of generalized right eigenvalues of quaternionic matrix pencils, and their applications.

Chapter 5 derives the definitions of the left and right eigenvalues of quaternionic matrix polynomials. Next, we present bounds of left and right eigenvalues of quaternionic matrix polynomials. A sufficient condition for the stability of a discrete-time quaternionic system is given. Furthermore, bounds for the absolute values of the left and right eigenvalues of quaternionic matrix polynomials are devised and illustrated for the matrix p-norm, where $p = 1, 2, \infty$, and F (Frobenius). The above results generalize bounds for the absolute values of the eigenvalues of complex matrix polynomials.

Chapter 6 gives the concept of perturbation bounds for right eigenvalues/generalized right eigenvalues of a quaternionic matrix/quaternionic matrix pencil. In particular, Bauer-Fike type theorems for right eigenvalues/generalized right eigenvalues of a diagonalizable quaternionic matrix/diagonalizable quaternionic matrix pencil are derived. Then, a relative perturbation bound for right eigenvalues of an invertible diagonalizable quaternionic matrix is given. Perturbation bounds of right eigenvalues of a quaternionic matrix and perturbation bounds for the zeros of quaternionic polynomials are presented.

Finally, in Chapter 7, we give conclusions of our research work and future prospect of this work.

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NOTATION

Symbols

$\mathbb{R}$	the field of real numbers
$\mathbb{C}$	the field of complex numbers
$\mathbb{C}_\infty$	the extended complex plane $\mathbb{C} \cup \{\infty\}$
$\mathbb{C}^+$	closed upper complex halfplane
$\mathbb{R}^+$	the set of positive real numbers
$\mathbb{H}$	the set of real quaternions
$\Im(z)$	the imaginary part of $z \in \mathbb{C}$
$\overline{q} = q^H$	the conjugate of $q \in \mathbb{H}$
$\lvert q \rvert$	the modulus of $q \in \mathbb{H}$
$\mathrm{Re}(q)$	the real part of $q \in \mathbb{H}$
$\mathrm{Im}(q)$	the imaginary part of $q \in \mathbb{H}$
$S_{\mathbb{H}^-}$	the set $\{q \in \mathbb{H} : \mathrm{Re}(q) < 0\}$
$S_{\mathbb{H}}$	the unit ball $\{q \in \mathbb{H} : \lvert q \rvert < 1\}$
$[q]$	the equivalence class of $q \in \mathbb{H}$
$\mathcal{K}^n$	the collections of all n-column vectors with entries in $\mathcal{K}$, where $\mathcal{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$
x^T	the transpose of $x \in \mathcal{K}^n$
$\overline{x}$	the conjugate of $x \in \mathcal{K}^n$
x^H	the transpose conjugate of $x \in \mathcal{K}^n$
$M_{m\times n}(\mathcal{K})$	the set of $m \times n$ matrices with entries in $\mathcal{K}$, abbreviated to $M_n(\mathcal{K})$ if $m = n$
I_n	the $n \times n$ identity matrix
0_n	the $n \times n$ zero matrix
A^{-1}	the inverse of $A \in M_n(\mathcal{K})$
A^T	the transpose of $A \in M_{m\times n}(\mathcal{K})$

$\overline{A}$	the conjugate of $A \in M_{m\times n}(\mathcal{K})$
A^H	the conjugate transpose of $A \in M_{m\times n}(\mathcal{K})$
$A \otimes B$	the Kronecker product of $A \in M_n(\mathbb{H})$ with $B \in M_n(\mathbb{H})$
$\mathrm{diag}(A_1, \dots, A_k)$	block diagonal matrix with the diagonal blocks $A_1, \dots, A_k \in M_n(\mathcal{K})$, abbreviated to $\mathrm{diag}(A_j)_{j=1}^k$
$\\|A\\|_p$	the norm of $A \in M_n(\mathcal{K})$, where $p = 1, 2, \infty$ and F (Frobenius)
$K_2(A)$	the condition number of $A \in M_n(\mathbb{H})$ with respect to the matrix 2-norm
Ψ_A	the complex adjoint matrix of $A \in M_n(\mathbb{H})$
$\Lambda(A)$	the set of eigenvalues of $A \in M_n(\mathbb{C})$
$\Lambda_l(A)$	the set of left eigenvalues of $A \in M_n(\mathbb{H})$
$\Lambda_r(A)$	the set of right eigenvalues of $A \in M_n(\mathbb{H})$
$\Lambda_s(A)$	the set of standard right eigenvalues of $A \in M_n(\mathbb{H})$
$\rho_l(A)$	the left spectral radius of $A \in M_n(\mathbb{H})$
$\rho_r(A)$	the right spectral radius of $A \in M_n(\mathbb{H})$
$p_l(z)$	the simple quaternionic polynomial of the form $\sum_{j=0}^m q_j z^j$, where z, $q_j \in \mathbb{H}$ $(0 \le j \le m)$
$p_r(z)$	the simple quaternionic polynomial of the form $\sum_{j=0}^m z^j q_j$, where z, $q_j \in \mathbb{H}$ $(0 \le j \le m)$
$q_l(z)$	the simple monic reversal quaternionic polynomial of $p_l(z)$
$q_r(z)$	the simple monic reversal quaternionic polynomial of $p_r(z)$
$Z_{\mathbb{H}}(p(z))$	the set of zeros of a simple quaternionic polynomial $p(z)$
$Z_{\mathbb{C}}(p(z))$	the set of complex zeros of a simple quaternionic polynomial $p(z)$
C_{p_l}	the corresponding companion matrix of the simple monic polynomial $p_l(z)$
C_{p_r}	the corresponding companion matrix of the simple monic polynomial $p_r(z)$
C_{q_l}	the corresponding companion matrix of the simple monic reversal polynomial $q_l(z)$
C_{q_r}	the corresponding companion matrix of the simple monic reversal polynomial $q_r(z)$

CHAPTER 1

INTRODUCTION

1.1. Introduction

The set of quaternions was discovered by the Irish mathematician Sir William Rowan Hamilton in 1843. It is an extension of the complex field. The set of quaternions is an associative but non-commutative algebra of rank four over $\mathbb{R}$. Quaternions are extensively used in programming video games, control theory, computer graphics, controlling spacecrafts, signal processing, quantum physics etc. (see, for example, [1,8,24,26,40,58]). Quaternions are used for describing rotations and orientations of objects in 3-dimensional space. For instance, spacecraft altitude-control systems are commanded in terms of quaternions (see, for example, [1,26] and the references therein). A quaternion is expressed by 4 real numbers whereas a 3×3 matrix requires 9 numbers. The composition of two rotations requires 16 multiplications and 12 additions in quaternion representation, but 27 multiplications and 18 additions in matrix representation [39].

The applications of complex eigenvalue problems have been studied in [4,7,10,11,14, 16,51,54,57]. The quaternionic eigenvalue problems occur in many scientific applications (see, for example, [2,3,9,13,18,27,31,32,43,44,52,53,55,56,61] and the references therein). The location of right eigenvalues of a quaternionic matrix plays an important role to find the stability of the linear differential equation with quaternionic matrix coefficients

$$\frac{\mathrm{d^m}}{\mathrm{dt^m}}\, w(t) + A_{m-1}\, \frac{\mathrm{d^{m-1}}}{\mathrm{dt^{m-1}}}\, w(t) + A_{m-2}\, \frac{\mathrm{d^{m-2}}}{\mathrm{dt^{m-2}}}\, w(t) + \cdots + A_0\, w(t) = 0, \tag{1.1}$$

and the quaternionic matrix difference equation

$$w(t+m) + A_{m-1}w(t+m-1) + \cdots + A_1 w(t+1) + A_0 w(t) = 0, \tag{1.2}$$

where $w(t) = (w_1(t), w_2(t), \ldots, w_n(t))^T \in \mathbb{H}^n$ with $w_i : \mathbb{R} \to \mathbb{H}$ $(t \in \mathbb{R}, 1 \leq i \leq n)$ ($\mathbb{H}$ and $\mathbb{H}^n$ are defined in Preliminaries 1.2) (see, for example, [43,44]). The location of left eigenvalues of a quaternionic matrix plays a crucial role to find bounds/location of zeros of quaternionic polynomials.

The quaternionic eigenvalue problems and location of left and right eigenvalues of a quaternionic matrix and their applications have been studied extensively (see, for example, [3, 6, 9, 18, 22, 27, 31, 32, 43, 44, 52, 60–63] and the references therein). Computation of zeros of quaternionic polynomials and their bounds have been studied extensively covering the theory as well as the applications (see, for example, [12, 19, 20, 23, 30, 41, 42, 45, 50]). An application of quaternionic quadratic equations is also found in solving homogeneous quaternionic linear second order differential equations with quaternion constant coefficients [28, 29]. The solution of quaternionic differential equation

$$\frac{\mathrm{d}^2}{\mathrm{dt}^2}\, u(t) - q_1\, \frac{\mathrm{d}}{\mathrm{dt}}\, u(t) - q_0\, u(t) = 0, \tag{1.3}$$

where $u : \mathbb{R} \to \mathbb{H}$, $q_0, q_1 \in \mathbb{H}$ and $t \in \mathbb{R}$, can be transformed by substituting $u(t) = \exp[\mathrm{z\,t}]\,(\mathrm{z} \in \mathbb{H})$ to the following quaternionic quadratic equation:

$$z^2 - q_1 z - q_0 = 0. \tag{1.4}$$

Generalizing (1.3), we observe that the problem of finding the solution of m-order linear differential equations with quaternion constant coefficients

$$\frac{\mathrm{d}^{\mathrm{m}}}{\mathrm{dt}^{\mathrm{m}}}\, u(t) - \sum_{k=1}^{m-1} q_k\, \frac{\mathrm{d}^k}{\mathrm{dt}^k}\, u(t) - q_0\, u(t) = 0, \tag{1.5}$$

where $q_j \in \mathbb{H}$, $(0 \le j \le m-1)$, can be transformed to the problem of finding the zeros of the corresponding m-order quaternionic polynomial,

$$P_l(z) = z^m - \sum_{j=0}^{m-1} q_j z^j. \tag{1.6}$$

To understand of the complexity of the quaternionic left and right eigenvalue problems is a challenge for mathematicians and physicists. Nowadays, the study of the eigenvalue problem for complex linear quaternionic operators [31] plays a fundamental role in solving quaternionic differential equations [28].

Many physical problems that are governed by differential operators are much simplified by applying the quaternionic matrix formalism and solving the corresponding quaternionic right eigenvalue problem. For example, the solutions of linear differential equations with quaternion constant coefficients

$$\frac{\mathrm{d}^{\mathrm{m}}}{\mathrm{dt}^{\mathrm{m}}}\, u(t) - q_{m-1}\, \frac{\mathrm{d}^{\mathrm{m}-1}}{\mathrm{dt}^{\mathrm{m}-1}}\, u(t) - q_{m-2}\, \frac{\mathrm{d}^{\mathrm{m}-2}}{\mathrm{dt}^{\mathrm{m}-2}}\, u(t) - \cdots - q_0\, u(t) = 0, \tag{1.7}$$

where $q_j \in \mathbb{H}$ $(0 \leq j \leq m-1)$, can be presented in terms of right eigenvalues and eigenvectors of the quaternionic matrix

$$\begin{bmatrix} q_{m-1} & q_{m-2} & \cdots & q_0 \\ 1 & 0 & \dots & 0 \\ . & . & \dots & . \\ . & . & \dots & . \\ 0 & \dots & 1 & 0 \end{bmatrix}.$$

This thesis mainly deals with localization results of eigenvalues of a quaternionic matrix, bounds for eigenvalues of quaternionic matrix polynomials, bounds of zeros of quaternionic polynomials, and perturbation bounds for right eigenvalues of a quaternionic matrix/quaternionic matrix pencil. We developed localization theorems for the left and right eigenvalues of a quaternionic matrix which include the Gerschgorin, Ostrowski, and Brauer type theorems for the left and right eigenvalues of a quaternionic matrix. We proposed inclusion regions for right eigenvalues of special matrices, viz., central closed quaternionic matrices, quaternionic Hermitian matrices, and quaternionic η-Hermitian matrices, where $\eta \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the standard quaternion imaginary units. A sufficient condition for the stability of a continuous-time quaternionic system is given. Bounds/location of zeros of quaternionic polynomials are proposed.

Thereafter inclusion regions for generalized right eigenvalues of a quaternionic matrix pencil and their applications are presented.

Next, the definitions of left and right eigenvalues of quaternionic matrix polynomials are proposed. We present bounds for the left and right eigenvalues of quaternionic matrix polynomials via localization theorems for left and right eigenvalues of a quaternionic matrix/quaternionic block matrix. Further, a sufficient condition for the stability of a discrete-time quaternionic system is given. Bounds for the absolute values of the left and right eigenvalues of quaternionic matrix polynomials are derived and illustrated for the quaternionic matrix p-norm, where $p = 1, 2, \infty$, and F (Frobenius). The above results generalize bounds for the absolute values of eigenvalues of complex matrix polynomials which give sharper bounds to the existing bounds for the case of $1, 2$, and ∞ matrix norms.

Finally, the concept of perturbation bounds for right eigenvalues/generalized right eigenvalues of a quaternionic matrix/quaternionic matrix pencil is developed. In particular, Bauer-Fike type theorems for right eigenvalues/generalized right eigenvalues of a diagonalizable quaternionic matrix/diagonalizable quaternionic matrix pencil are derived. Moreover, a relative perturbation bound for right eigenvalues of an invertible diagonalizable quaternionic matrix is discussed. A residual bound for right eigenvalues of a quaternionic Hermitian matrix is also considered. Perturbation bounds for zeros of quaternionic polynomials are presented.

This thesis is organized as follows. Chapter 2 derives location of the left and right eigenvalues of quaternionic matrices and bounds of zeros of quaternionic polynomials. Chapter 3 explores inclusion regions for zeros of quaternionic polynomials. Chapter 4 concerns localization theorems for generalized right eigenvalues of quaternionic matrix pencils and their applications. Chapter 5 is devoted to bounds of eigenvalues of quaternionic matrix polynomials. Perturbation analysis for quaternionic matrices and quaternionic polynomials are presented in Chapter 6. Finally, in Chapter 7, we give conclusions of this thesis and future prospect of our work.

1.2. Preliminaries

Throughout this thesis, $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers, respectively. We define the closed upper complex halfplane as

$$\mathbb{C}^+ := \{\alpha + \beta i : \alpha, \beta \in \mathbb{R},\ \beta \geq 0\}.$$

Define

$$\mathbb{R}^+ := \{\alpha : \alpha \in \mathbb{R}, \alpha > 0\}.$$

The set of real quaternions is defined by

$$\mathbb{H} := \{q = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} : a_0, a_1, a_2, a_3 \in \mathbb{R}\}$$

with $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$. This relation implies

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}.$$

The conjugate and the modulus of $q = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ are defined as

$$\overline{q} = q^H := a_0 - a_1\mathbf{i} - a_2\mathbf{j} - a_3\mathbf{k} \text{ and } |q| := \sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2},$$

respectively.

$\Im(a)$ denotes the imaginary part of $a \in \mathbb{C}$. The real and imaginary parts of a quaternion $q = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ are defined as

$$\text{Re}(q) := a_0 \text{ and } \text{Im}(q) := a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k},$$

respectively.

The stability region of a continuous-time quaternionic system is defined as

$$S_{\mathbb{H}^-} := \{q \in \mathbb{H} : \text{Re}(q) < 0\}. \tag{1.8}$$

Similarly, the stability region of a discrete-time quaternionic system is defined as

$$S_{\mathbb{H}} := \{q \in \mathbb{H} : |q| < 1\}. \tag{1.9}$$

Let $p, q \in \mathbb{H}$. Then p and q are said to be similar, denoted by $p \sim q$, if

$$p \sim q \Leftrightarrow \exists\, 0 \neq r \in \mathbb{H} \text{ such that } p = r^{-1}qr. \tag{1.10}$$

The set

$$[p] := \{u \in \mathbb{H} : u = \rho^{-1}\, p\, \rho \; for \; all \; 0 \neq \rho \in \mathbb{H}\} \tag{1.11}$$

is called an equivalence class of $p \in \mathbb{H}$.

It is known [61, Theorem 2.2] that

$$p \sim q \Leftrightarrow \text{Re}(p) = \text{Re}(q) \text{ and } |p| = |q|. \tag{1.12}$$

From (1.10), (1.11) and (1.12), $[p]$ can be written as

$$[p] := \{x \in \mathbb{H} : \text{Re}(x) = \text{Re}(p),\ |x| = |p|\}. \tag{1.13}$$

From (1.13), we have

$$\overline{p} \in [p].$$

The collections of all n-column vectors with entries in $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ are denoted by $\mathbb{R}^n$, $\mathbb{C}^n$ and $\mathbb{H}^n$, respectively.

For $x \in \mathcal{K}^n$, where $\mathcal{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, the transpose of x is x^T. If $x = [x_1, \ldots, x_n]^T$, the conjugate of x is defined as $\overline{x} := [\overline{x_1}, \ldots, \overline{x_n}]^T$ and the conjugate transpose of x is defined as $x^H := [\overline{x_1}, \ldots, \overline{x_n}]$.

Some elementary properties of the algebra of quaternions are listed below.

Proposition 1.1. *Let $p, q, r \in \mathbb{H}$. Then*

1. $pp^H = pp^H$*;*
2. $|p| = |p^H|$*;*
3. $|.|$ *is a norm on* $\mathbb{H}$*, i.e., for all* $p, q \in \mathbb{H}$ *we have:*

$$\begin{aligned} |p| &\geq 0 \text{ with equality if and only if } p = 0; \\ |p+q| &\leq |p| + |q|; \\ |pq| &= |qp| = |p||q|; \end{aligned}$$

4. $\mathbf{j}c\mathbf{j}^H = \mathbf{k}c\mathbf{k}^H = \bar{c}$ *for every* $c \in \mathbb{C}$*;*
5. $(pq)^H = q^H p^H$*;*
6. $p = p^H$ *if and only if* $p \in \mathbb{R}$*;*
7. $\alpha p = p\alpha$ *for every* $p \in \mathbb{H}$ *if and only if* $\alpha \in \mathbb{R}$*;*
8. *every* $p \in \mathbb{H} \setminus \{0\}$ *has an inverse* $p^{-1} = \frac{p^H}{|p|^2} \in \mathbb{H}$*; in more detail,*

$$p \times \left(\frac{p^H}{|p|^2}\right) = \left(\frac{p^H}{|p|^2}\right) \times p = 1;$$

9. $(pq)r = p(qr)$*;*
10. *in general,* $(p+q)^2 \neq p^2 + 2pq + q^2$*;*
11. $p^2 = -1$ *has infinitely many solutions over* $\mathbb{H}$*;*
12. *for every* $p \in \mathbb{H}$*,* p *can be uniquely written as* $p = c_1 + c_2\mathbf{j}$*, where* $c_1, c_2 \in \mathbb{C}$*.*

We now briefly consider vector norm to be used in the subsequent development.

Definition 1.2. A function $\|\cdot\| : \mathbb{H}^n \to \mathbb{R}$ is said to be a quaternion norm on $\mathbb{H}^n$ (or a quaternion vector norm) if $\|\cdot\|$ satisfies the following conditions:

- $\|y\| = 0 \Leftrightarrow y = 0$.
- $\|\alpha y\| = |\alpha|\|y\|$ for $\alpha \in \mathbb{H}$ and $y \in \mathbb{H}^n$.
- $\|y + z\| \leq \|y\| + \|z\|$ for $y, z \in \mathbb{H}^n$.

For $y, z \in \mathbb{H}^n$, define $\langle y, z\rangle := z^H y$ as an inner product and $\|y\|_2 := \sqrt{\langle y, y\rangle}$, the norm on $\mathbb{H}^n$. For $y \in \mathbb{H}^n$, the vector p-norm on $\mathbb{H}^n$ is defined as

$$\|y\|_p := \begin{cases} (\sum_{j=1}^n |y_j|^p)^{1/p}, & \text{for } 1 \leq p < \infty, \\ \max_{1\leq j\leq n} |y_j|, & \text{for } p = \infty. \end{cases}$$

1.2.1. Quaternionic matrices

The sets of $m \times n$ real, complex, and quaternionic matrices are denoted by $M_{m\times n}(\mathbb{R})$, $M_{m\times n}(\mathbb{C})$, and $M_{m\times n}(\mathbb{H})$, respectively. These sets are simply denoted by $M_n(\mathcal{K})$, $\mathcal{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, when $m = n$. I_n denotes the $n \times n$ identity matrix. For $A = (a_{ij}) \in M_{m\times n}(\mathcal{K})$, the conjugate, transpose and conjugate transpose of A are defined as $\overline{A} = (\overline{a_{ij}})$, $A^T = (a_{ji}) \in M_{n\times m}(\mathcal{K})$, and $A^H = (\overline{A})^T \in M_{n\times m}(\mathcal{K})$, respectively. Let $A_j \in M_n(\mathcal{K})$ $(1 \leq j \leq k)$. Then, we denote the block diagonal matrix by $\operatorname{diag}(A_1, A_2, \ldots, A_k)$, or by $\operatorname{diag}(A_j)$. For $\mu_j \in \mathbb{H}$ $(1 \leq j \leq n)$, define

$$\operatorname{diag}(\mu_j) := \operatorname{diag}(\mu_1, \mu_2, \ldots, \mu_n).$$

The Jordan block of size m associated with $\lambda \in \mathbb{H}$ is defined as

$$J_m(\lambda) := \begin{bmatrix} \lambda & 1 & 0 & \ldots & 0 \\ 0 & \lambda & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \lambda & 1 \\ 0 & 0 & \ldots & 0 & \lambda \end{bmatrix} \in M_m(\mathbb{H}), \quad \lambda \in \mathbb{H}. \tag{1.14}$$

For $A := (a_{ij}) \in M_n(\mathcal{K})$, the deleted absolute row and column sums of A are defined as

$$r_i(A) := \sum_{j=1,\, j\neq i}^{n} |a_{ij}| \text{ and } c_i(A) := \sum_{j=1,\, j\neq i}^{n} |a_{ji}| \qquad (1 \leq i \leq n), \tag{1.15}$$

respectively. Similarly, the absolute row and column sums of A are defined by

$$r'_i(A) := r_i(A) + |a_{ii}| \text{ and } c'_i(A) := c_i(A) + |a_{ii}| \qquad (1 \leq i \leq n), \tag{1.16}$$

respectively. Let $A, B \in M_{m\times n}(\mathbb{H})$. Then we have the following properties:

- $\alpha A = A\alpha$, for all $\alpha \in \mathbb{R}$.
- $(\alpha A + \beta B)^H = A^H\alpha^H + B^H\beta^H$, for all $\alpha, \beta \in \mathbb{H}$.
- $(A\alpha + B\beta)^H = \alpha^H A^H + \beta^H B^H$, for all $\alpha, \beta \in \mathbb{H}$.

- $(A^H)^H = A$.

Let $A := (a_{ij}) \in M_n(\mathcal{K})$ be partitioned into $k \times k$ real/complex/quaternionic blocks

$$A := (A_{rs}) := \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1k} \\ A_{21} & A_{22} & \dots & A_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ A_{k1} & A_{k2} & \dots & A_{kk} \end{bmatrix}_{n\times n}, \tag{1.17}$$

where $A_{r,s} \in M_{n_r \times n_s}(\mathcal{K})$ $(1 \le r, s \le k)$ is the (r, s) block of A such that $n_1 + \cdots + n_k = n$.

Definition 1.3. A matrix $A \in M_n(\mathbb{H})$ is said to be invertible (nonsingular) if there exists $B \in M_n(\mathbb{H})$ such that $AB = BA = I_n$, where I_n is the $n \times n$ identity matrix. If A is not invertible, then A is said to be singular.

Definition 1.4. Let $A \in M_n(\mathbb{H})$. Then the matrix A is said to be nilpotent if $A^k = 0$ for a least exponent k, where k is a positive integer and 0_n is the $n \times n$ zero matrix.

Definition 1.5. A function $\|.\| : M_{m\times n}(\mathbb{H}) \to \mathbb{R}$ is a quaternionic matrix norm on $M_{m\times n}(\mathbb{H})$ if it satisfies the following conditions:

- definiteness, $E \neq 0 \Rightarrow \|E\| \geq 0$;
- homogeneity, $\|\alpha E\| = |\alpha| \|E\|$;
- the triangle inequality, $\|E + F\| \leq \|E\| + \|F\|$,

where $E, F \in M_{m\times n}(\mathbb{H})$ are arbitrary matrices and $\alpha \in \mathbb{H}$.

For $A \in M_n(\mathcal{K})$, the 1-norm, ∞-norm, 2-norm (operator norm) and the Frobenius norm of $A \in M_n(\mathcal{K})$ are defined as

$$\begin{aligned} \|A\|_1 &:= \max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}| = \|A^H\|_\infty, \quad \|A\|_\infty := \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}| = \|A^H\|_1, \\ \|A\|_2 &:= \sup_{x\neq 0} \left\{ \frac{\|Ax\|_2}{\|x\|_2} : x \in \mathcal{K}^n \right\} = \|A^H\|_2, \text{ and } \|A\|_F := \left[\text{trace}\left(A^H A\right)\right]^{1/2}, \end{aligned}$$

respectively.

Definition 1.6. Let $A \in M_n(\mathbb{H})$ be an invertible matrix. Then the condition number of A with respect to the matrix 2-norm is defined as

$$K_2(A) := \|A\|_2 \, \|A^{-1}\|_2.$$

Definition 1.7. Let $A := (a_{ij}) \in M_n(\mathbb{H})$. Then A is said to be

1. Hermitian if and only if $A^H = A$;
2. normal if and only if $A^H A = AA^H$;
3. unitary if and only if $A^H A = AA^H = I_n$;
4. upper triangular if and only if $a_{ij} = 0$ for all $j < i$;
5. lower triangular if and only if $a_{ij} = 0$ for all $j > i$;
6. η-Hermitian if and only if $A^{\eta H} = A$, where $A^{\eta H} = -\eta A^H \eta$ and $\eta \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$;
7. η-anti-Hermitian if and only if $A^{\eta H} = -A$, where $A^{\eta H} = -\eta A^H \eta$ and $\eta \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$;
8. positive semidefinite if A is Hermitian matrix and $y^H Ay \geq 0$ for all nonzero vector $y \in \mathbb{H}^n$;
9. positive definite if A is Hermitian matrix and $y^H Ay > 0$ for all nonzero vector $y \in \mathbb{H}^n$.

Definition 1.8. Let $A \in M_n(\mathbb{H})$. Then A is said to be a central closed matrix if there exists an invertible matrix T such that

$$T^{-1}AT = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n),$$

where $\lambda_i \in \mathbb{R}$ $(1 \leq i \leq n)$.

Some properties of matrices over the complex field do not hold over the skew field of quaternions. In general, for $A \in M_{m\times n}(\mathbb{H})$ and $B \in M_{n\times p}(\mathbb{H})$, $\overline{AB} \neq \overline{A}\,\overline{B}$. We have the following facts for quaternionic matrices.

Proposition 1.9. *[61] Let $A \in M_{m\times n}(\mathbb{H})$ and let $B \in M_{n\times p}(\mathbb{H})$. Then*

1. $\left(\overline{A}\right)^T = \overline{A^T}$*;*
2. $(AB)^H = B^H A^H$*;*
3. $(AB)^{-1} = B^{-1}A^{-1}$ *if A and B are invertible;*
4. $(A^H)^{-1} = (A^{-1})^H$ *if A is invertible;*
5. *in general, $(AB)^T \neq B^T A^T$;*
6. *in general, $(\overline{A})^{-1} \neq \overline{(A^{-1})}$;*
7. *in general, $(A^T)^{-1} \neq (A^{-1})^T$.*

1.2.2. Complex adjoint matrix

Definition 1.10. Let $x \in \mathbb{H}^n$. Then x can be uniquely expressed as $x = x_1 + x_2\mathbf{j}$, where $x_1, x_2 \in \mathbb{C}^n$. Define a function $\psi : \mathbb{H}^n \to \mathbb{C}^{2n}$ by

$$\psi_x := \begin{bmatrix} x_1 \\ -\overline{x_2} \end{bmatrix}.$$

The vector ψ_x is called the complex adjoint vector of x. This function ψ is an injective linear transformation from $\mathbb{H}^n$ to $\mathbb{C}^{2n}$.

Definition 1.11. Let $A \in M_n(\mathbb{H})$. Then A can be uniquely expressed as $A = A_1 + A_2\mathbf{j}$, where $A_1, A_2 \in M_n(\mathbb{C})$. Define a function $\Psi : M_n(\mathbb{H}) \to M_{2n}(\mathbb{C})$ by

$$\Psi_A := \begin{bmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{bmatrix}. \tag{1.18}$$

The matrix Ψ_A is called the complex adjoint matrix of A. This function Ψ is an injective H-homomorphism.

Theorem 1.12. *[61] Let $A, B \in M_n(\mathbb{H})$ and let Ψ be the function which is defined in* (1.18). *Then*

1. $\Psi_{I_n} = I_{2n}$;
2. $\Psi_{AB} = \Psi_A\Psi_B$;
3. $\Psi_{\alpha A} = \alpha\Psi_A$, *where* $\alpha \in \mathbb{R}$;
4. $\Psi_{A+B} = \Psi_A + \Psi_B$;
5. $\Psi_{A^H} = (\Psi_A)^H$;
6. $\Psi_{A^{-1}} = (\Psi_A)^{-1}$ *if* A^{-1} *exists;*
7. Ψ_A *is unitary, invertible, diagonalizable, Hermitian or normal if and only if* A *is unitary, invertible, diagonalizable, Hermitian or normal, respectively.*

Lemma 1.13. *[34] Let $A \in M_n(\mathbb{H})$. Then we have*

$$\|A\|_2 := \max_{\|x\|_2 \neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \max_{\|y\|_2 \neq 0} \frac{\|\Psi_A y\|_2}{\|y\|_2} =: \|\Psi_A\|_2,$$

where $\|x\|_2 = \left(\sum_{i=1}^{n} \overline{x_i} x_i \right)^{\frac{1}{2}}$ *on* $\mathbb{H}^n$ *and* $\|y\|_2 = \left(\sum_{i=1}^{2n} \overline{y_i} y_i \right)^{\frac{1}{2}}$ *on* $\mathbb{C}^{2n}$.

1.2.3. The eigenvalue problem

We start this subsection with the definition of an eigenvalue of a complex matrix.

Definition 1.14. Let $A \in M_n(\mathbb{C})$ and let $\lambda \in \mathbb{C}$. Then λ is said to be an eigenvalue of A if $Ax = \lambda x$ for some nonzero $x \in \mathbb{C}^n$. The set

$$\Lambda(A) := \{\lambda \in \mathbb{C} : Ax = \lambda x \text{ for some nonzero } x \in \mathbb{C}^n\}$$

is called the complex spectrum of A.

For example, let $A = \begin{bmatrix} \mathbf{i} & 2+\mathbf{i} \\ 0 & 3\mathbf{i} \end{bmatrix}$. Then we have $\Lambda(A) = \{\mathbf{i}, 3\mathbf{i}\}$.

Unlike the complex case, there are two types of eigenvalues for quaternionic matrices, namely left and right eigenvalues which are defined as follows.

Definition 1.15. Let $A \in M_n(\mathbb{H})$ and let $\lambda \in \mathbb{H}$. Then λ is said to be a left eigenvalue of A if $Ay = \lambda y$ for some nonzero $y \in \mathbb{H}^n$. The set

$$\Lambda_l(A) := \{\lambda \in \mathbb{H} : Ay = \lambda y \text{ for some nonzero } y \in \mathbb{H}^n\}$$

is called the left spectrum of A.
Similarly, λ is said to be a right eigenvalue of A if $Ay = y\lambda$ for some nonzero $y \in \mathbb{H}^n$. The set

$$\Lambda_r(A) := \{\lambda \in \mathbb{H} : Ay = y\lambda \text{ for some nonzero } y \in \mathbb{H}^n\}$$

is called the right spectrum of A.

Due to the commutativity of quaternions with real numbers, real right eigenvalues of $A \in M_n(\mathbb{H})$ are also left eigenvalues of A and vice versa. Thus, the quaternionic matrix A may have at most n real right eigenvalues. Hence the quaternionic matrix A may have at most n real left eigenvalues.

It is known that every quaternionic matrix $A \in M_n(\mathbb{H})$ has exactly n complex right eigenvalues which are contained in $\mathbb{C}^+$. These are called the standard right eigenvalues of A.

Definition 1.16. Let $A \in M_n(\mathbb{H})$ and let $\lambda \in \mathbb{C}$. Then λ ($\Im(\lambda) \geq 0$) is said to be a standard right eigenvalue of A if $Ay = y\lambda$ for some nonzero $y \in \mathbb{H}^n$. The set

$$\Lambda_s(A) := \{\lambda \in \mathbb{C} : Ay = y\lambda \text{ for some nonzero } y \in \mathbb{H}^n,\ \Im(\lambda) \geq 0\}$$

is called the standard right spectrum of A.

Definition 1.17. Let $A \in M_n(\mathbb{H})$. Then the left spectral radius and the right spectral radius of A are given as

$$\rho_l(A) := \max\{|\lambda| : \lambda \in \Lambda_l(A)\}, \text{ and } \rho_r(A) := \max\{|\lambda| : \lambda \in \Lambda_r(A)\},$$

respectively.

By applying the determinant of a complex matrix and the complex adjoint matrix of a quaternionic matrix, we present the Cayley-Hamilton theorem for the quaternion case.

Theorem 1.18. *[61] Let $A \in M_n(\mathbb{H})$ and $P_A(\lambda) = \det(\lambda I_{2n} - \Psi_A)$, called the characteristic polynomial of A, where λ is a complex indeterminant. Then $P_A(A) = 0$ and $P_A(\lambda_0) = 0$ if and only if λ_0 is a right eigenvalue.*

Proposition 1.19. *[62] Let $A \in M_n(\mathbb{H})$ and let $\lambda \in \mathbb{H}$. Then λ is a left eigenvalue of A if and only if*

$$\det\left[\Psi_{(A-\lambda I_n)}\right] = 0.$$

We have the following relation between left and right eigenvalues of a square real matrix.

Theorem 1.20. *[61] Let A be a square real matrix. Then the left and right eigenvalues of A are same, i.e., $\Lambda_l(A) = \Lambda_r(A)$.*

We can also compute the right eigenvalues of a quaternionic matrix with the help of the standard right eigenvalues of that quaternionic matrix. Let λ_i $(1 \leq i \leq n)$ be the standard right eigenvalues of a matrix $A \in M_n(\mathbb{H})$. Then, the set of right eigenvalues of A is given as

$$\Lambda_r(A) := \cup_{i=1}^{n}[\lambda_i],$$

where $[\lambda_i]$ are equivalence classes of λ_i $(1 \leq i \leq n)$, respectively.

Proposition 1.21. *[27] Let $A \in M_n(\mathbb{H})$. Then A has exactly $2n$ complex right eigenvalues.*

Proposition 1.22. *[61] Let $A \in M_n(\mathbb{H})$. Then A has exactly n complex right eigenvalues which are contained in the closed upper complex halfplane $\mathbb{C}^+$. These right eigenvalues are called the standard right eigenvalues of A.*

Definition 1.23. Let $A \in M_n(\mathbb{H})$. Then A is said to be a diagonalizable matrix if there exists an invertible quaternionic matrix P such that

$$P^{-1}AP = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n),$$

where $\lambda_i \in \Lambda_s(A)$ $(1 \le i \le n)$.

Definition 1.24. Let $A \in M_n(\mathbb{H})$. Then the continuous-time quaternionic system

$$\frac{\mathrm{d}}{\mathrm{dt}}\, w(t) = A\, w(t)$$

is stable if and only if $\Lambda_r(A) \subset S_{\mathbb{H}^-}$ (defined in (1.8)).

Definition 1.25. Let $A \in M_n(\mathbb{H})$. Then the discrete-time quaternionic system

$$w(t+1) = A\, w(t)$$

is asymptotically stable if and only if $\Lambda_r(A) \subset S_{\mathbb{H}}$ (defined in (1.9)).

Next, we present the Jordan canonical form, singular-value decomposition, and Schur decomposition of a quaternionic matrix.

Proposition 1.26. *[59] Let $A \in M_n(\mathbb{H})$. Then there exists an invertible matrix $Y \in M_n(\mathbb{H})$ such that*

$$Y^{-1}AY = \operatorname{diag}(J_{m_1}(\lambda_1), J_{m_2}(\lambda_2), \ldots, J_{m_k}(\lambda_k)), \tag{1.19}$$

where $\lambda_i \in \mathbb{H}, \lambda_i \in \Lambda_s(A)$ $(1 \le i \le k)$ and $J_{m_i}(\lambda_i)$ are the $m_i \times m_i$ Jordan blocks with right eigenvalues λ_i, respectively. Moreover, the right hand side of (1.19) *is uniquely determined by A up to permutation of diagonal blocks, and up to replacement of each λ_j with any similar quaternion μ_j .*

Theorem 1.27. *[61] Let $A \in M_{m\times n}(\mathbb{H})$ be of rank r. Then there are two unitary matrices $U \in M_m(\mathbb{H})$ and $V \in M_n(\mathbb{H})$ such that*

$$U^H AV = \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix},$$

where $\Sigma_r = \operatorname{diag}(\sigma_1, \ldots, \sigma_r)$ and σ_i are the singular values of A.

Theorem 1.28. *[9] Let $A \in M_n(\mathbb{H})$. Then there exist matrices $T, V \in M_n(\mathbb{H})$ such that:*

1. *$V^H AV = T$, where V is an unitary matrix and T is an upper triangular matrix;*

2. *every diagonal entry of T is contained in the closed upper complex halfplane $\mathbb{C}^+$.*

Theorem 1.29. *[9] Let $A \in M_n(\mathbb{H})$ be normal. Then there exist an unitary matrix $V \in M_n(\mathbb{H})$ and a diagonal matrix $D \in M_n(\mathbb{H})$ such that:*

1. $V^H AV = D$;
2. *every diagonal entry of D is contained in the closed upper complex halfplane $\mathbb{C}^+$.*

We also need the following result for the development of our theory.

Proposition 1.30. *[46] Let $A \in M_m(\mathbb{H})$ and let $B \in M_n(\mathbb{H})$. Then the equation*

$$AX = XB, \quad X \in M_{m\times n}(\mathbb{H})$$

has only the trivial solution $X = 0$ if and only if $\Lambda_r(A) \cap \Lambda_r(B) = \emptyset$.

Further, we present some basic known facts on A and Ψ_A for the development of our theory.

Theorem 1.31. *[61] Let $A \in M_n(\mathbb{H})$. Then the following statements are equivalent:*

1. *A is invertible;*
2. *$Ax = 0$ has a unique solution;*
3. *$\det(\Psi_A) \neq 0$, i.e., Ψ_A is invertible;*
4. *A has no zero eigenvalue. More precisely, if $Ax = \lambda x$ or $Ax = x\lambda$ for some $\lambda \in \mathbb{H}$ and some vector $0 \neq x \in \mathbb{H}^n$, then $\lambda \neq 0$;*
5. *A is the product of elementary quaternionic matrices.*

We next derive the following theorem which gives a method for diagonalization of a quaternionic matrix.

Theorem 1.32. *[21] Let $A \in M_n(\mathbb{H})$. Then A is diagonalizable if and only if Ψ_A is diagonalizable. If Ψ_A is diagonalizable, then there exists an invertible matrix $T \in M_{2n}(\mathbb{C})$ such that*

$$T^{-1}\Psi_A T = \begin{bmatrix} D & 0 \\ 0 & \overline{D} \end{bmatrix} = \Psi_D \Leftrightarrow \Psi_A T = T\Psi_D,$$

where $D = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $\lambda_i \in \Lambda_s(A)\,(1 \leq i \leq n)$. Let

$$Y = \frac{1}{4}\begin{bmatrix} I_n & -\mathbf{j}I_n \end{bmatrix}(T + S_n^{-1}\overline{T}S_n)\begin{bmatrix} I_n \\ \mathbf{j}I_n \end{bmatrix},$$

where

$$S_n = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}.$$

Then $Y^{-1}AY = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, *where* Y *is an invertible quaternionic matrix.*

1.2.4. Quaternionic polynomials and their companion matrices

Due to the noncommutativity of quaternions, there are three types of quaternionic polynomials since the coefficients of polynomial can be taken to be on the left, on the right or on both sides of the indeterminant. However, throughout this thesis, we follow the following quaternionic polynomials:

$$p_l(z) := q_m z^m + q_{m-1} z^{m-1} + \cdots + q_1 z + q_0, \tag{1.20}$$

$$p_r(z) := z^m q_m + z^{m-1} q_{m-1} + \cdots + z q_1 + q_0, \tag{1.21}$$

where $q_j,\ z \in \mathbb{H}$, $(0 \leq j \leq m)$. The polynomials (1.20) and (1.21) are called "simple" and "monic" when $q_m = 1$. These polynomials play an important role in quaternion linear algebra since they are connected with linear difference and differential equations with quaternion coefficients.

The set of zeros of a quaternionic polynomial $p(z)$ is denoted by $Z_{\mathbb{H}}(p(z))$. The set of complex zeros of a quaternionic polynomial $p(z)$ is denoted by $Z_{\mathbb{C}}(p(z))$. For example, let us consider the quaternionic polynomial

$$p_l(z) := z^6 + \mathbf{j}z^5 + \mathbf{i}z^4 - z^2 - \mathbf{j}z - \mathbf{i}.$$

Then $Z_{\mathbb{C}}(p_l(z)) = \{1, -1, \mathbf{i}, -\mathbf{i}\}$ and

$$Z_{\mathbb{H}}(p_l(z)) = \{1, -1, [\mathbf{i}], (0.5 - 0.5\mathbf{i} - 0.5\mathbf{j} - 0.5\mathbf{k}), (-0.5 + 0.5\mathbf{i} - 0.5\mathbf{j} - 0.5\mathbf{k})\}.$$

The corresponding companion matrices of the simple monic polynomials $p_l(z)$ and $p_r(z)$ are given by

$$C_{p_l} := \begin{bmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \\ -q_0 & -q_1 & \ldots & -q_{m-1} \end{bmatrix} \text{ and } C_{p_r} := C_{p_l}^T,$$

respectively.

We now define the block forms of companion matrices C_{p_l} and C_{p_r} as follows.

$$C_{p_l} := \left[\begin{array}{c|ccc} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \\ \hline -q_0 & -q_1 & \ldots & -q_{m-1} \end{array}\right] := \begin{array}{c} \\ m-1 \\ 1 \end{array}\!\!\begin{array}{c} \begin{array}{cc} \quad 1 \quad & \quad m-1 \quad \end{array} \\ \left[\begin{array}{c|c} 0 & I \\ \hline C_{p_l}(m,1) & C_{p_l}(m,2:m) \end{array}\right] \end{array} \text{ and } C_{p_r} := C_{p_l}^T,$$

respectively, where $C_{p_l}(m,1) := -q_0$ and $C_{p_l}(m,2:m) := [-q_1 \ldots - q_{m-1}]$.

For $p, q \in \mathbb{H}$, define $p \times q := pq$. For $0 \neq p \in \mathbb{H}$ and $q \in \mathbb{H}$, define

$$\frac{1}{p} \times q := p^{-1} \times q := p^{-1}q, \text{ and } q \times \frac{1}{p} := q \times p^{-1} := qp^{-1}.$$

Consider $q_0 \neq 0$, i.e., $z = 0$ is not a zero of the simple monic polynomials $p_l(z)$ and $p_r(z)$. Define simple monic reversal polynomials of the polynomials $p_l(z)$ and $p_r(z)$ as follows.

$$(1.22) \quad q_l(z) := \frac{1}{q_0} \times p_l\left(\frac{1}{z}\right) \times z^m = z^m + q_0^{-1}q_1 z^{m-1} + \cdots + q_0^{-1}q_{m-1}z + q_0^{-1},$$

$$(1.23) \quad q_r(z) := z^m \times p_r\left(\frac{1}{z}\right) \times \frac{1}{q_0} = z^m + z^{m-1}q_1 q_0^{-1} + \cdots + zq_{m-1}q_0^{-1} + q_0^{-1},$$

respectively. Similarly, the corresponding companion matrices of the simple monic reversal polynomials $q_l(z)$ and $q_r(z)$ are denoted by C_{q_l} and C_{q_r}, respectively. We observe that the zeros of $q_l(z)$ and $q_r(z)$ are reciprocals of the zeros of $p_l(z)$ and $p_r(z)$, respectively.

Remark 1.33. If the simple quaternionic polynomial $p_l(z)$ is not monic, we can always factorize. For example, if $p_l(z) := q_m z^m + q_{m-1}z^{m-1} + \cdots + q_1 z + q_0$ with $q_m \neq 0$, then $p_l(z) = q_m\, p_l'(z)$ and $p_l'(z)$ is a simple monic polynomial.

Remark 1.34. To compute bounds for the zeros of the quaternionic polynomial $p_r(z) = \sum_{j=0}^m z^j q_j$, we define a quaternionic polynomial

$$P_l(z) := \sum_{j=0}^m \overline{q_j} z^j.$$

If z_0 is a zero of $P_l(z)$, then

$$\overline{P_l(z_0)} = \overline{\sum_{j=0}^m \overline{q_j} z^j} = \sum_{j=0}^m \overline{z}^j q_j = p_r(\overline{z_0}) = 0.$$

Thus, $\overline{z_0}$ is a zero of $p_r(z)$ and, therefore, it is sufficient to treat only one type of polynomial, preferably the polynomial $p_l(z)$.

Also, we need the following result for improvement of our theory.

Proposition 1.35. *[50] Let $p_l(z)$ be a simple monic polynomial over $\mathbb{H}$ of degree m. Then the set of zeros of $p_l(z)$ belongs to at most m equivalence classes of quaternions.*

Proposition 1.36. *[50] Let $\lambda \in \mathbb{H}$. Then λ is a zero of the simple monic polynomial $p_l(z)$ if and only if λ is a left eigenvalue of its corresponding companion matrix C_{p_l}.*

In general, a right eigenvalue of C_{p_l} is not necessarily a zero of the simple monic polynomial $p_l(z)$. For example, consider a simple monic polynomial $p_l(z) = z^2 + \mathbf{j}z + 2$. Then its companion matrix is given by

$$C_{p_l} = \begin{bmatrix} 0 & 1 \\ -2 & -\mathbf{j} \end{bmatrix}.$$

Here $\mathbf{i}$ is a right eigenvalue of C_{p_l}. However, $\mathbf{i}$ is not a zero of $p_l(z)$.

Analogous to Proposition 1.36, the following result is presented for $p_r(z)$.

Proposition 1.37. *Let $\lambda \in \mathbb{H}$. Then λ is a zero of the simple monic polynomial $p_r(z)$ if and only if λ is a left eigenvalue of its corresponding companion matrix C_{p_r}.*

Proof. Let λ be a left eigenvalue of C_{p_r}. Then there exists some nonzero $x \in \mathbb{H}^n$ such that $C_{p_r}x = \lambda x$. Let $x := [x_1, \dots, x_m]^T \in \mathbb{H}^n$. Then

$$\begin{bmatrix} 0 & \dots & 0 & -q_0 \\ 1 & & 0 & -q_1 \\ & \ddots & & \\ 0 & & 1 & -q_{m-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}. \tag{1.24}$$

(1.24) gives the following system of linear equations

$$\begin{aligned} -q_0 x_m &= \lambda x_1, \\ x_1 - q_1 x_m &= \lambda x_2, \\ x_2 - q_2 x_m &= \lambda x_3, \\ &\vdots \\ x_{m-1} - q_{m-1} x_m &= \lambda x_m. \end{aligned}$$

By solving the above system of linear equations, we have

$$\left(\lambda^m + \lambda^{m-1} q_{m-1} + \dots + \lambda q_1 + q_0\right) x_m = 0.$$

Since x cannot be the zero quaternionic vector, $x_m \neq 0$. Hence we conclude that

$$\lambda^m + \lambda^{m-1} q_{m-1} + \cdots + \lambda q_1 + q_0 = 0.$$

Thus, the left eigenvalue λ is a zero of $p_r(z)$. ■

Lemma 1.38. *[50] Let $\lambda \in \mathbb{H}$ be a left eigenvalue of C_{p_l}. Then it is also a right eigenvalue of C_{p_l}.*

Unlike the case of complex polynomials, quaternionic polynomials may have infinitely many zeros. For example, let us consider the following polynomial $p_l(z)$ over $\mathbb{H}$:

$$p_l(z) = z^6 + (\mathbf{i} + 3\mathbf{k})z^5 + (3 + \mathbf{j})z^4 + (5\mathbf{i} + 15\mathbf{k})z^3 + (-4 + 5\mathbf{j})z^2 + (6\mathbf{i} + 18\mathbf{k})z + (6\mathbf{j} - 12).$$

The set of zeros of $p_l(z)$ are given in [50] as follows.

$$Z_{\mathbb{H}}(p_l(z)) = \{-\mathbf{i} - 2\mathbf{k}, [\mathbf{i}\sqrt{3}], [\mathbf{i}\sqrt{2}], -0.6\mathbf{i} - 0.8\mathbf{k}\}.$$

We observe that the following relation between $Z_{\mathbb{H}}(p_l(z))$, $\Lambda_l(C_{p_l})$ and $\Lambda_r(C_{p_l})$.

- From Proposition 1.36, we have $Z_{\mathbb{H}}(p_l(z)) = \Lambda_l(C_{p_l})$. For instance, let

  $$p_l(z) := z^4 + 2z^3 - z^2 + 2z + 1.$$

  Then, the set of right eigenvalues of C_{p_l} is given by

  $$\Lambda_r(C_{p_l}) := \{-2.6180, -0.3820, [0.5000 + 0.8660\mathbf{i}]\}.$$

  From Theorem 1.20, we have $\Lambda_l(C_{p_l})) = \Lambda_r(C_{p_l})$. Thus, the set of zeros of $p_l(z)$ is given by

  $$Z_{\mathbb{H}}(p_l(z)) = \{-2.6180, -0.3820, [0.5000 + 0.8660\mathbf{i}]\}.$$

  Hence, $Z_{\mathbb{H}}(p_l(z)) = \Lambda_l(C_{p_l})$.
- Let $\lambda \in Z_{\mathbb{H}}(p_l(z))$. Then from Proposition 1.36, we have $\lambda \in \Lambda_l(C_{p_l})$. From Lemma 1.38, we obtain $\lambda \in \Lambda_r(C_{p_l})$. This implies that $Z_{\mathbb{H}}(p_l(z)) \subseteq \Lambda_r(C_{p_l})$. For example, let us consider the quaternionic polynomial

  $$p_l(z) := z^6 + \mathbf{j}z^5 + \mathbf{i}z^4 - z^2 - \mathbf{j}z - \mathbf{i}.$$

  Then, the set of right eigenvalues of C_{p_l} is given by

  $$\Lambda_r(C_{p_l}) := \{1, -1, [\mathbf{i}], [0.5 + 0.5\mathbf{i}], [-0.5 - 0.5\mathbf{i}]\}.$$

The set of zeros of $p_l(z)$ is given in [19] as follows.

$$Z_{\mathbb{H}}(p_l(z)) = \{1, -1, [\mathbf{i}], (0.5 - 0.5\mathbf{i} - 0.5\mathbf{j} - 0.5\mathbf{k}), (-0.5 + 0.5\mathbf{i} - 0.5\mathbf{j} - 0.5\mathbf{k})\}.$$

Since $0.5 - 0.5\mathbf{i} - 0.5\mathbf{j} - 0.5\mathbf{k}$ is similar to $0.5 + 0.5\mathbf{i}$ and $-0.5 + 0.5\mathbf{i} - 0.5\mathbf{j} - 0.5\mathbf{k}$ is similar to $-0.5 - 0.5\mathbf{i}$, we have

$$Z_{\mathbb{H}}(p_l(z)) \subseteq \Lambda_r(C_{p_l}).$$

- It is never true that $Z_{\mathbb{H}}(p_l(z)) \supset \Lambda_r(C_{p_l})$. From Proposition 1.35, it is known that all the zeros of $p_l(z)$ belong to at most m equivalence classes of standard right eigenvalues of C_{p_l}.

1.2.5. The generalized eigenvalue problem

Let $\mathbb{L}_1'(M_n(\mathbb{H}))$ be the space of matrix pencils over a quaternion division algebra. $\mathbf{L}_1' \in \mathbb{L}_1'(M_n(\mathbb{H}))$ is defined as

$$\mathbf{L}_1'(\lambda) := A + \lambda B, \tag{1.25}$$

where $\lambda \in \mathbb{H}$ and $A, B \in M_n(\mathbb{H})$. Throughout this thesis we consider the following three cases:

Case 1: when $\lambda \in \mathbb{R}$ and $A, B \in M_n(\mathbb{H})$,
Case 2: when $\lambda \in \mathbb{H}$ and $A, B \in M_n(\mathbb{H})$,
Case 3: when $\lambda \in \mathbb{C}$ and $A, B \in M_n(\mathbb{C})$.
Case 1. Let $\mathbb{L}_1(M_n(\mathbb{H}))$ be the space of matrix pencils over a quaternion division algebra. $\mathbf{L}_1 \in \mathbb{L}_1(M_n(\mathbb{H}))$ is defined as

$$\mathbf{L}_1(\lambda) := A + \lambda B, \tag{1.26}$$

where $A, B \in M_n(\mathbb{H})$ and λ commutes with the quaternionic matrices. This matrix pencil over a quaternion division algebra can be found in [33, 46, 47]. Now we turn to define generalized right eigenvalue of $\mathbf{L}_1 \in \mathbb{L}_1(M_n(\mathbb{H}))$ of the form (1.26) as follows.

Definition 1.39. Let $\mathbf{L}_1 \in \mathbb{L}_1(M_n(\mathbb{H}))$ be as in (1.26) and let $\mu \in \mathbb{H}$. Then μ is called a generalized right eigenvalue of $\mathbf{L}_1$ if

$$Ax = Bx\mu$$

for some nonzero $x \in \mathbb{H}^n$. Here x is called the right eigenvector corresponding to the generalized right eigenvalue μ. The set of generalized right eigenvalues of $\mathbf{L}_1$ is called right spectrum of $\mathbf{L}_1$, denoted by $\Lambda_r(\mathbf{L}_1)$.

Definition 1.40. Let $\mathbf{L}_1 \in \mathbb{L}_1(M_n(\mathbb{H}))$ be as in (1.26). Then the matrix pencil $\mathbf{L}_1$ is called regular if there exists $\alpha \in \mathbb{R}$ such that $A + \alpha\, B$ is an invertible matrix.

Let $\mathbb{P}_1(M_{2n}(\mathbb{C}))$ be the space of complex matrix pencils. $P_1 \in \mathbb{P}_1(M_{2n}(\mathbb{C}))$ is defines as

$$P_1(\mu) := \Psi_A + \mu\Psi_B, \tag{1.27}$$

where $A, B \in M_n(\mathbb{H})$ and $\mu \in \mathbb{C}$.

Then, we have the following relation between quaternionic matrix pencils and complex matrix pencils.

Lemma 1.41. *[33] Let $\mathbf{L}_1 \in \mathbb{L}_1(M_n(\mathbb{H}))$ be of the form $\mathbf{L}_1(\lambda) := A + \lambda B$, where λ commutes with the quaternionic matrices. Let $P_1 \in \mathbb{P}_1(M_{2n}(\mathbb{C}))$ be of the form $P_1(\mu) := \Psi_A + \mu\Psi_B$, where $\mu \in \mathbb{C}$. Then*

1. *the quaternionic matrix pencil $\mathbf{L}_1$ is regular if and only if the complex matrix pencil P_1 is regular;*
2. *the quaternionic matrix pencil $\mathbf{L}_1$ is regular if and only if a quaternionic matrix pencil $\mathbf{L}_2$ is regular, where $\mathbf{L}_2(\lambda) := B + \lambda A$.*

We now give a relation between the set of generalized right eigenvalues of a quaternionic matrix pencil and the set of eigenvalues of the complex adjoint matrix of that quaternionic matrix pencil as follows.

Proposition 1.42. *[33] Let $\mathbf{L} \in \mathbb{L}_1(M_n(\mathbb{H}))$ be of the form $\mathbf{L}(\lambda) := A + \lambda B$, where λ commutes with A and B. Let $P \in \mathbb{P}_1(M_{2n}(\mathbb{C}))$ be of the form $P(\mu) := \Psi_A + \mu\Psi_B$, where $\mu \in \mathbb{C}$. Then*

$$\begin{cases} \Lambda_r(\mathbf{L}) \cap \mathbb{C} = \Lambda(P), \\ \quad \Lambda_r(\mathbf{L}) = \{\rho^{-1}\lambda\rho : \lambda \in \Lambda(P),\ 0 \neq \rho \in \mathbb{H}\}. \end{cases} \tag{1.28}$$

Assume that $B = I$ in Proposition 1.42, we have the following corollary which also can be found from [33].

Corollary 1.43. *Let $A \in M_n(\mathbb{H})$. Then*

$$\begin{aligned} \Lambda_r(A) \cap \mathbb{C} &= \Lambda(\Psi_A), \\ \Lambda_r(A) &= \{\rho^{-1}\lambda\rho : \lambda \in \Lambda(\Psi_A),\ 0 \neq \rho \in \mathbb{H}\}. \end{aligned}$$

Case 2. We define generalized left eigenvalue of $\mathbf{L}_1' \in \mathbb{L}_1'(M_n(\mathbb{H}))$ of the form (1.25) as follows.

Definition 1.44. Let $\mathbf{L}_1' \in \mathbb{L}_1'(M_n(\mathbb{H}))$ be of the form (1.25) and let $\mu \in \mathbb{H}$. Then μ is called a generalized left eigenvalue of $\mathbf{L}_1'$ if

$$Ax = \mu Bx$$

for some nonzero $x \in \mathbb{H}^n$. Here x is called the left eigenvector corresponding to the generalized left eigenvalue μ. The set of generalized left eigenvalues of $\mathbf{L}_1'$ is called generalized left spectrum of $\mathbf{L}_1'$, denoted by $\Lambda_l\left(\mathbf{L}_1'\right)$.

1.2.6. The polynomial eigenvalue problem

Let $\mathcal{P}_m(M_n(\mathbb{C}))$ be the space of complex matrix polynomials. $\mathcal{P} \in \mathcal{P}_m(M_n(\mathbb{C}))$ is defined by

$$\mathcal{P}(\lambda) := \sum_{i=0}^{m} \lambda^i A_i,$$

where $A_i \in M_n(\mathbb{C})$ $(0 \leq i \leq m)$ and $\lambda \in \mathbb{C}$. Then the eigenvalue problem $\mathcal{P}(\lambda)x = 0$ is referred as a complex polynomial eigenvalue problem. The polynomial $\mathcal{P} \in \mathcal{P}_m(M_n(\mathbb{C}))$ is said to be regular if $\det(\mathcal{P}(\lambda)) \neq 0$ for some $\lambda \in \mathbb{C}$. The spectrum of a regular polynomial $\mathcal{P}$ is denoted by $\Lambda(\mathcal{P})$ and is defined by

$$\Lambda(\mathcal{P}) := \{\lambda \in \mathbb{C} : \det(\mathcal{P}(\lambda)) = 0\}.$$

Let $\mathbb{P}_m(M_{2n}(\mathbb{C}))$ be the space of complex matrix polynomials. $P \in \mathbb{P}_m(M_{2n}(\mathbb{C}))$ is defined as

$$P(\mu) := \sum_{i=0}^{m} \Psi_{A_i}\mu^i, \tag{1.29}$$

where $A_i \in M_n(\mathbb{H})$ $(0 \leq i \leq m)$ and $\mu \in \mathbb{C}$.

Let $\mathbb{L}'_m(M_n(\mathbb{H}))$ be the space of matrix polynomials over a quaternion division algebra. $\mathbf{L}' \in \mathbb{L}'_m(M_n(\mathbb{H}))$ is defined as

$$\mathbf{L}'(\xi) := \sum_{i=0}^{m} \xi^i \; A_i, \tag{1.30}$$

where $\xi \in \mathbb{H}$ and $A_i \in M_n(\mathbb{H})$ $(0 \leq i \leq m)$. Throughout this thesis we consider the following three cases:

Case 1: when $\xi \in \mathbb{R}$ and $A_i \in M_n(\mathbb{H})\,(0 \leq i \leq m)$,

Case 2: when $\xi \in \mathbb{H}$ and $A_i \in M_n(\mathbb{H})\,(0 \leq i \leq m)$,

Case 3: when $\xi \in \mathbb{C}$ and $A_i \in M_n(\mathbb{C})\,(0 \leq i \leq m)$.

Case 1. Let $\mathbb{L}_m(M_n(\mathbb{H}))$ be the space of matrix polynomials over a quaternion division algebra. $\mathbf{L} \in \mathbb{L}_m(M_n(\mathbb{H}))$ is defined as

$$\mathbf{L}(\lambda) := \sum_{i=0}^{m} \; A_i \lambda^i, \tag{1.31}$$

where $A_i \in M_n(\mathbb{H})$ $(0 \leq i \leq m)$ and λ commutes with the quaternionic coefficients of the matrix polynomial. This polynomial over a quaternion division algebra can be found in [46–48].

We now turn to define right eigenvalue of $\mathbf{L} \in \mathbb{L}_m(M_n(\mathbb{H}))$ of the form (1.31) as follows.

Definition 1.45. Let $\mathbf{L} \in \mathbb{L}_m(M_n(\mathbb{H}))$ be as in (1.31) and let $\mu \in \mathbb{H}$. Then μ is called a right eigenvalue of $\mathbf{L}$ if

$$A_0 x + A_1 x \mu + A_2 x \mu^2 + \cdots + A_m x \mu^m = 0$$

for some nonzero $x \in \mathbb{H}^n$. Here x is called the right eigenvector corresponding to the right eigenvalue μ. The set of right eigenvalues of $\mathbf{L}$ is called right spectrum of $\mathbf{L}$, denoted by $\Lambda_r(\mathbf{L})$.

Case 2. Left eigenvalue of $\mathbf{L}' \in \mathbb{L}'_m(M_n(\mathbb{H}))$ of the form (1.30) is defined as follows.

Definition 1.46. Let $\mathbf{L}' \in \mathbb{L}'_m(M_n(\mathbb{H}))$ be of the form (1.30) and let $\mu \in \mathbb{H}$. Then μ is called a left eigenvalue of $\mathbf{L}'$ if

$$A_0 x + \mu A_1 x + \mu^2 A_2 x + \cdots + \mu^m A_m x = 0$$

for some nonzero $x \in \mathbb{H}^n$. Here x is called the left eigenvector corresponding to the left eigenvalue μ. The set of left eigenvalues of $\mathbf{L}'$ is called left spectrum of $\mathbf{L}'$, denoted by $\Lambda_l(\mathbf{L}')$.

We write the linearization of the matrix polynomial $\mathbf{L} \in \mathbb{L}_m(M_n(\mathbb{H}))$ as follows.

- **For the right eigenvalues:** The polynomial $\mathbf{L} \in \mathbb{L}_m(M_n(\mathbb{H}))$ of the form (1.31) can be written in the form:

$$C_{\mathbf{L}} + \lambda X,$$

where $C_{\mathbf{L}}$, $X \in M_{mn}(\mathbb{H})$ are of the forms

$$C_{\mathbf{L}} := \begin{bmatrix} 0 & I_n & 0 & \dots & 0 \\ 0 & 0 & I_n & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & I_n \\ -A_0 & -A_1 & -A_2 & \dots & -A_{m-1} \end{bmatrix}, \quad X := \begin{bmatrix} I_n & 0 & 0 & \dots & 0 \\ 0 & I_n & 0 & \dots & 0 \\ 0 & 0 & I_n & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & A_m \end{bmatrix}$$

and λ commutes with quaternionic coefficients of matrix polynomial. When $A_m = I_n$, the identity matrix, the matrix polynomial (1.31) is said to be monic matrix polynomial and its linearization is given by

$$C_{\mathbf{L}} + \lambda E, \text{ where } E := I_{nm}.$$

We next find the corresponding linearization of the matrix polynomial $\mathbf{L}' \in \mathbb{L}'_m(M_n(\mathbb{H}))$ as follows.

- **For the left eigenvalues:** The polynomial $\mathbf{L}' \in \mathbb{L}'_m(M_n(\mathbb{H}))$ of the form (1.30) can be written in the linearization form:

$$C_{\mathbf{L}'} + \xi X,$$

where $\xi \in \mathbb{H}$, $C_{\mathbf{L}'}$, $X \in M_{mn}(\mathbb{H})$ are of the forms

$$C_{\mathbf{L}'} := \begin{bmatrix} 0 & 0 & 0 & \dots & -A_0 \\ I_n & 0 & 0 & \dots & -A_1 \\ & \ddots & \vdots & \dots & \vdots \\ 0 & 0 & I_n & & -A_{m-2} \\ 0 & 0 & \dots & I_n & -A_{m-1} \end{bmatrix}, \quad X := \begin{bmatrix} I_n & 0 & 0 & \dots & 0 \\ 0 & I_n & 0 & \dots & 0 \\ 0 & 0 & I_n & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & A_m \end{bmatrix}.$$

When $A_m = I_n$, the identity matrix, then the polynomial $\mathbf{L}' \in \mathbb{L}'_m(M_n(\mathbb{H}))$ of the form (1.30) is said to be monic matrix polynomial and its linearization is given by

$$C_{\mathbf{L}'} + \xi\, E, \text{ where } E := I_{nm}.$$

CHAPTER 2

LOCALIZATION THEOREMS OF MATRICES OVER A QUATERNION DIVISION ALGEBRA

In this chapter, Gerschgorin, Ostrowski, and Brauer type theorems are derived for the left and right eigenvalues of a quaternionic matrix. Generalizations of Gerschgorin type theorems are discussed for the left and right eigenvalues of a quaternionic matrix. Thereafter a sufficient condition for the stability of a continuous-time quaternionic system is given that generalizes the stability condition for a continuous-time complex system. Finally, a characterization of bounds for the zeros of quaternionic polynomials is presented.

2.1. Introduction

This chapter attempts to study localization theorems for matrices over a quaternion division algebra, which include the Ostrowski, Brauer, and Gerschgorin type theorems. Bounds for the zeros of quaternionic polynomials $p_l(z)$ and $p_r(z)$ (defined in (1.20) and (1.21)) are also considered.

In Section 2.2, we provide a general framework for localization theorems for quaternionic matrices. Let $M_n(\mathbb{H})$ be the space of all $n \times n$ quaternionic matrices. Then, for any $A := (a_{ij}) \in M_n(\mathbb{H})$, we prove a Ostrowski type theorem which states that all the left eigenvalues of A are located in the union of n balls

$$T_i(A) := \{z \in \mathbb{H} : |z - a_{ii}| \leq r_i(A)^{\gamma} c_i(A)^{1-\gamma}\},$$

where $r_i(A)$ and $c_i(A)$ are defined in (1.15) and $\forall\ \gamma \in [0, 1]$. We deduce a sufficient condition for invertibility of a quaternionic matrix. We proved that Ostrowski type theorem is also valid for right eigenvalues when all the diagonal entries of the quaternionic matrix A are real. We find that the Brauer type theorem, proved in [22] for the left eigenvalues in the case of deleted absolute column sums of a quaternionic matrix, is incorrect. We prove a corrected version of the Brauer type theorem. In addition, we derive some better results than [22, Theorems 6, 7] and [63, Theorem 4.3]. In the case of the generalized

Hölder inequality over the skew field of quaternions, we show that all the left eigenvalues of $A = (a_{ij}) \in M_n(\mathbb{H})$ are contained in the union of n generalized balls

$$B_i(A) := \{z \in \mathbb{H} : |z - a_{ii}| \leq (n-1)^{\frac{1-\gamma}{q}} r_i(A)^{\gamma} (n_i^{(p)}(A))^{1-\gamma}\},$$

where $\gamma \in [0, 1]$, $n_i^{(p)}(A) := \left(\sum_{j=1,\, j\neq i}^{n} |a_{ij}|^p\right)^{\frac{1}{p}}$, for any $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Further, we prove that all the right eigenvalues of the quaternionic matrix A with all real diagonal entries are contained in the union of n generalized balls $B_i(A)$.

In Section 2.3, we provide bounds for the zeros of quaternionic polynomials $p_l(z)$ and $p_r(z)$ by using the aforementioned localization theorems. Some recent developments on the location and computation of zeros of quaternionic polynomials $p_l(z)$ and $p_r(z)$ can be found in [12, 19, 20, 30, 41, 42, 45, 50]. As a consequence of the localization theorems for quaternionic matrices, we provide sharper bounds compared to the bound introduced by G. Opfer in [42] for the zeros of quaternionic polynomials $p_l(z)$ and $p_r(z)$. Finally, we provide sharper bounds for the zeros of quaternionic polynomials $p_l(z)$ and $p_r(z)$ in terms of powers of the companion matrices associated with the quaternionic polynomials $p_l(z)$ and $p_r(z)$. We show that the proposed bounds are better than that in [42].

2.2. Distribution for the left and right eigenvalues of quaternionic matrices

It is known from [47, Corollary 3.2] that a quaternionic matrix A and its conjugate transpose A^H have the same right eigenvalues. However, A and A^H may not have the same left eigenvalues. For example, the matrices

$$A = \begin{bmatrix} \mathbf{i} & 0 \\ 0 & \mathbf{j} \end{bmatrix} \text{ and } A^H = \begin{bmatrix} -\mathbf{i} & 0 \\ 0 & -\mathbf{j} \end{bmatrix}$$

do not have the same left eigenvalues. We now present the following lemma for left eigenvalues of A and A^H.

Lemma 2.1. *Let $A \in M_n(\mathbb{H})$ and let $\lambda \in \mathbb{H}$. Then λ is a left eigenvalue of A if and only if $\overline{\lambda}$ is a left eigenvalue of A^H.*

Proof. Let λ be a left eigenvalue of A. Then there exists $x(\neq 0) \in \mathbb{H}^n$ such that $(A - \lambda I_n)x = 0$. This can be written as $\Psi_{(A-\lambda I_n)}\psi_x = 0$. Hence it follows that λ is a left eigenvalue of A if and only if $\det\left[\Psi_{(A-\lambda I_n)}\right] = 0 \Leftrightarrow \det\left[\Psi^H_{(A-\lambda I_n)}\right] = 0 \Leftrightarrow \det\left[\Psi_{(A-\lambda I_n)^H}\right] = 0 \Leftrightarrow \det\left[\Psi_{(A^H-\overline{\lambda} I_n)}\right] = 0$. Thus, $\overline{\lambda}$ is a left eigenvalue of A^H. ■

The Gerschgorin type theorem is proved in [62] for the left eigenvalues in the case of deleted absolute row sums of a matrix $A \in M_n(\mathbb{H})$. However, Gerschgorin type theorem for the left eigenvalues has not established for the deleted absolute column sums of A. We state the following Gerschgorin type theorem for the deleted absolute column sums of A.

Theorem 2.2. *Let $A := (a_{ij}) \in M_n(\mathbb{H})$. Then all the left eigenvalues of A are located in the union of n Gerschgorin balls $\Omega_i(A) := \{z \in \mathbb{H} : |z - a_{ii}| \leq c_i(A)\}$ $(1 \leq i \leq n)$, i.e.,*

$$\Lambda_l(A) \subseteq \Omega(A) := \cup_{i=1}^n \Omega_i(A).$$

Proof. Let λ be a left eigenvalue of A. Then from Lemma 2.1, $\overline{\lambda}$ is a left eigenvalue of A^H. Then there exists some nonzero $x \in \mathbb{H}^n$ such that $A^H x = \overline{\lambda} x$. Let $x := [x_1, \ldots, x_n]^T \in \mathbb{H}^n$ and let x_t be an element of x such that $|x_t| \geq |x_i| \ \forall\ i$ $(1 \leq i \leq n)$. Then, $|x_t| > 0$. From the t-th equation of $A^H x = \overline{\lambda} x$, we have

$$\sum_{j=1}^{n} \overline{a_{jt}} x_j = \overline{\lambda} x_t.$$

This shows

$$|\lambda - a_{tt}| \leq \sum_{j=1,\, j\neq t}^{n} |a_{jt}| := c_t(A). \ ■$$

We now have the following localization theorem for the deleted absolute row and column sums of a matrix $A \in M_n(\mathbb{H})$ which is known as *Ostrowski type theorem.*

Theorem 2.3. (*Ostrowski type theorem for the left eigenvalues*) *Let $A := (a_{ij}) \in M_n(\mathbb{H})$ and let $\gamma \in [0, 1]$. Then all the left eigenvalues of A are located in the union of n balls $T_i(A) := \{z \in \mathbb{H} : |z - a_{ii}| \leq r_i(A)^{\gamma} c_i(A)^{1-\gamma}\}$ $(1 \leq i \leq n)$, i.e.,*

$$\Lambda_l(A) \subseteq T(A) := \cup_{i=1}^n T_i(A).$$

Proof. Let λ be a left eigenvalue of A. Then by [62, Theorem 6], for $\gamma \in [0,\ 1]$, we have

$$|\lambda - a_{ii}|^{\gamma} \leq r_i(A)^{\gamma} \qquad (1 \leq i \leq n). \tag{2.1}$$

Similarly, from Theorem 2.2, we obtain

$$(2.2) \qquad |\lambda - a_{ii}|^{1-\gamma} \leq c_i(A)^{1-\gamma} \qquad (1 \leq i \leq n).$$

Combining (2.1) and (2.2), we get

$$|\lambda - a_{ii}| \leq r_i(A)^{\gamma} c_i(A)^{1-\gamma} \qquad (1 \leq i \leq n).$$

Thus, all the left eigenvalues of A are located in the union of n balls $T_i(A)$. ■

Next, we derive Ostrowski type theorem for right eigenvalues of $A \in M_n(\mathbb{H})$ with all real diagonal entries.

Theorem 2.4. *Let $A := (a_{ij}) \in M_n(\mathbb{H})$ with $a_{ii} \in \mathbb{R}$ and let $\gamma \in [0,\, 1]$. Then all the right eigenvalues of A are located in the union of n balls*

$$G_i(A) := \left\{ z \in \mathbb{H} : |z - a_{ii}| \leq r_i(A)^{\gamma} c_i(A)^{1-\gamma} \right\} \quad (1 \leq i \leq n), \; \textit{i.e.,}$$

$$\Lambda_r(A) \subseteq G(A) := \cup_{i=1}^{n} G_i(A).$$

Proof. Let λ be a right eigenvalue of A. Then there exists some nonzero $x \in \mathbb{H}^n$ such that $Ax = x\lambda$. Let $x := [x_1, \ldots, x_n]^T \in \mathbb{H}^n$ and let x_t be an element of x such that $|x_t| \geq |x_i| \; \forall \; i \; (1 \leq i \leq n)$. From the t-th equation of $Ax = x\lambda$, we have

$$(2.3) \qquad a_{tt}x_t + \sum_{j=1,\, j\neq t}^{n} a_{tj}x_j \;\; = \;\; x_t\lambda.$$

Since $a_{tt} \in \mathbb{R}$, $a_{tt}x_t = x_t a_{tt}$. Proceeding as in the proof of Theorem 2.2, we obtain

$$(2.4) \qquad |\lambda - a_{tt}| \leq \sum_{j=1,\, j\neq t}^{n} |a_{jt}| =: r_t(A).$$

From [47, Corollary 2.7], λ is also a right eigenvalue of A^H. Then

$$(2.5) \qquad |\lambda - a_{tt}| \leq \sum_{j=1,\, j\neq t}^{n} |a_{tj}| =: c_t(A).$$

Let $\gamma \in [0, 1]$. Then from (2.4) and (2.5), we obtain

$$(2.6) \qquad |\lambda - a_{tt}|^{\gamma} \leq r_t^{\gamma}(A),$$

$$(2.7) \qquad |\lambda - a_{tt}|^{1-\gamma} \leq c_t^{1-\gamma}(A).$$

Combining (2.6) and (2.7), we get

$$|\lambda - a_{tt}| \leq r(A)_t^{\gamma}\, c(A)_t^{1-\gamma}. \; ■$$

Corollary 2.5. *For any* $A := (a_{ij}) \in M_n(\mathbb{H})$, $n \geq 2$ *and for any* $\gamma \in [0, 1]$. *Assume that*

$$|a_{ii}| > r_i(A)^{\gamma}\ r_i(A)^{1-\gamma} \quad \forall\ i\ (1 \leq i \leq n). \tag{2.8}$$

Then A *is invertible.*

Proof. On the contrary, suppose A is not invertible. Then by Theorem 1.31, there is a left eigenvalue $\lambda = 0$ of A. Now from Theorem 2.3, we obtain $|a_{ii}| \leq r_i(A)^{\gamma} c_i(A)^{1-\gamma}$. This contradicts our assumption (2.8). Hence A is invertible. ■

It is known that a matrix $A \in M_n(\mathbb{H})$ may have at most $2n$ complex right eigenvalues. From Theorem 2.4, all the complex right eigenvalues of a matrix $A = (a_{ij}) \in M_n(\mathbb{H})$ with all real diagonal entries lie in the union of n-discs

$$\mathcal{E}_i(A) := \{z \in \mathbb{C} : |z - a_{ii}| \leq r_i(A)^{\gamma} c_i(A)^{1-\gamma}\}\ \ (1 \leq i \leq n),\ \text{i.e.,}$$

$$\Lambda_c(A) \subseteq \mathcal{E}(A) := \cup_{i=1}^{n} \mathcal{E}_i(A), \tag{2.9}$$

where $\Lambda_c(A) := \{\lambda \in \mathbb{C} : Ax = x\lambda,\ 0 \neq x \in \mathbb{H}^n\}$.

The Brauer type theorem is proved in [22] for the left eigenvalues in the case of deleted absolute column sums of a matrix $A \in M_n(\mathbb{H})$, i.e., if $\lambda \in \Lambda_l(A)$, then its conjugate $\overline{\lambda}$ lies in the union of $\frac{n(n-1)}{2}$ ovals of Cassini. However, this is incorrect as the following example suggest:

Example 2.6. Let $A = \begin{bmatrix} \mathbf{i} & \mathbf{k} \\ 0 & \mathbf{j} \end{bmatrix}$. Then by [22, Theorem 5], oval of Cassini is given by $\{z \in \mathbb{H} : |z - \mathbf{i}|\ |z - \mathbf{j}| \leq 0\}$. Here, $\mathbf{i}$ is a left eigenvalue of A and its conjugate $-\mathbf{i}$ is not contained in the above oval of Cassini.

According to [22, Theorem 5], if $\lambda \in \Lambda_l(A)$, then $\overline{\lambda} \in \cup_{\substack{i,j=1, \\ i \neq j}}^{n} F_{ij}(A)$, where

$$F_{ij}(A) := \{z \in \mathbb{H} : |z - a_{ii}|\ |z - a_{jj}| \leq c_i(A) c_j(A)\} \qquad (1 \leq i, j \leq n; i \neq j).$$

However, this result is not necessarily true as

$$|\overline{\lambda} - a_{ii}|\ |\overline{\lambda} - a_{jj}| > c_i(A) c_j(A) \quad \forall\, i, j\, (1 \leq i, j \leq n; i \neq j)$$

which follows from Example 2.6. Now, we derive a corrected version of [22, Theorem 5] as follows.

Theorem 2.7. *Let $A := (a_{ij}) \in M_n(\mathbb{H})$. Then all the left eigenvalues of A are located in the union of $\frac{n(n-1)}{2}$ ovals of Cassini*

$$F_{ij}(A) := \{z \in \mathbb{H} : |z - a_{ii}|\ |z - a_{jj}| \leq c_i(A)c_j(A)\} \ \ (1 \leq i, j \leq n; i \neq j),\ i.e.,$$

$$\Lambda_l(A) \subseteq F(A) := \cup_{\substack{i,j=1,\\ i\neq j}}^{n} F_{ij}(A).$$

Proof. Let λ be a left eigenvalue of A. Then by Lemma 2.1, $\overline{\lambda}$ is a left eigenvalue of A^H, so that there exists some nonzero $x \in \mathbb{H}^n$ such that $A^H x = \overline{\lambda} x$. Let $x := [x_1, \ldots, x_n]^T \in \mathbb{H}^n$ and let x_s be an element of x such that $|x_s| \geq |x_i|\ \forall\ i\ (1 \leq i \leq n)$. Then, $|x_s| > 0$. Clearly, if all the other elements of x are zero, then the required result holds.

Let x_s and x_t be two nonzero elements of x such that $|x_s| \geq |x_t| \geq |x_i|\ \forall\ i\ (1 \leq i \leq n, i \neq s)$. From the s-th equation of $A^H x = \overline{\lambda} x$, we have

$$\sum_{j=1}^{n} \overline{a_{js}} x_j = \overline{\lambda} x_s,$$

which implies

$$(\overline{\lambda} - \overline{a_{ss}}) x_s = \sum_{j=1,\, j\neq s}^{n} \overline{a_{js}} x_j.$$

Thus

$$|\lambda - a_{ss}| \leq \left(\frac{|x_t|}{|x_s|} \right)\ c_s(A). \tag{2.10}$$

Similarly, from $A^H x = \overline{\lambda} x$, we obtain

$$|\lambda - a_{tt}| \leq \left(\frac{|x_s|}{|x_t|} \right)\ c_t(A). \tag{2.11}$$

Combining (2.10) and (2.11), we have

$$|\lambda - a_{ss}|\ |\lambda - a_{tt}| \leq c_s(A) c_t(A).$$

Hence, all the left eigenvalues of A are located in the union of $\frac{n(n-1)}{2}$ ovals of Cassini $F_{ij}(A)$ $(1 \leq i, j \leq n,\ i \neq j)$. ∎

Theorem 7 of [22] was stated for a central closed quaternionic matrix. Now we generalize this result for all quaternionic matrices as follows.

Theorem 2.8. *Let $A := (a_{ij}) \in M_n(\mathbb{H})$ and let $\gamma \in [0,\, 1]$. Then all the left eigenvalues of A are located in the union of $\dfrac{n(n-1)}{2}$ ovals of Cassini*

$$K_{ij}(A) := \left\{ z \in \mathbb{H} : |z - a_{ii}|\ |z - a_{jj}| \leq r_i(A)^{\gamma}\ r_j(A)^{\gamma}\ c_i(A)^{1-\gamma}\ c_j(A)^{1-\gamma} \right\}$$

$(1 \leq i, j \leq n; i \neq j)$, *i.e.*,

$$\Lambda_l(A) \subseteq K(A) := \cup_{\substack{i,j=1\\ i\neq j}}^{n} K_{ij}(A).$$

Proof. Let λ be a left eigenvalue of A. Then by [22, Theorem 4] and Theorem 2.7, for $\gamma \in [0, 1]$, we have

$$|\lambda - a_{ii}|^{\gamma} |\lambda - a_{jj}|^{\gamma} \leq r_i(A)^{\gamma} r_j(A)^{\gamma} \quad (1 \leq i, j \leq n; i \neq j), \tag{2.12}$$

and

$$|\lambda - a_{ii}|^{1-\gamma} |\lambda - a_{jj}|^{1-\gamma} \leq c_i(A)^{1-\gamma} c_j(A)^{1-\gamma} \quad (1 \leq i, j \leq n; i \neq j). \tag{2.13}$$

Combining (2.12) and (2.13), we have

$$|\lambda - a_{ii}||\lambda - a_{jj}| \leq r_i(A)^{\gamma} r_j(A)^{\gamma} c_i(A)^{1-\gamma} c_j(A)^{1-\gamma} \ (1 \leq i, j \leq n; i \neq j). \ \blacksquare$$

Corollary 2.9. *For any $A := (a_{ij}) \in M_n(\mathbb{H})$, $n \geq 2$ and for any $\gamma \in [0, 1]$. Assume that*

$$|a_{ii}||a_{jj}| > r_i(A)^{\gamma} r_j(A)^{\gamma} \ c_i(A)^{1-\gamma} c_j(A)^{1-\gamma} \quad \forall\, i, j\, (1 \leq i, j \leq n, \ i \neq j).$$

Then A is invertible.

Corollary 2.10. *Let $A := (a_{ij}) \in M_n(\mathbb{H})$. Then all the left eigenvalues of A are located in the union of $\frac{n(n-1)}{2}$ ovals of Cassini*

$$\Lambda_l(A) \subseteq \Phi(A) := \cup_{\substack{i,j=1\\ i\neq j}}^{n} \left\{ z \in \mathbb{H} : |z - a_{ii}| \ |z - a_{jj}| \leq \min\{r_i(A) r_j(A), \ c_i(A) c_j(A)\} \right\}.$$

Proof. Substituting $\gamma = 0, 1$ in Theorem 2.8, we obtain the following:

(a) $\Lambda_l(A) \subseteq E(A) := \cup_{\substack{i,j=1\\ i\neq j}}^{n} \left\{ z \in \mathbb{H} : |z - a_{ii}| \ |z - a_{jj}| \leq c_i(A) c_j(A) \right\}.$

(b) $\Lambda_l(A) \subseteq F(A) := \cup_{\substack{i,j=1\\ i\neq j}}^{n} \left\{ z \in \mathbb{H} : |z - a_{ii}| \ |z - a_{jj}| \leq r_i(A) r_j(A) \right\}.$

Combining (a) and (b), we get the required result. $\blacksquare$

The following result provides better estimation compare to Theorem 2.4.

Theorem 2.11. *Let $A := (a_{ij}) \in M_n(\mathbb{H})$ with $a_{ii} \in \mathbb{R}$ and let $\gamma \in [0, 1]$. Then all the right eigenvalues of A are located in the union of $\dfrac{n(n-1)}{2}$ ovals of Cassini $\mathcal{G}_{ij}(A) := \{z \in \mathbb{H} : |z - a_{ii}| \ |z - a_{jj}| \leq r_i(A)^{\gamma} \ r_j(A)^{\gamma} \ c_i(A)^{1-\gamma} \ c_j(A)^{1-\gamma}\}$ $(1 \leq i, j \leq n, \ i \neq j)$, i.e.,*

$$\Lambda_r(A) \subseteq \mathcal{G}(A) := \cup_{\substack{i,j=1\\ i\neq j}}^{n} \mathcal{G}_{ij}(A).$$

Proof. Proof follows from the proof method of Theorem 2.8 and by applying [63, Theorem 4.1, Corollary 4.1]. ■

From Theorem 2.11, all the complex right eigenvalues of a matrix $A := (a_{ij}) \in M_n(\mathbb{H})$ with $a_{ii} \in \mathbb{R}\ \forall\, i\,(1 \leq i \leq n)$ are contained in the union of $\frac{n(n-1)}{n}$ ovals of Cassini $\mathcal{F}_{ij}(A) := \{z \in \mathbb{C} : |z - a_{ii}|\,|z - a_{jj}| \leq r_i(A)^{\gamma} r_j(A)^{\gamma}\ c_i(A)^{1-\gamma} c_j(A)^{1-\gamma}\} \quad (1 \leq i, j \leq n,\ i \neq j)$, i.e.,

$$\Lambda_c(A) \subseteq \mathcal{F}(A) := \cup_{\substack{i,j=1\\ i\neq j}}^{n} \mathcal{F}_{ij}(A), \tag{2.14}$$

The following theorem shows that Theorem 2.8 is sharper than Theorem 2.3.

Theorem 2.12. *Let $A := (a_{ij}) \in M_n(\mathbb{H})$ with $n \geq 2$ and let $\gamma \in [0, 1]$. Then*

$$K(A) \subseteq T(A),$$

where $G(A)$ and $\mathcal{G}(A)$ are defined in Theorem 2.3 and Theorem 2.8, respectively.

Proof. Let $z \in K_{ij}(A)$ and fix any i and j, $(1 \leq i, j \leq n,\ i \neq j)$. Then from Theorem 2.8, we have

$$|z - a_{ii}|\ |z - a_{jj}| \leq r_i(A)^{\gamma} r_j(A)^{\gamma} c_i(A)^{1-\gamma} c_j(A)^{1-\gamma}. \tag{2.15}$$

Now the following two cases are possible.

Case 1: If $r_i(A)^{\gamma}\ r_j(A)^{\gamma} c_i(A)^{1-\gamma} c_j(A)^{1-\gamma} = 0$, then $z = a_{ii}$ or $z = a_{jj}$. However, from Theorem 2.3, we have $a_{ii} \in T_i(A)$ and $a_{jj} \in T_j(A)$. Thus $z \in T_i(A) \cup T_j(A)$.

Case 2: If $r_i(A)^{\gamma} r_j(A)^{\gamma} c_i(A)^{1-\gamma} c_j(A)^{1-\gamma} > 0$, then by (2.15)

$$\left(\frac{|z - a_{ii}|}{r_i(A)^{\gamma} c_i(A)^{1-\gamma}}\right)\left(\frac{|z - a_{jj}|}{r_j(A)^{\gamma} c_j(A)^{1-\gamma}}\right) \leq 1. \tag{2.16}$$

As the left side of (2.16) cannot exceed unity, then one of the factors of the left side can be at most unity, i.e., $z \in T_i(A)$ or $z \in T_j(A)$. Hence $z \in T_i(A) \cup T_j(A)$. Thus

$$K_{ij} \subseteq T_i(A) \cup T_j(A). \tag{2.17}$$

From Theorem 2.3 and Theorem 2.8, we obtain

$$K(A) := \cup_{\substack{i,j=1\\ i\neq j}}^{n} K_{ij}(A) \subseteq \cup_{\substack{i,j=1\\ i\neq j}}^{n} \{T_i(A) \cup T_j(A)\} = \cup_{k=1}^{n} T_k(A) =: T(A).\ ■$$

Similarly, we have the following relation between Theorem 2.11 and Theorem 2.4.

Theorem 2.13. *Let $A := (a_{ij}) \in M_n(\mathbb{H}), n \geq 2$ with $a_{ii} \in \mathbb{R}$ and let $\gamma \in [0,1]$. Then*

$$\mathcal{G}(A) \subseteq G(A),$$

where $G(A)$ and $\mathcal{G}(A)$ are defined in Theorem 2.4 and Theorem 2.11, respectively.

Proof. The proof is immediate from the proof method of Theorem 2.12 and by applying Theorems 2.4 and 2.11. ■

The following example verifies Theorem 2.13 for complex right eigenvalues of a matrix $A := (a_{ij}) \in M_n(\mathbb{H})$ with $a_{ii} \in \mathbb{R}\ \forall\, i\, (1 \leq i \leq n)$.

Example 2.14. Let $A = \begin{bmatrix} 3 & 1+\mathbf{i}+\mathbf{j}-\mathbf{k} & 2+3\mathbf{j}-\sqrt{3}\mathbf{k} \\ 5+\sqrt{2}\mathbf{j}+3\mathbf{k} & -2 & 3\mathbf{j}+4\mathbf{k} \\ 4+3\mathbf{j} & 2-\mathbf{i}-2\mathbf{k} & -5 \end{bmatrix}$. Substituting $\gamma = 1/4$ in (2.9), we get the following three discs:

$$\begin{aligned} \mathcal{E}_1(A) &:= \{z \in \mathbb{C} : |z-3| \leq 9.4533\}, \\ \mathcal{E}_2(A) &:= \{z \in \mathbb{C} : |z+2| \leq 6.0894\}, \\ \mathcal{E}_3(A) &:= \{z \in \mathbb{C} : |z+5| \leq 8.7389\}. \end{aligned}$$

Similarly, let $\gamma = 1/4$ in (2.14), we get the following three ovals of Cassini:

$$\begin{aligned} \mathcal{F}_{12}(A) &:= \{z \in \mathbb{C} : |z-3|\,|z+2| \leq 57.5649\}, \\ \mathcal{F}_{23}(A) &:= \{z \in \mathbb{C} : |z+2|\,|z+5| \leq 53.2145\}, \\ \mathcal{F}_{31}(A) &:= \{z \in \mathbb{C} : |z+5|\,|z-3| \leq 82.6108\}. \end{aligned}$$

In this example, there are six complex right eigenvalues λ_j $(1 \leq j \leq 6)$ which are shown in FIGURE 2.1. The set $\mathcal{F}(A) := \mathcal{F}_{12}(A) \cup \mathcal{F}_{23}(A) \cup \mathcal{F}_{31}(A)$ is represented by shaded region in FIGURE 2.1. From FIGURE 2.1, it is clear that $\mathcal{F}(A) \subset \mathcal{E}(A)$, where $\mathcal{E}(A) := \mathcal{E}_1(A) \cup \mathcal{E}_2(A) \cup \mathcal{E}_3(A)$.

For $A := (a_{ij}) \in M_n(\mathbb{H})$, define

$$n_i^{(p)}(A) := \left(\sum_{j=1,\, j \neq i}^{n} |a_{ij}|^p \right)^{\frac{1}{p}} \qquad (1 \leq i \leq n);\ p \in (1, \infty).$$

We are now ready to derive the following localization theorem for left eigenvalues of a quaternionic matrix.

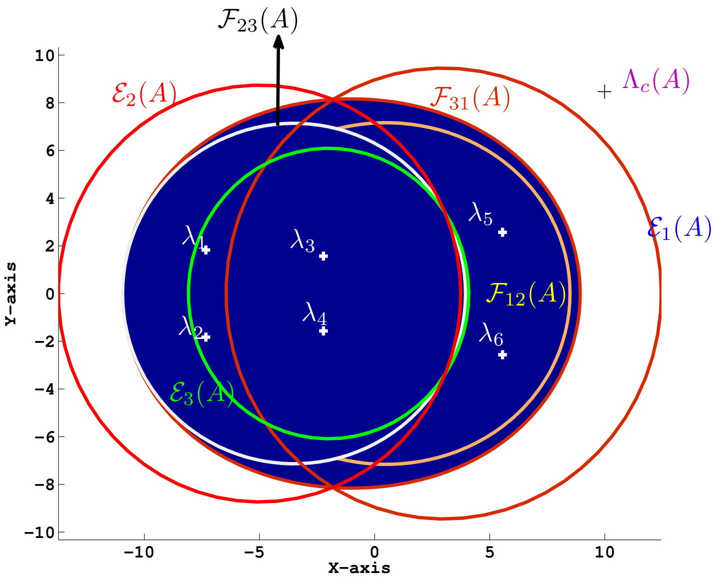


FIGURE 2.1. Location of complex right eigenvalues of A (Example 2.14) as $+$.

Theorem 2.15. *Let $A := (a_{ij}) \in M_n(\mathbb{H})$ and let $\gamma \in [0,1]$. Then all the left eigenvalues of A are contained in the union of n generalized balls*

$$B_i(A) := \left\{z \in \mathbb{H} : |z - a_{ii}| \leq (n-1)^{\frac{1-\gamma}{q}} r_i(A)^{\gamma} (n_i^{(p)}(A))^{1-\gamma}\right\} \ (1 \leq i \leq n), \ i.e.,$$

$$\Lambda_l(A) \subseteq B(A) := \cup_{i=1}^{n} B_i(A),$$

for any $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Let μ be a left eigenvalue of A. Then there exists some nonzero $x \in \mathbb{H}^n$ such that $Ax = \mu x$. Let $x := [x_1, \ldots, x_n]^T \in \mathbb{H}^n$ and let x_t be an element of x such that $|x_t| \geq |x_i| \ \forall \ i \ (1 \leq i \leq n)$. Then from $Ax = \mu x$, we have

$$a_{tt} x_t + \sum_{j=1,\, j \neq t}^{n} a_{tj} x_j \ = \ \mu x_t,$$

this implies

$$|\mu - a_{tt}||x_t| = \left| \sum_{j=1,\, j \neq t}^{n} a_{tj} x_j \right| \leq \sum_{j=1,\, j \neq t}^{n} |a_{tj}| \ |x_j|. \tag{2.18}$$

Applying the generalized Hölder inequality to (2.18), we have

$$|\mu - a_{tt}||x_t| \ \leq \ \left(\sum_{j=1,\, j \neq t}^{n} |a_{tj}|^p \right)^{\frac{1}{p}} \left(\sum_{j=1,\, j \neq t}^{n} |x_j|^q \right)^{\frac{1}{q}}.$$

Since $|x_t| \geq |x_i| \forall\, i\ (1 \leq i \leq n)$, we have

$$|\mu - a_{tt}||x_t| \;\; \leq \;\; n_t^{(p)}(A)\, ((n-1)|x_t|^q)^{\frac{1}{q}}\,, \text{i.e.,}$$

$$|\mu - a_{tt}| \leq n_t^{(p)}(A)\,(n-1)^{\frac{1}{q}}\,. \tag{2.19}$$

Similarly, using $|x_t| \geq |x_i|\ \forall\ i\ (1 \leq i \leq n)$ in (2.18), we get

$$|\mu - a_{tt}| \leq \sum_{j=1,\, j\neq t}^{n} |a_{tj}| = r_t(A). \tag{2.20}$$

Combining (2.19) and (2.20) for $\gamma \in [0,\, 1]$, we have

$$|\mu - a_{tt}|^{1-\gamma} \leq (n_t^{(p)}(A))^{1-\gamma}(n-1)^{\frac{1-\gamma}{q}} \text{ and } |\mu - a_{tt}|^{\gamma} \leq r_t(A)^{\gamma}, \text{ i.e.,} \tag{2.21}$$

$$|\mu - a_{tt}| \leq (n-1)^{\frac{1-\gamma}{q}}(n_t^{(p)}(A))^{1-\gamma} r_t(A)^{\gamma}. \;\; \blacksquare$$

Now, we present the following results from the literature:

- Assuming $p = q = 2$ and $\gamma = 1$ in Theorem 2.15. We obtain that all the left eigenvalues of $A := (a_{ij}) \in M_n(\mathbb{H})$ are contained in the union of n Greschgorin balls $B_i(A) := \{z \in \mathbb{H} : |z - a_{ii}| \leq r_i(A)\}$ $(1 \leq i \leq n)$, i.e.,

  $$\Lambda_l(A) \subseteq B(A) := \cup_{i=1}^{n} B_i(A).$$

  This can be found in [62, Theorem 6].
- Assuming $p = q = 2$ and $\gamma = 0$ in Theorem 2.15. We obtain that all the left eigenvalues of $A := (a_{ij}) \in M_n(\mathbb{H})$ are contained in the union of n balls $B_i(A) := \left\{z \in \mathbb{H} : |z - a_{ii}| \leq (n-1)^{\frac{1}{2}} n_i^{(2)}(A)\right\}$ $(1 \leq i \leq n)$, i.e.,

  $$\Lambda_l(A) \subseteq B(A) := \cup_{i=1}^{n} B_i(A).$$

  This can be seen in [60, Theorem 1].

We present a generalization of [62, Theorem 7] and [63, Theorem 3.1] by applying the generalized Hölder inequality over the skew field of quaternions. For a general matrix $A := (a_{ij}) \in M_n(\mathbb{H})$, all the right eigenvalues may not lie in the union of n generalized balls $B_i(A)\,(1 \leq i \leq n)$. On the other hand, we show that every connected region of the generalized balls $B_i(A)\,(1 \leq i \leq n)$ contains some right eigenvalues of A.

Theorem 2.16. *Let $A := (a_{ij}) \in M_n(\mathbb{H})$ and let $\gamma \in [0, 1]$. For every right eigenvalue μ of A there exists a nonzero quaternion β such that $\beta^{-1}\mu\beta$ (which is also a right eigenvalue) is contained in the union of n generalized balls*

$$B_i(A) := \left\{z \in \mathbb{H} : |z - a_{ii}| \leq (n-1)^{\frac{1-\gamma}{q}} r_i(A)^{\gamma} (n_i^{(p)}(A))^{1-\gamma}\right\} \ (1 \leq i \leq n), \ i.e.,$$

$$\left\{z^{-1}\mu z : 0 \neq z \in \mathbb{H}\right\} \cap \cup_{i=1}^n B_i(A) \neq \emptyset,$$

where $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Let μ be a right eigenvalue of A. Then there exists some nonzero $x \in \mathbb{H}^n$ such that $Ax = x\mu$. Let $x := [x_1, \ldots, x_n]^T \in \mathbb{H}^n$ and choose x_t from x as given in Theorem 2.15. Consider $\rho \in \mathbb{H}$ such that $x_t\mu = \rho x_t$. Then we have

$$|\rho - a_{tt}||x_t| = \left|\sum_{j=1,\, j\neq t}^{n} a_{tj}x_j\right| \leq \sum_{j=1,\, j\neq t}^{n} |a_{tj}|\ |x_j|. \tag{2.22}$$

Now from the proof method of Theorem 2.15, we have

$$|\rho - a_{tt}| \leq (n-1)^{\frac{1-\gamma}{q}} (n_t^{(p)}(A))^{1-\gamma} r_t(A)^{\gamma}. \ \blacksquare$$

We can see the following results from the literature:

- Substituting $p = q = 2$ and $\gamma = 1$ in Theorem 2.16, we obtain

  $$\{z^{-1}\mu z : 0 \neq z \in \mathbb{H}\} \cap \cup_{i=1}^n \{z \in \mathbb{H} : |z - a_{ii}| \leq r_i(A)\} \neq \emptyset.$$

  This can be seen in [62, Theorem 7].
- Substituting $p = q = 2$ and $\gamma = 0$ in Theorem 2.16, we get

  $$\{z^{-1}\mu z : 0 \neq z \in \mathbb{H}\} \cap \cup_{i=1}^n \left\{z \in \mathbb{H} : |z - a_{ii}| \leq \sqrt{n-1}\ n_i^{(2)}(A)\right\} \neq \emptyset.$$

  This can be found in [63, Theorem 3.1].

We next present a sufficient condition for the stability of a matrix $A \in M_n(\mathbb{H})$ for the case of a continuous-time quaternionic system.

Proposition 2.17. *Let $A := (a_{ij}) \in M_n(\mathbb{H})$ and let $\gamma \in [0, 1]$. Assume that*

$$Re(a_{ii}) + (n-1)^{\frac{1-\gamma}{q}} r_i(A)^{\gamma} (n_i^{(p)}(A))^{1-\gamma} < 0 \quad \forall\, i\, (1 \leq i \leq n), \tag{2.23}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ with $p, q \in (1, \infty)$. Then the matrix A is stable.

Proof. Let $\lambda \in \Lambda_r(A)$. Then from Theorem 2.16 there exists $0 \neq \rho \in \mathbb{H}$ such that $\rho^{-1}\lambda\rho \in \cup_{i=1}^{n} B_i(A)$. Without loss of generality, we assume $\rho^{-1}\lambda\rho \in B_l(A)$, i.e.,

$$|\rho^{-1}\lambda\rho - a_{ll}| \leq (n-1)^{\frac{1-\gamma}{q}} r_l(A)^{\gamma}(n_l^{(p)}(A))^{1-\gamma}.$$

Consider $\lambda := \lambda_1 + \lambda_2\mathbf{i} + \lambda_3\mathbf{j} + \lambda_4\mathbf{k}$ and $a_{ll} := a_l + b_l\mathbf{i} + c_l\mathbf{j} + d_l\mathbf{k}$. Then from (2.23), we obtain

$$(2.24) \quad |(\lambda_1 - a_l) + (\rho^{-1}\lambda_2\mathbf{i}\rho - b_l\mathbf{i}) + (\rho^{-1}\lambda_3\mathbf{j}\rho - c_l\mathbf{j}) + \xi_1| < -\mathrm{Re}(a_{ll}) = -a_l,$$

where $\xi_1 = \rho^{-1}\lambda_4\mathbf{k}\rho - d_l\mathbf{k}$. (2.24) is possible when $\lambda_1 < 0$, i.e., $\mathrm{Re}(\lambda) < 0$, hence $\lambda \in \mathbb{H}^-$. This shows that the matrix A is stable. ■

When all the diagonal entries of a matrix $A \in M_n(\mathbb{H})$ are real, then we have the following theorem.

Theorem 2.18. *Let $A := (a_{ij}) \in M_n(\mathbb{H})$ with $a_{ii} \in \mathbb{R}$ and let $\gamma \in [0,1]$. Then all the right eigenvalues of A are contained in the union of n generalized balls*

$$B_i(A) := \left\{z \in \mathbb{H} : |z - a_{ii}| \leq (n-1)^{\frac{1-\gamma}{q}} r_i(A)^{\gamma}(n_i^{(p)}(A))^{1-\gamma}\right\} \quad (1 \leq i \leq n), \text{ i.e.,}$$

$$\Lambda_r(A) \subseteq B(A) := \cup_{i=1}^{n} B_i(A),$$

where $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Let λ be a right eigenvalue of A. Then there exists some nonzero $x \in \mathbb{H}^n$ such that $Ax = x\lambda$. Let $x := [x_1, \ldots, x_n]^T \in \mathbb{H}^n$ and let x_t be an element of x such that $|x_t| \geq |x_i| \; \forall \, i \, (1 \leq i \leq n)$. Then $|x_t| > 0$. Thus from $Ax = x\lambda$, we have

$$a_{tt}x_t + \sum_{j=1,\, j\neq t}^{n} a_{tj}x_j \;=\; x_t\lambda.$$

Since $a_{tt} \in \mathbb{R}$, so $a_{tt}x_t = x_t a_{tt}$. Then from the proof method of Theorem 2.15, we have

$$|\lambda - a_{tt}| \leq (n-1)^{\frac{1-\gamma}{q}} (n_t^{(p)}(A))^{1-\gamma} r_t(A)^{\gamma}. \; ■$$

The above result has great significance as Hermitian and η-Hermitian matrices have all real diagonal entries. In general, η-Hermitian matrices arise widely in applications [17, 55, 56]. To that end, we state the following proposition when all diagonal entries of $A \in M_n(\mathbb{H})$ are real. In particular, this result gives a sufficient condition for the stability of a matrix $A \in M_n(\mathbb{H})$.

Proposition 2.19. *Let $A := (a_{ij}) \in M_n(\mathbb{H})$ with $a_{ii} \in \mathbb{R}$ and let $\gamma \in [0,1]$. Assume that*

$$a_{ii} + (n-1)^{\frac{1-\gamma}{q}} r_i(A)^{\gamma}(n_i^{(p)}(A))^{1-\gamma} < 0 \ \forall i \, (1 \leq i \leq n),$$

where $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then the matrix A is stable.

From Theorem 2.18, all the complex right eigenvalues of a matrix $A = (a_{ij}) \in M_n(\mathbb{H})$ with all real diagonal entries lie in the union of n-discs

$$D_i(A) := \{z \in \mathbb{C} : |z - a_{ii}| \leq (n-1)^{\frac{1-\gamma}{q}} r_i(A)^{\gamma}(n_i^{(p)}(A))^{1-\gamma}\} \ \ (1 \leq i \leq n), \text{ i.e.,}$$

$$\Lambda_c(A) \subseteq D(A) := \cup_{i=1}^{n} D_i(A). \tag{2.25}$$

However, if diagonal entries are from $\mathbb{C} \setminus \mathbb{R}$, then it is not necessary that all the complex right eigenvalues of A are contained in the union of n-discs $D_i(A)$ $(1 \leq i \leq n)$ as the following examples suggest.

Example 2.20. Let $A := \begin{bmatrix} 1-2\mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -2\mathbf{i} & -\mathbf{i} \\ 0 & \mathbf{k} & 3+\mathbf{i} \end{bmatrix}$. Then the set of complex right eigenvalues of A is given by

$$\Lambda_c(A) := \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6\},$$

where $\lambda_1 = -0.0164 + 2.0083\mathbf{i}$, $\lambda_2 = -0.0164 - 2.0083\mathbf{i}$, $\lambda_3 = 1 + 2\mathbf{i}$, $\lambda_4 = 1 - 2\mathbf{i}$, $\lambda_5 = 3.0164 + 1.0324\mathbf{i}$, and $\lambda_6 = 3.0164 + 1.0324\mathbf{i}$. For $\gamma = 1$ in (2.25), the discs $D_1(A), D_2(A)$, and $D_3(A)$ are given as follows:

$$\begin{aligned} D_1(A) &:= \{z \in \mathbb{C} : |z - 1 + 2\mathbf{i}| \leq 2\}, \\ D_2(A) &:= \{z \in \mathbb{C} : |z + 2\mathbf{i}| \leq 1\}, \text{ and} \\ D_3(A) &:= \{z \in \mathbb{C} : |z - 3 - \mathbf{i}| \leq 1\}. \end{aligned}$$

From FIGURE 2.2, it is clear that λ_1, λ_3, and λ_6 lie outside of the discs $D_1(A), D_2(A)$, and $D_3(A)$.

Example 2.21. Let $A = \begin{bmatrix} -4 & 1+\mathbf{j}+\sqrt{2}\mathbf{k} & \mathbf{j} \\ \mathbf{i}+\mathbf{j} & -10 & 2\mathbf{j}-\mathbf{k} \\ \mathbf{i}-2\mathbf{j}+2\mathbf{k} & \sqrt{3}+2\mathbf{j}-3\mathbf{k} & -8 \end{bmatrix}$. In this example, there are six complex right eigenvalues λ_j $(1 \leq j \leq 6)$ which are shown in FIGURE 2.3.

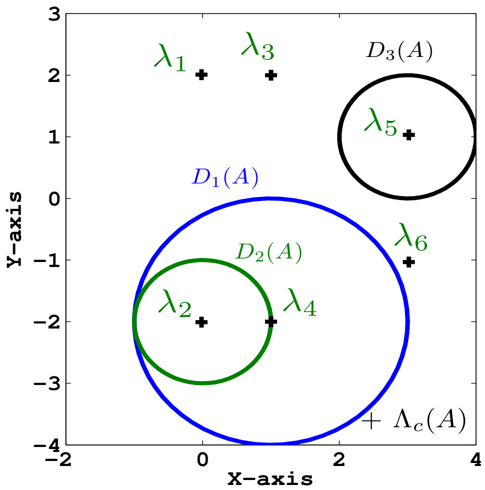


FIGURE 2.2. Location of complex right eigenvalues of A (Example 2.20) as $+$.

Substituting $\gamma = 1$ in (2.25), then all the complex right eigenvalues of the matrix A are contained in the union of three discs $D_1(A), D_2(A)$, and $D_3(A)$, where

$$\begin{aligned} D_1(A) &:= \{z \in \mathbb{C} : |z + 4| \leq 3\}, \\ D_2(A) &:= \{z \in \mathbb{C} : |z + 10| \leq \sqrt{2} + \sqrt{5}\}, \text{ and} \\ D_3(A) &:= \{z \in \mathbb{C} : |z + 8| \leq 7\}. \end{aligned}$$

From FIGURE 2.3, the standard right eigenvalues of A are λ_1, λ_3, and λ_5. Thus

$$\Lambda_r(A) = [\lambda_1] \cup [\lambda_3] \cup [\lambda_5].$$

Also, from FIGURE 2.3, we observe that $\mathrm{Re}(\lambda_i) \in \mathbb{H}^-$ $(i = 1, 3, 5)$. Hence

$$\mathrm{Re}(\lambda_1) = \mathrm{Re}(\rho^{-1}\lambda_1\rho),\ \mathrm{Re}(\lambda_2) = \mathrm{Re}(\tau^{-1}\lambda_2\tau),\ \mathrm{Re}(\lambda_3) = \mathrm{Re}(\nu^{-1}\lambda_3\nu)\ \ \forall\, \rho, \tau, \nu \in \mathbb{H}$$

Thus the matrix A is stable.

In general, similar quaternionic matrices may not have the same left eigenvalues, see, [62, Example 3.3]. However, the following result is true.

Proposition 2.22. *Let $A \in M_n(\mathbb{H})$ and let W be any invertible real matrix. Then A and WAW^{-1} have the same left eigenvalues.*

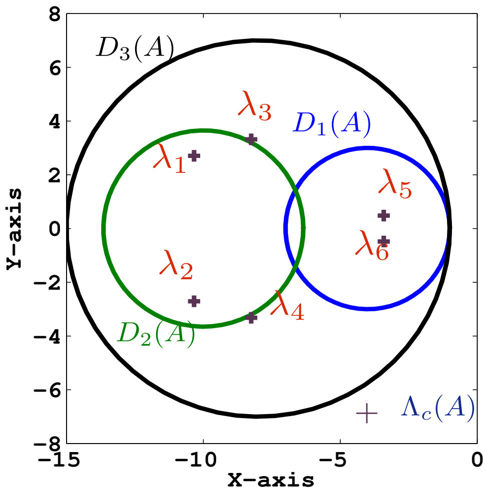


FIGURE 2.3. Location of complex right eigenvalues of A (Example 2.21) as $+$.

Proof. Let λ be a left eigenvalue of A. Then there exists some nonzero $x \in \mathbb{H}^n$ such that $Ax = \lambda x$. Let W be an invertible real matrix. Then

$$WAx = W\lambda x = \lambda Wx.$$

Now, $WAW^{-1}Wx = \lambda Wx$. Putting $Wx = y$. Then $WAW^{-1}y = \lambda y$. ■

Let $A := (a_{ij}) \in M_n(\mathbb{H})$. Suppose $W = \text{diag}(w_1, w_2, \ldots, w_n)$ with $w_i \in \mathbb{R}^+ \, (1 \leq i \leq n)$. Then

$$W^{-1}AW = \left(\frac{a_{ij}w_j}{w_i}\right) \text{ and } \Lambda_l(A) = \Lambda_l(W^{-1}AW).$$

Define

$$r_i^W(A) := \sum_{j=1,\, j\neq i}^{n} \frac{|a_{ij}|w_j}{w_i} \text{ and } c_i^W(A) := \sum_{j=1,\, j\neq i}^{n} \frac{|a_{ji}|w_i}{w_j} \quad (1 \leq i \leq n).$$

Applying Theorem 2.3 to $W^{-1}AW$, we get the following theorem which may be sharper than Theorem 2.3 depending upon the choice of W.

Theorem 2.23. *Let $A := (a_{ij}) \in M_n(\mathbb{H})$. Then all the left eigenvalues of A are contained in the union of n balls*

$$T_i^W(A) := \{z \in \mathbb{H} : |z - a_{ii}| \leq (r_i^W(A))^{\gamma}\, (c_i^W(A))^{1-\gamma}\} \;\; (1 \leq i \leq n), \textit{ i.e.,}$$

$$\Lambda_l(A) = \Lambda_l(W^{-1}AW) \subseteq T^W(A) := \cup_{i=1}^{n} T_i^W(A).$$

Since the above theorem holds for every $W = \text{diag}(w_1, w_2, \ldots, w_n)$, where $w_i \in \mathbb{R}^+$, we have

$$\Lambda_l(A) = \Lambda_l(W^{-1}AW) \subseteq \bigcap_{W \in M_n(S)} T^W(A) =: T^S(A),$$

where $M_n(S)$ is a set of invertible real diagonal matrices and $T^S(A)$ is called the minimal Ostrowski type set for the matrix A.

Substituting $\gamma = 1$ in Theorem 2.23, we obtain the following.

$$(\mathbf{a})\ \Lambda_l(A) = \Lambda_l(W^{-1}AW) \subseteq \eta^W(A) := \cup_{i=1}^n \eta_i^W(A),$$

where $\eta_i^W(A) := \left\{z \in \mathbb{H} : |z - a_{ii}| \leq r_i^W(A)\right\}$. Now from here, we get

$$\Lambda_l(A) = \Lambda_l(W^{-1}AW) \subseteq \bigcap_{W \in M_n(S)} \eta^W(A) =: \eta^S(A),$$

where $\eta^S(A)$ is called the first minimal Gerschgorin type set for the matrix A.

For $\gamma = 0$ in Theorem 2.23, we have the following.

$$(\mathbf{b})\ \Lambda_l(A) = \Lambda_l(W^{-1}AW) \subseteq \Omega^W(A) := \cup_{i=1}^n \Omega_i^W(A),$$

where $\Omega_i^W(A) := \left\{z \in \mathbb{H} : |z - a_{ii}| \leq c_i^W(A)\right\}$. Then

$$\Lambda_l(A) = \Lambda_l(W^{-1}AW) \subseteq \bigcap_{W \in M_n(S)} \Omega^W(A) =: \Omega^S(A),$$

where $\Omega^S(A)$ is called the second minimal Gerschgorin type set for the matrix A.

Equivalently, applying Theorem 2.8 to $W^{-1}AW$, we get the following theorem.

Theorem 2.24. *Let $A := (a_{ij}) \in M_n(\mathbb{H})$ and let $\gamma \in [0, 1]$. Then all the left eigenvalues of A are contained in the union of $\frac{n(n-1)}{2}$ ovals of Cassini $K_{ij}^W(A) := \{z \in \mathbb{H} : |z - a_{ii}|\,|z - a_{jj}| \leq \left(r_i^W(A)\right)^\gamma (r_j^W(A))^\gamma (c_i^W(A))^{1-\gamma} (c_j^W(A))^{1-\gamma}\}$ $(1 \leq i, j \leq n; i \neq j)$, i.e.,*

$$\Lambda_l(A) = \Lambda_l(W^{-1}AW) \subseteq K^W(A) := \cup_{\substack{i,j=1 \\ i \neq j}}^n K_{ij}^W(A).$$

Since Theorem 2.24 holds for every $W = \text{diag}(w_1, w_2, \ldots, w_n)$ with $w_i \in \mathbb{R}^+$. Then

$$\Lambda_l(A) = \Lambda_l(W^{-1}AW) \subseteq \bigcap_{W \in M_n(S)} K^W(A) =: K^S(A),$$

$K^S(A)$ is called the minimal Brauer type set for the matrix A.

Example 2.25. Let $A = \begin{bmatrix} \mathbf{j} & \mathbf{k} & \mathbf{2j} + \sqrt{\mathbf{5}}\mathbf{k} \\ 0 & \mathbf{i} + \mathbf{k} & \sqrt{\mathbf{2}}\mathbf{i} + \mathbf{j} - \mathbf{k} \\ 0 & 0 & \mathbf{2} - \mathbf{i} \end{bmatrix}$. Substituting $\gamma = 1$ in Theorem 2.3, we have the following three Gerschgorin type balls:

$$\begin{aligned} G_1(A) &:= \{z \in \mathbb{H} : |z - \mathbf{j}| \leq 4\}, \\ G_2(A) &:= \{z \in \mathbb{H} : |z - \mathbf{i} - \mathbf{k}| \leq 2\}, \text{ and} \\ G_3(A) &:= \{z \in \mathbb{H} : |z - \mathbf{2} + \mathbf{i}| \leq 0\}. \end{aligned}$$

If $W = \text{diag}(w_1, w_2, w_3)$ with $w_1 = 8,\ w_2 = 4, w_3 = 1$. Then by **(a)**, we have the following balls:

$$\begin{aligned} \eta_1^W(A) &:= \{z \in \mathbb{H} : |z - \mathbf{j}| \leq 7/8\}, \\ \eta_2^W(A) &:= \{z \in \mathbb{H} : |z - \mathbf{i} - \mathbf{k}| \leq 1/2\}, \text{ and} \\ \eta_3^W(A) &:= \{z \in \mathbb{H} : |z - \mathbf{2} + \mathbf{i}| \leq 0\}. \end{aligned}$$

Hence it is clear that $\eta_1^W(A) \subset G_1(A)$ and $\eta_2^W(A) \subset G_2(A)$.

For $\gamma = 1$, Theorem 2.8 gives the following ovals of Cassini:

$$\begin{aligned} K_{12}(A) &:= \{z \in \mathbb{H} : |z - \mathbf{j}|\,|z - \mathbf{i} - \mathbf{k}| \leq 8\}, \\ K_{23}A) &:= \{z \in \mathbb{H} : |z - \mathbf{i} - \mathbf{k}|\,|z - \mathbf{2} + \mathbf{i}| \leq 0\}, \text{ and} \\ K_{31}(A) &:= \{z \in \mathbb{H} : |z - \mathbf{2} + \mathbf{i}|\,|z - \mathbf{j}| \leq 0\}. \end{aligned}$$

Consider $W = \text{diag}(w_1, w_2, w_3)$ with $w_1 = w_2 = 6$, and $w_3 = 1$. Then by Theorem 2.24 with $\gamma = 1$, we obtain the following ovals of Cassini:

$$\begin{aligned} K_{12}^W(A) &:= \{z \in \mathbb{H} : |z - \mathbf{j}|\,|z - \mathbf{i} - \mathbf{k}| \leq 1/2\}, \\ K_{23}^W(A) &:= \{z \in \mathbb{H} : |z - \mathbf{i} - \mathbf{k}|\,|z - \mathbf{2} + \mathbf{i}| \leq 0\}, \text{ and} \\ K_{31}^W(A) &:= \{z \in \mathbb{H} : |z - \mathbf{2} + \mathbf{i}|\,|z - \mathbf{j}| \leq 0\}. \end{aligned}$$

Hence $K_{12}^W(A) \subset K_{12}(A)$. ■

2.3. Bounds for the zeros of quaternionic polynomials

First, in this section, we present bounds for the zeros of quaternionic polynomial $p_l(z)$ as follows, which is an extension of the result given in [15] for the case of the zeros of complex polynomials.

Theorem 2.26. *Let $p_l(z)$ be a simple monic polynomial over $\mathbb{H}$ of degree m. Then every zero $\tilde{z}$ of $p_l(z)$ satisfies the following inequality:*

$$\left(\max_{1\leq i\leq m}\left(r_i'(C_{q_l})^{\gamma}\, c_i'(C_{q_l})^{1-\gamma}\right)\right)^{-1} \leq |\tilde{z}| \leq \max_{1\leq i\leq m}\left(r_i'(C_{p_l})^{\gamma}\, c_i'(C_{p_l})^{1-\gamma}\right),$$

for every $\gamma \in [0,1]$.

Proof. From Proposition 1.36, zeros of $p_l(z)$ and left eigenvalues of C_{p_l} are same. Thus, if $\tilde{z}$ is a zero of $p_l(z)$, then $\tilde{z}$ is a left eigenvalue of C_{p_l}. By applying Theorem 2.3 (Ostrowski type theorem) to C_{p_l}, we obtain

$$|\tilde{z}| \leq \max_{1\leq i\leq m}\left(r_i'(C_{p_l})^{\gamma}\, c_i'(C_{p_l})^{1-\gamma}\right).$$

Proof for lower bounds: We use the respective upper bounds for the zeros of the simple monic reversal polynomial $q_l(z)$ for the desired lower bounds for the zeros of $p_l(z)$. ■

Corollary 2.27. *Let $p_l(z)$ be a simple monic polynomial over $\mathbb{H}$ of degree m. Then every zero $\tilde{z}$ of $p_l(z)$ satisfies the following inequalities:*

1. $\dfrac{|q_0|}{\max\limits_{1\leq i\leq (m-1)}\{1, |q_0|+|q_i|\}} \leq |\tilde{z}| \leq \max\limits_{1\leq i\leq (m-1)}\{|q_0|, 1+|q_i|\}.$
2. $\dfrac{|q_0|}{\max\left\{|q_0|, 1+\sum_{i=1}^{m-1}|q_i|\right\}} \leq |\tilde{z}| \leq \max\left\{1, \sum_{i=0}^{m-1}|q_i|\right\}.$

Proof. Substituting $\gamma = 0, 1$ in Theorem 2.26, we obtain the desired results. ■

Next, we derive the following lemma which gives a better bound than G. Opfer's bound [42, Theorem 4.2] for $|q_0| \geq 1$.

Lemma 2.28. *Assume that $|q_0| \geq 1$. Then $\alpha \leq \mathcal{T}$, where $\alpha := \max\limits_{1\leq i\leq m-1}\{|q_0|, 1+|q_i|\}$ and $\mathcal{T} := \max\left\{1, \sum_{i=0}^{m-1}|q_i|\right\}$.*

Proof. **Case 1:** If $|q_0| = 1$, then

$$\alpha = \max_{1\leq i\leq m-1}\{|q_0|, 1+|q_i|\} = \max_{1\leq i\leq m-1}\{1+|q_i|\}. \text{ Also}$$

$\mathcal{T} := \max\left\{1, \sum_{i=0}^{m-1} |q_i|\right\} = \max\left\{1, |q_0| + \sum_{i=1}^{m-1} |q_i|\right\} = 1 + \sum_{i=1}^{m-1} |q_i|$.

Case 2: If $|q_0| > 1$, then

$$\alpha = \max_{1\le i\le (m-1)} \{|q_0|, 1+|q_i|\} = |q_0| \text{ or } \max_{1\le i\le (m-1)} \{1+|q_i|\} \text{ and}$$

$\mathcal{T} := \max\{1, \sum_{i=0}^{m-1} |q_i|\} = \max\left\{1, |q_0| + \sum_{i=1}^{m-1} |q_i|\right\} = |q_0| + \sum_{i=1}^{m-1} |q_i|$. Thus $\alpha \le \mathcal{T}$. This completes the proof. ■

On the other hand, if $|q_0| < 1$, then $\alpha \le \mathcal{T}$ or $\alpha > \mathcal{T}$ follows from a simple monic polynomial $p_l'(z) := z^3 + (\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})z^2 - 2\mathbf{k}z + 0.5\mathbf{k}$. Then $\alpha = 4$ and $\mathcal{T} = 5.5$. Hence $\alpha < \mathcal{T}$ and further, if we consider $p_l''(z) = z^3 + +0.5\mathbf{j}z^2 + (0.2\mathbf{i} + 0.3\mathbf{j})z + 0.5\mathbf{i}$. Then $\alpha = 1.5$ and $\mathcal{T} = 1.36$. Hence $\alpha > \mathcal{T}$.

Theorem 2.29. *Let $w_i \in \mathbb{R}^+$ $(1 \le i \le m)$ and let $\gamma \in [0,1]$. Then every zero $\tilde{z}$ of the simple monic polynomial $p_l(z)$ satisfies the following inequality:*

$$\left[\max_{1\le i\le m} \left\{r_i'(WC_{q_l}W^{-1})^{\gamma}\, c_i'(WC_{q_l}W^{-1})^{1-\gamma}\right\}\right]^{-1} \le |\tilde{z}| \le \xi_1,$$

where $\xi_1 = \max\limits_{1\le i\le m} \{r_i'(WC_{p_l}W^{-1})^{\gamma}\, c_i'(WC_{p_l}W^{-1})^{1-\gamma}\}$ *and* $W := \operatorname{diag}(w_1, w_2, \ldots, w_m)$.

Proof. The companion matrix of $p_l(z)$ is given by

$$C_{p_l} = \begin{array}{c} \\ m-1 \\ 1 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 1 & m-1 \end{array} \\ \left[\begin{array}{c|c} 0 & I \\ \hline -q_0 & [-q_1 \ldots - q_{m-1}] \end{array}\right] \end{array}.$$

Then

$$WC_{p_l}W^{-1} = \begin{array}{c} \\ m-1 \\ 1 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 1 & m-1 \end{array} \\ \left[\begin{array}{c|c} 0 & \operatorname{diag}\left(\frac{w_1}{w_2}, \ldots, \frac{w_{m-1}}{w_m}\right) \\ \hline -\frac{w_m}{w_1} q_0 & -\frac{w_m}{w_2} q_1 \ldots - q_{m-1} \end{array}\right] \end{array}.$$

By Proposition 2.22, C_{p_l} and $WC_{p_l}W^{-1}$ have the same left eigenvalues. Rest of the proof follows from the proof method of Theorem 2.26. ■

Corollary 2.30. *Let $p_l(z)$ be a simple monic polynomial over $\mathbb{H}$ of degree m. Then every zero $\tilde{z}$ of $p_l(z)$ satisfies the following inequalities:*

1. $\left[\max\limits_{0\le j\le m-1} \left\{\dfrac{(|q_0|w_j + w_m|q_{m-j}|)}{|q_0|d_{j+1}}\right\}\right]^{-1} \le |\tilde{z}| \le \max\limits_{0\le j\le m-1} \left\{\dfrac{w_j + w_m|q_j|}{w_{j+1}}\right\}$, *where* $w_0 = 0$.
2. $\left[\max\limits_{1\le j\le m-1} \left\{\dfrac{w_j}{w_{j+1}}, \sum\limits_{i=0}^{m-1} \dfrac{w_m|q_i|}{|q_0|w_{i+1}}\right\}\right]^{-1} \le |\tilde{z}| \le \max\limits_{1\le j\le m-1} \left\{\dfrac{w_j}{w_{j+1}}, \sum\limits_{i=0}^{m-1} \dfrac{w_m|q_i|}{w_{i+1}}\right\}$.

Proof. Substituting $\gamma = 0, 1$ in Theorem 2.29, we get the desired results. ■

Let $w_j = w_m|q_j|$ $(1 \leq j \leq m-1)$ in the part (1) of Corollary 2.30. Then we obtain

$$|\tilde{z}| \leq \max_{1 \leq j \leq m-1} \left\{ \left|\frac{q_0}{q_1}\right|, 2\left|\frac{q_j}{q_{j+1}}\right| \right\}.$$

This is called the Kojima type bound for the zeros of the simple monic polynomial $p_l(z)$.

Bounds for the zeros of quaternionic polynomial $p_r(z)$: For computation of bounds of the zeros of $p_r(z)$, we define the following polynomial:

$$\tilde{p_l}(z) := \overline{p_r(\overline{z})} := \sum_{j=0}^{m} \overline{q_j} z^j.$$

Now, we discuss the following theorem which shows relation between the zeros of $p_r(z)$ and $\tilde{p_l}(z)$.

Theorem 2.31. *Let $\lambda \in \mathbb{H}$. Then λ is a zero of the simple monic polynomial $p_r(z)$ if and only if $\overline{\lambda}$ is a zero of the simple monic polynomial $\tilde{p_l}(z)$.*

Proof. The corresponding companion matrices of $p_r(z)$ and $\tilde{p_l}(z)$ are given by

$$C_{p_r} := C_{p_l}^T \text{ and } C_{\tilde{p_l}} := C_{p_r}^H,$$

respectively. By Lemma 2.1, if λ is a left eigenvalue of C_{p_r}, then $\overline{\lambda}$ is a left eigenvalue of $C_{p_r}^H = C_{\tilde{p_l}}$. By Propositions 1.36 and 1.37, the left eigenvalues of C_{p_r} and $C_{\tilde{p_l}}$ imply the zeros of $p_r(z)$ and $\tilde{p_l}(z)$, respectively. Hence if λ is a zero of $p_r(z)$, then $\overline{\lambda}$ is also a zero of $\tilde{p_l}(z)$. ■

Remark 2.32. Similar results can be obtained for the quaternionic polynomial $p_r(z)$ as well.

Bounds for the zeros of quaternionic polynomials by using the powers of companion matrices: First, we present some preliminary results for the powers of companion matrices C_{p_l} and C_{p_r}. In general, if λ is a left eigenvalue of a quaternionic matrix A, then λ^2 is not necessarily a left eigenvalue of A^2 follows from the following quaternionic matrix

$$A = \begin{bmatrix} 0 & \mathbf{i} \\ -\mathbf{i} & 0 \end{bmatrix}.$$

Then

$$\Lambda_l(A) := \left\{\mu : \mu = \alpha + \beta\mathbf{j} + \gamma\mathbf{k}, \alpha^2 + \beta^2 + \gamma^2 = 1\right\}.$$

Now, we have

$$A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \text{ So } \Lambda_l(A^2) := \{1\}.$$

Here, $\mathbf{j}$ is a left eigenvalue of A. However, $\mathbf{j}^2$ is not a left eigenvalue of A^2.

Now, we prove the following result for left eigenvalues of C_{p_l} and $C_{p_l}^t$ (t is a nonzero integer).

Proposition 2.33. *If λ is a left eigenvalue of C_{p_l} with respect to the eigenvector $x \in \mathbb{H}^n$, then λ^t (t is a nonzero integer) is a left eigenvalue of $C_{p_l}^t$ corresponding to the same eigenvector $x \in \mathbb{H}^n$.*

Proof. **Case (a)**: When t is a nonzero positive integer. Let λ be a left eigenvalue of C_{p_l}. Then, there exists $0 \neq x := [1, \lambda, \lambda^2, \ldots, \lambda^{m-1}]^T \in \mathbb{H}^n$ such that $C_{p_l}x = \lambda x$. Therefore,

$$\begin{aligned} C_{p_l}^2 x &= C_{p_l}(C_{p_l}x) = C_{p_l}x\lambda = x\lambda^2 \\ &\vdots \\ C_{p_l}^t x &= C_{p_l}^{t-1}(C_{p_l}x) = C_{p_l}^{t-1}x\lambda = \cdots = x\lambda^t = \lambda^t x. \end{aligned}$$

Thus, λ^t is a left eigenvalue of matrix $C_{p_l}^t$ corresponding to the same eigenvector $x \in \mathbb{H}^n$.

Case (b): When t is a negative integer. From **Case (a)**, we have $C_{p_l}x = x\lambda$. This implies $C_{p_l}^{-1}x = x\lambda^{-1}$. Therefore,

$$\begin{aligned} C_{p_l}^{-2} x &= C_{p_l}^{-1}(C_{p_l}^{-1}x) = C_{p_l}^{-1}x\lambda^{-1} = x\lambda^{-2} \\ &\vdots \\ C_{p_l}^t x &= C_{p_l}^{(t+1)}(C_{p_l}^{-1}x) = C_{p_l}^{(t+1)}x\lambda^{-1} = \cdots = x\lambda^t = \lambda^t x. \end{aligned}$$

Thus, λ^t is a left eigenvalue of $C_{p_l}^t$ with respect to the same eigenvector $x \in \mathbb{H}^n$. ■

Next, we state the following result for left eigenvalues of C_{p_r} and $C_{p_r}^t$ (t is a nonzero integer).

Proposition 2.34. *If λ is a left eigenvalue of C_{p_r} with respect to the eigenvector $x \in \mathbb{H}^n$, then λ^t (t is a nonzero integer) is a left eigenvalue of $C_{p_r}^t$ corresponding to the same eigenvector $x \in \mathbb{H}^n$.*

Proof. **Case (a)**: When t is a nonzero positive integer. Let λ be a left eigenvalue of C_{p_r}. Now from Lemma 2.1, $\overline{\lambda}$ is a left eigenvalue of $C_{p_r}^H$. Then there exists $0 \neq x := \left[1, \overline{\lambda}, (\overline{\lambda})^2, \dots, (\overline{\lambda})^{m-1}\right] \in \mathbb{H}^n$ such that $C_{p_r}^H x = \overline{\lambda} x = x\overline{\lambda}$. This gives

$$\begin{aligned}
\left(C_{p_r}^H\right)^2 x &= C_{p_r}^H(C_{p_r}^H x) = C_{p_r}^H x\overline{\lambda} = x(\overline{\lambda})^2 \\
&\vdots \\
\left(C_{p_r}^H\right)^t x &= \left(C_{p_r}^H\right)^{t-1}(C_{p_r}^H x) = \left(C_{p_r}^H\right)^{t-1} x\overline{\lambda} = \cdots = x(\overline{\lambda})^t = (\overline{\lambda})^t x.
\end{aligned}$$

Thus, $(\overline{\lambda})^t$ is a left eigenvalue of $\left(C_{p_r}^H\right)^t$. Then by Lemma 2.1, λ^t is a left eigenvalue of $C_{p_r}^t$.

Case (b): When t is a negative integer. From **Case (a)**, we have $C_{p_r}^H x = \overline{\lambda} x = x\overline{\lambda}$. This implies $(C_{p_r}^H)^{-1} x = x(\overline{\lambda})^{-1}$. Thus

$$\begin{aligned}
(C_{p_r}^H)^{-2} x &= (C_{p_r}^H)^{-1}\{(C_{p_r}^H)^{-1} x\} = (C_{p_r}^H)^{-1} x(\overline{\lambda})^{-1} = x(\overline{\lambda})^{-2} \\
&\vdots \\
(C_{p_r}^H)^t x &= (C_{p_r}^H)^{(t+1)}\{(C_{p_r}^H)^{-1} x\} = (C_{p_r}^H)^{(t+1)} x(\overline{\lambda})^{-1} = \cdots = x(\overline{\lambda})^t = (\overline{\lambda})^t x.
\end{aligned}$$

Thus, $(\overline{\lambda})^t$ is a left eigenvalue of $\left(C_{p_r}^H\right)^t$. Then by Lemma 2.1, λ^t is a left eigenvalue of $C_{p_r}^t$. ■

Further, we present a framework to find the powers of the companion matrix C_{p_l} which can be derived in a simple procedure as follows, keeping in view that quaternions do not commute.

Theorem 2.35. *Consider* $C_{p_l} = \begin{array}{c} \\ {\scriptstyle m-1} \\ {\scriptstyle 1} \end{array} \begin{array}{c} \begin{array}{cc} {\scriptstyle 1} & {\scriptstyle m-1} \end{array} \\ \left[\begin{array}{c|c} 0 & I \\ \hline C_{p_l}(m,1) & C_{p_l}(m,2:m) \end{array}\right] \end{array}.$

(a) *If $t < m$ is a positive integer, then*

$$C_{p_l}^t = \begin{array}{c} \\ {\scriptstyle m-t} \\ {\scriptstyle t} \end{array} \begin{array}{c} \begin{array}{cc} {\scriptstyle t} & {\scriptstyle m-t} \end{array} \\ \left[\begin{array}{c|c} 0 & I \\ \hline C & D \end{array}\right] \end{array}, \tag{2.26}$$

(b) *if $t \geq m$, then*

$$C_{p_l}^t = \begin{bmatrix} C_{p_l}^{t-(m-1)}(m,1:m) \\ C_{p_l}^{t-(m-2)}(m,1:m) \\ \vdots \\ C_{p_l}^{t-1}(m,1:m) \\ C_{p_l}^{t}(m,1:m) \end{bmatrix}_{m\times m}, \tag{2.27}$$

where

$$\begin{aligned} C_{p_l}^t(m,1) &:= C_{p_l}^{t-1}(m,m)C_{p_l}(m,1), \\ C_{p_l}^t(m,2:m) &:= C_{p_l}^{t-1}(m,1:m-1) + C_{p_l}^{t-1}(m,m)C_{p_l}(m,2:m), \end{aligned}$$

$$C := \begin{bmatrix} C_{p_l}(m,1:t) \\ C_{p_l}^2(m,1:t) \\ \vdots \\ C_{p_l}^t(m,1:t) \end{bmatrix}_{t\times t}, \text{ and } D := \begin{bmatrix} C_{p_l}(m,t+1:m) \\ C_{p_l}^2(m,t+1:m) \\ \vdots \\ C_{p_l}^t(m,t+1:m) \end{bmatrix}_{t\times(m-t)}.$$

Note that $C_{p_l}(k,1:m)$ denotes the k-th row of the matrix C_{p_l}.

Proof. Assuming $t = 1$, then (2.26) becomes

$$C_{p_l} = \begin{array}{c} {\scriptstyle m-1} \\ {\scriptstyle 1} \end{array}\!\!\overset{\begin{array}{cc} {\scriptstyle 1} & {\scriptstyle m-1} \end{array}}{\left[\begin{array}{c|c} 0 & I \\ \hline C_{p_l}(m,1) & C_{p_l}(m,2:m) \end{array}\right]},$$

where $C_{p_l}(m,1) := -q_0, C_{p_l}(m,2:m) := [-q_1 \ldots - q_{m-1}]$. Thus the theorem is true for $t = 1$. Now, let us consider C_{p_l} as

$$C_{p_l} = \begin{array}{c} {\scriptstyle k} \\ {\scriptstyle m-k} \end{array}\!\!\overset{\begin{array}{cc} {\scriptstyle m-k} & {\scriptstyle k} \end{array}}{\left[\begin{array}{c|c} A' & B' \\ \hline C' & D' \end{array}\right]}, \text{ where}$$

$A' := C_{p_l}(1:k,1:m-k), B' := C_{p_l}(k+1:m,m-k+1:m), C' := C_{p_l}(k+1:m,1:m-k), D' := C_{p_l}(k+1:m,m-k+1:m)$. For $t = k = 3$, we get

$$C_{p_l}^3 = \begin{array}{c} {\scriptstyle m-2} \\ {\scriptstyle 2} \end{array}\!\!\overset{\begin{array}{cc} {\scriptstyle 2} & {\scriptstyle m-2} \end{array}}{\left[\begin{array}{c|c} 0 & I \\ \hline C & D \end{array}\right]} \begin{array}{c} {\scriptstyle 2} \\ {\scriptstyle m-2} \end{array}\!\!\overset{\begin{array}{cc} {\scriptstyle m-2} & {\scriptstyle 2} \end{array}}{\left[\begin{array}{c|c} A' & B' \\ \hline C' & D' \end{array}\right]} = \begin{array}{c} {\scriptstyle m-2} \\ {\scriptstyle 2} \end{array}\!\!\overset{\begin{array}{cc} {\scriptstyle m-2} & {\scriptstyle 2} \end{array}}{\left[\begin{array}{c|c} C' & D' \\ \hline CA' + DC' & CB' + DD' \end{array}\right]}.$$

Note that in each step, size of the identity matrix I reduces by order 1 and the size of matrix C increases by order 1. Similarly, the matrix D increases by 1 row and decreases by 1 column. Finally, after rearranging and separating 0 and I matrices we get

$$\begin{array}{c c} & \begin{array}{cc} {\scriptstyle 2+1} & {\scriptstyle m-2-1} \end{array} \\ \begin{array}{c} {\scriptstyle m-2-1} \\ {\scriptstyle 2+1} \end{array} & \left[\begin{array}{c|c} 0 & I \\ \hline C & D \end{array}\right] \end{array},$$

where C and D are of size 3×3 and $3 \times (m-3)$, respectively. Assuming that the theorem is true for $t = k$. Then we have

$$C_{p_l}^{k+1} = C_{p_l}^{k} C_{p_l} = \begin{array}{c c} & \begin{array}{cc} {\scriptstyle m-k} & {\scriptstyle k} \end{array} \\ \begin{array}{c} {\scriptstyle m-k} \\ {\scriptstyle k} \end{array} & \left[\begin{array}{c|c} C' & D' \\ \hline CA' + DC' & CB' + DD' \end{array}\right] \end{array} = \begin{array}{c c} & \begin{array}{cc} {\scriptstyle k+1} & {\scriptstyle m-k-1} \end{array} \\ \begin{array}{c} {\scriptstyle m-k-1} \\ {\scriptstyle k+1} \end{array} & \left[\begin{array}{c|c} 0 & I \\ \hline C & D \end{array}\right] \end{array},$$

where the corresponding C and D matrices are given in the statement of the theorem. Similarly, we can prove for $t \geq m$. ■

In the case of quaternionic matrix, $C_{p_l} = C_{p_r}^T$ but $C_{p_r}^t \neq (C_{p_l}^t)^T$ for $t \geq 2$. It follows from the following example.

Example 2.36. Consider the following simple monic polynomials over $\mathbb{H}$:

$$p_l(z) = z^3 - \mathbf{k}z^2 + (\mathbf{k} - \mathbf{j})z + (\mathbf{i} + \mathbf{j}) \text{ and } p_r(z) = z^3 - z^2\mathbf{k} + z(\mathbf{k} - \mathbf{j}) + (\mathbf{i} + \mathbf{j}).$$

The corresponding companion matrices of $p_l(z)$ and $p_r(z)$ are given by

$$C_{p_l} = \begin{array}{c c} & \begin{array}{cc} {\scriptstyle 1} & {\scriptstyle 2} \end{array} \\ \begin{array}{c} {\scriptstyle 2} \\ {\scriptstyle 1} \end{array} & \left[\begin{array}{c|c} 0 & I \\ \hline C_{p_l}(3,1) & C_{p_l}(3,2:3) \end{array}\right] \end{array} \text{ and } C_{p_r} = C_{p_l}^T,$$

respectively, where $C_{p_l}(3,1) = -\mathbf{i} - \mathbf{j}$ and $C_{p_l}(3,2:3) := [\mathbf{j} - \mathbf{k}, \mathbf{k}]$. Then

$$C_{p_l}^2 = \begin{bmatrix} 0 & 0 & 1 \\ -\mathbf{i} - \mathbf{j} & \mathbf{j} - \mathbf{k} & \mathbf{k} \\ \mathbf{i} - \mathbf{j} & \mathbf{1} - \mathbf{2i} - \mathbf{j} & \mathbf{j} - \mathbf{k} - \mathbf{1} \end{bmatrix} \text{ and } C_{p_r}^2 = \begin{bmatrix} 0 & -\mathbf{i} - \mathbf{j} & \mathbf{j} - \mathbf{i} \\ 0 & \mathbf{j} - \mathbf{k} & \mathbf{1} - \mathbf{j} \\ 1 & \mathbf{k} & \mathbf{j} - \mathbf{k} - \mathbf{1} \end{bmatrix}.$$

This shows that $C_{p_r}^2 \neq (C_{p_l}^2)^T$.

Hence, we can write the similar algorithm to Theorem 2.35 for the case of $C_{p_r}^t$, $t \geq 2$.

Theorem 2.37. *Consider* $C_{p_r} = \begin{array}{c} \\ 1 \\ m-1 \end{array}\!\!\begin{array}{c} \begin{array}{cc} m-1 & 1 \end{array} \\ \left[\begin{array}{c|c} 0 & C_{p_r}(1,m) \\ \hline I & C_{p_r}(2:m,m) \end{array}\right] \end{array}$.

(a) *If $t < m$ is a positive integer, then*

$$C_{p_r}^t = \begin{array}{c} \\ t \\ m-t \end{array}\!\!\begin{array}{c} \begin{array}{cc} m-t & t \end{array} \\ \left[\begin{array}{c|c} 0 & C \\ \hline I & D \end{array}\right] \end{array}, \tag{2.28}$$

(b) *if $t \geq m$, then*

$$C_{p_r}^t = \left[\ C_{p_r}^{t-(m-1)}(1:m,m) \quad C_{p_r}^{t-(m-2)}(1:m,m) \quad \ldots \quad C_{p_r}^{t-1}(1:m,m) \quad C_{p_r}^{t}(1:m,m)\ \right]_{m\times m},$$

where

$$\begin{aligned} C &:= \left[C_{p_r}(1:t,m) \quad C_{p_r}^2(1:t,m) \quad \ldots \quad C_{p_r}^t(1:t,m)\right], \\ D &:= \left[C_{p_r}(t+1:m,m) \quad C_{p_r}^2(t+1:m,m) \quad \ldots \quad C_{p_r}^t(t+1:m,m)\right], \\ C_{p_r}^t(1,m) &:= C_{p_r}(1,m)\, C_{p_r}^{t-1}(m,m), \text{ and} \\ C_{p_r}^t(2:m,m) &:= C_{p_r}^{t-1}(1:m-1,m) + C_{p_r}(2:m,m)\, C_{p_r}^{t-1}(m,m). \end{aligned}$$

Proof. The proof follows from the proof method of Theorem 2.35. ■

Now from Example 2.36, we write

$$\tilde{p}_l(z) := \overline{p_r(\overline{z})} = z^3 + \mathbf{k}z^2 + (\mathbf{j}-\mathbf{k})z + (-\mathbf{i}-\mathbf{j}), \text{ and}$$

$$\tilde{p}_r(z) := \overline{p_l(\overline{z})} = z^3 + z^2\mathbf{k} + z(\mathbf{j}-\mathbf{k}) - (\mathbf{i}+\mathbf{j}).$$

Thus the companion matrices corresponding to $\tilde{p}_l(z)$ and $\tilde{p}_r(z)$ are given by

$$C_{\tilde{p}_l} = \overline{C_{p_l}} \text{ and } C_{\tilde{p}_r} = \overline{C_{p_r}},$$

respectively. Next,

$$C_{\tilde{p}_l}^2 = \begin{bmatrix} 0 & 0 & 1 \\ \mathbf{i}+\mathbf{j} & -\mathbf{j}+\mathbf{k} & -\mathbf{k} \\ \mathbf{i}-\mathbf{j} & \mathbf{1}+\mathbf{j} & \mathbf{k}-\mathbf{j}-\mathbf{1} \end{bmatrix} \text{ and } C_{\tilde{p}_r}^2 = \begin{bmatrix} 0 & \mathbf{i}+\mathbf{j} & \mathbf{j}-\mathbf{i} \\ 0 & -\mathbf{j}+\mathbf{k} & \mathbf{1}+\mathbf{2i}+\mathbf{j} \\ 1 & -\mathbf{k} & -\mathbf{1}-\mathbf{j}+\mathbf{k} \end{bmatrix}.$$

Then

(*a*) $\max\limits_{1\leq i\leq 3}\left[(r_i'(C_{p_l}^2))^{1/2}\right] = 2.3655$ and $\max\limits_{1\leq i\leq 3}\left[(r_i'(C_{\tilde{p}_r}^2))^{1/2}\right] = 1.9656.$

(b) $\max_{1\le i\le 3}\left[\left(r_i'\left(C_{p_r}^2\right)\right)^{1/2}\right] = 1.9319$ and $\max_{1\le i\le 3}\left[(r_i'(C_{\tilde{p_l}}^2))^{1/2}\right] = 2.1355.$

Now, we have

$$\begin{aligned}
\max_{1\le i\le 3}\left[(r_i'(C_{p_l}^2))^{1/2}\right] &\neq \max_{1\le i\le 3}\left[(r_i'(C_{\tilde{p_r}}^2))^{1/2}\right] \text{ and}\\
\max_{1\le i\le 3}\left[\left(r_i'\left(C_{p_r}^2\right)\right)^{1/2}\right] &\neq \max_{1\le i\le 3}\left[(r_i'(C_{\tilde{p_l}}^2))^{1/2}\right].
\end{aligned}$$

Hence, we have the following bounds for the zeros of $p_l(z)$ and $p_r(z)$ for $\gamma \in [0,\ 1]$.

Theorem 2.38. *Let $p_l(z)$ and $p_r(z)$ be the simple monic polynomials over $\mathbb{H}$ of degree m and let $C_{p_l}^t$ and $C_{p_r}^t$ $(t \ge 2)$ be the t-th power of the companion matrices C_{p_l} and C_{p_r}, corresponding to $p_l(z)$ and $p_r(z)$, respectively. Then, for $\gamma \in [0,1]$*

1. *bounds for every zero $\tilde{z}$ of $p_l(z)$ satisfies the following inequalities:*

$$(a)\ (\xi_1')^{-1} \le |\tilde{z}| \le \xi_1,$$

$$(b)\ (\xi_2')^{-1} \le |\tilde{z}| \le \xi_2,$$

where

$$\begin{aligned}
\xi_1 &= \max_{1\le i\le m}\left[\left(r_i'\left(C_{p_l}^t\right)\right)^{\gamma/t}\left(c_i'\left(C_{p_l}^t\right)\right)^{(1-\gamma)/t}\right],\\
\xi_1' &= \max_{1\le i\le m}\left[\left(r_i'\left(C_{q_l}^t\right)\right)^{\gamma/t}\left(c_i'\left(C_{q_l}^t\right)\right)^{(1-\gamma)/t}\right],\\
\xi_2 &= \max_{1\le i\le m}\left[\left(r_i'\left(C_{\tilde{p_r}}^t\right)\right)^{\gamma/t}\left(c_i'\left(C_{\tilde{p_r}}^t\right)\right)^{(1-\gamma)/t}\right],\\
\xi_2' &= \max_{1\le i\le m}\left[\left(r_i'\left(C_{\tilde{q_r}}^t\right)\right)^{\gamma/t}\left(c_i'\left(C_{\tilde{q_r}}^t\right)\right)^{(1-\gamma)/t}\right];
\end{aligned}$$

2. *bounds for every zero $\tilde{z}$ of $p_r(z)$ satisfies the following inequalities:*

$$(a)\ \left(\max_{1\le i\le m}\left[\left(r_i'\left(C_{q_r}^t\right)\right)^{\gamma/t}\left(c_i'\left(C_{q_r}^t\right)\right)^{(1-\gamma)/t}\right]\right)^{-1} \le |\tilde{z}| \le \xi_3,$$

$$(b)\ \left(\max_{1\le i\le m}\left[\left(r_i'\left(C_{\tilde{q_l}}^t\right)\right)^{\gamma/t}\left(c_i'\left(C_{\tilde{q_l}}^t\right)\right)^{(1-\gamma)/t}\right]\right)^{-1} \le |\tilde{z}| \le \xi_4,$$

where

$$\begin{aligned}
\xi_3 &= \max_{1\le i\le m}\left[\left(r_i'\left(C_{p_r}^t\right)\right)^{\gamma/t}\left(c_i'\left(C_{p_r}^t\right)\right)^{(1-\gamma)/t}\right],\\
\xi_3' &= \max_{1\le i\le m}\left[\left(r_i'\left(C_{q_r}^t\right)\right)^{\gamma/t}\left(c_i'\left(C_{q_r}^t\right)\right)^{(1-\gamma)/t}\right],\\
\xi_4 &= \max_{1\le i\le m}\left[\left(r_i'\left(C_{\tilde{p_l}}^t\right)\right)^{\gamma/t}\left(c_i'\left(C_{\tilde{p_l}}^t\right)\right)^{(1-\gamma)/t}\right],\\
\xi_4' &= \max_{1\le i\le m}\left[\left(r_i'\left(C_{\tilde{q_l}}^t\right)\right)^{\gamma/t}\left(c_i'\left(C_{\tilde{q_l}}^t\right)\right)^{(1-\gamma)/t}\right].
\end{aligned}$$

Proof. 1(a). Let λ be a left eigenvalue of C_{p_l}. Then by Proposition 2.33, λ^t ($t \geq 2$ is positive integer) is a left eigenvalue of $C_{p_l}^t$. Hence by applying Theorem 2.3, we get the desired result.

1(b). By Lemma 2.1, $\overline{\lambda}$ is a left eigenvalue of $C_{\tilde{p_r}}$ and by Proposition 2.34, $(\overline{\lambda})^t$ is a left eigenvalue of $(C_{\tilde{p_r}})^t$. Then from Theorem 2.3, we get the desired result.

For 2(a) and 2(b), the proofs are similar to the proof methods of 1(a) and 1(b), respectively. ■

Substituting $t = 2$ and $\gamma = 1$ in Theorem 2.38, we have the following corollary.

Corollary 2.39. *Let $p_l(z)$ and $p_r(z)$ be the simple monic polynomials over $\mathbb{H}$ of degree m. Then*

1. *bounds for every zero $\tilde{z}$ of $p_l(z)$ satisfies the following inequalities:*

$$(a)\ \frac{1}{\beta_1} \leq |\tilde{z}| \leq \alpha_1,$$

$$(b)\ \frac{1}{\beta_2} \leq |\tilde{z}| \leq \alpha_2,$$

where

$$\alpha_1 = \max\left\{1, \left(\sum_{j=0}^{m-1} |q_j|\right)^{1/2}, \left(\sum_{j=0}^{m-1} |q_{m-1}q_j - q_{j-1}|\right)^{1/2}\right\},$$

$$\alpha_2 = \max_{2\leq j\leq m-1}\Big\{ (|q_0| + |\overline{q_0}\ \overline{q_{m-1}}|)^{1/2}, (|q_1| + |\overline{q_1}\ \overline{q_{m-1}} - \overline{q_0}|)^{1/2},$$
$$(1 + |q_j| + |\overline{q_j}\ \overline{q_{m-1}} - \overline{q_{j-1}}|)^{1/2}\Big\},$$

$$\beta_1 = \max\left\{1, \left(\sum_{j=1}^{m-1} |q_0^{-1}q_j|\right)^{1/2}, \left(\sum_{j=0}^{m-1} |q_0^{-1}q_1q_0^{-1}q_{m-j} - q_0^{-1}q_{m-j+1}|\right)^{1/2}\right\},$$

$$\beta_2 = \max_{2\leq j\leq m-1}\Big\{ \left(|q_0^{-1}| + |\overline{q_0^{-1}}\ \overline{q_1q_0^{-1}}|\right)^{1/2}, \left(|q_{m-1}q_0^{-1}| + |\overline{q_{m-1}q_0^{-1}}\ \overline{q_1q_0^{-1}} - \overline{q_0^{-1}}|\right)^{1/2},$$
$$\left(1 + |q_{m-j}q_0^{-1}| + |\overline{q_{m-j}q_0^{-1}}\ \overline{q_1q_0^{-1}} - \overline{q_{m-j+1}q_0^{-1}}|\right)^{1/2}\Big\};$$

2. *bounds for every zero $\tilde{z}$ of $p_r(z)$ satisfies the following inequalities:*

$$(a)\ \frac{1}{\beta_3} \leq |\tilde{z}| \leq \alpha_3,$$

$$(b)\ \frac{1}{\beta_4} \leq |\tilde{z}| \leq \alpha_4,$$

where

$$\alpha_3 = \max_{2\le j\le m-1}\left\{ (|q_0| + |q_0\ q_{m-1}|)^{1/2}, (|q_1| + |q_1\ q_{m-1} - q_0|)^{1/2}, \right.$$
$$\left. (1 + |q_j| + |q_j\ q_{m-1} - q_{j-1}|)^{1/2} \right\},$$

$$\alpha_4 = \max\left\{1, \left(\sum_{j=0}^{m-1} |q_j|\right)^{1/2}, \left(\sum_{j=0}^{m-1} |\overline{q_{m-1}}\ \overline{q_j} - \overline{q_{j-1}}|\right)^{1/2}\right\},$$

$$\beta_3 = \max_{2\le j\le m-1}\left\{ \left(|q_0^{-1}| + |q_0^{-1}\ \ q_1 q_0^{-1}|\right)^{1/2}, \left(|q_{m-1}q_0^{-1}| + |q_{m-1}q_0^{-1}\, q_1 q_0^{-1} - q_0^{-1}|\right)^{1/2}, \right.$$
$$\left. \left(1 + |q_{m-j}q_0^{-1}| + |q_{m-j}q_0^{-1}\, q_1 q_0^{-1} - q_{m-j+1}q_0^{-1}|\right)^{1/2} \right\},$$

$$\beta_4 = \max\left\{1, \left(\sum_{j=1}^{m-1} |q_0^{-1} q_j|\right)^{1/2}, \left(\sum_{j=0}^{m-1} |\overline{q_0^{-1}q_1}\ \ \overline{q_0^{-1}q_{m-j}} - \overline{q_0^{-1}q_{m-j+1}}|\right)^{1/2}\right\},$$

$q_{-1} = 0 = q_{m+1}, q_m = 1.$

Proof. The proof follows from Theorem 2.38 and APPENDIX-A. ■

Example 2.40. Consider the following polynomials $p_l(z)$ and $p_r(z)$ over $\mathbb{H}$:

$p_l(z) = z^6 + (\mathbf{i} + 3\mathbf{k})z^5 + (3 + \mathbf{j})z^4 + (5\mathbf{i} + 15\mathbf{k})z^3 + (-4 + 5\mathbf{j})z^2 + (6\mathbf{i} + 18\mathbf{k})z + (6\mathbf{j} - 12),$

$p_r(z) = z^6 + z^5(\mathbf{i} + 3\mathbf{k}) + z^4(3 + \mathbf{j}) + z^3(5\mathbf{i} + 15\mathbf{k}) + z^2(-4 + 5\mathbf{j}) + z(6\mathbf{i} + 18\mathbf{k}) + (6\mathbf{j} - 12).$

The zeros of $p_l(z)$ are given in [50]. Moreover, we find the zeros of $p_r(z)$ by Niven's algorithm [41].

$\mathbf{Z}_{\mathbb{H}}(p_l(z)) =: z_1$	Abs(z_1)	$\mathbf{Z}_{\mathbb{H}}(p_r(z)) =: z_2$	Abs(z_2)
$-\mathbf{i} - 2\mathbf{k}$	2.2361	$-0.4\mathbf{i} - 2.2\mathbf{k}$	2.2361
$[\mathbf{i}\sqrt{3}]$	1.7321	$[\mathbf{i}\sqrt{3}]$	1.7321
$[\mathbf{i}\sqrt{2}]$	1.4142	$[\mathbf{i}\sqrt{2}]$	1.4142
$-0.6\mathbf{i} - 0.8\mathbf{k}$	1	$-\mathbf{k}$	1

TABLE 2.1. The zeros of $p_l(z)$ and $p_r(z)$ and their absolute values.

$z_1 :=$ the set of zeros of $p_l(z)$, $z_2 :=$ the set of zeros of $p_r(z)$,

Denote **LB:=** Lower Bound, **UB:=** Upper Bound.

Example 2.40	**LB**	**UB**
Corollary 2.27 (1)	0.4142	19.9737
Corollary 2.27 (2)	0.2766	60.9291
Theorem 2.26, $\gamma = 1/4$	0.3744	8.1415

TABLE 2.2. Lower and upper bounds for the zeros of $p_l(z)$ and $p_r(z)$.

Example 2.36	**LB**	**UB**
Corollary 2.39 1(a)	0.6156	2.3655
Corollary 2.39 1(b)	0.6078	1.9656
Corollary 2.39 2(a)	0.6078	1.9319
Corollary 2.39 2(b)	0.6436	2.1355

TABLE 2.3. Lower and upper bounds for the zeros of $p_l(z)$ and $p_r(z)$.

CHAPTER 3

LOCATION OF ZEROS OF POLYNOMIALS OVER A QUATERNION DIVISION ALGEBRA

In this chapter, inclusion regions for the left and right eigenvalues of quaternionic matrices are derived. The location of zeros of quaternionic polynomials is discussed via left eigenvalues of the companion matrices. The present investigation shows that the obtained ovals of Cassini are smaller than the existing ovals of Cassini for polynomials over the complex field under certain conditions.

3.1. Introduction

The work carried out in this chapter is based on Theorems 2.3, 2.4, 2.8, and 2.11. In this work, we discuss a theory for polynomials with quaternion coefficients. Thereafter we move to study similar results for polynomials with complex coefficients. Then, inclusion regions for the left and right eigenvalues of a quaternionic matrix are derived. Further, the location of zeros of the quaternionic polynomials $p_l(z)$ and $p_r(z)$ (defined in (1.20) and (1.21)) is discussed. The present work also focuses on inclusion regions for the zeros of the quaternionic polynomials $p_l(z)$ and $p_r(z)$ by applying Theorem 2.3 and Theorem 2.8. It is found that Theorem 2.3 and Theorem 2.8 give two sets; the first one describes the union of two balls and the second one is the union of a ball and an oval of Cassini. These sets can be reduced into smaller sets which depend on a parameter $\gamma \in [0, 1]$. By considering $\omega(z) = z^2 + q_{m-1}z$, where $q_{m-1}, z \in \mathbb{H}$, we present two ovals of Cassini and each contains all the zeros of the quaternionic polynomials $p_l(z)$ and $p_r(z)$. We show that the inclusion regions obtained in the present work are comparatively smaller than the inclusion regions developed in [37] under certain conditions.

3.2. Inclusion regions for left eigenvalues of quaternionic companion matrices

The corresponding companion matrices to the simple monic polynomials $p_l(z)$ and $p_r(z)$ are given by

$$C_{p_l} := \begin{bmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \\ -q_0 & -q_1 & \dots & -q_{m-1} \end{bmatrix} \text{ and } C_{p_r} := C_{p_l}^T,$$

respectively. Left eigenvalues of C_{p_l} are the zeros of the simple monic polynomial $p_l(z)$. Also, left eigenvalues of C_{p_r} are the zeros of the simple monic polynomial $p_r(z)$. However, right eigenvalues of C_{p_l} need not be zeros of the simple monic polynomial $p_l(z)$. Now define $\omega(z)$ as:

$$\omega(z) := \sum_{j \in S} \alpha_j z^j, \tag{3.1}$$

where $\alpha_j, z \in \mathcal{K}$, $\mathcal{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, and S is defined as the set of integers when $A \in M_n(\mathcal{K})$ is invertible; otherwise S is the set of nonnegative integers. Throughout this chapter we consider the following three cases:

Case 1: when $\alpha_j, z \in \mathbb{H}$, and $A \in M_n(\mathbb{H})$,

Case 2: when $\alpha_j \in \mathbb{R}, z \in \mathbb{H}$, and $A \in M_n(\mathbb{H})$,

Case 3: when $\alpha_j, z \in \mathbb{C}$, and $A \in M_n(\mathbb{C})$.

In general, if λ is a left eigenvalue of a matrix $A \in M_n(\mathbb{H})$, then λ^2 is not necessarily a left eigenvalue of A^2. If λ is a left eigenvalue of the quaternionic matrix A with corresponding eigenvector $x \in \mathbb{H}^n$, then λ^{-1} is a left eigenvalue of A^{-1}. However its corresponding eigenvector may be different to x. On the other hand, if λ is a left eigenvalue of C_{p_l} with corresponding eigenvector $y \in \mathbb{H}^n$, then λ^k is a left eigenvalue of $C_{p_l}^k$ with the same eigenvector y, where k is a nonzero integer. The above argument follows from Proposition 2.33. Now, we take **Case 1** for the development of our theory.

Case 1. We consider $\omega(z)$ as:

$$\omega(z) := \sum_{j \in S} \alpha_j z^j, \tag{3.2}$$

where $\alpha_j, z \in \mathbb{H}$ and S is defined as the set of integers when $A \in M_n(\mathbb{H})$ is invertible; otherwise S is the set of nonnegative integers. Then we have the following result.

Lemma 3.1. *Let C_{p_l} be a companion matrix of the simple monic polynomial $p_l(z)$ and let $\omega(z)$ be defined in* (3.2)*. If λ is a left eigenvalue of C_{p_l}, then $\omega(\lambda)$ is a left eigenvalue of $\omega(C_{p_l})$.*

Proof. Let λ be a left eigenvalue of C_{p_l}. Then $C_{p_l}x = \lambda x$ for some nonzero $x \in \mathbb{H}^n$. So

$$\omega(C_{p_l})x = \left(\sum_{j\in S} \alpha_j C_{p_l}^j\right) x.$$

From Proposition 2.33, we obtain

$$\omega(C_{p_l})x = \left(\sum_{j\in S} \alpha_j \lambda^j\right) x = \omega(\lambda)x. \blacksquare$$

We next present a generalization of Theorems 2.3 and 2.8 for a quaternionic companion matrix as follows.

Theorem 3.2. *Let $C_{p_l} := (c_{ij}) \in M_n(\mathbb{H})$ be a companion matrix of the simple monic polynomial $p_l(z)$ and let $\gamma \in [0, 1]$. Then all the left eigenvalues of C_{p_l} are located in the union of n sets $B_i(C_{p_l}, \omega) := \{z \in \mathbb{H} : |\omega(z)-(\omega(C_{p_l}))_{ii}| \leq r_i^{\gamma}(\omega(C_{p_l}))\ c_i^{1-\gamma}(\omega(C_{p_l}))\}$ $(1 \leq i \leq n)$, i.e.,*

$$\Lambda_l(C_{p_l}) \subseteq B(C_{p_l}, \omega) := \cup_{i=1}^n B_i(C_{p_l}, \omega),$$

where $\omega(z)$ is defined in (3.2).

Proof. Let λ be a left eigenvalue of C_{p_l}. Then from Lemma 3.1, $\omega(\lambda)$ is a left eigenvalue of $\omega(C_{p_l})$. Then there exists some nonzero $x \in \mathbb{H}^n$ such that $\omega(C_{p_l})x = \omega(\lambda)x$. Let $x := [x_1, \ldots, x_n]^T \in \mathbb{H}^n$ and let x_t be an element of x such that $|x_t| \geq |x_i|\ \forall\ i\ (1 \leq i \leq n)$. Then $|x_t| > 0$. From the t-th equation of $\omega(C_{p_l})x = \omega(\lambda)x$, we have

$$\sum_{j=1}^{n} \omega(C_{p_l})_{tj} x_j = \omega(\lambda)x_t.$$

Since $|x_t| \geq |x_i|\ \forall\ i\ (1 \leq i \leq n)$, then

$$|\omega(\lambda) - \omega(C_{p_l})_{tt}| \leq \sum_{j=1,\, j\neq t}^{n} |\omega(C_{p_l})_{tj}| := r_t(\omega(C_{p_l})). \tag{3.3}$$

From Lemma 2.1, $\overline{\lambda}$ is a left eigenvalue of A^H and hence $\overline{\omega(\lambda)}$ is a left eigenvalue of $(\omega(C_{p_l}))^H$. Similarly, we have

$$|\overline{\omega(\lambda)} - \overline{\omega(C_{p_l})_{tt}}| \;\; \leq \;\; \sum_{j=1,\, j\neq t}^{n} |\overline{\omega(C_{p_l})_{jt}}|.$$

This implies

$$|\omega(\lambda) - \omega(C_{p_l})_{tt}| \;\; \leq \;\; \sum_{j=1,\, j\neq t}^{n} |\omega(C_{p_l})_{jt}| := c_t(\omega(C_{p_l})). \tag{3.4}$$

Then for any $\gamma \in [0,1]$, (3.3) and (3.4) give

$$|\omega(\lambda) - \omega(C_{p_l})_{tt}|^{\gamma} \;\; \leq \;\; r_t^{\gamma}(\omega(C_{p_l})), \tag{3.5}$$

as well as

$$|\omega(\lambda) - \omega(C_{p_l})_{tt}|^{1-\gamma} \;\; \leq \;\; c_t^{1-\gamma}(\omega(C_{p_l})). \tag{3.6}$$

Combining (3.5) and (3.6), we obtain

$$|\omega(\lambda) - \omega(C_{p_l})_{tt}| \leq r_t^{\gamma}(\omega(C_{p_l}))\, c_t^{1-\gamma}(\omega(C_{p_l})). \;\; \blacksquare$$

From Theorems 2.3 and 3.2, it is clear that all the left eigenvalues of C_{p_l} are contained in $T(C_{p_l}) \cap B(C_{p_l}, \omega)$.

Theorem 3.3. *Let $C_{p_l} := (c_{ij}) \in M_n(\mathbb{H})$ be a companion matrix of the simple monic polynomial $p_l(z)$ and let $\gamma \in [0,1]$. Then all the left eigenvalues of C_{p_l} are located in the union of $\frac{n(n-1)}{2}$ sets $P_{ij}(C_{p_l}, \omega) := \{z \in \mathbb{H} : |\omega(z) - (\omega(C_{p_l}))_{ii}|\, |\omega(z) - (\omega(C_{p_l}))_{jj}| \leq r_i^{\gamma}(\omega(C_{p_l}))\; r_j^{\gamma}(\omega(C_{p_l}))\; c_i^{1-\gamma}(\omega(C_{p_l}))\; c_j^{1-\gamma}(\omega(C_{p_l}))\}$ $(1 \leq i, j \leq n; i \neq j)$, i.e.,*

$$\Lambda_l(C_{p_l}) \subseteq P(C_{p_l}, \omega) := \cup_{\substack{i,j=1 \\ i\neq j}}^{n} P_{ij}(C_{p_l}, \omega),$$

where $\omega(z)$ is defined in (3.2).

Proof. Let λ be a left eigenvalue of C_{p_l}. Then by Lemma 3.1, $\omega(\lambda)$ is a left eigenvalue of $\omega(C_{p_l})$. Then there exists some nonzero $x \in \mathbb{H}^n$ such that $\omega(C_{p_l})x = \omega(\lambda)x$. Let $x := [x_1, \ldots, x_n]^T \in \mathbb{H}^n$ and let x_s be an element of x such that $|x_s| \geq |x_i|\ \forall\ i\ (1 \leq i \leq n)$. Then $|x_s| > 0$. Clearly, if all the other elements of x are zero, then the result holds.

Now, suppose that there are at least two nonzero elements of x, and let x_t be an element with second largest absolute value, i.e., $|x_s| \geq |x_t| \geq |x_i| \ \forall \ i \ (1 \leq i \leq n, i \neq s)$, and $x_s \neq 0 \neq x_t$. From the s-th equation of $\omega(C_{p_l})x = \omega(\lambda)x$, we have

$$\sum_{j=1}^{n} \omega(C_{p_l})_{sj} x_j = \omega(\lambda) x_s,$$

which implies

$$(\omega(\lambda) - \omega(C_{p_l})_{ss}) x_s = \sum_{j=1,\, j\neq s}^{n} \omega(C_{p_l})_{sj} x_j.$$

Thus

$$|\omega(\lambda) - \omega(C_{p_l})_{ss}| \leq \left(\frac{|x_t|}{|x_s|}\right) \ r_s(\omega(C_{p_l})). \tag{3.7}$$

Similarly, from the t-th equation of $\omega(C_{p_l})x = \omega(\lambda)x$, we obtain

$$|\omega(\lambda) - \omega(C_{p_l})_{tt}| \leq \left(\frac{|x_s|}{|x_t|}\right) \ r_t(\omega(C_{p_l})). \tag{3.8}$$

Combining (3.7) and (3.8), we have

$$|\omega(\lambda) - \omega(C_{p_l})_{ss}| \ |\omega(\lambda) - \omega(C_{p_l})_{tt}| \leq r_s(\omega(C_{p_l}))\, r_t(\omega(C_{p_l})) \tag{3.9}$$

$(1 \leq s, t \leq n, s \neq t)$.

Now, from Lemma 2.1, $\overline{\lambda}$ is a left eigenvalue of $C_{p_l}^H$ and hence $\overline{\omega(\lambda)}$ is a left eigenvalue of $(\omega(C_{p_l}))^H$. Similarly, we can obtain

$$|\omega(\lambda) - \omega(C_{p_l})_{ss}| \ |\omega(\lambda) - \omega(C_{p_l})_{tt}| \leq c_s(\omega(C_{p_l}))\, c_t(\omega(C_{p_l})) \tag{3.10}$$

$(1 \leq s, t \leq n, s \neq t)$.

For any $\gamma \in [0, 1]$, (3.9) and (3.10) yield

$$|\omega(\lambda) - \omega(C_{p_l})_{ss}|^{\gamma} \ |\omega(\lambda) - \omega(C_{p_l})_{tt}|^{\gamma} \leq r_s^{\gamma}(\omega(C_{p_l}))\, r_t^{\gamma}(\omega(C_{p_l})), \tag{3.11}$$

and

$$|\omega(\lambda) - \omega(C_{p_l})_{ss}|^{1-\gamma} \ |\omega(\lambda) - \omega(C_{p_l})_{tt}|^{1-\gamma} \leq c_s^{1-\gamma}(\omega(C_{p_l}))\, c_t^{1-\gamma}(\omega(C_{p_l})). \tag{3.12}$$

Combining (3.11) and (3.12), we obtain

$$|\omega(\lambda) - \omega(C_{p_l})_{ss}| \ |\omega(\lambda) - \omega(C_{p_l})_{tt}| \leq r_s^{\gamma}(\omega(C_{p_l}))\, r_t^{\gamma}(\omega(C_{p_l}))\, c_s^{1-\gamma}(\omega(C_{p_l}))\, c_t^{1-\gamma}(\omega(C_{p_l})).$$

Hence, all the left eigenvalues of A are located in the union of $\frac{n(n-1)}{2}$ sets $P_{ij}(C_{p_l}, \omega)$ $(1 \leq i, j \leq n, \ i \neq j)$. ■

From Theorems 2.8 and 3.3, we conclude that all the left eigenvalues of C_{p_l} are contained in $K(C_{p_l}) \cap P(C_{p_l}, \omega)$.

In general, similar quaternionic matrices may not have the same left eigenvalues, see, [62, Example 3.3]. However, if $A \in M_n(\mathbb{H})$ and W is any invertible real matrix, then A and WAW^{-1} have the same left eigenvalues (proved in Chapter 2).

Also, in general, $\omega(WAW^{-1}) \neq W\omega(A)W^{-1}$, where W is any quaternionic invertible matrix. For example, let

$$A = \begin{bmatrix} \mathbf{i}+\mathbf{j} & \mathbf{k} \\ \mathbf{j} & 1-\mathbf{j} \end{bmatrix},$$

where $W = \begin{bmatrix} \mathbf{j} & 0 \\ 0 & \mathbf{k} \end{bmatrix}$ and $W^{-1} = \begin{bmatrix} -\mathbf{j} & 0 \\ 0 & -\mathbf{k} \end{bmatrix}$. Suppose $\omega(z) = z^2 + (\mathbf{j}+\mathbf{k})z$. Then

$$\omega(WAW^{-1}) \neq W\omega(A)W^{-1}.$$

However, if W is any invertible real matrix, then we have the following proposition.

Proposition 3.4. *Let $A \in M_n(\mathbb{H})$ and let W be a real invertible matrix. Then $\omega(WAW^{-1}) = W\omega(A)W^{-1}$, where $\omega(z)$ be defined in (3.2).*

Proof. We have

$$\omega(WAW^{-1}) = \sum_{j\in S} \alpha_j (WAW^{-1})^j.$$

Since $\alpha_j W = W\alpha_j \ \forall\, j\ (0 \leq j \leq m)$, $\alpha_0 WW^{-1} = W\alpha_0 W^{-1}$, and $(W\alpha_0 W^{-1})^t = W\alpha_0^t W^{-1}$ (t is a nonzero integer), we obtain

$$\omega(WAW^{-1}) = W\left(\sum_{j\in S} \alpha_j A^j\right) W^{-1} = W\omega(A)W^{-1}. \ \blacksquare$$

Remark 3.5. From Proposition 2.22, we observe that left eigenvalues of quaternionic matrices A and WAW^{-1} are same, where W is any invertible real matrix. Applying Theorems 2.3 and 2.8 to WAW^{-1} instead of A, we obtain different and potentially smaller inclusion regions.

Remark 3.6. By applying Theorems 3.2 and 3.3 to $\omega(WC_{p_l}W^{-1}) = W\omega(C_{p_l})W^{-1}$ instead of $\omega(C_{p_l})$, we obtain different and potentially sharper inclusion regions.

Remark 3.7. Similar results can be obtained for the quaternionic companion matrix C_{p_r} as well.

3.3. Inclusion regions for right eigenvalues of quaternionic matrices

Case 2. We consider $\omega(z)$ as:

$$\omega(z) := \sum_{j \in S} \alpha_j z^j, \tag{3.13}$$

where $\alpha_j \in \mathbb{R}, z \in \mathbb{H}$ and S is defined as the set of integers when $A \in M_n(\mathbb{H})$ is invertible; otherwise S is the set of nonnegative integers. Thus, an extension of the Gerschgorin type theorem [62, Theorem 7] for right eigenvalues of a quaternionic matrix to the quaternionic matrix $\omega(A)$ is as follows.

Theorem 3.8. *Let $A = (a_{ij}) \in M_n(\mathbb{H})$. For every right eigenvalue λ of A there exists a nonzero quaternion β such that $\beta^{-1}\omega(\lambda)\beta$ (which is a right eigenvalue of $\omega(A)$) is contained in the union of n sets*

$$\mathcal{B}_i(A, \omega) := \{z \in \mathbb{H} : |\omega(z) - (\omega(A))_{ii}| \leq r_i(\omega(A))\} \ \ (1 \leq i \leq n), \ \textit{i.e.},$$

$$\{z^{-1}\omega(\lambda)z : 0 \neq z \in \mathbb{H}\} \cap \cup_{i=1}^{n} \mathcal{B}_i(A, \omega) \neq \emptyset,$$

where $\omega(z)$ is defined in (3.13).

Proof. Let λ be a right eigenvalue of A. Then, $\omega(\lambda)$ is a right eigenvalue of the matrix $\omega(A)$, so that there exists some nonzero $x \in \mathbb{H}^n$ such that $\omega(A)x = x\omega(\lambda)$. Let $x := [x_1, \dots, x_n]^T \in \mathbb{H}^n$ and let x_t be an element of x such that $|x_t| \geq |x_i| \ \forall \ i \ (1 \leq i \leq n)$. Then $|x_t| > 0$. Thus from $\omega(A)x = xw(\lambda)$, we have

$$(\omega(A))_{tt}x_t + \sum_{j=1,\, j \neq t}^{n} (\omega(A))_{tj}x_j = x_t\omega(\lambda).$$

Since $x_t \neq 0$, consider $\rho \in \mathbb{H}$ such that $x_t\lambda = \rho x_t$. Then $x_t\lambda^j = \rho^j x_t$, where $j \in S$; the set of integers when $A \in M_n(\mathbb{H})$ is invertible; otherwise S is the set of nonnegative integers. Thus

$$(\omega(A))_{tt}x_t + \sum_{j=1,\, j \neq t}^{n} (\omega(A))_{tj}x_j = \omega(\rho)x_t,$$

which implies

$$(\omega(\rho) - \omega(A)_{tt})x_t = \sum_{j=1,\, j \neq t}^{n} (\omega(A))_{tj}x_j.$$

Since $|x_t| \geq |x_i| \; \forall\, i \, (1 \leq i \leq n)$, we obtain

$$|\omega(\rho) - (\omega(A))_{tt}| \leq \sum_{j=1,\, j\neq t}^{n} |(\omega(A))_{tj}| := r_t(\omega(A)). \quad \blacksquare$$

Remark 3.9. It is known that if λ is a right eigenvalue of $A \in M_n(\mathbb{H})$, then λ^k (k is an integer when A is nonsingular, whereas k is a nonzero positive integer when A is singular) is a right eigenvalue of A^k. Thus similar results to Theorems 3.2 and 3.3 can be obtained for right eigenvalues of $A = (a_{ij}) \in M_n(\mathbb{H})$, where $(\omega(A))_{ii} \in \mathbb{R}$ for all i $(1 \leq i \leq n)$ with $\omega(z)$ as defined in (3.13). Since similar quaternionic matrices have the same right eigenvalues. Thus we can obtain potentially sharper inclusion sets by applying similar results to the matrix WAW^{-1} or $\omega(WAW^{-1}) = W\omega(A)W^{-1}$, where W is any invertible real matrix.

3.4. Inclusion regions for zeros of quaternionic polynomials

Inclusion regions for zeros of the simple monic polynomials $p_l(z)$ and $p_r(z)$ are discussed in this section. We first define the following notation:

$\xi_1^{(\gamma)}$:= largest element of the set

$$\{|q_0|^{1-\gamma}, (1+|q_1|)^{1-\gamma}, \ldots, (1+|q_{m-2}|)^{1-\gamma}\},$$

$\xi_2^{(\gamma)}$:= second largest element of the set

$$\{|q_0|^{1-\gamma}, (1+|q_1|)^{1-\gamma}, \ldots, (1+|q_{m-2}|)^{1-\gamma}\},$$

$$\xi_3^{(\gamma)} := \left(\sum_{j=0}^{m-2} |q_j|\right)^{\gamma}, \text{ and } \; \xi_4^{(\gamma)} := \left(\sum_{j=0}^{m-3} |q_j|\right)^{\gamma},$$

where $\gamma \in [0,1]$.

Then by applying Theorems 2.3 and 2.8 to C_{p_l}, we have the following result.

Theorem 3.10. *Let $p_l(z)$ be a simple monic polynomial over $\mathbb{H}$ of degree m and let $\gamma \in [0,1]$. Then all the zeros of $p_l(z)$ are contained in each of the following sets:*

$$\begin{aligned} T^{(\gamma)} &:= \{z \in \mathbb{H} : |z| \leq \xi_1^{(\gamma)}\} \cup \{z \in \mathbb{H} : |z + q_{m-1}| \leq \xi_3^{(\gamma)}\}, \textit{ and} \\ K^{(\gamma)} &:= \{z \in \mathbb{H} : |z| \leq \sqrt{\xi_1^{(\gamma)}\xi_2^{(\gamma)}}\} \cup \{z \in \mathbb{H} : |z|\; |z + q_{m-1}| \leq \xi_1^{(\gamma)}\xi_3^{(\gamma)}\}. \end{aligned}$$

Each of sets $T^{(\gamma)}$ is the union of two balls, whereas the set $K^{(\gamma)}$ for each $\gamma \in [0,1]$ is the union of a ball and an oval of Cassini. Also, $K^{(\gamma)} \subseteq T^{(\gamma)}\ \forall\, \gamma \in [0,1]$ which was proved in Theorem 2.12.

Case 1. We consider $\omega(z)$ as:

$$\omega(z) := \sum_{j\in S} \alpha_j z^j, \tag{3.14}$$

where $\alpha_j, z \in \mathbb{H}$ and S is defined as the set of integers when $A \in M_n(\mathbb{H})$ is invertible; otherwise S is the set of nonnegative integers. Now, we take $\omega(z)$ as follows:

$$\omega(z) := z^2 + q_{m-1}z.$$

To find $\omega(C_{p_l})$, we first construct $C_{p_l}^2$;

$$C_{p_l}^2 = \begin{bmatrix} 0 & 0 & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ -q_0 & -q_1 & -q_2 & \cdots & -q_{m-1} \\ q_{m-1}q_0 & q_{m-1}q_1 - q_0 & q_{m-1}q_2 - q_1 & \cdots & q_{m-1}^2 - q_{m-2} \end{bmatrix}.$$

Therefore, we obtain

$$C_{p_l}^2 + q_{m-1}C_{p_l} = \begin{bmatrix} 0 & q_{m-1} & 1 & & & \\ 0 & 0 & q_{m-1} & \ddots & & \\ 0 & 0 & 0 & \ddots & 1 & \\ & & & & q_{m-1} & 1 \\ -q_0 & -q_1 & -q_2 & \cdots & -q_{m-2} & 0 \\ 0 & -q_0 & -q_1 & \cdots & -q_{m-3} & -q_{m-2} \end{bmatrix}.$$

Throughout this chapter for $\gamma \in [0,1]$, we adopt the following notation.

$$\begin{aligned} \beta_1^{(\gamma)} &:= |q_0|^{(1-\gamma)}\,(1+|q_{m-1}|)^\gamma, \\ \beta_2^{(\gamma)} &:= (|q_0|+|q_1|+|q_{m-1}|)^{(1-\gamma)}\,(1+|q_{m-1}|)^\gamma, \\ \beta_j^{(\gamma)} &:= (|q_{j-2}|+|q_{j-1}|+|q_{m-1}|+1)^{(1-\gamma)}\,(1+|q_{m-1}|)^\gamma;\ \ (3 \le j \le m-2), \\ \beta_{m-1}^{(\gamma)} &:= \xi_4^{(\gamma)}\,(|q_{m-1}|+|q_{m-3}|+1)^{(1-\gamma)}. \end{aligned}$$

$$
(3.15) \qquad \begin{cases} \eta_1^{(\gamma)} \text{ and } \eta_2^{(\gamma)} \text{ be the largest and second largest elements of a set } S, \\ \text{where } S := \{\beta_1^{(\gamma)}, \beta_2^{(\gamma)}, \ldots, \beta_{m-2}^{(\gamma)}\}, \\ \eta_3^{(\gamma)} = \frac{1}{2}\left(|q_{m-2}| + \sqrt{|q_{m-2}|^2 + 4\eta_1^{(\gamma)}\beta_{m-1}^{(\gamma)}} \right). \end{cases}
$$

By applying Theorem 3.2 to $\omega(C_{p_l})$, we have the following theorem which derives inclusion regions for zeros of $p_l(z)$.

Theorem 3.11. *Let $p_l(z)$ be a simple monic polynomial over $\mathbb{H}$ of degree m and let $\gamma \in [0,1]$. Then all the zeros of $p_l(z)$ are contained in $\mathcal{G}_1^{(\gamma)} \cup \mathcal{G}_2^{(\gamma)}$, where*

$$
\begin{aligned}
\mathcal{G}_1^{(\gamma)} &:= \{z \in \mathbb{H} : |\omega(z)| \le \eta_1^{(\gamma)}\}, \\
\mathcal{G}_2^{(\gamma)} &:= \{z \in \mathbb{H} : |\omega(z) + q_{m-2}| \le \beta_{m-1}^{(\gamma)}\},
\end{aligned}
$$

and $\omega(z) := z^2 + q_{m-1}z$ with $z, q_{m-1} \in \mathbb{H}$.

Proof. Applying Theorem 3.2 to $\omega(C_{p_l})$ and using $\eta_1^{(\gamma)}$ and $\beta_{m-1}^{(\gamma)}$, we have the desired result. ■

The union of $\mathcal{G}_1^{(\gamma)}$ and $\mathcal{G}_2^{(\gamma)}$ can easily be enclosed in one of the two ovals of Cassini which is as follows.

Theorem 3.12. *Let $p_l(z)$ be a simple monic polynomial over $\mathbb{H}$ of degree m and let $\gamma \in [0,1]$. Then all the zeros of $p_l(z)$ are contained in each of the following ovals of Cassini:*

$$
\begin{aligned}
\mathcal{K}_1^{(\gamma)} &:= \{z \in \mathbb{H} : |z|\ |z + q_{m-1}| \le \max\{\eta_1^{(\gamma)}, |q_{m-2}| + \beta_{m-1}^{(\gamma)}\}\}, \\
\mathcal{K}_2^{(\gamma)} &:= \{z \in \mathbb{H} : |z^2 + q_{m-1}z + q_{m-2}| \le \max\{|q_{m-2}| + \eta_1^{(\gamma)}, \beta_{m-1}^{(\gamma)}\}\}.
\end{aligned}
$$

Proof. To prove the theorem, we prove the following conditions:

$$
\mathcal{G}_1^{(\gamma)} \cup \mathcal{G}_2^{(\gamma)} \subseteq \mathcal{K}_1^{(\gamma)} \ \text{ and } \ \mathcal{G}_1^{(\gamma)} \cup \mathcal{G}_2^{(\gamma)} \subseteq \mathcal{K}_2^{(\gamma)}.
$$

Clearly $\mathcal{G}_1^{(\gamma)} \subseteq \mathcal{K}_1^{(\gamma)}$. Consider $z \in \mathcal{G}_2^{(\gamma)}$ implies $|z^2 + q_{m-1}z + q_{m-2}| \le \beta_{m-1}^{(\gamma)}$. Then $|z^2 + q_{m-1}z| \le |q_{m-2}| + \beta_{m-1}^{(\gamma)}$. This implies $z \in \mathcal{K}_1^{(\gamma)}$. Hence $\mathcal{G}_2^{(\gamma)} \subseteq \mathcal{K}_1^{(\gamma)}$. Thus from the above, we have $\mathcal{G}_1^{(\gamma)} \cup \mathcal{G}_2^{(\gamma)} \subseteq \mathcal{K}_1^{(\gamma)}$.

For the 2nd part: It is clear that $\mathcal{G}_2^{(\gamma)} \subseteq \mathcal{K}_2^{(\gamma)}$. Let $z \in \mathcal{G}_1^{(\gamma)}$. Then $|z^2 + q_{m-1}z| \le \eta_1^{(\gamma)}$. This shows that $|z^2 + q_{m-1}z + q_{m-2}| - |q_{m-2}| \le \eta_1^{(\gamma)}$. This implies $|z^2 + q_{m-1}z + q_{m-2}| \le |q_{m-2}| + \eta_1^{(\gamma)}$. Hence $z \in \mathcal{K}_2^{(\gamma)}$ implies $\mathcal{G}_1^{(\gamma)} \subseteq \mathcal{K}_2^{(\gamma)}$. Therefore, $\mathcal{G}_1^{(\gamma)} \cup \mathcal{G}_2^{(\gamma)} \subseteq \mathcal{K}_2^{(\gamma)}$. ■

We give the following example to illustrate our theory.

Example 3.13. Consider the following quaternionic polynomial $p_l(z)$:

$$p_l(z) = z^6 + \mathbf{j}z^5 + \mathbf{i}z^4 - z^2 - \mathbf{j}z - \mathbf{i}.$$

The zeros of $p_l(z)$ are given in [19] as follows:

$$Z_{\mathbb{H}}(p_l) = \{1,\ -1,\ [\mathbf{i}],\ \frac{1}{2}(1 - \mathbf{i} - \mathbf{j} - \mathbf{k}),\ \frac{1}{2}(-1 + \mathbf{i} - \mathbf{j} - \mathbf{k})\}.$$

The corresponding companion matrix to $p_l(z)$ is given as

$$C_{p_l} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \mathbf{i} & \mathbf{j} & 1 & 0 & -\mathbf{i} & \mathbf{j} \end{bmatrix}.$$

We have the following expressions:

$$\beta_1^{(\gamma)} = 2^\gamma, \tag{3.16}$$

$$\beta_2^{(\gamma)} = \beta_4^{(\gamma)} = \beta_5^{(\gamma)} = \eta_2^{(\gamma)} = 3^{1-\gamma}\, 2^\gamma, \tag{3.17}$$

$$\beta_3^{(\gamma)} = \eta_1^{(\gamma)} = 4^{1-\gamma}\, 2^\gamma, \tag{3.18}$$

$$\eta_3^{(\gamma)} = \frac{1}{2}\left[1 + \sqrt{1 + 2^{5-2\gamma}\, 3^\gamma}\right]. \tag{3.19}$$

Substituting $\gamma = 1/2$ in Theorem 3.12, then we have the following ovals of Cassini:

$$\begin{aligned} \mathcal{K}_1^{(1/2)} &:= \{z \in \mathbb{H} : |z|\ |z + \mathbf{j}| \leq 3.4495\}, \\ \mathcal{K}_2^{(1/2)} &:= \{z \in \mathbb{H} : |z^2 + \mathbf{j}z + \mathbf{i}| \leq 3.8284\}. \end{aligned}$$

From the above, it is clear that all the zeros of $p_l(z)$ are contained in the ovals of Cassini $\mathcal{K}_1^{(1/2)}$ and $\mathcal{K}_2^{(1/2)}$.

We now present the following inclusion regions for zeros of $p_l(z)$ by applying Theorem 3.3.

Theorem 3.14. *Let $p_l(z)$ be a simple monic polynomial over $\mathbb{H}$ of degree m and let $\gamma \in [0,1]$. Then all the zeros of $p_l(z)$ are contained in $\Gamma_1^{(\gamma)} \cup \Gamma_2^{(\gamma)} \cup \Gamma_3^{(\gamma)}$, where*

$$\begin{aligned}
\Gamma_1^{(\gamma)} &:= \{z \in \mathbb{H} : |\omega(z)| \leq \sqrt{\eta_1^{(\gamma)}\ \eta_2^{(\gamma)}}\},\\
\Gamma_2^{(\gamma)} &:= \{z \in \mathbb{H} : |\omega(z) + q_{m-2}| \leq \xi_4^{(\gamma)} \sqrt{\beta_{m-1}^{(\gamma)}}\},\\
\Gamma_3^{(\gamma)} &:= \{z \in \mathbb{H} : |\omega(z)|\ |\omega(z) + q_{m-2}| \leq \eta_1^{(\gamma)}\beta_{m-1}^{(\gamma)}\},
\end{aligned}$$

and $\omega(z) = z^2 + q_{m-1}z$ with $z, q_{m-1} \in \mathbb{H}$.

Proof. Applying Theorem 3.3 to $\omega(z)$, where $\omega(z) = z^2 + q_{m-1}z$ and from the definitions of $\eta_1^{(\gamma)}, \eta_2^{(\gamma)}, \xi_4^{(\gamma)}$, and $\beta_{m-1}^{(\gamma)}$, we have the desired result. ■

Before deriving next inclusion regions for the zeros of the simple monic polynomial $p_l(z)$, we require the following lemma.

Lemma 3.15. *Let Υ_1, Υ_2, and Υ_3 be the three sets such that*

$$\begin{aligned}
\Upsilon_1 &:= \{z \in \mathbb{H} : |z|\,|z+a| \leq \delta\},\ \Upsilon_2 := \{z \in \mathbb{H} : |z|^2 - |a|\,|z| \leq \delta\},\ \textit{and}\\
\Upsilon_3 &:= \{z \in \mathbb{H} : |z| \leq \frac{1}{2}(|a| + \sqrt{|a|^2 + 4\delta})\},
\end{aligned}$$

where $a \in \mathbb{H}$ and $\delta > 0$. Then $\Upsilon_1 \subseteq \Upsilon_2 := \Upsilon_3$.

Proof. Consider $z \in \Upsilon_1$. Then, $|z||z+a| \leq \delta$ implies $|z|^2 - |a||z| \leq \delta$. Hence $z \in \Upsilon_2$. This shows $\Upsilon_1 \subseteq \Upsilon_2$. Next, we assume that $z \in \Upsilon_2 \Leftrightarrow |z|^2 - |a||z| \leq \delta$, then

$$\left[|z| - \frac{1}{2}\left(|a| + \sqrt{|a|^2 + 4\delta}\right)\right]\left[|z| - \frac{1}{2}\left(|a| - \sqrt{|a|^2 + 4\delta}\right)\right] \leq 0. \tag{3.20}$$

Since $|z| \geq \frac{1}{2}\left(|a| - \sqrt{|a|^2 + 4\delta}\right)$, then from (3.20), we have

$$|z| \leq \frac{1}{2}\left(|a| + \sqrt{|a|^2 + 4\delta}\right).$$

Hence $z \in \Upsilon_3$. Thus $\Upsilon_2 = \Upsilon_3$. ■

We give the following inclusion sets for the zeros of the quaternionic polynomial $p_l(z)$.

Theorem 3.16. *Let $p_l(z)$ be a simple monic polynomial over $\mathbb{H}$ of degree m and let $\gamma \in [0,1]$. Then all the zeros of $p_l(z)$ are contained in each of the following ovals of Cassini:*

$$\Omega_1^{(\gamma)} := \{z \in \mathbb{H} : |z|\ |z + q_{m-1}| \leq \max\{\sqrt{\eta_1^{(\gamma)}\eta_2^{(\gamma)}}, \eta_3^{(\gamma)}, |q_{m-2}| + \sqrt{\beta_{m-1}^{(\gamma)}}\}\},$$

$$\Omega_2^{(\gamma)} := \{z \in \mathbb{H} : |z^2 + q_{m-1}z + q_{m-2}| \leq \max\{|q_{m-2}| + \sqrt{\eta_1^{(\gamma)}\eta_2^{(\gamma)}}, \eta_3^{(\gamma)}, \sqrt{\beta_{m-1}^{(\gamma)}}\}\},$$

where $\eta_1^{(\gamma)}, \eta_2^{(\gamma)}$*, and* $\eta_3^{(\gamma)}$ *are defined in* (3.15).

Proof. To prove the theorem, we prove the following conditions:

$$\Gamma_1^{(\gamma)} \cup \Gamma_2^{(\gamma)} \cup \Gamma_3^{(\gamma)} \subseteq \Omega_1^{(\gamma)} \text{ and } \Gamma_1^{(\gamma)} \cup \Gamma_2^{(\gamma)} \cup \Gamma_3^{(\gamma)} \subseteq \Omega_2^{(\gamma)}.$$

Clearly $\Gamma_1^{(\gamma)} \subseteq \Omega_1^{(\gamma)}$. Consider $z \in \Gamma_2^{(\gamma)}$, this implies $|z^2 + q_{m-1}z + q_{m-2}| \leq \beta_{m-1}^{(\gamma)}$. This shows that $|z^2 + q_{m-1}z| \leq |q_{m-2}| + \beta_{m-1}^{(\gamma)}$. Then $z \in \Omega_1^{(\gamma)}$. Hence $\Gamma_2^{(\gamma)} \subseteq \Omega_1^{(\gamma)}$.

Next we assume $z \in \Gamma_3^{(\gamma)}$. This implies $|\omega(z)|\,|\omega(z) + q_{m-2}| \leq \eta_1^{(\gamma)}\beta_{m-1}^{(\gamma)}$. Hence $(|\omega(z)|^2 - |\omega(z)||q_{m-2}|) \leq \eta_1^{(\gamma)}\beta_{m-1}^{(\gamma)}$. Now, from Lemma 3.15, we obtain $|\omega(z)| \leq \eta_3^{(\gamma)}$. Hence $z \in \Omega_1^{(\gamma)}$. Thus from the above, we have $\Gamma_1^{(\gamma)} \cup \Gamma_2^{(\gamma)} \cup \Gamma_3^{(\gamma)} \subseteq \Omega_1^{(\gamma)}$.

On the other hand, it is clear that $\Gamma_2^{(\gamma)} \subseteq \Omega_2^{(\gamma)}$. Considering $z \in \Gamma_1^{(\gamma)}$ it implies $|z^2 + q_{m-1}z| \leq \sqrt{\eta_1^{(\gamma)}\eta_2^{(\gamma)}}$. Then we write $|z^2 + q_{m-1}z + q_{m-2} - q_{m-2}| \leq \sqrt{\eta_1^{(\gamma)}\eta_2^{(\gamma)}}$. This shows $|z^2 + q_{m-1}z + q_{m-2}| \leq |q_{m-2}| + \sqrt{\eta_1^{(\gamma)}\eta_2^{(\gamma)}}$ which implies $z \in \Omega_2^{(\gamma)}$. Thus $\Gamma_1^{(\gamma)} \subseteq \Omega_2^{(\gamma)}$. Now, let $z \in \Gamma_3^{(\gamma)}$. Then $|\omega(z)|\,|\omega(z) + q_{m-2}| \leq \eta_1^{(\gamma)}\beta_{m-1}^{(\gamma)}$. Thus

$$(|\omega(z) + q_{m-2}| - |q_{m-2}|)|\omega(z) + q_{m-2}| \leq \eta_1^{(\gamma)}\beta_{m-1}^{(\gamma)}.$$

From the above discussion and by Lemma 3.15, we have

$$|\omega(z) + q_{m-2}| \leq \eta_3^{(\gamma)}.$$

Thus, we obtain $z \in \Omega_2^{(\gamma)}$. Therefore $\Gamma_1^{(\gamma)} \cup \Gamma_2^{(\gamma)} \cup \Gamma_3^{(\gamma)} \subseteq \Omega_2^{(\gamma)}$. ■

To illustrate Theorem 3.16, we take Example 3.13. Substituting $\gamma = 1/2$ in Theorem 3.16, we have the following ovals of Cassini from Example 3.13:

$$\Omega_1^{(1/2)} = \{z \in \mathbb{H} : |z|\ |z + \mathbf{j}| \leq 6.1723\}, \text{ and}$$

$$\Omega_2^{(1/2)} = \{z \in \mathbb{H} : |z^2 + \mathbf{j}z + \mathbf{i}| \leq 7.1723\}.$$

Here it is also clear that all the zeros of $p_l(z)$ (defined in Example 3.13) are located in the ovals of Cassini $\Omega_1^{(1/2)}$ and $\Omega_2^{(1/2)}$.

Theorems 3.12 and 3.16 give the following corollary.

Corollary 3.17. *Let* $p_l(z)$ *be a simple monic polynomial over* $\mathbb{H}$ *of degree* m*. Then all the zeros of* $p_l(z)$ *are contained in ovals of Cassini* $\mathcal{K}_1^{(\gamma)} \cap \Omega_1^{(\gamma)}$ $\forall \gamma \in [0,1]$ *and also in ovals of Cassini* $\mathcal{K}_2^{(\gamma)} \cap \Omega_2^{(\gamma)}$ $\forall\, \gamma \in [0,1]$.

Inclusion regions for the zeros of the quaternionic polynomial $p_r(z)$: Since left eigenvalues of $C_{p_l}^T = C_{p_r}$ are the zeros of the simple monic polynomial $p_r(z)$. We also need some machinery for the development of inclusion regions for the zeros of $p_r(z)$. Now, let us take $\omega(z)$ as

$$\omega(z) := z^2 + \overline{q_{m-1}}\ z,$$

where $z, q_{m-1} \in \mathbb{H}$. To find $\omega_3(C_{p_l})$, we first construct $\overline{C_{p_l}}^2$:

$$\overline{C_{p_l}}^2 = \begin{bmatrix} 0 & 0 & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ -\overline{q_0} & -\overline{q_1} & -\overline{q_2} & \dots & -\overline{q_{m-1}} \\ \overline{q_{m-1}}\ \overline{q_0} & \overline{q_{m-1}}\ \overline{q_1} - \overline{q_0} & \overline{q_{m-1}}\ \overline{q_2} - \overline{q_1} & \dots & \overline{q_{m-1}}^2 - \overline{q_{m-2}} \end{bmatrix}.$$

Thus, we have

$$\overline{C_{p_l}}^2 + \overline{q_{m-1}}\ \overline{C_{p_l}} = \begin{bmatrix} 0 & \overline{q_{m-1}} & 1 & & & \\ 0 & 0 & \overline{q_{m-1}} & \ddots & & \\ 0 & 0 & 0 & \ddots & 1 & \\ & & & & \overline{q_{m-1}} & 1 \\ -\overline{q_0} & -\overline{q_1} & -\overline{q_2} & \dots & -\overline{q_{m-2}} & 0 \\ 0 & -\overline{q_0} & -\overline{q_1} & \dots & -\overline{q_{m-3}} & -\overline{q_{m-2}} \end{bmatrix}.$$

It is known from Lemma 2.1 that if λ is a left eigenvalue of a quaternionic matrix A, then $\overline{\lambda}$ is a left eigenvalue of A^H. Thus if λ is a left eigenvalue of the companion matrix C_{p_r}, then $\overline{\lambda}$ is a left eigenvalue of $C_{p_r}^H = \overline{C_{p_l}}$. Therefore $\overline{\lambda}^2 + \overline{q_{m-1}}\ \overline{\lambda}$ is a left eigenvalue of $\overline{C_{p_l}}^2 + \overline{q_{m-1}}\ \overline{C_{p_l}}$.

Remark 3.18. Almost similar results can be obtained for the zeros of the simple monic polynomial $p_r(z)$ with $\omega(z) := z^2 + \overline{q_{m-1}}\ z$ by using the ideas given in this section for the zeros of the simple monic polynomial $p_l(z)$.

3.4.1. Inclusion regions for zeros of complex polynomials

Case 3. Let us consider $\omega(z)$ as:

$$\omega(z) := \sum_{j \in S} \alpha_j z^j, \tag{3.21}$$

where $\alpha_j, z \in \mathbb{C}$ and S is defined as the set of integers when $A \in M_n(\mathbb{C})$ is invertible; otherwise S is the set of nonnegative integers. Throughout this section, we use the polynomial $\omega(z)$ (defined in (3.21)).

We take $p_l(z)$ and $p_r(z)$ over the complex field. Then $p_l(z) = p_r(z) = p(z)$ (say). Hence the complex polynomial is defined as

$$p(z) := q_m z^m + q_{m-1} z^{m-1} + \cdots + q_1 z + q_0, \tag{3.22}$$

where $q_j,\ z \in \mathbb{C}$ $(0 \leq j \leq m)$. The polynomial (3.22) is called monic if $q_m = 1$. Then, the corresponding companion matrix of the monic polynomial $p(z)$ is given by

$$C_p := \begin{bmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \\ -q_0 & -q_1 & \cdots & -q_{m-1} \end{bmatrix}.$$

Remark 3.19. Similar results to Theorems 2.3 and 2.8 can be obtained in the case of complex matrices.

Remark 3.20. Similar results to Theorems 3.2 and 3.3 can be obtained in the case of complex matrices with $\omega(z)$.

Remark 3.21. Similar results to Theorems 3.10, 3.11, 3.12, 3.14, and 3.16 can be obtained for the zeros of the complex monic polynomial $p(z)$.

Now, we have the following corollary for the zeros of complex polynomials from Corollary 3.17.

Corollary 3.22. *Let $p(z)$ be a monic polynomial over $\mathbb{C}$ of degree m. Then all the zeros of $p(z)$ are contained in oval of Cassini $\mathcal{K}_1^{(\gamma)} \cap \Omega_1^{(\gamma)}$ $\forall \gamma \in [0,1]$ and also in oval of Cassini $\mathcal{K}_2^{(\gamma)} \cap \Omega_2^{(\gamma)}$ $\forall\, \gamma \in [0,1]$.*

We now state the following results from the above remarks which are available in the literature.

- Assuming $\gamma = 0$ in Theorem 3.10 for the case of complex monic polynomial $p(z)$, we observe that all the zeros of $p(z)$ are contained in each of the following sets:

$$\begin{aligned} T^{(0)} &:= \{z \in \mathbb{C} : |z| \leq \xi_1^{(0)}\} \cup \{z \in \mathbb{C} : |z + q_{m-1}| \leq 1\}, \\ K^{(0)} &:= \{z \in \mathbb{C} : |z| \leq \sqrt{\xi_1^{(0)}\xi_2^{(0)}}\} \cup \{z \in \mathbb{C} : |z|\ |z + q_{m-1}| \leq \xi_1^{(0)}\}. \end{aligned}$$

  This can be seen in [37, Theorem 3.1].

- Substituting $\gamma = 0$ in Theorem 3.11 for the case of complex monic polynomial $p(z)$, we obtain that all the zeros of $p(z)$ are contained in $\mathcal{G}_1^{(0)} \cup \mathcal{G}_2^{(0)}$, where

$$\mathcal{G}_1^{(0)} := \{z \in \mathbb{C} : |\omega(z)| \leq \eta_1^{(0)}\},\ \mathcal{G}_2^{(0)} := \{z \in \mathbb{C} : |\omega(z) + q_{m-2}| \leq \beta_{m-1}^{(0)}\}$$

  with $\omega(z) := z^2 + q_{m-1}z,\ q_{m-1}, z \in \mathbb{C}$. This can be found in [37, Theorem 3.2].

- Let $\gamma = 0$ in Theorem 3.12 for the case of complex monic polynomial $p(z)$, we see that all the zeros of $p(z)$ are contained in the oval of Cassini defined by

$$\mathcal{K}_1^{(0)} := \{z \in \mathbb{C} : |z|\ |z + q_{m-1}| \leq \max\{\eta_1^{(0)}, |q_{m-2}| + \beta_{m-1}^{(0)}\}\},$$

  and also in the oval of Cassini defined as

$$\mathcal{K}_2^{(0)} := \{z \in \mathbb{C} : |z^2 + q_{m-1}z + q_{m-2}| \leq \max\{|q_{m-2}| + \eta_1^{(0)}, \beta_{m-1}^{(0)}\}\}.$$

  This can be seen in [37, Theorem 3.3].

- Assuming $\gamma = 0$ in Theorem 3.14 for the case of complex monic polynomial $p(z)$, we obtain that all the zeros of $p(z)$ are contained in $\Gamma_1^{(0)} \cup \Gamma_2^{(0)} \cup \Gamma_3^{(0)}$, where

$$\begin{aligned} \Gamma_1^{(0)} &:= \{z \in \mathbb{C} : |\omega(z)| \leq \sqrt{\eta_1^{(0)}\ \eta_2^{(0)}}\}, \\ \Gamma_2^{(0)} &:= \{z \in \mathbb{C} : |\omega(z) + q_{m-2}| \leq \sqrt{\beta_{m-1}^{(0)}}\}, \\ \Gamma_3^{(0)} &:= \{z \in \mathbb{C} : |\omega(z)|\ |\omega(z) + q_{m-2}| \leq \quad \eta_1^{(0)}\beta_{m-1}^{(0)}\} \end{aligned}$$

  with $\omega(z) := z^2 + q_{m-1}z,\ q_{m-1}, z \in \mathbb{C}.$ This can be found in [37, Theorem 3.4].

- Assuming $\gamma = 0$ in Theorem 3.16, then all the zeros of the complex monic polynomial $p(z)$ are contained in the oval of Cassini defined as

$$\Omega_1^{(0)} := \{z \in \mathbb{C} : |z|\ |z + q_{m-1}| \leq \max\{\sqrt{\eta_1^{(0)}\eta_2^{(0)}}, \eta_3^{(0)}, |q_{m-2}| + \sqrt{\beta_{m-1}^{(0)}}\}\},$$

  and also in the oval of Cassini given by

$$\Omega_2^{(0)} := \{z \in \mathbb{C} : |z^2 + q_{m-1}z + q_{m-2}| \leq \max\{|q_{m-2}| + \sqrt{\eta_1^{(0)}\eta_2^{(0)}}, \eta_3^{(0)}, \sqrt{\beta_{m-1}^{(0)}}\}\},$$

where $\eta_3^{(0)} := \frac{1}{2}\left(|q_{m-2}| + \sqrt{|q_{m-2}|^2 + 4\eta_1^{(0)}\beta_{m-1}^{(0)}}\right)$. This is stated in [37, Theorem 3.5].

Comparisons with the existing results: The inclusion regions $T^{(0)}, K^{(0)}, G_1^{(0)}, G_2^{(0)}$, $\mathcal{K}_1^{(0)}, \mathcal{K}_2^{(0)}, \Gamma_1^{(0)}, \Gamma_2^{(0)}, \Gamma_3^{(0)}, \Omega_1^{(0)}, \Omega_2^{(0)}$ are also given in [37]. Now, we compare these inclusion regions with $T^{(\gamma)}, G_1^{(\gamma)}, G_2^{(\gamma)}, \mathcal{K}_1^{(\gamma)}, \mathcal{K}_2^{(\gamma)}, \Gamma_1^{(\gamma)}, \Gamma_2^{(\gamma)}, \Gamma_3^{(\gamma)}\Omega_1^{(\gamma)}, \Omega_2^{(\gamma)}\ \forall \gamma \in [0,1]$, under certain conditions which are as follows:

(*a*) Let $\xi_3^{(\gamma)} \leq 1\ \forall\gamma \in [0,1]$. Since $\xi_1^{(\gamma)} \leq \xi_1^{(0)}\ \forall \gamma \in [0,1]$, then $T^{(\gamma)} \subseteq T^{(0)}\ \forall\gamma \in [0,1]$. Further, we have seen that $\xi_1^{(\gamma)}\xi_3^{(\gamma)} \leq \xi_1^{(0)}$ and $\xi_2^{(\gamma)} \leq \xi_2^{(0)}\ \forall \gamma \in [0,1]$, then $K^{(\gamma)} \subseteq K^{(0)}\ \forall \gamma \in [0,1]$.

(*b*) If $1 + |q_{m-1}| \leq \eta_1^{(0)}$, then $\eta_1^{(\gamma)} \leq \eta_1^{(0)}\ \forall \gamma \in [0,1]$. Further, if $\xi_4^{(\gamma)} \leq 1\ \forall \gamma \in [0,1]$, then $\beta_{m-1}^{(\gamma)} \leq \beta_{m-1}^{(0)}$. Consequently, $\mathcal{G}_1^{(\gamma)} \subseteq \mathcal{G}_1^{(0)}$ and $\mathcal{G}_2^{(\gamma)} \subseteq \mathcal{G}_2^{(0)}\ \forall \gamma \in [0,1]$.

(*c*) Considering $1 + |q_{m-1}| \leq \eta_1^{(0)}$ and $\xi_4^{(\gamma)} \leq 1\ \forall\gamma \in [0,1]$, then $\max\{\eta_1^{(\gamma)}, |q_{m-2}| + \beta_{m-1}^{(\gamma)}\} \leq \max\{\eta_1^{(0)}, |q_{m-2}| + \beta_{m-1}^{(0)}\}$, and $\max\{|q_{m-2}| + \eta_1^{(\gamma)}, \beta_{m-1}^{(\gamma)}\} \leq \max\{|q_{m-2}| + \eta_1^{(0)}, \beta_{m-1}^{(0)}\}\ \forall \gamma \in [0,1]$. Thus $\mathcal{K}_1^{(\gamma)} \subseteq \mathcal{K}_1^{(0)}$ and $\mathcal{K}_2^{(\gamma)} \subseteq \mathcal{K}_2^{(0)}\ \forall \gamma \in [0,1]$.

(*d*) Supposing $1 + |q_{m-1}| \leq \eta_1^{(0)}$ and $1 + |q_{m-1}| \leq \eta_2^{(0)}$, then $\eta_1^{(\gamma)} \leq \eta_1^{(0)}$ and $\eta_2^{(\gamma)} \leq \eta_2^{(0)}\ \forall \gamma \in [0,1]$. Consequently, $\sqrt{\eta_1^{(\gamma)}\eta_2^{(\gamma)}} \leq \sqrt{\eta_1^{(0)}\eta_2^{(0)}}\ \forall \gamma \in [0,1]$. Also if $\xi_4^{(\gamma)} \leq 1\ \forall \gamma \in [0,1]$, then $\xi_4^{(\gamma)}\sqrt{\beta_{m-1}^{(\gamma)}} \leq \sqrt{\beta_{m-1}^{(0)}}$ and $\eta_1^{(\gamma)}\beta_{m-1}^{(\gamma)} \leq \eta_1^{(0)}\beta_{m-1}^{(0)}$. Hence $\Gamma_1^{(\gamma)} \subseteq \Gamma_1^{(0)}$, $\Gamma_2^{(\gamma)} \subseteq \Gamma_2^{(0)}$, and $\Gamma_3^{(\gamma)} \subseteq \Gamma_3^{(0)}\ \forall \gamma \in [0,1]$.

(*e*) If $1 + |q_{m-1}| \leq \eta_1^{(0)}$, then $\eta_1^{(\gamma)} \leq \eta_1^{(0)}$. Similarly, if $\xi_4^{(\gamma)} \leq 1\ \forall \gamma \in [0,1]$, then $\beta_{m-1}^{(\gamma)} \leq \beta_{m-1}^{(0)}$. Consequently, $\eta_3^{(\gamma)} \leq \eta_3^{(0)}$. Hence $\Omega_1^{(\gamma)} \subseteq \Omega_1^{(0)}$ and $\Omega_2^{(\gamma)} \subseteq \Omega_2^{(0)}\ \forall \gamma \in [0,1]$.

We see the following observations:

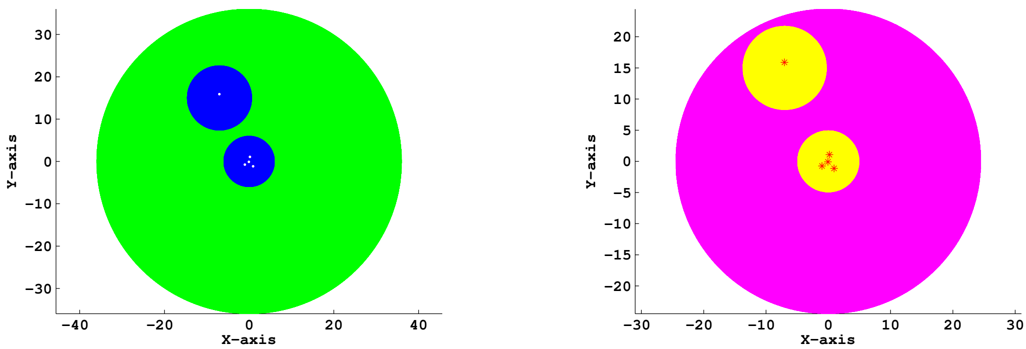
- If $1 + |q_{m-1}| \leq \eta_1^{(0)}$ and $\xi_4^{(\gamma)} \leq 1\ \forall\gamma \in [0,1]$, then $\mathcal{K}_1^{(\gamma)} \subseteq \mathcal{K}_1^{(0)}$, $\mathcal{K}_2^{(\gamma)} \subseteq \mathcal{K}_2^{(0)}$, $\Omega_1^{(\gamma)} \subseteq \Omega_1^{(0)}$, and $\Omega_2^{(\gamma)} \subseteq \Omega_2^{(0)}\ \forall \gamma \in [0,1]$.
- If $\xi_3^{(\gamma)} \leq (1+\rho)^\gamma\ \forall \gamma \in (0,1]$, then our inclusion regions $K^{(\gamma)}\ \forall\gamma \in (0,1]$ are sharper than the inclusion region $K^{(0)}$ [37, Theorem 3.1].
- If $\xi_4^{(\gamma)} \leq (1 + k\rho)^\gamma\ \forall \gamma \in (0,1]; k = 2, 3$, then ovals of Cassini $\mathcal{K}_1^{(\gamma)} \cap \Omega_1^{(\gamma)}\ \forall\gamma \in (0,1]$ and $\mathcal{K}_2^{(\gamma)} \cap \Omega_2^{(\gamma)}\ \forall\gamma \in (0,1]$ are sharper than the ovals of Cassini [37, Corollary 3.1].

Finally, we compare our results with A. Melman's results given in [37]. Then each of the sets $T^{(\gamma)}$ and $K^{(\gamma)}$ (for the case of complex polynomials) can be shown that they are smaller than the sets $T^{(0)}$ and $K^{(0)}$, respectively. For example, consider the complex polynomial

$$p(z) = z^5 + (7-15i)z^4 + (14+7i)z^3 + 3z^2 + 35z + 2 + 3i. \tag{3.23}$$

$$\begin{aligned}
T^{(0)} &= \{z \in \mathbb{C} : |z| \leq 35\} \cup \{z \in \mathbb{C} : |z + 7 - 15i| \leq 1\},\\
K^{(0)} &= \{z \in \mathbb{C} : |z| \leq \sqrt{(35)\,(16.6526)}\} \cup \{z \in \mathbb{C} : |z|\,|z + 7 - 15i| \leq 35\},\\
T^{(\gamma)} &= \{z \in \mathbb{C} : |z| \leq (35)^{1-\gamma}\} \cup \{z \in \mathbb{C} : |z + 7 - 15i| \leq (58.2580)^{\gamma}\},\\
K^{(\gamma)} &= \{z \in \mathbb{C} : |z| \leq \sqrt{(35)^{1-\gamma}\,(16.6526)^{1-\gamma}}\} \cup \{z \in \mathbb{C} : |z|\,|z + 7 - 15i| \leq \chi\},
\end{aligned}$$

where $\chi = (35)^{1-\gamma}\,(58.2580)^{\gamma}$.



(a) ■ $T^{(0)}$, ■ $T^{(1/2)}$; the sets $T^{(0)}$ and $T^{(1/2)}$ are shaded in green and blue color, respectively. The white dots are the zeros of $p(z)$.

(b) ■ $K^{(0)}$, ■ $K^{(1/2)}$; the sets $K^{(0)}$ and $K^{(1/2)}$ are shaded in magenta and yellow color, respectively. The star symbols are represented as the zeros of $p(z)$.

FIGURE 3.1. Location of zeros of $p(z)$.

CHAPTER 4

LOCALIZATION THEOREMS FOR QUATERNIONIC MATRIX PENCILS

In this chapter, a general framework for defining and analyzing the generalized right eigenvalues of a quaternionic matrix pencil are developed. Inclusion regions for the generalized right eigenvalues of a quaternionic matrix pencil and their applications are presented.

4.1. Introduction and preliminaries

Quaternionic matrix pencils and their corresponding canonical forms have been derived in [46,47]. In this chapter, we derive localization theorems for generalized right eigenvalues of a quaternionic matrix pencil with some properties. Location of zeros of quaternionic polynomials is presented.

Throughout this chapter, we adopt the following basic facts: Let $\mathbb{L}_1(M_n(\mathbb{H}))$ be the space of matrix pencils over a quaternion division algebra. $\mathbf{L}_1 \in \mathbb{L}_1(M_n(\mathbb{H}))$ is defined as

$$\mathbf{L}_1(\lambda) := A + \lambda B, \tag{4.1}$$

where $A, B \in M_n(\mathbb{H})$ and λ commutes with the quaternionic matrices. This matrix pencil over a quaternion division algebra can be found in [33, 46–48].

Now we define generalized right eigenvalue of $\mathbf{L}_1 \in \mathbb{L}_1(M_n(\mathbb{H}))$ of the form (4.1) which is as follows.

Definition 4.1. Let $\mathbf{L}_1 \in \mathbb{L}_1(M_n(\mathbb{H}))$ be as in (4.1) and let $\mu \in \mathbb{H}$. Then μ is called a generalized right eigenvalue of the matrix pencil $\mathbf{L}_1$ if

$$Ax = Bx\mu$$

for some nonzero $x \in \mathbb{H}^n$. Here x is called the right eigenvector corresponding to the generalized right eigenvalue μ. The set of generalized right eigenvalues of $\mathbf{L}_1$ is called right spectrum of $\mathbf{L}_1$, denoted by $\Lambda_r(\mathbf{L}_1)$.

Regular matrix pencil over a quaternion division algebra is defined as follows.

Definition 4.2. Let $\mathbf{L}_1 \in \mathbb{L}_1(M_n(\mathbb{H}))$ be as in (4.1). Then the matrix pencil $\mathbf{L}_1$ is called regular if there exists $\alpha \in \mathbb{R}$ such that $A + \alpha\ B$ is an invertible matrix.

Let $\mathbb{P}_1(M_{2n}(\mathbb{C}))$ be the space of complex matrix pencils. $P_1 \in \mathbb{P}_1(M_{2n}(\mathbb{C}))$ is defined as

$$P_1(\mu) := \Psi_A + \mu\Psi_B, \tag{4.2}$$

where $A, B \in M_n(\mathbb{H})$ and $\mu \in \mathbb{C}$.

4.2. Location of generalized right eigenvalues

We give the following theorem with the help of the complex adjoint matrix.

Theorem 4.3. *Let $\mathbf{L}_1 \in \mathbb{L}_1(M_n(\mathbb{H}))$ be as in (4.1). Then the matrix pencil $\mathbf{L}_1$ is a regular if and only if $P_1 \in \mathbb{P}_1(M_{2n}(\mathbb{C}))$ is a regular complex matrix pencil.*

Proof. Consider $\mathbf{L}_1 \in \mathbb{L}_1(M_n(\mathbb{H}))$ is a regular matrix pencil, then there exists $\alpha \in \mathbb{R}$ such that $\mathbf{L}_1(\alpha) = A + \alpha B$ is an invertible matrix pencil. Now from Theorem 1.12, the corresponding complex matrix pencil $\Psi_{\mathbf{L}_1(\alpha)} = \Psi_A + \alpha\Psi_B$ is an invertible complex matrix pencil. Thus, $P_1(\mu)$ is a regular complex matrix pencil.

Conversely let $P_1(\mu)$ be a regular complex matrix pencil, then

$$\det[P_1(\eta)] \neq 0 \text{ for some } \eta \in \mathbb{C}, \text{ i.e.,}$$

$$\det[\Psi_A + \eta\Psi_B] \neq 0.$$

Thus, there exists a real number λ_0 (say) such that

$$\det[\Psi_A + \lambda_0\Psi_B] \neq 0,$$

i.e., $\Psi_A + \lambda_0\Psi_B = \Psi_{(A+\lambda_0 B)}$ is an invertible complex pencil. Then by Theorem 1.12, the matrix pencil $A + \lambda_0 B$ is invertible. Hence $\mathbf{L}_1 \in \mathbb{L}_1(M_n(\mathbb{H}))$ is a regular quaternionic matrix pencil. ■

Proposition 4.4. *Let $\mathbf{L}_1 \in \mathbb{L}_1(M_n(\mathbb{H}))$ be as in (4.1) and let $\mu \in \mathbb{H}$ be a generalized right eigenvalue of the matrix pencil $\mathbf{L}_1$. Let $0 \neq \rho \in \mathbb{H}$. Then $\rho^{-1}\mu\rho$ is a generalized right eigenvalue of $\mathbf{L}_1$.*

Proof. Suppose μ is a generalized right eigenvalue of the matrix pencil $\mathbf{L}_1$ corresponding to an eigenvector $x \in \mathbb{H}^n$, then

$$Ax = Bx\mu. \tag{4.3}$$

If $\rho \in \mathbb{H} \setminus \{0\}$, then from (4.3), we have

$$Ax\rho = Bx\rho(\rho^{-1}\mu\rho).$$

Thus $\rho^{-1}\mu\rho$ is also a generalized right eigenvalue of the matrix pencil $\mathbf{L}_1$ corresponding to an eigenvector $x\rho$. ■

Complex matrix pencils: Let $A, B \in M_n(\mathbb{C})$. Let $\mathcal{L}(z) := A - zB$ be complex matrix pencil. The complex eigenvalue problem $\mathcal{L}(\lambda)x = 0$, is called the generalized complex eigenvalue problem. Denote the set of $n \times n$ matrix pencils by $\tilde{\mathcal{L}}$ $(M_n(\mathbb{C}))$. $\mathcal{L} \in \tilde{\mathcal{L}}$ $(M_n(\mathbb{C}))$ is said to be regular if $\det(\mathcal{L}(\lambda)) \neq 0$ for some $\lambda \in \mathbb{C}$. The spectrum of a regular matrix pencil $\mathcal{L}$ is given by

$$\Lambda(\mathcal{L}) := \{\lambda \in \mathbb{C} : Ax = \lambda Bx = Bx\lambda \text{ for some nonzero } x \in \mathbb{C}^n\}.$$

We define generalized standard right eigenvalue of a regular matrix pencil $\mathbf{L}_1 \in \mathbb{L}_1(M_n(\mathbb{H}))$ as follows.

Definition 4.5. Let $\mathbf{L}_1 \in \mathbb{L}_1(M_n)(\mathbb{H})$ be as in (4.1). Then we define the set of the generalized standard right eigenvalues of a regular matrix pencil $\mathbf{L}_1$ as

$$\Lambda_s(\mathbf{L}) := \{\alpha \in \mathbb{C}_\infty : Ax = Bx\alpha,\ 0 \neq x \in \mathbb{H}^n,\ \Im(\alpha) \geq 0\}, \text{ where } \mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}.$$

The above definition generalizes the definition of standard right eigenvalues of a single quaternionic matrix [5, 27].

It is known that if B is not invertible, then Ψ_B is not invertible. Hence $P_1(\mu) = \Psi_A + \mu\Psi_B$ has an infinite eigenvalue. Consequently, $\mathbf{L}_1(\lambda) = A + \lambda B$ has an infinite right eigenvalue. However, if B is an invertible matrix, then $P_1(\mu) = \Psi_A + \mu\Psi_B$ has exactly $2n$ finite eigenvalues. Then, $\mathbf{L}_1(\lambda) = A + \lambda B$ has exactly $2n$ finite complex right eigenvalues. Thus, we have the following observations:

- If $\mathbf{L}_1$ is regular, then $\mathbf{L}_1$ has $2n$ right eigenvalues in $\mathbb{C}_\infty$.
- If B is an invertible matrix, then $\mathbf{L}_1$ has exactly $2n$ finite complex right eigenvalues.

We now present the Gerschgorin-type localization of eigenvalues of complex matrix pencils as follows.

Theorem 4.6. *[25] Let $\mathcal{L} \in \tilde{\mathcal{L}}\ (M_n(\mathbb{C}))$ be a complex regular matrix pencil. Then all the eigenvalues of $\mathcal{L}$ are contained in the union of n regions*

$$\mathcal{G}_i(\mathcal{L}) := \left\{ z \in \mathbb{C} : |zb_{ii} - a_{ii}| \leq \sum_{j=1,\, j\neq i}^{n} |a_{ij} - zb_{ij}| \right\} (1 \leq i \leq n), i.e.$$

$$\Lambda(\mathcal{L}) \subseteq \cup_{i=1}^{n} \mathcal{G}_i(\mathcal{L}).$$

Similarly, we define n regions over the skew field of quaternions as follows.

$$G_i(\mathbf{L}_1) := \left\{ z \in \mathbb{H} : |zb_{ii} - a_{ii}| \leq \sum_{j=1,\, j\neq i}^{n} |a_{ij} - zb_{ij}| \right\} (1 \leq i \leq n), \tag{4.4}$$

where $\mathbf{L}_1(\lambda) := A + \lambda B$ is defined in (4.1).

But, in the case of quaternionic matrix pencil, a generalized right eigenvalue is not necessarily contained in the union of n regions $G_i(\mathbf{L}_1)$ as the following example suggest.

Example 4.7. A generalized right eigenvalue is not necessarily contained in a region. Consider the quaternionic matrix pencil $\mathbf{L}_1(\lambda) = A + \lambda B$, where

$$A = \begin{bmatrix} 2\mathbf{i} & 0 \\ 0 & 1 \end{bmatrix},\ B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

and λ commutes with A and B. For finding generalized right eigenvalue, let

$$\det[\mu \Psi_B - \Psi_A] = 0,$$

where $\mu \in \mathbb{C}$. Then $(\mu + 2\mathbf{i})(\mu - 2\mathbf{i}) = 0$ has two zeros corresponding to the values $\mu = 2\mathbf{i}$ and $-2\mathbf{i}$ with multiplicity 1. From (4.4), we have the following regions:

$$\begin{aligned} G_1(\mathbf{L}_1) &:= \{z \in \mathbb{H} : |z - 2\mathbf{i}| \leq 0\}, \text{ and} \\ G_2(\mathbf{L}_1) &:= \{z \in \mathbb{H} : |-1| \leq 1\}. \end{aligned}$$

In particular, $-2\mathbf{i}$ is a generalized right eigenvalue of $\mathbf{L}_1$. However it is not contained in any regions $G_i(\mathbf{L}_1)$, $i = 1, 2$.

Moreover, we present the generalized Gerschgorin type theorem for a quaternionic matrix pencil as follows.

Theorem 4.8. (*Generalized Gerschgorin type theorem for generalized right eigenvalues*) *Let* $\mathbf{L}_1 \in \mathbb{L}_1(M_n(\mathbb{H}))$ *be a quaternionic regular matrix pencil. For every generalized right eigenvalue* μ *of* $\mathbf{L}_1$ *there exists a nonzero quaternion* α *such that* $\alpha^{-1}\mu\alpha$ *(which is also a generalized right eigenvalue) is contained in the union of* n *regions*

$$D_i(\mathbf{L}_1) := \{z \in \mathbb{H} : |b_{ii}z - a_{ii}| \leq |z|\, r_i(B) + r_i(A)\} \ \ (1 \leq i \leq n), \text{ i.e.,}$$

$$\{z^{-1}\mu z : 0 \neq z \in \mathbb{H}\} \cap \cup_{i=1}^{n} D_i(\mathbf{L}_1) \neq \emptyset.$$

In particular, when μ *is real, then it is contained in the union of the regions* $D_i(\mathbf{L}_1)$.

Proof. Let λ be a generalized right eigenvalue of the quaternionic matrix pencil $\mathbf{L}_1$. Then $Ax = Bx\mu$ for some nonzero $x = [x_1, \ldots, x_n]^T \in \mathbb{H}^n$. Let x_t be an element of x such that $|x_t| \geq |x_i| \ \forall\, i \ (1 \leq i \leq n)$. Then $|x_t| > 0$. From the t-th equation of $Ax = Bx\mu$, we have

$$\begin{aligned} \sum_{j=1}^{n} a_{tj}x_j &= \sum_{j=1}^{n} b_{tj}x_j\mu \\ a_{tt}x_t + \sum_{j=1,\, j\neq t}^{n} a_{tj}x_j &= b_{tt}x_t\mu + \sum_{j=1,\, j\neq t}^{n} b_{tj}x_j\mu. \end{aligned}$$

Since $x_t \neq 0$, let $x_t\mu = \xi x_t$, i.e., ξ is similar to μ. Then

$$\begin{aligned} a_{tt}x_t - b_{tt}\xi x_t &= \sum_{j=1,\, j\neq t}^{n} b_{tj}x_j\mu - \sum_{j=1,\, j\neq t}^{n} a_{tj}x_j \\ (a_{tt} - b_{tt}\xi)x_t &= \sum_{j=1,\, j\neq t}^{n} b_{tj}x_j\mu - \sum_{j=1,\, j\neq t}^{n} a_{tj}x_j, \end{aligned}$$

which yields, by the triangle inequality,

$$\begin{aligned} |a_{tt} - b_{tt}\xi||x_t| &= |\sum_{j=1,\, j\neq t}^{n} b_{tj}x_j\mu - \sum_{j=1,\, j\neq t}^{n} a_{tj}x_j| \\ &\leq \sum_{j=1,\, j\neq t}^{n} |b_{tj}x_j\mu| + \sum_{j=1,\, j\neq t}^{n} |a_{tj}x_j| \\ |a_{tt} - b_{tt}\xi||x_t| &\leq \sum_{j=1,\, j\neq t}^{n} (|b_{tj}\mu| + |a_{tj}|)|x_j| \\ |a_{tt} - b_{tt}\xi| &\leq \sum_{j=1,\, j\neq t}^{n} (|b_{tj}\mu| + |a_{tj}|) \\ |a_{tt} - b_{tt}\xi| &\leq |\mu|\, r_t(B) + r_t(A). \end{aligned}$$

Note that ξ is also a generalized right eigenvalue of $\mathbf{L}_1$ and $|\mu| = |\xi|$. Hence, ξ lies in the union of n regions $D_i(\mathbf{L}_1)$. ■

Theorem 4.9. *Let $\mathbf{L}_1 \in \mathbb{L}_1(M_n(\mathbb{H}))$ be a quaternionic regular matrix pencil and let all the diagonal entries of quaternionic matrices A and B are real. Then all the generalized right eigenvalues of $\mathbf{L}_1$ are contained in the union of n regions*

$$D_i(\mathbf{L}_1) := \{z \in \mathbb{H} : |b_{ii}z - a_{ii}| \leq |z|\, r_i(B) + r_i(A)\} \ \ (1 \leq i \leq n),$$

which means

$$\Lambda_r(\mathbf{L}_1) \subseteq \cup_{i=1}^{n} D_i(\mathbf{L}_1).$$

Proof. The proof is immediate from the proof method of Theorem 4.8 by using the fact that $ab = ba \ \forall\, a \in \mathbb{R}$ and $\forall\, b \in \mathbb{H}$, so we skip the proof. ■

Before going to locate the zeros of quaternionic polynomials, we first recall the quaternionic polynomials $p_l(z)$ and $p_r(z)$ from (1.20) and (1.21) as follows.

$$p_l(z) \ := \ q_m z^m + q_{m-1} z^{m-1} + \cdots + q_1 z + q_0, \tag{4.5}$$

$$p_r(z) \ := \ z^m q_m + z^{m-1} q_{m-1} + \cdots + z q_1 + q_0, \tag{4.6}$$

where $q_j,\ z \in \mathbb{H},\ (0 \leq j \leq m)$. The polynomial $p_l(z)$ is associated with the quaternionic matrix pencil

$$T_1(\lambda) := \lambda E - C_{p_l}, \tag{4.7}$$

where λ commutes with the quaternionic matrices

$$C_{p_l} := \begin{bmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \\ -q_0 & -q_1 & \dots & -q_{m-1} \end{bmatrix} \text{ and } E := \begin{bmatrix} 1 & 0 & \dots & 0 \\ & \ddots & & \vdots \\ 0 & \dots & 1 & 0 \\ 0 & \dots & \dots & q_m \end{bmatrix}.$$

This is known as a linearization of $p_l(z)$. We now turn to find the relation between the zeros of $p_l(z)$ and generalized right eigenvalues of the quaternionic matrix pencil $T_1(\lambda) = \lambda E - C_{p_l}$.

Theorem 4.10. *Let $T_1(\lambda)$ be a quaternionic matrix pencil as in (4.7). For every generalized right eigenvalue μ of T_1 there exists a nonzero quaternion β such that $\beta^{-1}\mu\beta$ is a zero of the quaternionic polynomial $p_l(z)$.*

Proof. Let μ be a generalized right eigenvalue of T_1. Then, there exists some nonzero $x \in \mathbb{H}^n$ such that $C_{p_l}x = Ex\mu$. Let $x := [x_1, \ldots, x_m]^T \in \mathbb{H}^n$. Then

$$\begin{bmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \\ -q_0 & -q_1 & \ldots & -q_{m-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ & \ddots & & \vdots \\ 0 & \ldots & 1 & 0 \\ 0 & \ldots & \ldots & q_m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \mu. \tag{4.8}$$

(4.8) gives the following system of linear equations

$$\begin{aligned} x_2 &= x_1\mu, \\ x_3 &= x_2\mu, \\ &\vdots \\ x_m &= x_{m-1}\mu, \\ -q_0x_1 - q_1x_2 - q_2x_3 - \cdots - q_{m-2}x_{m-1} - q_{m-1}x_m &= q_mx_m\mu. \end{aligned}$$

By solving the above system of linear equations, we obtain

$$-q_0x_1 - q_1x_1\mu - q_2x_1\mu^2 - \cdots - q_{m-2}x_1\mu^{m-2} - q_{m-1}x_1\mu^{m-1} = q_mx_1\mu^m. \tag{4.9}$$

Since x cannot be the zero quaternionic vector, $x_m \neq 0$ and hence multiplying by x_1^{-1} on the both sides of (4.9), we obtain

$$-q_0 - q_1x_1\mu x_1^{-1} - q_2x_1\mu^2x_1^{-1} - \cdots - q_{m-2}x_1\mu^{m-2}x_1^{-1} - q_{m-1}x_1\mu^{m-1}x_1^{-1} - q_mx_1\mu^mx_1^{-1} = 0.$$

Putting $x_1\mu x_1^{-1} = \rho$, we obtain

$$q_0 + q_1\rho + q_2\rho^2 + \cdots + q_{m-2}\rho^{m-2} + q_{m-1}\rho^{m-1} + q_m\rho^m = 0.$$

Hence ρ is a zero of $p_l(z)$. ■

Let $C_{p_l} := (c_{ij}) \in M_n(\mathbb{H})$ and let $E := (e_{ij}) \in M_n(\mathbb{H})$. We define the following regions.

$$D_i(T_1) := \{z \in \mathbb{H} : |e_{ii}z - c_{ii}| \leq |z|\, r_i(E) + r_i(C_{p_l})\} \ \ (1 \leq i \leq n),$$

where $T_1(\lambda) := \lambda E - C_{p_l}$ with λ commutes with the quaternionic matrices. Now, we have the following observations:

- From Theorem 4.10, if μ is a generalized right eigenvalue of the quaternionic matrix pencil T_1 (associated with $p_l(z)$), then $\beta^{-1}\mu\beta$ $(0 \neq \beta \in \mathbb{H})$ is a zero of $p_l(z)$. From Theorem 4.8, we obtain

$$\beta^{-1}\mu\beta \in \cup_{i=1}^{n} D_i(T_1), \text{ i.e.,}$$

$$\{z^{-1}\mu z : 0 \neq z \in \mathbb{H}\} \cap \cup_{i=1}^{n} D_i(T_1) \neq \emptyset.$$

- Let $q_m, q_{m-1} \in \mathbb{R}$ and let μ be a generalized right eigenvalue of T_1. Then from Theorem 4.9, μ as well as $\beta^{-1}\mu\beta$ (zero of $p_l(z)$) are contained in the union of n regions $D_i(T_1)$.

Remark 4.11. Similar results can be obtained for $p_r(z)$ as well.

CHAPTER 5

BOUNDS FOR EIGENVALUES OF MATRIX POLYNOMIALS OVER A QUATERNION DIVISION ALGEBRA

In this chapter, the definitions of the left and right eigenvalues of quaternionic matrices are extended to quaternionic matrix polynomials. Localization theorems are discussed for the left and right eigenvalues of a quaternionic block matrix. Furthermore, bounds for the absolute values of the left and right eigenvalues of quaternionic matrix polynomials are devised and illustrated for the matrix p-norm, where $p = 1, 2, \infty$, and F (Frobenius). The above bounds generalize the bounds on the absolute values of the eigenvalues of complex matrix polynomials which give sharper bounds to the existing bounds for the case of $1, 2$, and ∞ matrix norms.

5.1. Introduction and preliminaries

The bounds for the absolute values of the eigenvalues of complex matrix polynomials have been described in [15]. The location of the eigenvalues when computed with an iterative method, see e.g. [54], is of pivotal importance. The applications on matrices, matrix pencils, and matrix polynomials over a quaternion division algebra are discussed in [44, 47, 48]. The literature available on the theory of quaternionic matrix polynomials is limited [43, 44] and are restricted to the stability of various systems with quaternionic matrix coefficients. Therefore, it is of prime significance to carry out further research on quaternionic matrix polynomials to characterize the stability of systems with quaternionic matrix coefficients. The stability analysis of a given system depends on the behavior of right eigenvalues of quaternionic matrix polynomials. For understanding the stability of a given system it is required to analyze the location of the right eigenvalues of quaternion matrix polynomials. There are two types of eigenvalues for the case of matrix pencils and matrix polynomials over a quaternionic division algebra. In view of these facts, in this chapter, we have extended the ideas developed in Chapter 2 to obtain a general framework for matrix polynomials over a quaternion division algebra. A systematic procedure for finding the left and right eigenvalues and their corresponding eigenvectors for matrix polynomials over a quaternion division algebra is derived.

Besides, bounds for the absolute values of the left and right eigenvalues are obtained for quaternionic matrix polynomials which generalize bounds for the absolute values of the eigenvalues of complex matrix polynomials for 1, 2, and ∞-matrix norms. Specifically, bounds for the absolute values of the left and right eigenvalues of quaternionic matrix polynomials are presented by using localization theorems for the left and right eigenvalues of a quaternionic block matrix.

Some of the results (given in Subsection 5.3.2) directly generalize the bounds for the zeros of the right quaternionic polynomials. In this work, we show that some of our bounds for the absolute values of the eigenvalues of complex matrix polynomials are sharper than the bounds given in [15, Lemma 2.3 (2.1), Corollary 2.4 (2.5) (2.6)]. Moreover, we develop an algorithm to derive the above results for the powers of the block companion matrices which give better and sharper results than the bounds obtained via block companion matrices for the left and right eigenvalues of quaternionic matrix polynomials. Our bounds so obtained via the powers of block companion matrices are better than the bounds illustrated in [15, Lemma 2.3].

Throughout this chapter, we adopt the following notation and terminology: For $A, B \in M_n(\mathbb{H})$, Kronecker product of A and B is defined as

$$A \otimes B := (a_{ij}B).$$

Let $A \in M_n(\mathcal{K})$, $\mathcal{K} := \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, be partitioned into $k \times k$ complex/quaternionic blocks

$$A := (A_{ij}) := \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1k} \\ A_{21} & A_{22} & \dots & A_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ A_{k1} & A_{k2} & \dots & A_{kk} \end{bmatrix}_{n\times n}, \tag{5.1}$$

where $A_{i,j} \in M_{n_i\times n_j}(\mathcal{K})$, $(1 \le i, j \le k)$, is the (i, j) block of A such that $n_1 + \cdots + n_k = n$.

Let us define

$$
\begin{aligned}
r_i^{(p)}(A) &:= \sum_{j=1,\, j\neq i}^{k} \left\|A_{ij}\right\|_p, \quad c_i^{(p)}(A) := \sum_{j=1,\, j\neq i}^{k} \left\|A_{ji}\right\|_p, \\
R_i^{(p)}(A) &:= r_i^{(p)}(A) + \|A_{ii}\|_p, \; C_i^{(p)}(A) := c_i^{(p)}(A) + \|A_{ii}\|_p,
\end{aligned}
$$

where $p = 1, 2$ and ∞.

5.2. Inclusion regions for the eigenvalues of quaternionic block matrices

We first derive inclusion regions for the left and right eigenvalues of a quaternionic block matrix.

Theorem 5.1. *Let $A := (A_{ij}) \in M_n(\mathbb{H})$ be a block matrix as in (5.1). Then all the left and right eigenvalues of A are contained in ball $\mathcal{E} := \cap_{p=1,2,\infty}\mathcal{E}^{(p)}$, where*

$$
\mathcal{E}^{(p)} := \cup_{i=1}^{k}\mathcal{E}_i^{(p)} \text{ with } \mathcal{E}_i^{(p)} := \{z \in \mathbb{H} : \; |z| \leq R_i^{(p)}(A)\}, \tag{5.2}
$$

similarly, all the left and right eigenvalues of A are contained in ball $\mathcal{F} := \cap_{p=1,2,\infty}\mathcal{F}^{(p)}$, where

$$
\mathcal{F}^{(p)} := \cup_{i=1}^{k}\mathcal{F}_i^{(p)} \text{ with } \mathcal{F}_i^{(p)} := \{z \in \mathbb{H} : \; |z| \leq C_i^{(p)}(A)\}, \tag{5.3}
$$

$p = 1, 2, \infty$ and $(1 \leq i \leq k)$.

Proof. The proof of the first part is similar to that of the complex case , for $p = 1, 2$ and ∞.

Second part: For $p = 2$, let λ be a right eigenvalue of the quaternionic block matrix A. Then from [47, Corollary 2.7], λ is also a right eigenvalue of A^H. Thus $A^H y = y\lambda$ with nonzero $y = [y_1^T, y_2^T, \ldots, y_k^T]^T \in \mathbb{H}^n$, where $y_i \in \mathbb{H}^{n_i}$. Let y_t be an element of y such that $\|y_t\|_2 \geq \|y_i\|_2$ for all $i \, (1 \leq i \leq k)$. Then $A^H y = y\lambda$ implies

$$
\sum_{j=1}^{k} A_{jt}^H y_j = y_t\lambda.
$$

Taking 2-norm and applying $\|y_t\|_2 \geq \|y_i\|_2$ for all $i\,(1 \leq i \leq k)$, $\left\|A_{jt}^H\right\|_2 = \|A_{jt}\|_2$, we obtain

$$|\lambda| \leq \sum_{j=1}^{k} \|A_{jt}\|_2 .$$

Similar proofs can be obtained for $p = 1$ and ∞.

Let λ be a left eigenvalue of a quaternionic matrix A. Then from Lemma 2.1, $\overline{\lambda}$ is a left eigenvalue of A^H. Now analogue proof can be obtained for the desired results. ■

Theorem 5.2. *Let $A := (A_{ij}) \in M_n(\mathbb{H})$ be a block matrix as in (5.1) and let $\gamma \in [0,1]$. Then all the left and right eigenvalues of A are contained in ball $\mathcal{T} := \cap_{p=1,2,\infty}\mathcal{T}^{(p)}$, where*

$$\mathcal{T}^{(p)} := \cup_{i=1}^{k}\mathcal{T}_i^{(p)} \text{ with } \mathcal{T}_i^{(p)} := \{z \in \mathbb{H} : \ |z| \leq R_i^{(p)}(A)^{\gamma}\ C_i^{(p)}(A)^{1-\gamma}\},$$

$p = 1, 2, \infty$ and $(1 \leq i \leq k)$.

Proof. The proof is immediate from Theorem 5.1, so we skip the proof. ■

We now state an inclusion region for the left and right eigenvalues of a general quaternionic matrix which follows immediately from Theorem 5.2. Thus we have the following corollary.

Corollary 5.3. *Let $A := (a_{ij}) \in M_n(\mathbb{H})$ and let $\gamma \in [0,1]$. Then all the left and right eigenvalues of A are contained in the union of n-balls $\mathcal{G}_i(A) := \{z \in \mathbb{H} : |z| \leq r_i'(A)^{\gamma}\ c_i'(A)^{1-\gamma}\}$ $(1 \leq i \leq n)$, i.e.,*

$$\Lambda_l(A),\, \Lambda_r(A) \subseteq \mathcal{G}(A) := \cup_{i=1}^{n}\mathcal{G}_i(A),$$

where $r_i'(A)$ and $c_i'(A)$ are defined in (1.16).

Now, we present a sufficient condition for the stability of the discrete-time quaternionic system

$$w(t+1) = Aw(t),$$

where $w : \mathbb{R} \to \mathbb{H}$, $t \in \mathbb{R}$ and $A \in M_n(\mathbb{H})$.

Proposition 5.4. *Let $A = (a_{ij}) \in M_n(\mathbb{H})$. Assume that*

$$\omega_i(A)^{\gamma}\ \tau_i(A)^{1-\gamma} < 1\ \forall\, i\,(1 \leq i \leq n).$$

Then A is stable.

Proof. The proof is immediate from Corollary 5.3 and Definition 1.25. ■

It is known that all the right eigenvalues of a quaternionic Hermitian matrix A are real. Thus $\lambda_s(A) = \Lambda_r(A)$. By applying this argument, we have the following result.

Theorem 5.5. *Let $A = (a_{ij}) \in M_n(\mathbb{H})$ be a Hermitian matrix. Then*

$$\text{trace}(A) = \sum_{i=1}^{n} \lambda_i(A),$$

where $\lambda_i \in \Lambda_r(A)$, $(1 \leq i \leq n)$.

Proof. Let A be a quaternionic Hermitian matrix. Then from Theorem 1.29, we have

$$A = VDV^H,$$

where $V \in M_n(\mathbb{H})$ is an unitary matrix and D is a diagonal matrix with the standard right eigenvalues λ_i $(1 \leq i \leq n)$ of A. Applying mapping Ψ and taking trace, we obtain

$$\text{trace}\left(\Psi_A\right) = \text{trace}\left(\Psi_{UDU^H}\right) = \text{trace}\left(\Psi_U \Psi_D \Psi_{U^H}\right) = \text{trace}\left(\Psi_D\right).$$

If all the diagonal entries of $A \in M_n(\mathbb{H})$ are real, then $\text{trace}(\Psi_A) = 2\text{trace}(A)$. ■

Lemma 5.6. *Let $A = (a_{ij}) \in M_n(\mathbb{H})$. Then $\|A\|_2 \leq \|A\|_F$.*

Proof. By the definition of matrix 2-norm and from Theorem 5.5, we have

$$\|A\|_2^2 = \rho_r(AA^H) \leq \sum_{i=1}^{n} \lambda_i(AA^H) = \text{trace}(AA^H) = \|A\|_F^2. \ ■$$

Lemma 5.7. *Let $A = (a_{ij}) \in M_n(\mathbb{H})$. Then*

$$\rho_l(A),\ \rho_r(A) \leq \|A\|_F.$$

Proof. The proof is immediate from [62, Theorem 3] and Lemma 5.6, so we skip the proof. ■

Lemma 5.8. *Let $A = (a_{ij}) \in M_n(\mathbb{H})$. Then*

$$\|A\|_2^2 = \left\|A^H\right\|_2^2 = \left\|A^H A\right\|_2 = \left\|AA^H\right\|_2.$$

Proof. By the definition of the operator norm of A

$$\|A\|_2^2 = \rho_r\left(A^H A\right) = \rho\left(\Psi_{(A^H A)}\right) = \sqrt{\rho\left(\Psi_{(A^H A)}\Psi_{(A^H A)}\right)} = \sqrt{\rho\left(\Psi_{(A^H A)(A^H A)}\right)}$$

$$\|A\|_2^2 = \sqrt{\rho_r\left((A^H A)(A^H A)\right)} = \|A^H A\|_2.$$

Analogue proof can be derived for the remaining parts. ■

Theorem 5.9. *Let $A = (A_{ij}) \in M_n(\mathbb{H})$ be a block matrix as in (5.1). Then*

$$\rho_l(A),\, \rho_r(A) \leq \left(\|A\|_F^2 - \left[\max_{1\leq i\leq k} |\xi_i(A) - \xi_i(A^T)| \right]^2 \right)^{1/2},$$

where $\xi_i(A) := \sum_{j=1,\, j\neq i}^{k} \|A_{ij}\|_F$.

Proof. We prove the equivalent statement

$$\rho_l(A),\, \rho_r(A) \leq \left[\|A\|_F^2 - \left(\left|\xi_i(A) - \xi_i(A^T)\right| \right)^2 \right]^{1/2} \quad (1 \leq i \leq k).$$

Let $\xi_i(A) \neq 0$, $\xi_i(A^T) \neq 0$ and let $W_w = \mathrm{diag}(I_1, \ldots, I_{i-1}, W_i, I_{i+1}, \ldots, I_k)$ with

$$W_i = \mathrm{diag}(w, w, \ldots, w),$$

where $0 \neq w \in \mathbb{R}^+$, in the i-th position. By [63, Lemma 2.3] and Proposition 2.22, we have

$$\rho_r(W_w A W_w^{-1}) = \rho_r(A) \text{ and } \rho_l(W_w A W_w^{-1}) = \rho_l(A).$$

Then by Lemma 5.7, we obtain the following.

$$[\rho_l(A)]^2, [\rho_r(A)]^2 \leq \left\|W_w A W_w^{-1}\right\|_F^2 = \|A\|_F^2 - \xi_i(A)^2 - \xi_i(A^T)^2 + \Phi_1, \tag{5.4}$$

where $\Phi_1 = w^2\xi_i(A)^2 + w^{-2}\xi_i(A^T)^2$. Suppose $f(w) := \|A\|_F^2 - \xi_i(A)^2 - \xi_i(A^T)^2 + w^2\xi_i(A)^2 + w^{-2}\xi_i(A^T)^2$, then its first and second derivatives are given by $f'(w) = 2w\xi_i(A)^2 - 2w^{-3}\xi_i(A^T)^2$ and $f''(w) = 2\xi_i(A)^2 + 6w^{-4}\xi_i(A^T)^2$, respectively. Then $f'(w) = 0$ yields

$$w^2 = \pm \left[\frac{\xi_i(A^T)}{\xi_i(A)} \right],$$

and $f''(w) = +\text{ve}$. Hence $f(w)$ has minimum value at $w^2 = \left[\frac{\xi_i(A^T)}{\xi_i(A)}\right]$. Therefore

$$\rho_l(A),\, \rho_r(A) \leq \left[\|A\|_F^2 - \left(|\xi_i(A) - \xi_i(A^T)| \right)^2 \right]^{1/2}. \ ■$$

Corollary 5.10. *Let $A := (a_{ij}) \in M_n(\mathbb{H})$. Then*

$$\rho_l(A),\, \rho_r(A) \leq \left(\|A\|_F^2 - \left[\max_{1\leq i\leq n} |M_i(A) - M_i(A^T)| \right]^2 \right)^{1/2},$$

where $M_i(A) := \left(\sum_{j=1,\, j\neq i}^{n} |a_{ij}|^2 \right)^{1/2}$.

5.3. Quaternionic matrix polynomials

Let $\mathcal{P}_m(M_n(\mathbb{C}))$ be the space of complex matrix polynomials. $\mathcal{P} \in \mathcal{P}_m(M_n(\mathbb{C}))$ is defined by

$$\mathcal{P}(\lambda) := \sum_{i=0}^{m} \lambda^i A_i, \tag{5.5}$$

where $A_i \in M_n(\mathbb{C})$ $(0 \leq i \leq m)$ and $\lambda \in \mathbb{C}$. Then the eigenvalue problem $\mathcal{P}(\lambda)x = 0$ is referred as a complex polynomial eigenvalue problem. The polynomial $\mathcal{P} \in \mathcal{P}_m(M_n(\mathbb{C}))$ is said to be regular if $\det(\mathcal{P}(\lambda)) \neq 0$ for some $\lambda \in \mathbb{C}$. The spectrum of a regular polynomial $\mathcal{P}$ is denoted by $\Lambda(\mathcal{P})$, and is defined by

$$\Lambda(\mathcal{P}) := \{\lambda \in \mathbb{C} : \det(\mathcal{P}(\lambda)) = 0\}.$$

The above space of complex matrix polynomials can be extended to the space of matrix polynomials over a quaternion division algebra. Quaternionic matrix polynomials are derived in [44, 46–48], but for the case of right eigenvalues, where the matrix coefficients commutes with the variable identity of quaternionic matrix polynomial. Now in this section, we define a general framework for matrix polynomials over a quaternion division algebra to discuss the left and right eigenvalues of quaternionic matrix polynomials.

Let $\mathbb{L}'_m(M_n(\mathbb{H}))$ be the space of matrix polynomials over a quaternion division algebra. $\mathbf{L}' \in \mathbb{L}'_m(M_n(\mathbb{H}))$ is defined as

$$\mathbf{L}'(\xi) := \sum_{i=0}^{m} \xi^i \, A_i, \tag{5.6}$$

where $A_i \in M_n(\mathbb{H})$ $(0 \leq i \leq m)$ and $\xi \in \mathbb{H}$. Throughout this chapter we consider the following three cases:

Case 1: when $\xi \in \mathbb{R}$ and $A_i \in M_n(\mathbb{H})\,(0 \leq i \leq m)$,

Case 2: when $\xi \in \mathbb{H}$ and $A_i \in M_n(\mathbb{H})\,(0 \leq i \leq m)$,

Case 3: when $\xi \in \mathbb{C}$ and $A_i \in M_n(\mathbb{C})\,(0 \leq i \leq m)$.

Case 1. Let $\mathbb{L}_m(M_n(\mathbb{H}))$ be the space of matrix polynomials over a quaternion division algebra. $\mathbf{L} \in \mathbb{L}_m(M_n(\mathbb{H}))$ is defined as

$$\mathbf{L}(\lambda) := \sum_{i=0}^{m} A_i \lambda^i, \tag{5.7}$$

where $A_i \in M_n(\mathbb{H})$ $(0 \leq i \leq m)$ and λ commutes with the quaternionic coefficients of the matrix polynomial. This polynomial over a quaternion division algebra can be found in [46–48].

Now we turn to define the right eigenvalue of $\mathbf{L} \in \mathbb{L}_m(M_n(\mathbb{H}))$ of the form (5.7) which is as follows.

Definition 5.11. Let $\mathbf{L} \in \mathbb{L}_m(M_n(\mathbb{H}))$ be as in (5.7) and let $\mu \in \mathbb{H}$. Then μ is called a right eigenvalue of $\mathbf{L}$ if

$$A_0x + A_1x\mu + A_2x\mu^2 + \cdots + A_mx\mu^m = 0$$

for some nonzero $x \in \mathbb{H}^n$. Here x is called the right eigenvector corresponding to right eigenvalue μ. The set of right eigenvalues of $\mathbf{L}$ is called right spectrum of $\mathbf{L}$, denoted by $\Lambda_r(\mathbf{L})$.

Case 2. We define the left eigenvalue of $\mathbf{L}' \in \mathbb{L}'_m(M_n(\mathbb{H}))$ which is as follows.

Definition 5.12. Let $\mathbf{L}' \in \mathbb{L}'_m(M_n(\mathbb{H}))$ be of the form (5.6) and let $\mu \in \mathbb{H}$. Then μ is called a left eigenvalue of $\mathbf{L}'$ if

$$A_0x + \mu A_1x + \mu^2A_2x + \cdots + \mu^mA_mx = 0$$

for some nonzero $x \in \mathbb{H}^n$. Here x is called the left eigenvector corresponding to left eigenvalue μ. The set of left eigenvalues of $\mathbf{L}'$ is called left spectrum of $\mathbf{L}'$, denoted by $\Lambda_l(\mathbf{L}')$.

5.3.1. Right eigenvalues of quaternionic matrix polynomials and their bounds

In this subsection, first we give the linearization form of the quaternionic matrix polynomial $\mathbf{L}$ (defined in (5.7)) by using the standard linearization technique given in [10] for the right and left eigenvalues of $\mathbf{L}$.

- **For the right eigenvalues:** The polynomial $\mathbf{L} \in \mathbb{L}_m(M_n(\mathbb{H}))$ be as in (5.7) can be written in the form:

$$C_{\mathbf{L}} + \lambda X,$$

where $C_{\mathbf{L}},\, X \in M_{mn}(\mathbb{H})$ are of the forms

$$C_{\mathbf{L}} := \begin{bmatrix} 0 & I_n & 0 & \dots & 0 \\ 0 & 0 & I_n & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & I_n \\ -A_0 & -A_1 & -A_2 & \dots & -A_{m-1} \end{bmatrix}, \quad X := \begin{bmatrix} I_n & 0 & 0 & \dots & 0 \\ 0 & I_n & 0 & \dots & 0 \\ 0 & 0 & I_n & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & A_m \end{bmatrix}$$

and λ commutes with the quaternionic coefficients of the matrix polynomial. When $A_m = I_n$, the identity matrix, then the matrix polynomial (5.7) is said to be monic matrix polynomial and linearization form of it is given by

$$C_{\mathbf{L}} + \lambda E, \text{ where } E := I_{nm}.$$

Let $\mathbb{P}_m(M_{2n}(\mathbb{C}))$ be the space of complex matrix polynomials. $P \in \mathbb{P}_m(M_{2n}(\mathbb{C}))$ is given by

$$P(\mu) := \sum_{i=0}^{m} \Psi_{A_i} \mu^i, \tag{5.8}$$

where $A_i \in M_n(\mathbb{H})$, $\Psi_{A_i} \in M_{2n}(\mathbb{C})$ $(0 \leq i \leq m)$ and $\mu \in \mathbb{C}$.

Theorem 5.13. *Let $\mathbf{L} \in \mathbb{L}_m(M_n(\mathbb{H}))$ be as in (5.7) and let $(C_{\mathbf{L}}, X)$ be a matrix pencil obtained from the linearization of $\mathbf{L}$. Then they have the same right eigenvalues.*

Proof. Let $\mu \in \mathbb{H}$ be a right eigenvalue of the quaternionic matrix pencil $(C_{\mathbf{L}}, X)$, then there exists $x := [x_1^T, x_2^T, \dots, x_m^T]^T \in \mathbb{H}^n$ with $x_i \in \mathbb{H}^n$, $i = 1, 2, \dots, m$ such that

$$\begin{bmatrix} 0 & I_n & 0 & \dots & 0 \\ 0 & 0 & I_n & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & I_n \\ -A_0 & -A_1 & -A_2 & \dots & -A_{m-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{m-1} \\ x_m \end{bmatrix} = \begin{bmatrix} I_n & 0 & 0 & \dots & 0 \\ 0 & I_n & 0 & \dots & 0 \\ 0 & 0 & I_n & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & A_m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{m-1} \\ x_m \end{bmatrix} \mu, \tag{5.9}$$

then (5.9) implies

$$\begin{aligned} x_2 &= x_1\mu, \\ x_3 &= x_2\mu, \\ &\vdots \\ x_m &= x_{m-1}\mu, \\ -A_0x_1 - A_1x_2 - \cdots - A_{m-1}x_m &= A_mx_m\mu. \end{aligned}$$

Consecutive substitutions of the $(m-1)^{st}$ system of equations give

$$x_j := x_1\mu^{j-1}, \quad j = 2, 3, \ldots, m. \tag{5.10}$$

Substituting (5.10) in the last system of equation and solving, we obtain

$$A_0x_1 + A_1x_1\mu + \cdots + A_{m-2}x_1\mu^{m-2} + A_{m-1}x_1\mu^{m-1} + A_mx_1\mu^m = 0.$$

Hence μ is a right eigenvalue of the quaternionic matrix polynomial $\mathbf{L}$. ■

Regular matrix pencil over the quaternion skew filed has been defined in [33] and now, we extend it for matrix polynomial over the quaternion skew field.

Definition 5.14. Let $\mathbf{L} \in \mathbb{L}_m(M_n(\mathbb{H}))$ be as in (5.7). Then the matrix polynomial $\mathbf{L}$ is called regular if there exists $\alpha \in \mathbb{R}$ such that $A_0 + \alpha\ A_1 + \cdots + \alpha^m\ A_m$ is an invertible matrix.

Theorem 5.15. *Let $\mathbf{L} \in \mathbb{L}_m(M_n(\mathbb{H}))$ be of the form (5.7). Then the matrix polynomial $\mathbf{L}$ is a regular polynomial if and only if $P \in \mathbb{P}_m(M_{2n}(\mathbb{C}))$ is a regular complex matrix polynomial.*

Proof. Consider $\mathbf{L} \in \mathbb{L}_m(M_n(\mathbb{H}))$ is a regular matrix polynomial, then there exists $\alpha \in \mathbb{R}$ such that $\mathbf{L}(\alpha) = A_0 + \alpha A_1 + \cdots + \alpha^m A_m$ is an invertible matrix polynomial. From Theorem 1.12, the corresponding complex matrix polynomial $\Psi_{\mathbf{L}(\alpha)} = \Psi_{A_0} + \alpha\Psi_{A_1} + \cdots + \alpha^m\Psi_{A_m}$ is an invertible complex matrix polynomial. Thus, $P(\mu)$ is a regular complex matrix polynomial.

Conversely, let $P(\mu)$ be a regular complex matrix polynomial, then

$$\det[P(\eta)] \neq 0, \text{ for some } \eta \in \mathbb{C}, \text{ i.e.,}$$

$$\det[\Psi_{A_0} + \eta\Psi_{A_1} + \cdots + \eta^m\Psi_{A_m}] \neq 0.$$

Thus, there exists a real number λ_0 (say) such that

$$\det[\Psi_{A_0} + \lambda_0 \Psi_{A_1} + \cdots + \lambda_0^m \Psi_{A_m}] \neq 0,$$

i.e., $\Psi_{A_0} + \lambda_0 \Psi_{A_1} + \cdots + \lambda_0^m \Psi_{A_m} = \Psi_{(A_0 + \lambda_0 A_1 + \cdots + \lambda_0^m A_m)}$ is an invertible complex matrix polynomial. Then by Theorem 1.12, the matrix polynomial $A_0 + \lambda_0 A_1 + \cdots + \lambda_0^m A_m$ is an invertible quaternionic matrix polynomial. Hence $\mathbf{L} \in \mathbb{L}_m(M_n(\mathbb{H}))$ is a regular quaternionic matrix polynomial. ■

Proposition 5.16. *Let $\mathbf{L} \in \mathbb{L}_m(M_n(\mathbb{H}))$ be as in (5.7) and let $\mu \in \mathbb{H}$ be a right eigenvalue of $\mathbf{L}$. Assume that $0 \neq \rho \in \mathbb{H}$, then $\rho^{-1}\mu\rho$ is also a right eigenvalue of $\mathbf{L}$.*

Proof. Suppose x is a quaternionic eigenvector of the matrix polynomial $\mathbf{L}$ corresponding to the right eigenvalue μ of the matrix polynomial $\mathbf{L}$, then

$$A_0 x + A_1 x\mu + A_2 x\mu^2 + \cdots + A_m x\mu^m = 0. \tag{5.11}$$

If $\rho \in \mathbb{H} \setminus \{0\}$, then from (5.11), we have

$$A_0 x\rho + A_1 x\rho(\rho^{-1}\mu\rho) + A_2 x\rho(\rho^{-1}\mu^2\rho) + \cdots + A_m x\rho(\rho^{-1}\mu^m\rho) = 0.$$

Thus $\rho^{-1}\mu\rho$ is also a right eigenvalue of the matrix polynomial $\mathbf{L}$ corresponding to an eigenvector $x\rho$. ■

Theorem 5.17. *Let $\mathbf{L} \in \mathbb{L}_m(M_n(\mathbb{H}))$ be as in (5.7). Then*

$$\Lambda_r(\mathbf{L}) \cap \mathbb{C} = \Lambda(P), \tag{5.12}$$

$$\Lambda_r(\mathbf{L}) = \{\rho^{-1}\mu\rho \in \mathbb{H} : \mu \in \Lambda(P),\ 0 \neq \rho \in \mathbb{H}\}. \tag{5.13}$$

Proof. Consider $A_i \in M_n(\mathbb{H})$ for $i = 0, 1, \ldots, m$ and $x \in \mathbb{H}^n$. We write A_i and x as $A_i = A_i{}' + A_i^{''}\mathbf{j}, 0 \leq i \leq m$ and $x = x_1 + x_2\mathbf{j}$, where $A_i^{'}, A_i^{''} \in M_n(\mathbb{C})$ and $x_1, x_2 \in \mathbb{C}^n$. If $\mu \in \Lambda_r(\mathbf{L}) \cap \mathbb{C}$, then there exists $x \in \mathbb{H}^n$ such that $A_0 x + A_1 x\mu + A_2 x\mu^2 + \cdots + A_m x\mu^m = 0$ which is equivalent to the complex system. Hence from Theorem 1.12, we obtain

$$\begin{bmatrix} A_0^{'} & A_0^{''} \\ -\overline{A_0^{''}} & \overline{A_0^{'}} \end{bmatrix} \begin{bmatrix} x_1 \\ -\overline{x_2} \end{bmatrix} + \mu \begin{bmatrix} A_1^{'} & A_1^{''} \\ -\overline{A_1^{''}} & \overline{A_1{}'} \end{bmatrix} \begin{bmatrix} x_1 \\ -\overline{x_2} \end{bmatrix} + \cdots + \mu^m \ \Phi_2 \begin{bmatrix} x_1 \\ -\overline{x_2} \end{bmatrix} = 0, \tag{5.14}$$

where $\Phi_2 = \begin{bmatrix} A'_m & A''_m \\ -\overline{A''_m} & \overline{A'_m} \end{bmatrix}$ and also

$$\begin{bmatrix} A'_0 & A''_0 \\ -\overline{A''_0} & \overline{A'_0} \end{bmatrix} \begin{bmatrix} x_2 \\ \overline{x_1} \end{bmatrix} + \overline{\mu} \begin{bmatrix} A'_1 & A''_1 \\ -\overline{A''_1} & \overline{A_1'} \end{bmatrix} \begin{bmatrix} x_2 \\ \overline{x_1} \end{bmatrix} + \cdots + \overline{\mu}^m \begin{bmatrix} A'_m & A''_m \\ -\overline{A''_m} & \overline{A'_m} \end{bmatrix} \begin{bmatrix} x_2 \\ \overline{x_1} \end{bmatrix} = 0;$$

thus $\mu,\ \overline{\mu} \in \Lambda(P)$.

Conversely, suppose $\mu \in \Lambda(P)$, then there exists $x := [x_1^T, x_2^T]^T \in \mathbb{C}^{2n}$ such that

$$\begin{bmatrix} A'_0 & A''_0 \\ -\overline{A''_0} & \overline{A'_0} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \mu \begin{bmatrix} A'_1 & A''_1 \\ -\overline{A''_1} & \overline{A_1'} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \cdots + \mu^m \begin{bmatrix} A'_m & A''_m \\ -\overline{A''_m} & \overline{A'_m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0,$$

which is equivalent to the following systems

$$(5.15) \qquad \begin{cases} A'_0 x_1 + A''_0 x_2 + \mu A'_1 x_1 + \mu A''_1 x_2 + \cdots + \mu^m A'_m x_1 + \mu^m A''_m x_2 = 0, \\ -\overline{A''_0} x_1 + \overline{A'_0} x_2 - \mu \overline{A''_1} x_1 + \mu \overline{A'_1} x_2 + \cdots - \mu^m \overline{A''_m} x_1 + \mu^m \overline{A'_m} x_2 = 0. \end{cases}$$

Suppose that $x = x_1 - \overline{x_2}\,\mathbf{j}$, then (5.15) is equivalent to $A_0 x + A_1 x\mu + A_2 x\mu^2 + \cdots + A_m x \mu^m = 0$, hence $\mu \in \Lambda_r(\mathbf{L}) \cap \mathbb{C}$.

For (5.13), consider $\mu \in \Lambda\left(\Psi_{C_{\mathbf{L}}}, \Psi_X\right) = \Lambda\left(P\right)$, then for every $0 \neq \rho \in \mathbb{H}$, it is clear that $\rho^{-1}\mu\rho \in \Lambda_r(C_{\mathbf{L}}, X) = \Lambda_r\left(\mathbf{L}\right)$. Conversely; consider $\lambda \in \Lambda_r(C_{\mathbf{L}}, X) = \Lambda_r\left(\mathbf{L}\right)$, then by [5, Lemma 9], there exists $0 \neq \rho \in \mathbb{H}$ such that $\rho\mu\rho^{-1} = \mu_1 \in \Lambda_r(C_{\mathbf{L}}, X) \cap \mathbb{C}$. Applying (5.12), we have $\mu_1 \in \Lambda(P)$ and also $\mu = \rho^{-1}\mu_1\rho$. ■

We now state two corollaries to Theorem 5.17 which are also available in [33].

Corollary 5.18. *Let $\mathbf{L} \in \mathbb{L}_1(M_n(\mathbb{H}))$ be of the form $\mathbf{L}(\lambda) := A + \lambda B$, where λ commutes with the quaternionic matrices and let $P \in \mathbb{P}_1(M_{2n}(\mathbb{C}))$ be of the form $P(\mu) := \Psi_A + \mu\Psi_B$, where $\mu \in \mathbb{C}$. Then*

$$(5.16) \qquad \begin{cases} \Lambda_r(\mathbf{L}) \cap \mathbb{C} = \Lambda(P), \\ \qquad \Lambda_r(\mathbf{L}) = \{\rho^{-1}\lambda\rho : \lambda \in \Lambda(P),\ 0 \neq \rho \in \mathbb{H}\}. \end{cases}$$

Proof. Assume that $m = 1$ in Theorem 5.17, then we have the desired result. ■

Also we give another corollary of Theorem 5.17 which can be found from [33].

Corollary 5.19. *Let $A \in M_n(\mathbb{H})$. Then*

$$\begin{aligned} \Lambda_r(A) \cap \mathbb{C} &= \Lambda(\Psi_A), \\ \Lambda_r(A) &= \{\rho^{-1}\lambda\rho : \lambda \in \Lambda(\Psi_A),\ 0 \neq \rho \in \mathbb{H}\}. \end{aligned}$$

Proof. Assume that $B = I_n$ in Corollary 5.18, we have the desired result. ■

Remark 5.20. From (5.14), it is also clear that $\mu \in \mathbb{H}$ is a right eigenvalue of the quaternionic matrix polynomial $\mathbf{L}$ if and only if $\det(P(\mu)) = 0$.

Let $\mathbb{P}_m(M_{2n}(\mathbb{C}))$ be the space of complex matrix polynomials of degree $(\leq m)$, and let $P \in \mathbb{P}_m(M_{2n}(\mathbb{C}))$ be of the form (5.8). If A_m is invertible, then P has $2mn$ finite eigenvalues, i.e., $\mathbf{L}$ does not have any infinite (∞) right eigenvalue as there is one one correspondence between the space of matrix polynomials of degree m over a quaternion division algebra and the space of its conjugate matrix polynomial P of degree m over the complex field. Also, if A_m is not invertible, then P has at least one infinite (∞) eigenvalue. In general, if A_m is not invertible, then the quaternionic matrix polynomial $\mathbf{L}$ has at least one infinite right eigenvalue, whereas if A_0 is not invertible, then 0 is a right eigenvalue of the quaternionic matrix polynomial $\mathbf{L}$. Therefore, for upper bounds of the absolute values of the right eigenvalues of $\mathbf{L}$ require A_m to be invertible and the lower bounds of the absolute values of the right eigenvalues of $\mathbf{L}$ require A_0 to be invertible. Hence, now onwards, we assume A_0 and A_m are invertible quaternionic matrices.

Let d be the degree of $\det[P(\mu)]$, where $\mu \in \mathbb{H}$. Then the d zeros of $\det[P(\mu)]$ are called the finite eigenvalues of P. If $d < 2mn$, then we say that $P(\mu)$ has $2mn - d$ infinite eigenvalues. Let $\mathbf{L} \in \mathbb{L}_m(M_n(\mathbb{H}))$ be a regular polynomial and if A_m is not invertible, then the corresponding adjoint complex matrix Ψ_{A_m} is not invertible. Thus, the complex matrix polynomial $P(\mu)$ has at least one infinite (∞) eigenvalue and hence the quaternionic matrix polynomial $\mathbf{L}$ has at least one infinite (∞) right eigenvalue.

Example 5.21. *Consider the quaternionic matrix pencil $\mathbf{L}(\lambda) = A + \lambda B$, where λ commutes with the quaternionic matrices and $A, B \in M_n(\mathbb{H})$ of the forms*

$$A = \begin{bmatrix} \mathbf{i}+\mathbf{j} & 2+\mathbf{k} \\ 1-\mathbf{j} & 3-\mathbf{i}-\mathbf{j} \end{bmatrix}, \; B = \begin{bmatrix} 0 & \mathbf{j}+\mathbf{k} \\ 0 & 0 \end{bmatrix}.$$

Since B is not invertible, hence Ψ_B is also not invertible. The complex matrix pencil $\Psi_A + \mu\Psi_B$, $\mu \in \mathbb{C}$ has at least one infinite (∞) eigenvalue. Hence $\mathbf{L}$ has an infinite (∞) right eigenvalue. In this case, the set of right eigenvalues of $\mathbf{L}$ is $\{[-1 + 2.5495\mathbf{i}], \infty\}$.

Now, consider A_m is invertible. Then from [5, 27] the block companion matrix C_V has exactly mn right eigenvalues which are complex numbers with non-negative imaginary parts. Also, we observe that if A_m is not invertible and $\mathbf{L}$ is regular, then $P \in \mathbb{P}_m(M_{2n}(\mathbb{C}))$ has the corresponding matrix polynomial has exactly mn right eigenvalues which belong to $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$ with non-negative imaginary parts. Those right eigenvalues are said to be the standard right eigenvalues of $\mathbf{L}$. Thus, we define the standard right eigenvalues of $\mathbf{L} \in \mathbb{L}_m(M_n(\mathbb{H}))$ as follows.

Definition 5.22. Let $\mathbf{L} \in \mathbb{L}_m(M_n)(\mathbb{H})$ be as in (5.7). Then we define the set of standard right eigenvalues of a regular matrix polynomial $\mathbf{L}$ as

$$\Lambda_s(\mathbf{L}) := \{\alpha \in \mathbb{C}_\infty : A_0 x + A_1 x\alpha + \cdots + A_m x\alpha^m = 0,\ 0 \neq x \in \mathbb{H}^n,\ \Im(\alpha) \geq 0\},$$

$$\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}.$$

The above definition generalizes the definition of standard right eigenvalues of a single quaternionic matrix [5, 27].

Now, we state a framework to find bounds for the right eigenvalues of quaternionic matrix polynomials.

- To find bounds of the right eigenvalues of the quaternionic matrix polynomial $\mathbf{L} \in \mathbb{L}_m(M_n(\mathbb{H}))$, we introduce two new quaternionic matrix polynomials associated with $\mathbf{L} \in \mathbb{L}_m(M_n(\mathbb{H}))$:

$$\mathbf{L}_V(\lambda) := \lambda^m I_n + \sum_{i=0}^{m-1} \lambda^i\, V_i, \tag{5.17}$$

where $V_i := A_m^{-1} A_i$, $i = 0, 1, \ldots, m-1$, so that $\mathbf{L}(\lambda) = A_m \mathbf{L}_V(\lambda)$, and

$$\mathbf{L}_S(\lambda) := \lambda^m I_n + \sum_{i=1}^{m} S_i \lambda^{m-i}, \tag{5.18}$$

where $S_i := A_0^{-1} A_i$, $i = 1, \ldots, m$, now $A_0^{-1}\lambda^m \mathbf{L}(1/\lambda) = \mathbf{L}_S(\lambda)$. The polynomials $\mathbf{L}$ and $\mathbf{L}_V$ have the same right eigenvalues, whereas the right eigenvalues of the polynomial $\mathbf{L}_S$ are the reciprocal of the right eigenvalues of the polynomial $\mathbf{L}$. The block companion matrices C_V and C_S corresponding to the monic matrix polynomials $\mathbf{L}_V$ and $\mathbf{L}_S$ are given by

$$C_V := \begin{array}{c} \scriptstyle n(m-1) \\ \scriptstyle n \end{array} \overset{\begin{array}{cc} \scriptstyle n & \scriptstyle n(m-1) \end{array}}{\left[\begin{array}{c|c} 0 & I \\ \hline -V_0 & -\Delta_V \end{array}\right]} \text{ and } C_S := \begin{array}{c} \scriptstyle n(m-1) \\ \scriptstyle n \end{array} \overset{\begin{array}{cc} \scriptstyle n & \scriptstyle n(m-1) \end{array}}{\left[\begin{array}{c|c} 0 & I \\ \hline -S_m & -\Delta_S \end{array}\right]},$$

respectively, where $\Delta_V,\ \Delta_S \in M_{n\times n(m-1)}(\mathbb{H})$ are of the forms

$$\Delta_V = \begin{bmatrix} V_1 & V_2 & \dots & V_{m-1} \end{bmatrix},\ \Delta_S = \begin{bmatrix} S_{m-1} & S_{m-2} & \dots & S_1 \end{bmatrix}.$$

Moreover, the right eigenvalues of the monic matrix polynomials $\mathbf{L}_V(\lambda), \mathbf{L}_S(\lambda)$ and the right eigenvalues of the block companion matrices C_V, C_S are same, respectively.

Bounds for the right eigenvalues of quaternionic matrix polynomials: Now onwards, we define

$$V := \begin{bmatrix} V_0 & V_1 & \dots & V_{m-1} \end{bmatrix} \in M_{n\times mn}(\mathbb{H}),\ \ S := \begin{bmatrix} S_m & S_{m-1} & \dots & S_1 \end{bmatrix} \in M_{n\times mn}(\mathbb{H}),$$

where $V_i,\ S_j \in M_n(\mathbb{H})$, $(0 \le i \le m-1)$, $(1 \le j \le m)$.

Theorem 5.23. *Let $\mathbf{L} \in \mathbb{L}_m(M_n(\mathbb{H}))$ be as in (5.7). Then for any right eigenvalue μ of the polynomial $\mathbf{L}$ satisfies the following inequalities:*

$$\frac{1}{\beta_1} \le |\mu| \le \alpha_1, \tag{5.19}$$

$$\frac{1}{\beta_2} \le |\mu| \le \alpha_2, \tag{5.20}$$

$$\frac{1}{\beta_3} \le |\mu| \le \alpha_3, \tag{5.21}$$

where $\gamma \in [0,1]$,

$$\begin{aligned}
\alpha_1 &:= \min_{p=1,2,\infty} \left\{ \max\left(\|V_0\|_p, 1 + \max_{1\le j\le m-1} (\|V_j\|_p) \right) \right\},\\
\beta_1 &:= \min_{p=1,2,\infty} \left\{ \max\left(\|S_m\|_p, 1 + \max_{1\le j\le m-1} (\|S_j\|_p) \right) \right\},\\
\alpha_2 &:= \min_{p=1,2,\infty} \left\{ \max\left(1, \sum_{j=0}^{m-1} \|V_j\|_p \right) \right\},\\
\beta_2 &:= \min_{p=1,2,\infty} \left(\max\left\{ 1, \sum_{j=1}^{m} \|S_j\|_p \right\} \right),
\end{aligned}$$

$$\begin{aligned}
\alpha_3 &:= \min_{p=1,2,\infty} \left\{ \max_{1\le i\le m} \left(R_i^{(p)}(C_V)^{\gamma}\ C_i^{(p)}(C_V)^{1-\gamma} \right) \right\}, \text{ and}\\
\beta_3 &:= \min_{p=1,2,\infty} \left\{ \max_{1\le i\le m} \left(R_i^{(p)}(C_S)^{\gamma}\ C_i^{(p)}(C_S)^{1-\gamma} \right) \right\}.
\end{aligned}$$

Proof. The proofs are immediate from Theorems 5.1 and 5.2, so we skip the proofs. ■

It is known that similar quaternionic matrices have the same right eigenvalues. Therefore, we can apply a similarity transformation to obtain different and potentially tighter bounds. So, we now define the following matrices.

$$C_V' := WC_VW^{-1},\ C_S' := WC_SW^{-1},\ W := w_1I_n \oplus (W' \otimes I_n) = w_1I_n \oplus \cdots \oplus w_mI_n,$$

$W' := \operatorname{diag}(w_2I_n, w_3I_n, \ldots, w_mI_n)$, where w_i are positive integers. The matrix C_V' can also be written as

$$C_V' := WC_VW^{-1} \tag{5.22}$$

with $W := \begin{array}{c} \\ n \\ n(m-1) \end{array}\begin{array}{c} \begin{array}{cc} n(m-1) & n \end{array} \\ \left[\begin{array}{c|c} C_1 & 0 \\ \hline C_2 & C_3 \end{array}\right] \end{array}$ and $W^{-1} := \begin{array}{c} \\ n \\ n(m-1) \end{array}\begin{array}{c} \begin{array}{cc} n(m-1) & n \end{array} \\ \left[\begin{array}{c|c} C_1' & 0 \\ \hline C_2' & C_3' \end{array}\right] \end{array}$,

where

$$C_1 := \begin{bmatrix} w_1I_n & 0 & \ldots & 0 \end{bmatrix},\ C_1' := \begin{bmatrix} \frac{1}{w_1}I_n & 0 & \ldots & 0 \end{bmatrix},$$

$$C_2 := \begin{bmatrix} 0 & w_2I_n & 0 & \ldots & 0 \\ \vdots & 0 & w_3I_n & & 0 \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & 0 & \ldots & w_{m-1}I_n \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix},\ C_2' := \begin{bmatrix} 0 & \frac{1}{w_2}I_n & 0 & \ldots & 0 \\ \vdots & 0 & \frac{1}{w_3}I_n & & 0 \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & 0 & \ldots & \frac{1}{w_{m-1}}I_n \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix},$$

$$C_3 := \begin{bmatrix} 0 \\ 0 \\ \vdots \\ w_mI_n \end{bmatrix},\ \text{and } C_3' := \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \frac{1}{w_m}I_n \end{bmatrix}.$$

By using above arguments and, Theorems 5.1 and 5.2, we have the following bounds.

Theorem 5.24. *Let $w_j \in \mathbb{R}^+$, $(1 \le j \le m)$, with $w_m = 1$. Then for any right eigenvalue μ of $\mathbf{L} \in \mathbb{L}_m(M_n(\mathbb{H}))$ satisfies the following inequalities:*

$$\frac{1}{\beta_1} \le |\mu| \le \alpha_1, \tag{5.23}$$

$$\frac{1}{\beta_2} \le |\mu| \le \alpha_2, \tag{5.24}$$

$$\frac{1}{\beta_3} \le |\mu| \le \alpha_3, \tag{5.25}$$

where $\gamma \in [0, 1]$,

$$\begin{aligned}
\alpha_1 &:= \min_{p=1,2,\infty} \left\{ \max\left(\frac{\|V_0\|_p}{w_1}, \max_{1\le j\le m-1} \left(\frac{w_j}{w_{j+1}} + \frac{\|V_j\|_p}{w_{j+1}} \right) \right) \right\}, \\
\beta_1 &:= \min_{p=1,2,\infty} \left\{ \max\left(\frac{\|S_m\|_p}{w_1}, \max_{1\le j\le m-1} \left(\frac{w_j}{w_{j+1}} + \frac{\|S_{m-j}\|_p}{w_{j+1}} \right) \right) \right\}, \\
\alpha_2 &:= \min_{p=1,2,\infty} \left\{ \max\left(\max_{1\le j\le m-1} \left(\frac{w_j}{w_{j+1}} \right), \sum_{i=0}^{m-1} \frac{\|V_i\|_p}{w_{i+1}} \right) \right\}, \\
\beta_2 &:= \min_{p=1,2,\infty} \left\{ \max\left(\max_{1\le j\le m-1} \left(\frac{w_j}{w_{j+1}} \right), \sum_{i=0}^{m-1} \frac{\|S_{m-i}\|_p}{w_{i+1}} \right) \right\}, \\
\alpha_3 &:= \min_{p=1,2,\infty} \left\{ \max_{1\le i\le m} \left(R_i^{(p)}(C'_V)^{\gamma}\, C_i^{(p)}(C'_V)^{1-\gamma} \right) \right\}, \text{ and} \\
\beta_3 &:= \min_{p=1,2,\infty} \left\{ \max_{1\le i\le m} \left(R_i^{(p)}(C'_S)^{\gamma}\, C_i^{(p)}(C'_S)^{1-\gamma} \right) \right\}.
\end{aligned}$$

Proof. The block companion matrix of the monic matrix polynomial $\mathbf{L}_V(\lambda)$ is given by

$$C_V := \begin{matrix} & n & n(m-1) \\ n(m-1) & \\ n & \end{matrix} \left[\begin{array}{c|c} 0 & I \\ \hline -V_0 & -\Delta_V \end{array} \right].$$

Now (5.22) yields

$$C'_V := W C_V W^{-1}$$

$$= \begin{matrix} & n(m-1) & n \\ n & \\ n(m-1) & \end{matrix} \left[\begin{array}{c|c} C_1 & 0 \\ \hline C_2 & C_3 \end{array} \right] \begin{matrix} & n & n(m-1) \\ n(m-1) & \\ n & \end{matrix} \left[\begin{array}{c|c} 0 & I \\ \hline -V_0 & -\Delta_V \end{array} \right] \begin{matrix} & n(m-1) & n \\ n & \\ n(m-1) & \end{matrix} \left[\begin{array}{c|c} C'_1 & 0 \\ \hline C'_2 & C'_3 \end{array} \right]$$

$$C'_V = \begin{matrix} & n(m-1) & n \\ n & \\ n(m-1) & \end{matrix} \left[\begin{array}{c|c} C''_1 & 0 \\ \hline C''_2 & C''_3 \end{array} \right], \text{ where } C''_1 = \begin{bmatrix} 0 & \frac{w1}{w2} I_n & 0 & \dots & 0 \end{bmatrix},$$

$$C_2'' = \begin{bmatrix} 0 & 0 & \frac{w_2}{w_3}I_n & 0 & \dots & 0 \\ \vdots & \vdots & 0 & \frac{w_3}{w_4}I_n & & 0 \\ \vdots & \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & 0 & 0 & \dots & \frac{w_{m-2}}{w_{m-1}}I_n \\ -\frac{w_m}{w_1}V_0 & -\frac{w_m}{w_2}V_1 & -\frac{w_m}{w_3}V_2 & -\frac{w_m}{w_4}V_3 & \dots & -\frac{w_m}{w_{m-1}}V_{m-2} \end{bmatrix}, \text{ and } C_3'' := \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \frac{w_{m-1}}{w_m}I_n \\ -V_{m-1} \end{bmatrix}.$$

Upper bounds of the right eigenvalues of the matrix polynomial $\mathbf{L}$ are obtained by applying Theorems 5.1 and 5.2 to the matrix C_V'.

Also, lower bounds of the right eigenvalues of the matrix polynomial $\mathbf{L}$ are obtained by applying Theorems 5.1 and 5.2 to the matrix C_S'. ■

Substituting $w_j = \|V_j\|_p$, $(p = 1, 2, \infty)$, in the part (1) of Theorem 5.24 , we obtain

$$|\mu| \leq \min_{p=1,2,\infty} \left\{ \max \left(\frac{\|V_0\|_p}{\|V_1\|_p}, \ 2 \max_{1 \leq j \leq m-1} \frac{\|V_j\|_p}{\|V_{j+1}\|_p} \right) \right\}. \tag{5.26}$$

Now we discuss the different bounds for the right eigenvalues of quaternionic matrix polynomials.

Theorem 5.25. *Let* $\mathbf{L} \in \mathbb{L}_m(M_n(\mathbb{H}))$ *be as in (5.7). Then for any right eigenvalue* μ *of* $\mathbf{L}$ *satisfies the following inequality:*

$$\frac{1}{\eta^{C_S}} \leq |\mu| \leq \eta^{C_V},$$

where

$$\eta^{C_\alpha} := \left[\|C_\alpha\|_F^2 - \left\{ \max_{1 \leq i \leq m} \left| \xi_i\left(C_\alpha\right) - \xi_i\left(C_\alpha^T\right) \right| \right\}^2 \right]^{1/2}, \alpha \in \{V, S\}.$$

Proof. The proof is immediate from Theorem 5.9, so we omit the proof. ■

Theorem 5.26. *Let* $\mathbf{L} \in \mathbb{L}_m(M_n(\mathbb{H}))$ *be as in (5.7). Then for any right eigenvalue* μ *of* $\mathbf{L}$ *satisfies the following inequality:*

$$\frac{1}{\eta^{C_S}} \leq |\mu| \leq \eta^{C_V}, \textit{ where}$$

$$\eta^{C_\alpha} := \left[\|C_\alpha\|_F^2 - \left\{ \max_{1 \leq i \leq m} \left| M_i\left(C_\alpha\right) - M_i\left(C_\alpha^T\right) \right| \right\}^2 \right]^{1/2}, \alpha \in \{V, S\}.$$

Proof. The proof is immediate from Corollary 5.10, so we omit the proof. ■

Next, we give upper bounds for the left and right spectral radius of quaternionic matrices in term of quaternionic matrix norms.

Theorem 5.27. *Let $A \in M_n(\mathbb{H})$. If $\|.\|_\beta$, $(\beta = 1, 2, \infty, F)$, are the quaternionic matrix norms, then*

$$\rho_l(A),\ \rho_r(A) \leq \|A\|_\beta.$$

Proof. The proof is similar to that of the complex case. ■

By applying Theorem 5.27, we extend bounds for the eigenvalues of a complex matrix polynomial [15] to bounds for the right eigenvalues of a quaternionic matrix polynomial which are as follows.

Theorem 5.28. *Let $\mathbf{L} \in \mathbb{L}_m(M_n(\mathbb{H}))$ be as in (5.7). Then for any right eigenvalue μ of $\mathbf{L}$ satisfies the following inequalities:*

1. $\max\left(\|S_m\|_1, 1 + \max\limits_{1 \leq i \leq m-1} (\|S_i\|_1)\right)^{-1} \leq |\mu| \leq \max\left(\|V_0\|_1, 1 + \max\limits_{1 \leq i \leq m-1} (\|V_i\|_1)\right)$,
2. $\max\left(1, \|S\|_\infty\right)^{-1} \leq |\mu| \leq \max\left(1, \|V\|_\infty\right)$,
3. $\left(\|I_n + SS^H\|_2\right)^{-1/2} \leq |\mu| \leq \left(\|I_n + VV^H\|_2\right)^{1/2}$.

Proof. The first two bounds are follows from Theorem 5.27 (for $\beta = 1, \infty$). Now for the third bound, applying Theorem 5.27 ($\beta = 2$), we have

$$(\|C_S\|_2)^{-1} \leq |\mu| \leq \|C_V\|_2.$$

To find $\|C_V\|_2$, we write

$$M := \begin{bmatrix} 0 & I_n & 0 & \dots & 0 \\ 0 & 0 & I_n & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & I_n \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \text{ and } N := \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \\ -V_0 & -V_1 & -V_2 & \dots & -V_{m-1} \end{bmatrix}.$$

Then $M^H N = N^H M = 0$. Thus by Lemma 5.8, we have

$$\begin{aligned} \|C_V\|_2^2 &= \|C_V^H C_V\|_2 = \|(M+N)^H (M+N)\|_2 = \|M^H M + N^H N\|_2 \\ &\leq \|I_{mn} + N^H N\|_2 = \|I_{mn} + NN^H\|_2 = \|I_n + VV^H\|_2. \end{aligned}$$

For finding $\|C_S\|_2$ analogue proof can be given, so we skip the proof. ■

Theorem 5.29. *Let $w_j \in \mathbb{R}^+$, $(1 \leq j \leq m)$, with $w_m = 1$. Then for any right eigenvalue μ of $\mathbf{L} \in \mathbb{L}_m(M_n(\mathbb{H}))$ satisfies the following inequalities:*

1. $$\left[\max_{1\le j\le m-1}\left\{\frac{\|S_m\|_1}{w_1}, \frac{w_j}{w_{j+1}}+\frac{\|S_{m-j}\|_1}{w_{j+1}}\right\}\right]^{-1} \le |\mu| \le \max_{1\le j\le m-1}\left\{\frac{\|V_0\|_1}{w_1}, \frac{w_j}{w_{j+1}}+\frac{\|V_j\|_1}{w_{j+1}}\right\}.$$

2. $$\tfrac{1}{\beta_1} \le |\mu| \le \alpha_1,$$

where

$$\alpha_1 := \max_{1\le j\le m-1}\left(\frac{w_j}{w_{j+1}}, \left\|\left[\frac{V_0}{w_1},\dots,\frac{V_{m-2}}{w_{m-1}}, V_{m-1}\right]\right\|_\infty\right),$$

$$\beta_1 := \max_{1\le j\le m-1}\left(\frac{w_j}{w_{j+1}}, \left\|\left[\frac{S_m}{w_1},\dots,\frac{S_2}{w_{m-1}}, S_1\right]\right\|_\infty\right).$$

3. $$\left[\max_{1\le j\le m-1}\left(\frac{w_j^2}{w_{j+1}^2}\right)+\sum_{j=0}^{m-1}\frac{\|S_{m-j}\|_2^2}{w_{j+1}^2}\right]^{-1/2} \le |\mu| \le \left[\max_{1\le j\le m-1}\left(\frac{w_j^2}{w_{j+1}^2}\right)+\sum_{j=0}^{m-1}\frac{\|V_j\|_2^2}{w_{j+1}^2}\right]^{1/2}.$$

Proof. The proofs are immediate from the proof method of Theorem 5.28 and using Lemma 5.27 on the matrices C'_V and C'_S. ■

The following theorem is a direct application of Corollary 5.3 which gives bounds for the right eigenvalues of quaternionic matrix polynomials.

Theorem 5.30. *Let* $\mathbf{L} \in \mathbb{L}_m(M_n(\mathbb{H}))$ *be as in (5.7) and let* $\gamma \in [0,1]$. *Then for any right eigenvalue* μ *of* $\mathbf{L}$ *satisfies the following inequality:*

$$\left[\max_{1\le i\le mn}\left\{r'_i(C_S)^\gamma\, t'_i(C_S)^{1-\gamma}\right\}\right]^{-1} \le |\mu| \le \max_{1\le i\le mn}\left\{r'_i(C_V)^\gamma\, c'_i(C_V)^{1-\gamma}\right\}.$$

By the definition of the right spectral radius, we have for any $A \in M_n(\mathbb{H})$

$$\|A\|_2^2 = \rho_r(A^HA) \le \left\|A^HA\right\|_1 \le \left\|A^H\right\|_1 \|A\|_1$$
$$\|A\|_2^2 \le \|A\|_\infty \|A\|_1.$$

Now, we have two cases as follows.

Case 1: If $\|A\|_1 \le \|A\|_\infty$, then $\|A\|_2 \le \|A\|_\infty$.

Case 2: If $\|A\|_\infty \le \|A\|_1$, then $\|A\|_2 \le \|A\|_1$.

Thus if $A \in \mathbb{S} :=$ { symmetric, skew symmetric, Hermitian, skew-Hermitian, η-Hermitian, η-anti-Hermitian}, then we have

$$\|A\|_2 \le \|A\|_\infty = \|A\|_1.$$

It is clear that if $V_j\,(0 \le j \le m-1) \in \mathbb{S}$, then $\|V_j\|_2 \le \|V_j\|_q\,(\beta = 1,\infty)$. Thus from Theorem 5.23, we obtain

$$\begin{aligned}
\min_{p=1,2,\infty}\left\{\max\left(\|V_0\|_p, 1+\max_{1\le j\le m-1}(\|V_j\|_p)\right)\right\} &= \max\left(\|V_0\|_2, 1+\max_{1\le j\le m-1}(\|V_j\|_2)\right),\\
\min_{p=1,2,\infty}\left\{\max\left(1, \sum_{j=0}^{m-1}\|V_j\|_p\right)\right\} &= \max\left(1, \sum_{j=0}^{m-1}\|V_j\|_2\right),\\
\min_{p=1,2,\infty}\left\{\max_{1\le i\le m}\left(R_i^{(p)}(C_V)^{\gamma}\; C_i^{(p)}(C_V)^{1-\gamma}\right)\right\} &= \max_{1\le i\le m}\left(R_i^{(2)}(C_V)^{\gamma}\; C_i^{(2)}(C_V)^{1-\gamma}\right).
\end{aligned}$$

Other bounds can also be written in reduced form for the above structured matrices.

Bounds for the right eigenvalues of quaternionic matrix polynomials using powers of block companion matrix: We first define the colon notation. Let $A := (A_{ij}) \in M_n(\mathbb{H})$ be a block matrix as in (5.1), then $A[s,:]$ designates the s^{th} block row of the block matrix A as in (5.1) as follows:

$$A[s,:] = \begin{bmatrix} A_{s1} & \dots & A_{sk}\end{bmatrix} \in M_{n\times nk}(\mathbb{H}),$$

and the s^{th} block column of the block matrix A as in (5.1) is specified by

$$A[:,s] := \begin{bmatrix} A_{1s} \\ \vdots \\ A_{ks}\end{bmatrix} \in M_{nk\times n}(\mathbb{H}),$$

where $1 \le s \le k$. Consider the integers p, q, r satisfy $1 \le p \le q \le k,\ 1 \le r \le k$. Define

$$A[r, p:q] := \begin{bmatrix} A_{rp} & \dots & A_{rq}\end{bmatrix} \in M_{n\times n(q-p+1)}(\mathbb{H}).$$

Similarly if $1 \le p \le q \le k,\ 1 \le c \le k$, then we define

$$A[p:q, c] := \begin{bmatrix} A_{pc} \\ \vdots \\ A_{qc}\end{bmatrix} \in M_{n(q-p+1)\times n}(\mathbb{H}).$$

Now, we present a framework to find the powers of the quaternionic block companion matrix $C_{\mathbf{L}}$ with the help of above colon notation as follows.

Theorem 5.31. *Consider*
$$C_{\mathbf{L}} = \begin{array}{c} \\ n(m-1) \\ n \end{array} \begin{array}{c} \begin{array}{cc} n & n(m-1) \end{array} \\ \left[\begin{array}{c|c} 0 & I \\ \hline -A_0 & -\Delta \end{array}\right] \end{array}$$
and let $t < m$ be a positive integer, then

$$C_{\mathbf{L}}^t = \begin{array}{c} \\ n(m-t) \\ nt \end{array} \begin{array}{c} \begin{array}{cc} nt & n(m-t) \end{array} \\ \left[\begin{array}{c|c} 0 & I \\ \hline C & D \end{array}\right] \end{array}. \tag{5.27}$$

If $t \geq m$, then

$$C_{\mathbf{L}}^t = \begin{bmatrix} C_{\mathbf{L}}^{t-(m-1)}[m, 1:m] \\ C_{\mathbf{L}}^{t-(m-2)}[m, 1:m] \\ \vdots \\ C_{\mathbf{L}}^{t-1}[m, 1:m] \\ C_{\mathbf{L}}^{t}[m, 1:m] \end{bmatrix}_{nm\times nm}, \tag{5.28}$$

where

$$C = \begin{bmatrix} C_{\mathbf{L}}[m, 1:t] \\ C_{\mathbf{L}}^2[m, 1:t] \\ \vdots \\ C_{\mathbf{L}}^t[m, 1:t] \end{bmatrix}_{nt\times nt}, \quad D = \begin{bmatrix} C_{\mathbf{L}}[m, t+1:m] \\ C_{\mathbf{L}}^2[m, t+1:m] \\ \vdots \\ C_{\mathbf{L}}^t[m, t+1:m] \end{bmatrix}_{nt\times n(m-t)},$$

$$\Delta = \begin{bmatrix} A_1 & A_2 & \ldots & A_m \end{bmatrix}, \; C_{\mathbf{L}}^t[m,1] := C_{\mathbf{L}}^{t-1}[m,m]C_{\mathbf{L}}[m,1], \text{ and}$$
$$C_{\mathbf{L}}^t[m, 2:m] := C_{\mathbf{L}}^{t-1}[m, 1:m-1] + C_{\mathbf{L}}^{t-1}[m,m]C_{\mathbf{L}}[m, 2:m].$$

Proof. The proof is similar to the proof method of Theorem 2.35 and by using above notation. ■

We can now derive the following results similar to the complex case. Consider $A \in M_n(\mathbb{H})$, then

$$\rho_r(A) = \rho_r\left(A^t\right)^{1/t} \leq \left\|A^t\right\|_\beta^{1/t} \leq \|A\|_\beta, \; t = 1, 2, \ldots,$$

where $\|.\|_\beta$ is any quaternionic matrix norm. Also we have

$$\rho_r(A) = \lim_{t\to\infty} \left\|A^t\right\|_\beta^{1/t}.$$

The following bound for the right eigenvalues of a quaternionic matrix polynomial is derived by using powers of the corresponding quaternionic block companion matrices.

Theorem 5.32. *Let* $\mathbf{L} \in \mathbb{L}_m(M_n(\mathbb{H}))$ *be as in (5.7) and let* $\gamma \in [0,1]$. *Then for any right eigenvalue* μ *of* $\mathbf{L}$ *satisfies the following inequality:*

$$\frac{1}{\xi_1'} \leq |\mu| \leq \xi_1,$$

where

$$\begin{aligned} \xi_1 &= \min_{p=1,2,\infty} \left\{ \max_{1 \leq i \leq m} \left(\left(R_i^{(p)}(C_V^t) \right)^{\gamma/t} \left(C_i^{(p)}(C_V^t) \right)^{\frac{1-\gamma}{t}} \right) \right\}, \\ \xi_1' &= \min_{p=1,2,\infty} \left\{ \max_{1 \leq i \leq m} \left(\left(R_i^{(p)}(C_S^t) \right)^{\gamma/t} \left(C_i^{(p)}(C_S^t) \right)^{\frac{1-\gamma}{t}} \right) \right\}. \end{aligned}$$

Proof. The proof follows from Theorem 5.2, so we skip the proof. ■

Next, we derive bounds for the right eigenvalues of quaternionic matrix polynomials by using the matrix $1, 2$ and ∞-norms and powers of corresponding quaternionic block companion matrices which are as follows.

Theorem 5.33. *Let* $\mathbf{L} \in \mathbb{L}_m(M_n(\mathbb{H}))$ *be as in (5.7). Then for any right eigenvalue* μ *of* $\mathbf{L}$ *satisfies the following inequality:*

$$\left(\|C_S^t\|_1 \right)^{-1/t} \leq |\mu| \leq \left(\|C_V^t\|_1 \right)^{1/t}, \tag{5.29}$$

$$\left(\|C_S^t\|_\infty \right)^{-1/t} \leq |\mu| \leq \left(\|C_V^t\|_\infty \right)^{1/t}, \tag{5.30}$$

$$\left(\|C_S^t\|_2 \right)^{-1/t} \leq |\mu| \leq \left(\|C_V^t\|_2 \right)^{1/t}. \tag{5.31}$$

Proof. Consider λ is a right eigenvalue of the block companion matrix C_V, then λ^t ($t \geq 2$ is a positive integer) is a right eigenvalue of the matrix C_V^t. Thus by proof method of Theorem 5.27, we have

$$\rho_r(C_V) \leq \|C_V^t\|_\beta^{1/t}, \quad (\beta = 1, 2, \infty). \tag{5.32}$$

Similarly, we have also for C_S

$$\rho_r(C_S) \leq \|C_S^t\|_\beta^{1/t}, \quad (\beta = 1, 2, \infty). \tag{5.33}$$

Thus, from (5.32) and (5.33), we have the required results. ■

Now, we define

$$U := \begin{bmatrix} U_0 & U_1 & \dots & U_{m-1} \end{bmatrix}, \; \mathcal{L} := \begin{bmatrix} \mathcal{L}_m & \mathcal{L}_{m-1} & \dots & \mathcal{L}_1 \end{bmatrix},$$

where $U_j = V_{m-1}V_j - V_{j-1}$ and $\mathcal{L}_{j+1} = S_1 S_{j+1} - S_{j+2}$, $(0 \leq j \leq m-1)$ with $V_{-1} = S_{m+1} = 0$.

Substituting $t = 2$ in Theorem 5.33 and applying APPENDIX-B, we obtain the following corollaries.

Corollary 5.34. *Let* $\mathbf{L} \in \mathbb{L}_m(M_n(\mathbb{H}))$ *be as in (5.7). Then for any right eigenvalue* μ *of* $\mathbf{L}$ *satisfies the following inequalities:*

$$\frac{1}{\beta_4} \leq |\mu| \leq \alpha_4, \tag{5.34}$$

$$\max\left(1, \|S\|_\infty, \|\mathcal{L}\|_\infty\right)^{-1/2} \leq |\mu| \leq \max\left(1, \|V\|_\infty, \|U\|_\infty\right)^{1/2}, \tag{5.35}$$

where

$$\begin{aligned} \alpha_4 &:= \left[\max\left(\|V_0\|_1 + \|U_0\|_1, \|V_1\|_1 + \|U_1\|_1, 1 + \max_{2\leq i\leq m-1}\left(\|V_i\|_1 + \|U_i\|_1\right)\right)\right]^{1/2}, \\ \beta_4 &:= \left[\max\left(\|S_m\|_1 + \|\mathcal{L}_m\|_1, \|S_{m-1}\|_1 + \|\mathcal{L}_{m-1}\|_1, 1 + \max_{1\leq i\leq m-2}\left(\|S_i\|_1 + \|\mathcal{L}_i\|_1\right)\right)\right]^{1/2}. \end{aligned}$$

Corollary 5.35. *Let* $\mathbf{L} \in \mathbb{L}_m(M_n(\mathbb{H}))$ *be as in (5.7). Then for any right eigenvalue* μ *of* $\mathbf{L}$ *satisfies the following inequality:*

$$\left(\left\|I_{2n} + TT^H\right\|_2\right)^{-1/4} \leq |\mu| \leq \left(\left\|I_{2n} + DD^H\right\|_2\right)^{1/4},$$

where $D = \begin{bmatrix} V & 0 \\ 0 & U \end{bmatrix}$ *and* $T = \begin{bmatrix} S & 0 \\ 0 & \mathcal{L} \end{bmatrix}$.

Proof. Let us consider $P = C_V^2 = X + Y + Z$, where

$$X := \begin{bmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \\ U_0 & U_1 & \dots & U_{m-1} \end{bmatrix}, Y := \begin{bmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ -V_0 & -V_1 & \dots & -V_{m-1} \\ 0 & 0 & \dots & 0 \end{bmatrix}, Z := \begin{bmatrix} 0 & 0 & I_n & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & I_n \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

Now, we have $\left\|XY^H\right\|_2 = \left\|XZ^H\right\|_2 = \left\|YX^H\right\|_2 = \left\|YZ^H\right\|_2 = \left\|ZX^H\right\|_2 = \left\|ZY^H\right\|_2 = 0.$ Thus

$$\begin{aligned} \|P\|_2^2 = \left\|PP^H\right\|_2 &\leq \left\|XX^H + YY^H + ZZ^H\right\|_2 \\ &\leq \left\|I_{mn} + XX^H + YY^H\right\|_2 = \left\|I + DD^H\right\|_2 \\ \|P\|_2 &\leq \left(\left\|I_{2n} + DD^H\right\|_2\right)^{1/2}. \end{aligned}$$

If μ is any right eigenvalue of C_U, then

$$|\mu| \leq \left\|C_V^2\right\|_2^{1/2} \leq \left(\left\|I_{2n} + DD^H\right\|_2\right)^{1/4}.$$

Analogue proof can be given for the lower bound. ■

We give the following example to illustrate our theory.

Example 5.36. Consider the quaternionic matrix polynomial $\mathbf{L} \in \mathbb{L}_2(M_2(\mathbb{H}))$ of the form

$$\mathbf{L}(\lambda) := A_2\lambda^2 + A_1\lambda + A_0, \text{ where}$$

$$A_0 = \begin{bmatrix} 1+\mathbf{i} & 1+\mathbf{i} \\ 1 & 1-\mathbf{i} \end{bmatrix}, \ A_1 = \begin{bmatrix} 11+2\mathbf{i}+3\mathbf{j}+\mathbf{k} & 7+3\mathbf{i}-\mathbf{j}+8\mathbf{k} \\ 1-8\mathbf{i}-3\mathbf{j}-7\mathbf{k} & 9+5\mathbf{i}+\mathbf{j}-\mathbf{k} \end{bmatrix}, A_2 = \begin{bmatrix} 2+\mathbf{i} & 2-\mathbf{i} \\ 2 & 1-\mathbf{i} \end{bmatrix}.$$

Then

$$\begin{aligned} V_0 &= A_2^{-1}A_0 = \begin{bmatrix} 0.5-0.5\mathbf{i} & 1-2\mathbf{i} \\ -0.5+0.5\mathbf{i} & -2+\mathbf{i} \end{bmatrix}, \\ V_1 &= A_2^{-1}A_1 = \begin{bmatrix} -5.5-13.5\mathbf{i}-4\mathbf{j}-13\mathbf{k} & 4+9\mathbf{i}+2\mathbf{j}-6\mathbf{k} \\ -3.5+15.5\mathbf{i}-7\mathbf{j}+12\mathbf{k} & 7-6\mathbf{i}-8\mathbf{j}+5\mathbf{k} \end{bmatrix}, \\ S_1 &= A_0^{-1}A_1 = \begin{bmatrix} 3+\mathbf{i}-4\mathbf{j}+4\mathbf{k} & 12-6\mathbf{i}+7\mathbf{k} \\ 3.5-5.5\mathbf{i}+6\mathbf{j}-5\mathbf{k} & -7+4\mathbf{i}+3.5\mathbf{j}-2.5\mathbf{k} \end{bmatrix}, \\ S_2 &= A_0^{-1}A_2 = \begin{bmatrix} 2-\mathbf{i} & 1-2\mathbf{i} \\ -0.5+0.5\mathbf{i} & -0.5+0.5\mathbf{i} \end{bmatrix}. \end{aligned}$$

The right spectrum of $\mathbf{L}(\lambda)$ is

$$\Lambda_r(\mathbf{L}) = [1.5719+23.2242\mathbf{i}] \cup [-3.0550+4.1765\mathbf{i}] \cup [-0.0806-0.0730\mathbf{i}] \cup [0.0637+0.0421\mathbf{i}].$$

Also $\max\limits_{\lambda_i \in \Lambda_r(\mathbf{L})} |\lambda_i| = 23.2773$ and $\min\limits_{\lambda_i \in \Lambda_r(\mathbf{L})} |\lambda_i| = 0.0764$.

Let **LB:=** Lower Bound, **UB:=** Upper Bound. Thus Theorem 5.23 is verified. Also we have verified our results for the case of the 1-norm, 2-norm and the ∞-norm.

5.3.2. Left eigenvalues of quaternionic matrix polynomials and their bounds

In this subsection, we state that if the variable of a matrix polynomial is a quaternion identity, then we always be able to find the bounds for the left eigenvalues of quaternionic matrix polynomials. So, if we consider any left matrix polynomial $\mathbf{L}'$ as given in (5.6)

Example 5.36	**LB**	**UB**
Theorem 5.23		
(5.19)	0.0464	34.7804
(5.20)	0.0419	36.9768
(5.21), $\gamma = 1/4$	0.0452	35.3178
Theorem 5.25	0.0469	34.1084
Theorem 5.26	0.0608	24.2378
Theorem 5.28		
(1)	0.0412	42.0443
(2)	0.0383	35.7120
(3)	0.0480	33.9391

TABLE 5.1. Lower and upper bounds for right eigenvalues of $\mathbf{L}(\lambda)$.

including the one define in (5.7), then the similar bounds as in the previous section for the right eigenvalues of $\mathbf{L}$ can be derived for the left eigenvalues of $\mathbf{L}'$.

Let $\mathbf{L}' \in \mathbb{L}'_m(M_n(\mathbb{H}))$ of the form (5.6). Then, we have the following linearization form of the matrix polynomial $\mathbf{L}'$.

- **For the left eigenvalues:** The matrix polynomial $\mathbf{L}' \in \mathbb{L}'_m(M_n(\mathbb{H}))$ of the form (5.6) can be written in the linearization form:

$$C_{\mathbf{L}'} + \xi\, X,$$

where $\xi \in \mathbb{H}$, $C_{\mathbf{L}'}$, $X \in M_{mn}(\mathbb{H})$ are of the forms

$$C_{\mathbf{L}'} := \begin{bmatrix} 0 & 0 & 0 & \dots & -A_0 \\ I_n & 0 & 0 & \dots & -A_1 \\ & \ddots & \vdots & \dots & \vdots \\ 0 & 0 & I_n & & -A_{m-2} \\ 0 & 0 & \dots & I_n & -A_{m-1} \end{bmatrix}, \quad X := \begin{bmatrix} I_n & 0 & 0 & \dots & 0 \\ 0 & I_n & 0 & \dots & 0 \\ 0 & 0 & I_n & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & A_m \end{bmatrix}.$$

When $A_m = I_n$, the identity matrix, then the matrix polynomial $\mathbf{L}' \in \mathbb{L}'_m(M_n(\mathbb{H}))$ of the form (5.6) is said to be monic matrix polynomial and the corresponding linearization form of it is given by

$$C_{\mathbf{L}'} + \xi\, E, \text{ where } E := I_{nm}.$$

Theorem 5.37. *The quaternionic matrix polynomial* $\mathbf{L}'$ *of the form* (*5.6*) *and the pencil* $(C_{\mathbf{L}'}, X)$ *have same left eigenvalues.*

Proof. Let $\mu \in \mathbb{H}$ be a left eigenvalue of the quaternionic matrix pencil $(C_{\mathbf{L}'}, X)$. Then there exists $x := [x_1^T, x_2^T, \dots, x_m^T]^T$ with $x_i \in \mathbb{H}^n$, $i = 1, 2, \dots, m$ such that

$$\begin{bmatrix} 0 & 0 & 0 & \dots & -A_0 \\ I_n & 0 & 0 & \dots & -A_1 \\ & \ddots & \vdots & & \vdots \\ 0 & 0 & I_n & & -A_{m-2} \\ 0 & 0 & \dots & I_n & -A_{m-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{m-1} \\ x_m \end{bmatrix} = \mu \begin{bmatrix} I_n & 0 & 0 & \dots & 0 \\ 0 & I_n & 0 & \dots & 0 \\ 0 & 0 & I_n & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & A_m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{m-1} \\ x_m \end{bmatrix}, \tag{5.36}$$

then (5.36) implies

$$\begin{aligned} -A_0 x_m &= \mu x_1, \\ x_1 - A_1 x_m &= \mu x_2, \\ x_2 - A_2 x_m &= \mu x_3, \\ &\vdots \\ x_{m-2} - A_{m-2} x_m &= \mu x_{m-1}, \\ x_{m-1} - A_{m-1} x_m &= \mu A_m x_m. \end{aligned}$$

By solving the m^{st} system of equations, we obtain

$$A_0 x_m + \mu A_1 x_m + \mu^2 A_2 x_m + \dots + \mu^m A_m x_m = 0.$$

Hence μ is a left eigenvalue of the quaternionic matrix polynomial $\mathbf{L}'$. ■

Theorem 5.38. *Let* $\mathbf{L}' \in \mathbb{L}_m(M_n(\mathbb{H}))$ *be of the form* (*5.6*)*. Then* $\mu \in \mathbb{H}$ *is a left eigenvalue of* $\mathbf{L}'$ *if and only if* $\det[\Psi_{(A_0 + \mu A_1 + \dots + \mu^m A_m)}] = 0$.

Proof. Consider μ is a left eigenvalue of $\mathbf{L}$, then there exists a nonzero $x \in \mathbb{H}^n$ such that $A_0 x + \mu A_1 x + \mu^2 A_2 x + \dots + \mu^m A_m x = 0$. Applying the mapping Ψ, we have

$$\Psi_{(A_0 + \mu A_1 + \dots + \mu^m A_m)} \Psi_x = 0.$$

It follows that μ is a left eigenvalue of $\mathbf{L}'$ if and only if

$$\det[\Psi_{(A_0 + \mu A_1 + \dots + \mu^m A_m)}] = 0. \blacksquare$$

We now state a framework to find bounds for the left eigenvalues of quaternionic matrix polynomials.

- To find the bounds of the left eigenvalues of the quaternionic matrix polynomial $\mathbf{L}'$, we again introduce the new quaternionic matrix polynomials associated with $\mathbf{L}'$:

$$\mathbf{L}'_{V''}(\xi) := \xi^m I_n + \sum_{i=0}^{m-1} \xi^i V''_i, \tag{5.37}$$

where $V''_i := A_i A_m^{-1}$, $i = 0, 1, \ldots, m-1$, so that $\mathbf{L}'(\xi) = \mathbf{L}'_{V''}(\xi) A_m$, and

$$\mathbf{L}'_{S''}(\xi) := \xi^m I_n + \sum_{i=1}^{m} \xi^{m-i} S''_i, \tag{5.38}$$

where $S''_i := A_i A_0^{-1}$, $i = 1, \ldots, m$, now $\xi^m \mathbf{L}'(1/\xi) A_0^{-1} = \mathbf{L}'_{S''}(\xi)$. The matrix polynomials $\mathbf{L}'$ and $\mathbf{L}'_{V''}$ have the same left eigenvalues, whereas the left eigenvalues of the matrix polynomial $\mathbf{L}'_{S''}$ are the reciprocal of the left eigenvalues of the matrix polynomial $\mathbf{L}'$. The block companion matrices $C_{V''}$ and $C_{S''}$ corresponding to the monic matrix polynomials $\mathbf{L}'_{V''}$ and $\mathbf{L}'_{S''}$ are given by

$$C_{V''} := \begin{array}{c} \\ n \\ n(m-1) \end{array} \begin{array}{c} \begin{array}{cc} n(m-1) & n \end{array} \\ \left[\begin{array}{c|c} 0 & -V''_0 \\ \hline I & -\Delta_{V''} \end{array}\right] \end{array} \text{ and } C_{S''} := \begin{array}{c} \\ n \\ n(m-1) \end{array} \begin{array}{c} \begin{array}{cc} n(m-1) & n \end{array} \\ \left[\begin{array}{c|c} 0 & -S''_m \\ \hline I & -\Delta_{S''} \end{array}\right] \end{array},$$

respectively, where

$$\begin{aligned} \Delta_{V''} &= \begin{bmatrix} V_1''^T & V_2''^T & \ldots & V''_{m-1}{}^T \end{bmatrix}^T \in M_{n(m-1)\times n}(\mathbb{H}), \\ \Delta_{S''} &= \begin{bmatrix} S''_{m-1}{}^T & S''_{m-2}{}^T & \ldots & S_1''^T \end{bmatrix}^T \in M_{n(m-1)\times n}(\mathbb{H}). \end{aligned}$$

Moreover, the left eigenvalues of the monic matrix polynomials $\mathbf{L}'_{V''}(\xi), \mathbf{L}'_{S''}(\xi)$ and the left eigenvalues of the block companion matrices $C_{V''}, C_{S''}$ are same, respectively.

Theorem 5.39. *Consider* $C_{\mathbf{L}'} = \begin{array}{c} \\ n \\ n(m-1) \end{array}\begin{array}{c} \begin{array}{cc} n(m-1) & n \end{array} \\ \left[\begin{array}{c|c} 0 & -A_0 \\ \hline I & -\Delta \end{array}\right] \end{array}$ *and let* $t < m$ *be a positive integer, then*

$$C_{\mathbf{L}'}^t = \begin{array}{c} \\ nt \\ n(m-t) \end{array}\begin{array}{c} \begin{array}{cc} n(m-t) & nt \end{array} \\ \left[\begin{array}{c|c} 0 & C \\ \hline I & D \end{array}\right] \end{array}. \tag{5.39}$$

If $t \geq m$*, then we write*

$$C_{\mathbf{L}'}^t = \left[\begin{array}{ccccc} C_{\mathbf{L}'}^{t-(m-1)}[1:m,m] & C_{\mathbf{L}'}^{t-(m-2)}[1:m,m] & \dots & C_{\mathbf{L}'}^{t-1}[1:m,m] & C_{\mathbf{L}'}^{t}[1:m,m] \end{array}\right],$$

where $C_{\mathbf{L}'}^t$ *are* $nm \times nm$ *matrices,*

$\Delta := \begin{bmatrix} A_1^T & A_2^T & \dots & A_m^T \end{bmatrix}^T$, $C := \begin{bmatrix} C_{\mathbf{L}'}[1:t,m] & C_{\mathbf{L}'}^2[1:t,m] & \dots & C_{\mathbf{L}'}^t[1:t,m] \end{bmatrix}$,

$$\begin{aligned} D &:= \begin{bmatrix} C_{\mathbf{L}'}[t+1:m,m] & C_{\mathbf{L}'}^2[t+1:m,m] & \dots & C_{\mathbf{L}'}^t[t+1:m,m] \end{bmatrix}, \\ C_{\mathbf{L}'}^t[1,m] &:= C_{\mathbf{L}'}[1,m]\ C_{\mathbf{L}'}^{t-1}[m,m], \text{ and} \\ C_{\mathbf{L}'}^t[2:m,m] &:= C_{\mathbf{L}'}^{t-1}[1:m-1,m] + C_{\mathbf{L}'}[2:m,m]\ C_{\mathbf{L};}^{t-1}[m,m]. \end{aligned}$$

Proof. The proof is similar to the proof method of Theorem 2.37 and by using colon notation of Subsection 5.3.1. ■

Remark 5.40. Similar results can be obtained for the left eigenvalues of the quaternionic matrix polynomial $\mathbf{L}' \in \mathbb{L}'_m(M_n(\mathbb{H}))$.

Next, we define the right quaternionic polynomial as

$$p_r(z) := z^m q_m + z^{m-1} q_{m-1} + \cdots + z q_1 + q_0, \tag{5.40}$$

where $z,\ q_j \in \mathbb{H}$ $(0 \leq j \leq m)$. The polynomial (5.40) is called monic if $q_m = 1$. Then we write the following remark.

Remark 5.41. If we take $n = 1$, then the quaternionic matrix polynomial $\mathbf{L}'$ will be reduced into the right quaternionic polynomial $p_r(z)$.

Now we have the following observations.

- By Remark 5.41, Theorems 5.25 and 5.26 for left eigenvalue of the matrix polynomial $\mathbf{L}'$ yield the bounds for the zeros of the right monic quaternionic polynomial $p_r(z)$.
- Theorem 5.30 for left eigenvalue of the matrix polynomial $\mathbf{L}'$ yields the bounds for the zeros of the right monic quaternionic polynomial $p_r(z)$ which can be found in Theorem 2.26 for the zero of the right monic quaternionic polynomial $p_r(z)$.

The following corollary is a particular case of Theorem 5.28 for the bounds of the left eigenvalues of the quaternionic matrix polynomial $\mathbf{L}' \in \mathbb{L}'_m(M_n(\mathbb{H}))$ which generalizes the bounds for the zeros of right quaternionic polynomials and these can be seen in Corollary 2.27.

Corollary 5.42. *Let $\mathbf{L}' \in \mathbb{L}'_m(M_n(\mathbb{H}))$ be as in (5.6). Then for any left eigenvalue μ of $\mathbf{L}'$ satisfies the following inequalities:*

$$\left(1 + \max_{1\le i\le m} \|S_i''\|_1\right)^{-1} \le |\mu| \le 1 + \max_{0\le i\le m-1} \|V_i''\|_1, \tag{5.41}$$

$$\max\left(1, \sum_{i=1}^{m} \|S_i''\|_\infty\right)^{-1} \le |\mu| \le \max\left(1, \sum_{i=0}^{m-1} \|V_i''\|_\infty\right), \tag{5.42}$$

$$\left(1 + \sum_{i=1}^{m} \|S_i''\|_2^2\right)^{-1/2} \le |\mu| \le \left(1 + \sum_{i=0}^{m-1} \|V_i''\|_2^2\right)^{1/2}. \tag{5.43}$$

5.4. Bounds for eigenvalues of complex matrix polynomials

We show that the bounds obtained in Subsections 5.3.1 and 5.3.2 for the left and right eigenvalues are similar to the bounds for the eigenvalues of complex matrix polynomials. Our framework for the bounds of the eigenvalues of quaternionic matrix polynomials are same with the bounds on the eigenvalue of complex matrix polynomials. That is the bounds for the left and right eigenvalues of quaternionic matrix polynomials are same with the bounds for the eigenvalues of a complex matrix polynomial. Bounds for the eigenvalues of a complex matrix polynomial and their applications are given in [4, 15, 38]. If A_m is not invertible in the matrix polynomial $\mathcal{P} \in \mathcal{P}_m(M_n(\mathbb{C}))$ given in (5.5), then $\mathcal{P}(z)$ has an infinite eigenvalue. While if A_0 is not invertible, then 0 is an eigenvalue of the matrix polynomial $\mathcal{P}(z)$. Therefore, for upper bounds of the absolute values of the eigenvalues of $\mathcal{P}(z)$ require A_m to be invertible and lower bounds of the absolute values of eigenvalues of $\mathcal{P}(z)$ require A_0 to be invertible.

For finding the upper and lower bounds for the absolute values of the eigenvalues of the complex matrix polynomial $\mathcal{P}(z)$, we state the following remarks from Section 5.2 and Subsection 5.3.1.

Remark 5.43. It is seen that Theorems 5.1 and 5.2 are true for the eigenvalues of complex block matrices. Then Theorems 5.23 and 5.24 are true for the eigenvalues of complex matrix polynomials.

From Remark 5.43, it is clear that bounds (5.19) and (5.20) for the absolute values of the eigenvalues of complex matrix polynomials are tighter than the bounds given in [15, Lemma 2.3 (2.1), Corollary 2.4 (2.5) (2.6)]. Now we have the following observations from Section 5.2 and Subsection 5.3.1.

- From Remark 5.43 it shows that Theorems 5.32 is true for the eigenvalues of complex matrix polynomials.
- Theorem 5.33 is true for the eigenvalues of complex matrix polynomials for $1, 2$ and ∞-matrix norms over the complex field.
- It is obvious that Theorem 5.9 is also true for complex block matrices. Hence Theorem 5.25 holds for the eigenvalues of complex matrix polynomials.
- It is also seen that Corollary 5.10 is true for complex matrices. Hence, Theorem 5.26 is valid for the eigenvalues of complex matrix polynomials.

If we take bounds for the absolute values of the eigenvalues of complex matrix polynomials by using powers of block companion matrix, then the bounds (5.29), (5.30) and (5.31) for complex matrix polynomials are sharper than the bounds given in [15, Lemma 2.3].

Example 5.44. Consider the complex matrix polynomial

$$\mathcal{P}(z) := A_5 z^5 + A_4 z^4 + A_3 z^3 + A_2 z^2 + A_1 z + A_0,$$

where

$$A_0 = \begin{bmatrix} 2+i & 13-2i \\ -2+13i & 4 \end{bmatrix}, A_1 = \begin{bmatrix} 23i & 7+3i \\ -7+3i & 31+i \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 2 & 1+2i \\ 1-2i & 23i \end{bmatrix}, A_3 = \begin{bmatrix} 1 & 9-7i \\ 7+9i & 2 \end{bmatrix},$$
$$A_4 = \begin{bmatrix} 7+i & 3i \\ 3 & 25 \end{bmatrix}, A_5 = \begin{bmatrix} 11+2i & 10+8i \\ 8-10i & 6-7i \end{bmatrix}.$$

The spectrum of $\mathcal{P}(z)$ is given by

$$\begin{aligned}\Lambda(\mathcal{P}) = &\Big\{3.3319 - 1.0719i, 1.0701 - 1.3366i, -0.6994 - 1.0580i, -1.3205 + 0.0574i, \\ &-0.8939 + 0.5348i, 0.1973 + 1.1657i, 0.7705 - 0.5866i, 0.2632 + 0.6958i, \\ &0.4865 + 0.0727i, -0.4532 + 0.0166i\Big\},\end{aligned}$$

$$\max_{\lambda_i \in \Lambda(\mathcal{P}(z))} |\lambda_i| = 3.5001, \text{ and } \min_{\lambda_i \in \Lambda(\mathcal{P}(z))} |\lambda_i| = 0.4535.$$

As we have already explained in this section that Theorems 5.23, 5.32 and 5.33 are true for the eigenvalues of complex matrix polynomials. Thus we now give the following tables for the absolute values of the eigenvalues of $(\mathcal{P}(z))$.

Example 5.44	**LB**	**UB**
Theorem 5.23		
(5.19)	0.2962	8.6818
(5.20)	0.1105	23.1505
(5.21) , $\gamma = 1/4$	0.2280	7.6143

TABLE 5.2. Lower and upper bounds for eigenvalues of $\mathcal{P}(z)$.

Example 5.44	**LB**	**UB**
[15, Lemma 2.3]		
(2.1)	0.2501	8.7210
(2.2)	0.1049	23.0150
(2.3)	0.2407	11.3218
Theorem 5.33, t=2		
(5.29)	0.3299	6.0610
(5.30)	0.2304	9.3571
(5.31)	0.3473	6.6054
Theorem 5.33, t=3		
(5.29)	0.3528	5.2268
(5.30)	0.2954	6.9081
(5.31)	0.3709	5.4567
Theorem 5.33, t=4		
(5.29)	0.3853	4.7508
(5.30)	0.3315	5.8152
(5.31)	0.4086	4.8856

TABLE 5.3. Lower and upper bounds for eigenvalues of $\mathcal{P}(z)$.

CHAPTER 6

PERTURBATION BOUNDS FOR RIGHT EIGENVALUES OF QUATERNIONIC MATRICES AND THEIR APPLICATIONS

This chapter deals with the concept of perturbation bounds for right eigenvalues/generalized right eigenvalues of a quaternionic matrix/quaternionic matrix pencil. In particular, Bauer-Fike type theorems for right eigenvalues/generalized right eigenvalues of a diagonalizable quaternionic matrix/diagonalizable quaternionic matrix pencil are derived. Other perturbation bounds for right eigenvalues of a quaternionic matrix are discussed. Furthermore, the location of standard right eigenvalues of a quaternionic matrix and a sufficient condition for the stability of a perturbed quaternionic matrix are given. Perturbation bounds for the zeros of quaternionic polynomials are derived. Finally, we give numerical examples to illustrate our results.

6.1. Introduction

The goal of this chapter is to derive Bauer-Fike type theorems for right eigenvalues/generalized right eigenvalues, perturbation analysis for quaternionic matrices, location of right eigenvalues of perturbed quaternionic matrices, and perturbation bounds for the zeros of quaternionic polynomials.

Bauer-Fike theorems are the standard results in the perturbation theory over the complex field. The applications of the Bauer-Fike theorems over the complex field are given in [7, 11, 35, 36, 51]. In general, quaternionic matrix similarity is meaningless for left eigenvalues. However, there are many literatures on matrix similarity and diagonalization for right eigenvalues of a quaternionic matrix. Many results, like the Jordan canonical form, Schur decomposition, singular-value decomposition, diagonalizable of a quaternionic matrix have been extended from the complex field to the skew field of quaternions by different authors. For instance, see [9, 21, 46, 49, 61]. However, an extension of Bauer-Fike theorem, perturbation analysis on quaternionic matrices, perturbation analysis of zeros of quaternionic polynomials have not yet been studied. Perturbation analysis over the skew field of quaternions is important in quantum physics, control theory, and mechanics (see, for example, [1, 26, 43, 49]).

In this chapter, we extend the Bauer-Fike theorems over the complex field to the skew field of quaternions. Specifically, Bauer-Fike type theorems for right eigenvalues/generalized right eigenvalues of a diagonalizable quaternionic matrix/diagonalizable quaternionic matrix pencil are derived. Other perturbation results for right eigenvalues of a quaternionic matrix are given via Jordan canonical form and block diagonal decomposition of a quaternionic matrix. Meanwhile, localization theorems for right eigenvalues of a quaternionic matrix and a sufficient condition for the stability of a perturbed quaternionic matrix are derived. Perturbation bounds for the zeros of the quaternionic polynomials $p_l(z)$ and $p_r(z)$ (defined in Subsection 1.2.4) are given.

6.2. Perturbation analysis on quaternionic matrices

We derive the following lemmas for the development of our theory.

Lemma 6.1. *Let $a, b \in \mathbb{C}^+$. Then $|a-b| \leq |\overline{a}-b|$.*

Proof. Suppose $a = x+iy$ and $b = p+iq$, where $x, p \in \mathbb{R}$ and $y, q \in \mathbb{R}^+ \cup \{0\}$. We have

$$|a-b| = |x+iy-(p+iq)| = |(x-p)+i(y-q)|, \text{ and} \tag{6.1}$$

$$|\overline{a}-b| = |x-iy-(p+iq)| = |(x-p)+i(y+q)|. \tag{6.2}$$

It is known that if $y, q \in \mathbb{R}^+ \cup \{0\}$, then $y+q \geq y-q$. Hence from (6.1) and (6.2), we have $|a-b| \leq |\overline{a}-b|$. ■

Lemma 6.2. *Consider $\alpha_k, \beta_k, \gamma_k \in \mathbb{R}, (k=1,2)$, such that $\alpha_1^2+\beta_1^2+\gamma_1^2 = 1$ and $\alpha_2^2+\beta_2^2+\gamma_2^2 = 1$. Then*

$$\alpha_1\alpha_2+\beta_1\beta_2+\gamma_1\gamma_2 \leq 1.$$

Proof. We have that

$$\begin{aligned}(\alpha_1-\alpha_2)^2+(\beta_1-\beta_2)^2+(\gamma_1-\gamma_2)^2 &= 2-2\left(\alpha_1\alpha_2+\beta_1\beta_2+\gamma_1\gamma_2\right) \geq 0, \text{ i.e.,} \\ \alpha_1\alpha_2+\beta_1\beta_2+\gamma_1\gamma_2 &\leq 1. \ ■\end{aligned}$$

Lemma 6.3. *Let $\lambda, \mu \in \mathbb{C}^+$ and let $0 \neq \rho, \eta \in \mathbb{H}$. Then*

$$|\lambda - \mu| \leq \left|\rho^{-1}\lambda\rho - \eta^{-1}\mu\eta\right|.$$

Proof. Consider $\lambda = \lambda' + \lambda''\mathbf{i}$, $\mu = \mu' + \mu''\mathbf{i}$, $\rho^{-1}\mathbf{i}\rho = \alpha_1\mathbf{i} + \beta_1\mathbf{j} + \gamma_1\mathbf{k}$, and $\eta^{-1}\mathbf{i}\eta = \alpha_2\mathbf{i} + \beta_2\mathbf{j} + \gamma_2\mathbf{k}$, where $\alpha_k, \beta_k, \gamma_k \in \mathbb{R}$ $(k = 1, 2)$ with $\alpha_1^2 + \beta_1^2 + \gamma_1^2 = \alpha_2^2 + \beta_2^2 + \gamma_2^2 = 1$. Then

$$P = |\lambda - \mu|^2 = (\lambda' - \mu')^2 + (\lambda'')^2 + (\mu'')^2 - 2\lambda''\mu''. \tag{6.3}$$

Also, $Q = |\rho^{-1}\lambda\rho - \eta^{-1}\mu\eta|^2 = (\lambda' - \mu')^2 + (\lambda''\alpha_1 - \mu''\alpha_2)^2 + (\lambda''\beta_1 - \mu''\beta_2)^2 + (\lambda''\gamma_1 - \mu''\gamma_2)^2 = (\lambda' - \mu')^2 + (\lambda'')^2[\alpha_1^2 + \beta_1^2 + \gamma_1^2] + (\mu'')^2[\alpha_2^2 + \beta_2^2 + \gamma_2^2] - 2\lambda''\mu''[\alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2]$. From Lemma 6.2, we have

$$\alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2 < 1.$$

Thus $P \leq Q$, i.e.,

$$|\lambda - \mu| \leq \left|\rho^{-1}\lambda\rho - \eta^{-1}\mu\eta\right|. \blacksquare$$

Lemma 6.4. *Let $A \in M_n(\mathbb{H})$ such that $A^t = 0_n$ for any positive integer t, where 0_n is the $n \times n$ zero matrix. Then $\Psi_A^t = 0_{2n}$, where 0_{2n} is the $2n \times 2n$ zero matrix.*

Proof. Consider $A \in M_n(\mathbb{H})$ such that $A^t = 0_n$ for any positive integer t. Then by taking the complex adjoint matrix of $A^t = 0_n$ and by applying Theorem 1.12, we have

$$\Psi_{A^t} = \Psi_{0_n} \Rightarrow \Psi_A^t = 0_{2n}. \blacksquare$$

Lemma 6.5. *Let $T \in M_n(\mathbb{H})$ be partitioned as follows:*

$$T = \begin{array}{c} \\ p \\ q \end{array}\!\!\begin{array}{c} \begin{array}{cc} p & q \end{array} \\ \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \end{array}.$$

Define the linear transformation $\phi : M_{p\times q}(\mathbb{H}) \to M_{p\times q}(\mathbb{H})$ by

$$\phi(X) = T_{11}X - XT_{22},$$

where $X \in M_{p\times q}(\mathbb{H})$. Then ϕ is invertible if and only if $\Lambda_r(T_{11}) \cap \Lambda_r(T_{22}) = \emptyset$. If ϕ is invertible and $Y \in M_n(\mathbb{H})$ is defined by

$$Y = \begin{bmatrix} I_p & Z \\ 0 & I_q \end{bmatrix}, \qquad \phi(Z) = -T_{12}$$

then $Y^{-1}TY = \text{diag}(T_{11}, T_{22})$.

Proof. Let $X \in M_{p\times q}(\mathbb{H})$. Then singular value decomposition of X is given by

$$U^H XV = \begin{array}{c} \\ r \\ p-r \end{array}\begin{array}{c} \begin{array}{cc} r & q-r \end{array} \\ \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix} \end{array}, \tag{6.4}$$

where $\Sigma_r = \text{diag}(\sigma_i)$ and r =rank(X). Assuming $\phi(X) = 0$ for $X \neq 0$. Substituting (6.4) into the quaternionic matrix equation $T_{11}X = XT_{22}$, we have

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

where $U^H T_{11} U := (A_{ij})$ and $V^H T_{22} V := (B_{ij})$. By comparing blocks we see that $A_{21} = 0$, $B_{12} = 0$, and $\Lambda_r(A_{11}) = \Lambda_r(B_{11})$. Consequently,

$$\emptyset \neq \Lambda_r(A_{11}) = \Lambda_r(B_{11}) \subseteq \Lambda_r(T_{11}) \cap \Lambda_r(T_{22}).$$

On the other hand, if $\lambda \in \Lambda_r(T_{11}) \cap \Lambda_r(T_{22})$, then there exist nonzero $x, y \in \mathbb{H}^n$ such that

$$T_{11}x = x\lambda, \quad T_{22}^H y = y\lambda^H.$$

Then $y^H T_{22} = \lambda y^H$ and hence $\phi(xy^H) = 0$. Finally, if ϕ is invertible, then the quaternionic matrix Z exists and

$$\begin{aligned} Y^{-1}TY &= \begin{bmatrix} I & -Z \\ 0 & I \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \begin{bmatrix} I & Z \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} T_{11} & T_{11}Z - ZT_{22} + T_{12} \\ 0 & T_{22} \end{bmatrix} = \begin{bmatrix} T_{11} & 0 \\ 0 & T_{22} \end{bmatrix}. \end{aligned} \blacksquare$$

We next establish block diagonal decomposition of a quaternionic matrix.

Theorem 6.6. (*Block diagonal decomposition*) *Suppose*

$$U^H AU = T = \begin{bmatrix} T_{11} & T_{12} & \dots & T_{1q} \\ 0 & T_{22} & \dots & T_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & T_{qq} \end{bmatrix}$$

is a Schur decomposition of $A \in M_n(\mathbb{H})$ *and assume that the* T_{ii} *are square. If* $\Lambda_r(T_{ii}) \cap \Lambda_r(T_{jj}) = \emptyset$ *whenever* $i \neq j$*, then there exists an invertible matrix* $Y \in M_n(\mathbb{H})$ *such that*

$$(QY)^{-1}A(QY) = \text{diag}(T_{11}, \ldots, T_{qq}).$$

Proof. The proof is immediate by applying Lemma 6.5 and induction. ■

6.2.1. Bauer-Fike type theorem for the right eigenvalues

We first derive Bauer-Fike type theorem for diagonalizable quaternionic matrices which is as follows.

Theorem 6.7. *Let* $A \in M_n(\mathbb{H})$ *be a diagonalizable matrix, i.e.,* $A = Y\Lambda Y^{-1}$, $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$*, where* λ_i *are the standard right eigenvalues of* A *and* Y *be an invertible quaternionic matrix. Let* $\Delta A \in M_n(\mathbb{H})$*. If* μ *is a standard right eigenvalue of* $A + \Delta A$*, then*

$$\text{dist}\big(\mu, \Lambda_s(A)\big) := \min_{\lambda_i \in \Lambda_s(A)} \big\{|\lambda_i - \mu|\big\} \leq K_2(Y)\, \|\Delta A\|_2.$$

Moreover, we have

$$\text{dist}\big(\xi, \Lambda_r(A)\big) := \inf_{\eta_j \in \Lambda_r(A)} \big\{|\eta_j - \xi|\big\} \leq K_2(Y)\, \|\Delta A\|_2,$$

where $\xi \in \Lambda_r(A + \Delta A)$ *and* $K_2(\cdot)$ *is the condition number with respect to the matrix 2-norm.*

Proof. Let $\lambda_i \neq \mu$ for any i. Since μ is a standard right eigenvalue of $A + \Delta A$, then there exists $x \neq 0 \in \mathbb{H}^n$ such that $(A + \Delta A)x = x\mu$. This system is equivalent to the complex system

$$\Psi_{A+\Delta A}\psi_x = \mu\psi_x$$

which implies

$$(\Psi_{A+\Delta A} - \mu I_{2n})\psi_x = 0.$$

From this we have

$$(\Psi_A + \Psi_{\Delta A} - \mu I_{2n})\psi_x = 0.$$

The above system can be written as $(\mu I_{2n} - \Psi_{Y\Lambda Y^{-1}})\psi_x = \Psi_{\Delta A}\psi_x$. Further we can write $\Psi_Y(\mu I_{2n} - \Psi_\Lambda)\Psi_{Y^{-1}}\psi_x = \Psi_{\Delta A}\psi_x$. Then $(\mu I_{2n} - \Psi_\Lambda)(\Psi_Y)^{-1}\psi_x = (\Psi_Y)^{-1}\Psi_{\Delta A}\psi_x$. Thus

$(\Psi_Y)^{-1}\psi_x = (\mu I_{2n} - \Psi_\Lambda)^{-1}[(\Psi_Y)^{-1}\Psi_{\Delta A}\Psi_Y](\Psi_Y)^{-1}\psi_x$. Taking matrix 2-norm (operator norm) on both sides, we get

$$\begin{aligned}
\|(\Psi_Y)^{-1}\psi_x\|_2 &\leq \|(\mu I_{2n} - \Psi_\Lambda)^{-1}\|_2 \; \|(\Psi_Y)^{-1}\Psi_{\Delta A}\Psi_Y]\|_2 \; \|(\Psi_Y)^{-1}\psi_x\|_2 \\
1 &\leq \|(\mu I_{2n} - \Psi_\Lambda)^{-1}\|_2 \; \|(\Psi_Y)^{-1}\|_2 \; \|\Psi_{\Delta A}\|_2 \; \|\Psi_Y\|_2 \\
1 &\leq \max_{\lambda_i \in \Lambda_{\mathbf{s}}(A)} \left\{ \frac{1}{|\lambda_i - \mu|}, \frac{1}{|\overline{\lambda_i} - \mu|} \right\} \; \|(\Psi_Y)^{-1}\|_2 \; \|\Psi_{\Delta A}\|_2 \; \|\Psi_Y\|_2.
\end{aligned}$$

From Lemma 6.1, we have

$$\begin{aligned}
1 &\leq \max_{\lambda_i \in \Lambda_{\mathbf{s}}(A)} \left\{ \frac{1}{|\lambda_i - \mu|} \right\} \|(\Psi_Y)^{-1}\|_2 \; \|\Psi_{\Delta A}\|_2 \; \|\Psi_Y\|_2 \\
1 &\leq \frac{1}{\min\limits_{\lambda_i \in \Lambda_s}(A)\{|\lambda_i - \mu|\}} \; K_2(\Psi_Y)\|\Psi_{\Delta A}\|_2 \\
\min_{\lambda_i \in \Lambda_{\mathbf{s}}(A)} \big\{|\lambda_i - \mu|\big\} &\leq K_2(\Psi_Y) \; \|\Psi_{\Delta A}\|_2.
\end{aligned}$$

Now from Lemma 1.13, we obtain

$$\min_{\lambda_i \in \Lambda_s(A)} \big\{|\lambda_i - \mu|\big\} \leq K_2(Y)\|\Delta A\|_2.$$

Thus Lemma 6.3 yields

$$\min_{\lambda_i \in \Lambda_s(A)} \{|\lambda_i - \mu|\} = \inf_{\eta_j \in \Lambda_r(A)} \{|\eta_j - \xi|\},$$

where $\xi \in \Lambda_r(A + \Delta A)$. Hence

$$\inf_{\eta_j \in \Lambda_r(A)} \{|\eta_j - \xi|\} \leq K_2(Y) \; \|\Delta A\|_2. \; \blacksquare$$

In particular, when $A \in M_n(\mathbb{H})$ is normal, Theorem 6.7 leads to the following corollary.

Corollary 6.8. *Let $A \in M_n(\mathbb{H})$ be a normal matrix and let μ be a standard right eigenvalue of the perturbed quaternionic matrix $A + \Delta A$. Then*

$$\mathrm{dist}(\mu, \Lambda_s(A)) := \min_{\lambda_i \in \Lambda_s(A)} \big\{|\lambda_i - \mu|\big\} \leq \|\Delta A\|_2.$$

Moreover, we have

$$\mathrm{dist}\big(\xi, \Lambda_r(A)\big) := \min_{\eta_j \in \Lambda_r(A)} \big\{|\eta_j - \xi|\big\} \leq \|\Delta A\|_2,$$

where $\xi \in \Lambda_r(A + \Delta A)$.

We next have the following theorem for a relative perturbation bound.

Theorem 6.9. *Let $A \in M_n(\mathbb{H})$ be diagonalizable and invertible, i.e., $A = Y\Lambda Y^{-1}$, $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, where λ_i are the standard right eigenvalues of A and Y be an invertible quaternionic matrix. Let $\Delta A \in M_n(\mathbb{H})$. If μ is a standard right eigenvalue of $A + \Delta A$, then*

$$\operatorname{dist}\big(\mu, \Lambda_s(A)\big) := \min_{\lambda_i \in \Lambda_s(A)} \left\{ \frac{|\lambda_i - \mu|}{|\lambda_i|} \right\} \leq K_2(Y)\, \|A^{-1}\Delta A\|_2.$$

Moreover, we have

$$\operatorname{dist}\big(\xi, \Lambda_r(A)\big) := \inf_{\eta_j \in \Lambda_r(A)} \left\{ \frac{|\eta_j - \xi|}{|\eta_j|} \right\} \leq K_2(Y)\, \|A^{-1}\Delta A\|_2,$$

where $\xi \in \Lambda_r(A + \Delta A)$ and $K_2(Y) = \|Y\|_2\, \|Y^{-1}\|_2$ is the condition number of Y with respect to the matrix 2-norm.

Proof. Let $\lambda_i \neq \mu$ for any i. Since μ is a standard right eigenvalue of $A + \Delta A$, there exists $x \neq 0 \in \mathbb{H}^n$ such that $(A + \Delta A)x = x\mu$. This system is equivalent to the complex system

$$\Psi_{A+\Delta A}\psi_x = \mu\psi_x.$$

Since A is an invertible matrix, multiplying by $-\Psi_{A^{-1}}$ from left, we obtain

$$-\Psi_{A^{-1}}\left(\Psi_A + \Psi_{\Delta A}\right)\psi_x = -\mu\Psi_{A^{-1}}\psi_x$$

$$\left(-I_{2n} - \Psi_{A^{-1}\Delta A}\right)\psi_x = -\mu\Psi_{A^{-1}}\psi_x$$

$$\left(\mu\Psi_{A^{-1}} - I_{2n}\right) = \Psi_{A^{-1}\Delta A}\psi_x.$$

The matrix A is diagonalizable, i.e., $A = Y\Lambda Y^{-1}$, we obtain

$$(\mu\Psi_{Y\Lambda^{-1}Y^{-1}} - I_{2n})\psi_x = \Psi_{A^{-1}\Delta A}\psi_x$$

$$\Psi_Y(\mu\Psi_{\Lambda^{-1}} - I_{2n})\Psi_{Y^{-1}}\psi_x = \Psi_{A^{-1}\Delta A}\psi_x.$$

Thus by calculation, we get

$$(\Psi_Y)^{-1}\psi_x = (\mu\Psi_{\Lambda^{-1}} - I_{2n})^{-1}[(\Psi_Y)^{-1}\Psi_{A^{-1}\Delta A}\Psi_Y](\Psi_Y)^{-1}\psi_x.$$

By applying the proof method of Theorem 6.7, we have

$$\operatorname{dist}\big(\mu, \Lambda_s(A)\big) := \min_{\lambda_i \in \Lambda_s(A)} \left\{ \frac{|\lambda_i - \mu|}{|\lambda_i|} \right\} \leq K_2(Y)\, \|A^{-1}\Delta A\|_2.$$

As well as

$$\operatorname{dist}\big(\xi, \Lambda_r(A)\big) := \inf_{\eta_j \in \Lambda_r(A)} \left\{ \frac{|\eta_j - \xi|}{|\eta_j|} \right\} \leq K_2(Y)\, \|A^{-1}\Delta A\|_2,$$

where $\xi \in \Lambda_r(A + \Delta A)$. ■

Before going to present a localization theorem of standard right eigenvalues of the perturbed quaternionic matrix $A+\Delta A$. We see that a right eigenvalue of the perturbed quaternionic matrix $A+\Delta A$ is not necessarily contained in the union of n-discs

$$\Omega_i(A) := \{z \in \mathbb{C} : |z-\lambda_i| \leq K_2(Y)\ \|\Delta A\|_2\} \quad (1 \leq i \leq n), \tag{6.5}$$

where A is diagonalizable, i.e., $A = Y\mathrm{diag}(\lambda_i)Y^{-1}$ and $\lambda_i \in \Lambda_s(A)$. For example, consider a quaternionic matrix $A = \begin{bmatrix} 1+\mathbf{i} & 0 \\ 0 & \mathbf{i} \end{bmatrix}$. Let $\Delta A = \begin{bmatrix} \epsilon & 0 \\ 0 & \epsilon \end{bmatrix}$. Then $A+\Delta A = \begin{bmatrix} 1+\mathbf{i}+\epsilon & 0 \\ 0 & \mathbf{i}+\epsilon \end{bmatrix}$ and $\|\Delta A\|_2 = \epsilon$. Since A is normal, $K_2(Y) = 1$. We set $\epsilon = 10^{-3}$. From (6.5), we get the following two discs:

$$\Omega_1(A) = \left\{z \in \mathbb{C} : |z-1-\mathbf{i}| \leq 10^{-3}\right\} \text{ and } \Omega_2(A) = \left\{z \in \mathbb{C} : |z-\mathbf{i}| \leq 10^{-3}\right\}.$$

The perturbed quaternionic matrix $A+\Delta A$ has two standard right eigenvalues $1+\mathbf{i}+\epsilon$ and $\mathbf{i}+\epsilon$. In particular, $1-\mathbf{i}+\epsilon$ is also a right eigenvalue of $A+\Delta A$. However it is not contained in any discs. Fortunately, we can show that all the standard right eigenvalues of $A+\Delta A$ are contained in the union of n-discs $\Omega_i(A)$ which is as follows.

Theorem 6.10. *Let $A \in M_n(\mathbb{H})$ be a diagonalizable matrix, i.e., $A = Y\Lambda Y^{-1}$, $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, where λ_i are the standard right eigenvalues of A and Y be an invertible quaternionic matrix. Let $\Delta A \in M_n(\mathbb{H})$. Then all the standard right eigenvalues of the perturbed matrix $A+\Delta A$ are contained in the union of n-discs $\Omega_i(A) := \{z \in \mathbb{C} : |z-\lambda_i| \leq K_2(Y)\ \|\Delta A\|_2\}$ $(1 \leq i \leq n)$, i.e.,*

$$\Lambda_s(A+\Delta A) \subseteq \Omega(A) := \cup_{i=1}^{n} \Omega_i(A).$$

Proof. The proof follows direct from Theorem 6.7. ■

The following result is a sufficient condition for the stability of the perturbed quaternionic matrix $A+\Delta A$.

Proposition 6.11. *Let $A \in M_n(\mathbb{H})$ be a diagonalizable, i.e., $A = Y\Lambda Y^{-1}$, $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, where λ_i are the standard right eigenvalues of A and Y be an invertible quaternionic matrix. Let $\Delta A \in M_n(\mathbb{H})$. Assume that*

$$Re(\lambda_i) + K_2(Y)\ \|\Delta A\|_2 < 0 \quad \forall\, i\, (1 \leq i \leq n). \tag{6.6}$$

Then the perturbed quaternionic matrix $A+\Delta A$ is stable.

Proof. Let λ be any standard right eigenvalue of A. Then

$$\mathrm{Re}(\lambda) = \mathrm{Re}(\rho^{-1}\lambda\rho) \;\forall\; \rho \in \mathbb{H},$$

i.e., the real part of the standard right eigenvalue λ and the real part of corresponding non standard right eigenvalues are same. Thus, the proof follows from the proof method of Proposition 2.17 and using Theorem 6.10 with Definition 1.24. ■

We next turn to prove the Bauer-Fike type theorem for central closed matrices.

Theorem 6.12. *Let $A \in M_n(\mathbb{H})$ be a central closed matrix, i.e., $A = Y\Lambda Y^{-1}$, $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, where λ_i are the standard right eigenvalues of A. Let $\Delta A \in M_n(\mathbb{H})$. If μ is a standard right eigenvalue of $A + \Delta A$, then*

$$\mathrm{dist}\big(\mu, \Lambda_s(A)\big) := \min_{\lambda_i \in \Lambda_{\mathbf{s}}(A)} \big\{|\lambda_i - \mu|\big\} \leq K_2(Y)\; \|\Delta A\|_2.$$

Moreover,

$$\mathrm{dist}\big(\xi, \Lambda_r(A)\big) := \inf_{\lambda_i \in \Lambda_{\mathbf{r}}(A)} \big\{|\lambda_i - \xi|\big\} \leq K_2(Y)\; \|\Delta A\|_2,$$

where ξ is a right eigenvalue of $A + \Delta A$ and $K_2(\cdot)$ is the condition number with respect to the matrix 2-norm.

Proof. Let $\lambda_i \neq \mu$ for any i. Then it is not trivial. Since μ is a standard right eigenvalue of $A + \Delta A$, there exists $x \neq 0 \in \mathbb{H}^n$ such that $(A + \Delta A)x = x\mu$. By the proof method of Theorem 6.7, we have

$$1 \;\leq\; \|(\mu I_{2n} - \Psi_\Lambda)^{-1}\|_2\; \|(\Psi_Y)^{-1}\|_2\; \|\Psi_{\Delta A}\|_2\; \|\Psi_Y\|_2.$$

From Definition 1.8, λ_i $(1 \leq i \leq n)$ are real. Hence

$$\begin{aligned}
1 \;&\leq\; \max_{\lambda_i \in \Lambda_{\mathbf{s}}(A)} \left\{\frac{1}{|\lambda_i - \mu|}\right\}\; \|(\Psi_Y)^{-1}\|_2\; \|\Psi_{\Delta A}\|_2\; \|\Psi_Y\|_2 \\
1 \;&\leq\; \frac{1}{\min\limits_{\lambda_i \in \Lambda_s(A)} \big\{|\lambda_i - \mu|\big\}}\; K_2(\Psi_Y)\|\Psi_{\Delta A}\|_2 \\
\min_{\lambda_i \in \Lambda_{\mathbf{s}}(A)} \big\{|\lambda_i - \mu|\big\} \;&\leq\; K_2(\Psi_Y)\; \|\Psi_{\Delta A}\|_2.
\end{aligned}$$

By Lemma 1.13, we obtain

$$\min_{\lambda_i \in \Lambda_s(A)} \big\{|\lambda_i - \mu|\big\} \leq K_2(Y)\; \|\Delta A\|_2.$$

Since $|\lambda_i - \mu| = |\lambda_i - \rho^{-1}\mu\rho|,\ 0 \neq \rho \in \mathbb{H}$. Hence

$$\inf_{\lambda_i \in \Lambda_r(A)} \{|\lambda_i - \eta|\} \leq K_2(Y)\ \|\Delta A\|_2,$$

where $\eta \in \Lambda_r(A + \Delta A)$. ■

The following theorem is a direct consequence of Theorem 6.12 .

Theorem 6.13. *Let $A \in M_n(\mathbb{H})$ be a central closed matrix, i.e., $A = Y\Lambda Y^{-1}$, $\Lambda =$ $\mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, where λ_i are the standard right eigenvalues of A. Let $\Delta A \in M_n(\mathbb{H})$. Then all the right eigenvalues of $A + \Delta A$ are contained in the union of n-balls $\mathcal{G}_i(A) :=$ $\{z \in \mathbb{H} : |z - \lambda_i| \leq K_2(Y)\ \|\Delta A\|_2\}\ (1 \leq i \leq n)$, i.e.,*

$$\Lambda_r(A + \Delta A) \subseteq \mathcal{G}(A) := \cup_{i=1}^{n} \mathcal{G}_i(A).$$

Proof. The proof follows from Theorem 6.12. ■

Since real numbers commute with quaternions. Therefore, real left eigenvalues of a matrix $A \in M_n(\mathbb{H})$ are also right eigenvalues of A and vice versa. By applying this argument, we have the following result.

Theorem 6.14. *Let $A \in M_n(\mathbb{H})$ be a Hermitian matrix. For some $\tilde{\mu} \in \mathbb{R}$ and $\tilde{x} \in \mathbb{H}^n$ with $\|\tilde{x}\|_2 = 1$, define residual vector $r = A\tilde{x} - \tilde{\mu}\tilde{x}$. Then $|\tilde{\mu} - \mu| \leq \|r\|_2$ for some $\mu \in \Lambda_r(A)$.*

Proof. Since $\mu \in \Lambda_r(A)$, $\mu \in \Lambda_l(A)$. Now if $\tilde{\mu} \notin \Lambda_r(A)$, then $\tilde{\mu} \notin \Lambda_l(A)$. Hence $(A - \tilde{\mu} I_n)^{-1}$ exists. So we can write $r = A\tilde{x} - \tilde{\mu}\tilde{x}$ as follows:

$$\tilde{x} = (A - \tilde{\mu} I_n)^{-1} r.$$

Since A is a Hermitian matrix. Therefore, from Theorem 1.29, A is unitarily diagonalizable, i.e., $V^{-1}AV = \mathrm{diag}(\mu_i)\ (1 \leq i \leq n)$, where V is a quaternionic unitary matrix. Now, by applying the proof method of Theorem 6.7, we have the required result

$$\min_{\mu_i \in \Lambda_{\mathbf{s}}(A)} \left\{|\tilde{\mu} - \mu_i|\right\} \leq \|r\|_2. \ ■$$

6.2.2. Perturbation bounds for non-diagonalizable quaternionic matrices

First, in this subsection, we derive a perturbation result on quaternionic matrices via block diagonal decomposition of quaternionic matrices.

Theorem 6.15. *Let $A \in M_n(\mathbb{H})$. Consider $A = YTY^{-1}$, $T = \operatorname{diag}(V_1, \ldots, V_k)$, where $V_i = \Lambda_i + N_i \in M_{n_i}(\mathbb{H})$ is upper triangular, Λ_i is diagonal and N_i is strict upper triangular for $i = 1, 2, \ldots, k$. If μ is a standard right eigenvalue of $A + \Delta A$, then there exists $\lambda_j \in \Lambda_s(A)$ such that*

$$|\lambda_j - \mu| \leq \max\left(\chi, \chi^{1/n_j}\right),$$

where $\chi = \|Y^{-1}\Delta AY\|_2 \sum_{t=0}^{n_j-1} \|N_j\|_2^t$, and $N_j^{n_j} = 0_{n_j}$ with $N_j^{n_j-1} \neq 0_{n_j}$.

Proof. Let $\mu \notin \Lambda_s(A)$. Since μ is a standard right eigenvalue of $A + \Delta A$, there exists $x \neq 0 \in \mathbb{H}^n$ such that $(A + \Delta A)x = x\mu$. Then from the proof method of Theorem 6.7, we have

$$1 \leq \|(\mu I_{2n} - \Psi_T)^{-1}\|_2 \, \|Y^{-1}\Delta AY\|_2. \tag{6.7}$$

We have that

$$\frac{1}{\|(\mu I_{2n} - \Psi_T)^{-1}\|_2} = \frac{1}{\max\limits_{1\leq i\leq k} \|(\mu I_{2n_i} - \Psi_{V_i})^{-1}\|_2} = \frac{1}{\|(\mu I_{2n_j} - \Psi_{V_j})^{-1}\|_2}. \tag{6.8}$$

Define

$$\tau = \frac{1}{\|(\mu I_{2n_j} - \Psi_{\Lambda_j})^{-1}\|_2} = \frac{1}{\max\left\{\frac{1}{|\mu-\lambda_j|}, \frac{1}{|\mu-\overline{\lambda_j}|}\right\}} = |\mu - \lambda_j|. \tag{6.9}$$

Since $N_j^{n_j} = 0$, from Lemma 6.4, we have $\Psi_{N_j}^{n_j} = 0$. As N_j is a strict upper triangular quaternionic matrix, Ψ_{N_j} is also a strict upper triangular. Hence $\left(\mu I_{2n_j} - \Psi_{\Lambda_j}\right)\Psi_{N_j} = \Psi_{N_j}\left(\mu I_{2n_j} - \Psi_{\Lambda_j}\right)$. Therefore,

$$\left[\left(\mu I_{2n_j} - \Psi_{\Lambda_j}\right)^{-1}\Psi_{N_j}\right]^{n_j} = 0.$$

Consequently, we get

$$\left(\mu I_{2n_j} - \Psi_{V_j}\right)^{-1} = \sum_{t=0}^{n_j-1} (-1)^t \left[\left(\mu I_{2n_j} - \Psi_{\Lambda_j}\right)^{-1}\Psi_{N_j}\right]^t \left(\mu I_{2n_j} - \Psi_{\Lambda_j}\right)^{-1}. \tag{6.10}$$

Then, we have

$$\|\left(\mu I_{2n} - \Psi_T\right)^{-1}\|_2 = \|\left(\mu I_{2n_j} - \Psi_{V_j}\right)^{-1}\|_2 \leq \frac{1}{\tau}\sum_{t=0}^{n_j-1}\left[\frac{\|N_j\|_2}{\tau}\right]^t. \tag{6.11}$$

If $\tau > 1$, then

$$\|\left(\mu I_{2n_j} - \Psi_{V_j}\right)^{-1}\|_2 \leq \frac{1}{\tau}\sum_{t=0}^{n_j-1}\|N_j\|_2^t. \tag{6.12}$$

From (6.7), (6.9), and (6.12), we obtain

$$1 \leq \frac{1}{\tau} \sum_{t=0}^{n_j-1} \|N_j\|_2^t \, (\|Y^{-1}\Delta A Y\|_2). \tag{6.13}$$

Setting $\chi = \|Y^{-1}\Delta A Y\|_2 \sum_{t=0}^{n_j-1} \|N_j\|_2^t$. Thus $\tau \leq \chi$.

If $\tau \leq 1$, then

$$\| \left(\mu I_{2n} - \Psi_T\right)^{-1} \|_2 = \| \left(\mu I_{2n_j} - \Psi_{V_j}\right)^{-1} \|_2 \leq \frac{1}{\tau^{n_j}} \sum_{t=0}^{n_j-1} \|N_j\|_2^t. \tag{6.14}$$

Hence, (6.7), (6.9), and (6.14) yield

$$1 \leq \frac{1}{\tau^{n_j}} \sum_{t=0}^{n_j-1} \|N_j\|_2^t \, (\|Y^{-1}\Delta A Y\|_2). \tag{6.15}$$

Thus $\tau^{n_j} \leq \chi$ and hence $\tau \leq \chi^{\frac{1}{n_j}}$. From the above it is clear that

$$|\lambda_j - \mu| \leq \max\left(\chi, \chi^{1/n_j}\right).$$

where $\chi = \|Y^{-1}\Delta A Y\|_2 \sum_{t=0}^{n_j-1} \|N_j\|_2^t$. ■

Next, we present a perturbation result on quaternionic matrices via the Jordan canonical form of a quaternionic matrix.

Theorem 6.16. *Let $A \in M_n(\mathbb{H})$ with $Y^{-1}AY = J = \operatorname{diag}(J_{m_i}(\lambda_i))$, where $J_{m_i}(\lambda_i)$ $(1 \leq i \leq t)$ are Jordan blocks of A. Let $\Delta A \in M_n(\mathbb{H})$. If μ is a standard right eigenvalue of $A + \Delta A$, then*

$$\min_{1\leq i\leq t} \left\{ \frac{1}{\|(J_{m_i}(\lambda_i) - \mu I_{m_i})^{-1}\|_2}, \frac{1}{\|(\overline{J_{m_i}(\lambda_i)} - \mu I_{m_i})^{-1}\|_2} \right\} \leq K_2(Y) \, \|\Delta A\|_2,$$

where $K_2(\cdot)$ is the condition number with respect to the matrix 2-norm.

Proof. If μ is not a standard right eigenvalue of any Jordan block matrices $J_{m_i}(\lambda_i)$, then it is not trivial. Since μ is a standard right eigenvalue of $A + \Delta A$, there exists $x \neq 0 \in \mathbb{H}^n$ such that

$$(A + \Delta A)x \;\; = \;\; x\mu,$$

and this system is equivalent to the complex system

$$\Psi_{A+\Delta A}\psi_x = \mu\psi_x \Leftrightarrow (\Psi_{A+\Delta A} - \mu I_{2n})\psi_x = 0 \Leftrightarrow (\Psi_A + \Psi_{\Delta A} - \mu I_{2n})\psi_x = 0.$$

Since A has a Jordan canonical form via the invertible quaternionic matrix Y. Hence

$$\begin{aligned}
(\mu I_{2n} - \Psi_{Y\operatorname{diag}(J_{m_i}(\lambda_i))Y^{-1}})\psi_x &= \Psi_{\Delta A}\psi_x \\
\Psi_Y(\mu I_{2n} - \Psi_{\operatorname{diag}(J_{m_i}(\lambda_i))})\Psi_{Y^{-1}}\psi_x &= \Psi_{\Delta A}\psi_x \\
(\mu I_{2n} - \Psi_{\operatorname{diag}(J_{m_i}(\lambda_i))})(\Psi_Y)^{-1}\psi_x &= (\Psi_Y)^{-1}\Psi_{\Delta A}\psi_x.
\end{aligned}$$

This implies that $(\Psi_Y)^{-1}\psi_x = (\mu I_{2n} - \Psi_{\operatorname{diag}(J_{m_i}(\lambda_i))})^{-1}[(\Psi_Y)^{-1}\Psi_{\Delta A}\Psi_Y](\Psi_Y)^{-1}\psi_x$.

Taking matrix 2-norm on both sides of the above equation, we obtain

$$\|(\Psi_Y)^{-1}\psi_x\|_2 \leq \|(\mu I_{2n} - \Psi_{\operatorname{diag}(J_{m_i}(\lambda_i))})^{-1}\|_2 \, \|[(\Psi_Y)^{-1}\Psi_{\Delta A}\Psi_Y]\|_2 \, \|(\Psi_Y)^{-1}\psi_x\|_2$$

$$\begin{aligned}
1 &\leq \|(\mu I_{2n} - \Psi_{\operatorname{diag}(J_{m_i}(\lambda_i))})^{-1}\|_2 \, \|(\Psi_Y)^{-1}\|_2 \, \|\Psi_{\Delta A}\|_2 \, \|\Psi_Y\|_2 \\
1 &\leq \max_{1\leq i\leq t}\{\|(J_{m_i}(\lambda_i) - \mu I_{m_i})^{-1}\|_2, \|(\overline{J_{m_i}(\lambda_i)} - \mu I_{m_i})^{-1}\|_2\} \, \|(\Psi_Y)^{-1}\|_2\|\Psi_{\Delta A}\|_2 \, \|\Psi_Y\|_2 \\
1 &\leq \max_{1\leq i\leq t}\{\|(J_{m_i}(\lambda_i) - \mu I_{m_i})^{-1}\|_2, \|(\overline{J_{m_i}(\lambda_i)} - \mu I_{m_i})^{-1}\|_2\} \, \|(\Psi_Y)^{-1}\|_2 \, \|\Psi_{\Delta A}\|_2 \, \|\Psi_Y\|_2.
\end{aligned}$$

The above inequality can be rewritten as

$$\frac{1}{\max\limits_{1\leq i\leq t}\left\{\|(J_{m_i}(\lambda_i) - \mu I_{m_i})^{-1}\|_2, \|(\overline{J_{m_i}(\lambda_i)} - \mu I_{m_i})^{-1}\|_2\right\}} \leq \|(\Psi_Y)^{-1}\|_2 \, \|\Psi_{\Delta A}\|_2 \, \|\Psi_Y\|_2,$$

i.e.,

$$\min_{1\leq i\leq t}\left\{\frac{1}{\|(J_{m_i}(\lambda_i) - \mu I_{m_i})^{-1}\|_2}, \frac{1}{\|(\overline{J_{m_i}(\lambda_i)} - \mu I_{m_i})^{-1}\|_2}\right\} \leq K_2(\Psi_Y) \, \|\Psi_{\Delta A}\|_2.$$

Then, from Lemma 1.13, we have

$$\min_{1\leq i\leq t}\left\{\frac{1}{\|(J_{m_i}(\lambda_i) - \mu I_{m_i})^{-1}\|_2}, \frac{1}{\|(\overline{J_{m_i}(\lambda_i)} - \mu I_{m_i})^{-1}\|_2}\right\} \leq K_2(Y) \, \|\Delta A\|_2. \quad \blacksquare$$

Remark 6.17. Obviously from Lemma 5.6, all the results in the 2-norm (operator norm) imply that for the Frobenius norm.

We give the following localization theorem for standard right eigenvalues of the perturbed quaternionic matrix $A + \Delta A$ when A is not a diagonalizable matrix.

Theorem 6.18. *Let $A \in M_n(\mathbb{H})$ with $Y^{-1}AY = J = \operatorname{diag}(J_{m_i}(\lambda_i))$, where $J_{m_i}(\lambda_i)$ $(1 \leq i \leq t)$ are Jordan blocks of A. Let $\Delta A \in M_n(\mathbb{H})$. Then all the standard right eigenvalues of $A + \Delta A$ are contained in the union of t-sets $P_i(A) := T_i(A) \cup K_i(A)$, where*

$$\begin{aligned} T_i(A) &:= \{z \in \mathbb{C} : \|(J_{m_i}(\lambda_i) - \mu I_{m_i})^{-1}\|_2^{-1} \leq K_2(Y)\|\Delta A\|_2\}, \\ K_i(A) &:= \{z \in \mathbb{C} : \|(\overline{J_{m_i}(\lambda_i)} - \mu I_{m_i})^{-1}\|_2^{-1} \leq K_2(Y)\ \|\Delta A\|_2\}, \text{ i.e.,} \\ \Lambda_s(A + \Delta A) \subseteq P(A) &:= (\cup_{i=1}^t T_i(A)) \cup (\cup_{i=1}^t K_i(A)). \end{aligned}$$

Proof. The proof follows from Theorem 6.16. ■

6.3. Perturbation bounds for the zeros of quaternionic polynomials

Theorem 6.19. *Let $p_l(z) = z^m + \sum_{k=0}^{m-1} q_k z^k$ be a quaternionic simple monic polynomial. Let $C_{p_l} = Y\operatorname{diag}(V_1, V_2, \ldots, V_t)Y^{-1}$ with $V_i = \Lambda_i + N_i \in M_{n_i}(\mathbb{H})$ is upper triangular, Λ_i is diagonal and N_i is strict upper triangular for $i = 1, 2, \ldots, t$. Assume that $\widehat{p_l}(z) = z^m + \sum_{k=0}^{m-1} \widehat{q_k} z^k$ is a perturbation of $p_l(z)$ with $\widehat{q_k} = q_k + \Delta q_k$, $|\Delta q_k| \leq \epsilon$, $(0 \leq k \leq m-1)$. Then for any complex zero $\widehat{z_k} \in Z_{\mathbb{C}}(\widehat{p_l}(z))$, there exists a complex zero $z_j \in Z_{\mathbb{C}}(p_l(z))$ such that*

$$|\widehat{z_k} - z_j| \leq \max(\chi, \chi^{1/n_j}),$$

where $\chi := \|Y^{-1}\Delta C_{p_l} Y\|_2 \sum_{\eta=0}^{n_j-1} \|N_j\|_2^\eta$, and $\Delta C_{p_l} := -e_m[\Delta q_0, \ldots, \Delta q_{m-1}]$ with $e_m := [0, \ldots, 0, 1]^T \in \mathbb{R}^m$.

Proof. Let us consider the corresponding companion matrix C_{p_l} to the simple monic polynomial $p_l(z)$ such that $C_{p_l} = Y\operatorname{diag}(V_1, V_2, \ldots, V_t)Y^{-1}$. Since $\Delta C_{p_l} = -e_m[\Delta q_0, \ldots, \Delta q_{m-1}]$, therefore

$$C_{p_l} + \Delta C_{p_l} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -\widehat{q_0} & -\widehat{q_1} & -\widehat{q_2} & \ldots & -\widehat{q_{m-1}} \end{bmatrix}.$$

It is known that the left eigenvalues of C_{p_l} and the zeros of $p_l(z)$ are same. Also the left spectrum of C_{p_l} falls in the right spectrum of C_{p_l}. Thus by the proof method of Theorem 6.15, we get the desired result. ■

Theorem 6.20. *Let $p_l(z) = z^m + \sum_{k=0}^{m-1} q_k z^k$ be a quaternionic simple monic polynomial and its companion matrix $C_{p_l} = YDY^{-1}, D = \operatorname{diag}(\lambda_1, \ldots, \lambda_m), \lambda_t \in \Lambda_s(A), t = 1, 2, \ldots, m$. Assume that $\widehat{p_l}(z) = z^m + \sum_{k=0}^{m-1} \widehat{q_k} z^k$ is a perturbation of $p_l(z)$ with $\widehat{q_k} = q_k + \Delta q_k$, $|\Delta q_k| \leq \epsilon$, $(0 \leq k \leq m-1)$. Then the zeros of $p_l(z)$ and $\widehat{p_l}(z)$ can be given as*

$$\inf_{\substack{\widehat{z_i} \in Z_{\mathbb{H}}(\widehat{p_l}(z)) \\ z_j \in Z_{\mathbb{H}}(p_l(z))}} |\widehat{z_i} - z_j| \leq K_2(Y) \, \|\Delta C_{p_l}\|_2,$$

where $\Delta C_{p_l} := -e_m[\Delta q_0, \ldots, \Delta q_{m-1}]$ with $e_m := [0, \ldots, 0, 1]^T \in \mathbb{R}^m$.

Proof. By applying Theorem 6.7 and the proof method of Theorem 6.19, we get the desired result. ■

Remark 6.21. Similar results can be obtained for the zeros of $p_r(z)$ as well.

6.4. Bauer-Fike type theorem for generalized right eigenvalues

Let $\mathbb{L}_1(M_n(\mathbb{H}))$ be the space of matrix pencils over a quaternion division algebra. $\mathbf{L}_1 \in \mathbb{L}_1(M_n(\mathbb{H}))$ is defined as

$$\mathbf{L}_1(\lambda) := A + \lambda B, \tag{6.16}$$

where $A, B \in M_n(\mathbb{H})$ and λ commutes with the quaternionic matrices. The generalized right eigenvalue of $\mathbf{L}_1 \in \mathbb{L}_1(M_n(\mathbb{H}))$ of the form (6.16) is defined as follows.

Definition 6.22. Let $\mathbf{L}_1 \in \mathbb{L}_1(M_n(\mathbb{H}))$ be as in (6.16) and let $\mu \in \mathbb{H}$. Then μ is called a generalized right eigenvalue of $\mathbf{L}_1$ if

$$Ax = Bx\mu$$

for some nonzero $x \in \mathbb{H}^n$. Here x is called the right eigenvector corresponding to the generalized right eigenvalue μ. The set of generalized right eigenvalues of $\mathbf{L}_1$ is called generalized right spectrum of $\mathbf{L}_1$, denoted by $\Lambda_r(\mathbf{L}_1)$.

Definition 6.23. Let $\mathbf{L}_1 \in \mathbb{L}_1(M_n(\mathbb{H}))$ be as in (6.16). Then the matrix pencil $\mathbf{L}_1$ is called regular if there exists $\alpha \in \mathbb{R}$ such that $A + \alpha\ B$ is an invertible matrix.

Next, we give the definition of generalized standard right eigenvalues of quaternionic matrix pencils.

Definition 6.24. Let $\mathbf{L}_1 \in \mathbb{L}_1(M_n)(\mathbb{H})$ be as in (6.16). Then we define a set of generalized standard right eigenvalues of a regular matrix pencil $\mathbf{L}_1$ as

$$\Lambda_s(\mathbf{L}_1) := \{\alpha \in \mathbb{C}_\infty : Ax = Bx\alpha,\ 0 \neq x \in \mathbb{H}^n,\ \Im(\alpha) \geq 0\},\ \mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}.$$

We have the quaternionic generalized right eigenvalue problem

$$Ax = Bx\mu.$$

If B is an invertible matrix, then $B^{-1}Ax = x\mu$. Moreover, $AB^{-1}(Bx) = (Bx)\mu$. Putting $Bx = y \in \mathbb{H}^n$, $AB^{-1}y = y\mu$. Hence,

$$\Lambda_r(\mathbf{L}_1) = \Lambda_r(A^{-1}B) = \Lambda_r(AB^{-1}).$$

Theorem 6.25. *Let $A + \lambda B$ (defined in (6.16)) be a quaternionic matrix pencil and let $\mu \in \mathbb{H}$ be a generalized right eigenvalue of $A + \lambda B$ with eigenvector $x \in \mathbb{H}^n$. Then μ is also a generalized right eigenvalue of the quaternionic matrix pencil $P^HAQ + \lambda P^HBQ$ with eigenvector $Q^{-1}x \in \mathbb{H}^n$, where $P, Q \in M_n(\mathbb{H})$ are invertible matrices.*

Proof. Let μ be a generalized right eigenvalue of the quaternionic matrix pencil $A + \lambda B$. Then

$$Ax = Bx\mu \tag{6.17}$$

for some nonzero $x \in \mathbb{H}^n$. If P and Q are invertible quaternionic matrices, then (6.17) is equivalent to the following generalized right eigenvalue problem

$$P^HAQ(Q^{-1}x) = P^HBQ(Q^{-1}x)\mu. \blacksquare$$

Theorem 6.26. *Let $\mathbf{L}_1 \in \mathbb{L}_1(M_n(\mathbb{H}))$ be a quaternionic matrix pencil such that A and B are Hermitian matrices and B be a positive definite. Then there exists an invertible quaternionic matrix Q such that $Q^HAQ = \operatorname{diag}(\mu_i)$ with μ_i are the generalized standard right eigenvalues of $\mathbf{L}_1$ and $Q^HBQ = I_n$.*

Proof. Let λ be a generalized right eigenvalue of $\mathbf{L}_1$. Then there exists some nonzero $x \in \mathbb{H}^n$ such that

$$Ax = Bx\lambda. \tag{6.18}$$

Since B is a Hermitian positive definite matrix, B has a positive definite square root $B^{\frac{1}{2}}$. Then the generalized right eigenvalue problem (6.18) is equivalent to the following quaternionic eigenvalue problem.

$$B^{-\frac{1}{2}}AB^{-\frac{1}{2}}y = y\lambda, \tag{6.19}$$

where $y = (B^{-\frac{1}{2}}x) \in \mathbb{H}^n$. Also, we can see that

$$(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^H = (B^{-\frac{1}{2}})^H A^H (B^{-\frac{1}{2}})^H = B^{-\frac{1}{2}}AB^{-\frac{1}{2}}.$$

Hence $B^{-\frac{1}{2}}AB^{-\frac{1}{2}}$ is a Hermitian matrix. From Theorem 1.29, there exists a unitary matrix $V \in M_n(\mathbb{H})$ such that

$$B^{-\frac{1}{2}}AB^{-\frac{1}{2}} = V\operatorname{diag}(\mu_i)V^H,$$

where $\mu_i \in \Lambda_r(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})$. Setting $Q = B^{-\frac{1}{2}}V$, $Q^HAQ = \operatorname{diag}(\mu_i)$ and $Q^HBQ = I$. Thus from (6.19) and Theorem 6.25, we have the required result. ■

Now, we develop Bauer-Fike type theorem for diagonalizable regular quaternionic matrix pencils.

Theorem 6.27. *Let $\mathbf{L}_1 \in \mathbb{L}_1(M_n(\mathbb{H}))$ be a diagonalizable regular quaternionic matrix pencil with $\Lambda_s(\mathbf{L}_1) = \{\mu_1, \mu_2, \dots, \mu_n\}$, i.e., $\mathbf{L}_1(\lambda) := A + \lambda B = P(D + \lambda I_n)Q$, where $P, Q \in M_n(\mathbb{H})$ are invertible matrices, $D = \operatorname{diag}(\mu_1, \mu_2, \dots, \mu_n)$ and λ commutes with the quaternionic matrices. If μ is a standard generalized right eigenvalue of the perturbed quaternionic matrix pencil $\mathbf{L}_1 + \Delta\mathbf{L}_1 \in \mathbb{L}_1(M_n(\mathbb{H}))$, then*

$$\min_{\mu_i \in \Lambda_s(\mathbf{L}_1)} |\mu_i - \mu| \leq \|P^{-1}\|_2 \; \|Q^{-1}\|_2 \; \|E\|_2,$$

where $(\mathbf{L}_1 + \Delta\mathbf{L}_1)\lambda := (A + \Delta A) + \lambda(B + \Delta B)$ and $E := \mu\Psi_{\Delta B} - \Psi_{\Delta A}$.

Proof. If $\mu \in \Lambda_s(\mathbf{L}_1)$, then the result follows. Assume that $\mu \notin \Lambda_s(\mathbf{L}_1)$. Since $\mu \in \Lambda_s(\mathbf{L}_1 + \Delta\mathbf{L}_1)$, there exists $0 \neq x \in \mathbb{H}^n$ such that

$$(A + \Delta A)x = (B + \Delta B)x\mu.$$

This system is equivalent to the complex system

$$\Psi_{A+\Delta A}\psi_x = \mu\Psi_{B+\Delta B}\psi_x$$

$$(\Psi_A - \mu\Psi_B)\psi_x = (\mu\Psi_{\Delta B} - \Psi_{\Delta A})\psi_x$$

$$(\Psi_{PDQ} - \mu\Psi_{PI_nQ})\psi_x = (\mu\Psi_{\Delta B} - \Psi_{\Delta A})\psi_x$$

$$\Psi_P(\Psi_D - \mu I_{2n})\Psi_Q\psi_x = (\mu\Psi_{\Delta B} - \Psi_{\Delta A})\psi_x$$

$$(\Psi_D - \mu I_{2n})\Psi_Q\Psi_x = \Psi_P^{-1}(\mu\Psi_{\Delta B} - \Psi_{\Delta A})\psi_x$$

$$\Psi_Q\psi_x = (\Psi_D - \mu I_{2n})^{-1}\Psi_P^{-1}(\mu\Psi_{\Delta B} - \Psi_{\Delta A})\Psi_Q^{-1}(\Psi_Q\psi_x).$$

Taking the matrix 2-norm on both sides, we have

$$\|\Psi_Q\psi_x\|_2 \leq \|(\Psi_D - \mu I_{2n})^{-1}\|_2 \; \|\Psi_P^{-1}\|_2 \; \|(\mu\Psi_{\Delta B} - \Psi_{\Delta A})\|_2 \; \|\Psi_Q^{-1}\|_2 \; \|\Psi_Q\psi_x\|_2$$

$$1 \leq \max_{\mu_i \in \Lambda_s(\mathbf{L}_1)} \left\{ \frac{1}{|\mu_i - \mu|}, \frac{1}{|\overline{\mu_i} - \mu|} \right\} \|\Psi_P^{-1}\|_2 \; \|(\mu\Psi_{\Delta B} - \Psi_{\Delta A})\|_2 \; \|\Psi_Q^{-1}\|_2.$$

By Lemmas 1.13 and 6.1 and putting $E := \mu\Psi_{\Delta B} - \Psi_{\Delta A}$, we have the following expression.

$$1 \leq \frac{1}{\min_{\mu_i \in \Lambda_s(\mathbf{L}_1)}\{|\mu_i - \mu|\}} \|P^{-1}\|_2 \; \|Q^{-1}\|_2 \; \|E\|_2$$

$$\min_{\mu_i \in \Lambda_s(\mathbf{L}_1)} \{|\mu_i - \mu|\} \leq \|P^{-1}\|_2 \; \|Q^{-1}\|_2 \; \|E\|_2. \quad \blacksquare$$

Numerical Examples: Here, we give some numerical examples to illustrate our results.

Example 6.28. Let us consider a quaternionic matrix

$$A = \begin{bmatrix} 2\mathbf{i} & -2\mathbf{j} & \mathbf{j}+\mathbf{k} \\ -\mathbf{k} & 2 & -1 \\ -\mathbf{j} & 1-\mathbf{i} & 1 \end{bmatrix}.$$

Then the complex adjoint matrix of A is

$$\Psi_A = \begin{bmatrix} 2\mathbf{i} & 0 & 0 & 0-2 & 1+\mathbf{i} & \\ 0 & 2 & -1 & -\mathbf{i} & 0 & 0 \\ 0 & 1-\mathbf{i} & 1 & -1 & 0 & 0 \\ 0 & 2 & -1+\mathbf{i} & -2\mathbf{i} & 0 & 0 \\ -\mathbf{i} & 0 & 0 & 0 & 2 & -1 \\ 1 & 0 & 0 & 0 & 1+\mathbf{i} & 1 \end{bmatrix}.$$

Since the complex adjoint matrix Ψ_A is diagonalizable, from Theorem 1.12, the matrix A is diagonalizable. The standard right eigenvalues of A are $1, 1+\mathbf{i}$ and $1+\mathbf{i}$. Hence from Theorem 1.32, there is an invertible quaternionic matrix Y such that

$$Y = \begin{bmatrix} 1+\mathbf{i} & \mathbf{i} & \mathbf{i} \\ \mathbf{j} & \mathbf{j} & 0 \\ -\mathbf{k} & -\mathbf{k} & -\mathbf{j} \end{bmatrix}.$$

Here $\Lambda_s(A) = \{1, 1+\mathbf{i}\}$, where $1+\mathbf{i}$ is with multiplicity 2. Then

$$Y^{-1}AY = \text{diag}(1, 1+\mathbf{i}, 1+\mathbf{i}).$$

Suppose a perturbation matrix $\Delta A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \epsilon \\ \epsilon & \epsilon & \epsilon \end{bmatrix}$. Take $\epsilon = 10^{-10}$. Then

$$A + \Delta A = \begin{bmatrix} 2\mathbf{i} & -2\mathbf{j} & \mathbf{j}+\mathbf{k} \\ -\mathbf{k} & 2 & -1+\epsilon \\ -\mathbf{j}+\epsilon & 1-\mathbf{i}+\epsilon & 1+\epsilon \end{bmatrix}.$$

Therefore $\|\Delta A\|_2 = \epsilon\sqrt{(1+\sqrt{2})}$ and $\Lambda_s(A+\Delta A) = \{1+1.0001\mathbf{i}, 1.0001+0.9999\mathbf{i}\}$, where $1+1.0001\mathbf{i}$ is with multiplicity 2. The condition number of the quaternionic matrix Y is $K_2(Y) = 10.2193$ and

$$\min_{\lambda_i \in \Lambda_{\mathbf{s}}(A)} \{|\lambda_i - \mu|\} = \inf_{\eta_i \in \Lambda_{\mathbf{r}}(A),\ \xi \in \Lambda_{\mathbf{r}}(A+\Delta A)} \{|\eta_i - \xi|\} = 0.0001.$$

Example 6.29. Consider a quaternionic matrix $A = \begin{bmatrix} 1 & -\mathbf{i} & -\mathbf{j} & \mathbf{k} \\ \mathbf{i} & 1 & -2\mathbf{k} & \mathbf{j} \\ \mathbf{j} & 2\mathbf{k} & 7 & -\mathbf{i} \\ -\mathbf{k} & -\mathbf{j} & \mathbf{i} & 1 \end{bmatrix}$. Then the complex adjoint matrix of A is

$$\Psi_A = \begin{bmatrix} 1 & -\mathbf{i} & 0 & 0 & 0 & 0 & -1 & \mathbf{i} \\ \mathbf{i} & 1 & 0 & 0 & 0 & 0 & 2-\mathbf{i} & 1 \\ 0 & 0 & 7 & -\mathbf{i} & 1 & 2\mathbf{i} & 0 & 0 \\ 0 & 0 & \mathbf{i} & 1 & -\mathbf{i} & -1 & 0 & 0 \\ 0 & 0 & 1 & \mathbf{i} & 1 & \mathbf{i} & 0 & 0 \\ 0 & 0 & -2\mathbf{i} & -1 & -\mathbf{i} & 1 & 0 & 0 \\ -1 & 2\mathbf{i} & 0 & 0 & 0 & 0 & 7 & \mathbf{i} \\ -\mathbf{i} & 1 & 0 & 0 & 0 & 0 & -\mathbf{i} & 1 \end{bmatrix}.$$

The set of standard right eigenvalues of A is

$$\Lambda_s(A) = \{-1, 1, 2, 8\}.$$

Since Ψ_A is Hermitian, from Theorem 1.12, A is Hermitian. Also from Theorem 1.29, the matrix A is unitarily diagonalizable, i.e., $U^H AU = \text{diag}(-1, 1, 2, 8)$, where $U \in M_n(\mathbb{H})$

is an unitary matrix. Since A is Hermitian, so $\Lambda_s(A) = \Lambda_r(A)$. Consider a perturbation matrix

$$\Delta A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon \\ 0 & 0 & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon & \epsilon \end{bmatrix}. \text{ Take } \epsilon = 10^{-3}. \text{ Then } A + \Delta A = \begin{bmatrix} 1 & -\mathbf{i} & -\mathbf{j} & \mathbf{k} \\ \mathbf{i} & 1 & -2\mathbf{k} & \mathbf{j}+\epsilon \\ \mathbf{j} & 2\mathbf{k} & 7+\epsilon & -\mathbf{i}+\epsilon \\ -\mathbf{k}+\epsilon & -\mathbf{j}+\epsilon & \mathbf{i}+\epsilon & 1+\epsilon \end{bmatrix}.$$

Therefore $\|\Delta A\|_2 = 0.0024$ and

$$\Lambda_s(A + \Delta A) = \{1.0003 + 0.0001\mathbf{i}, 2.0005 + 0.0005\mathbf{i}, -0.9997 + 0.0003\mathbf{i}, 8.0009\}.$$

Since the matrix A is unitarily diagonalizable, $K_2(Y) = 1$. Moreover, it is clear that A is also a central closed as well as a normal matrix.

Example 6.30. Define $t_0 := 2\mathbf{i}$, $t_1 := \mathbf{j}$, $t_2 := \mathbf{k}$. Then from [42], the quaternionic Vandermonde matrix is defined as

$$A := \begin{bmatrix} 1 & 1 & 1 \\ t_0 & t_1 & t_2 \\ t_0^2 & t_1^2 & t_2^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 2\mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & -1 & -1 \end{bmatrix}.$$

The complex adjoint matrix of A is

$$\Psi_A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 2\mathbf{i} & 0 & 0 & 0 & 1 & \mathbf{i} \\ -4 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & -1 & \mathbf{i} & -2\mathbf{i} & 0 & 0 \\ 0 & 0 & 0 & -4 & -1 & -1 \end{bmatrix}.$$

The eigenvalues of Ψ_A are $0.8014+1.70007\mathbf{i}$, $0.8014-1.70007\mathbf{i}$, $-0.4552+1.9952\mathbf{i}$, $-0.4552-1.9952\mathbf{i}$, $-0.3462 + 1.0469\mathbf{i}$, $-0.3462 - 1.0469\mathbf{i}$. Hence the complex adjoint matrix Ψ_A is diagonalizable. Then from Theorem 1.12, A is diagonalizable. The set of standard right eigenvalues of A is given as $\Lambda_s(A) = \{0.8014 + 1.70007\mathbf{i}, -0.4552 + 1.9952\mathbf{i}, -0.3462 + 1.0469\mathbf{i}\}$. Then there exists an invertible matrix $Y \in M_n(\mathbb{H})$ such that

$$Y^{-1}AY = \text{diag}(0.8014 + 1.70007\mathbf{i}, -0.4552 + 1.9952\mathbf{i}, -0.3462 + 1.0469\mathbf{i}).$$

Suppose a perturbation matrix

$$\Delta A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \epsilon & 0 & \epsilon \end{bmatrix}. \text{ Take } \epsilon = 10^{-3}. \text{ Then } A + \Delta A = \begin{bmatrix} 1 & 1 & 1 \\ 2\mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4+\epsilon & -1 & -1+\epsilon \end{bmatrix}.$$

Therefore $\|\Delta A\|_2 = 1.4 \times 10^{-3}$ and

$$\Lambda_s(A + \Delta A) = \{0.8016 + 1.7009\mathbf{i},\ -0.4549 + 1.9950\mathbf{i},\ -0.3457 + 1.0470\mathbf{i}\}.$$

The condition number of the invertible matrix Y is $K_2(Y) \geq 1$. Also

$$\min_{\lambda_i \in \Lambda_{\mathbf{s}}(A)} \{|\lambda_i - \mu|\} = \inf_{\substack{\eta_i \in \Lambda_{\mathbf{r}}(A) \\ \xi \in \Lambda_{\mathbf{r}}(A+\Delta A)}} \{|\eta_i - \xi|\} = 0.0002.$$

$K_2(Y)\|\Delta A\|_2 \geq 0.0014.$

CHAPTER 7

CONCLUSIONS AND SCOPE FOR FUTURE WORK

We have derived localization theorems for the left and right eigenvalues of a quaternionic matrix. In particular, we have presented the Gerschgorin, Ostrowski, and Brauer type theorems for the left and right eigenvalues of a quaternionic matrix. Thereafter we have given a sufficient condition for the stability of a continuous-time quaternionic system. Furthermore, we have derived bounds/location of zeros of quaternionic polynomials.

We have developed various properties of a quaternionic matrix pencil. We have derived localization theorems for generalized right eigenvalues of a quaternionic matrix pencil and their applications.

The definitions of the left and right eigenvalues of quaternionic matrix polynomials have proposed. We have given a sufficient condition for the stability of a discrete-time quaternionic system. We have presented bounds for the absolute values of the left and right eigenvalues of quaternionic matrix polynomials and illustrated for the matrix p-norm, where $p = 1, 2, \infty$, and F (Frobenius).

Next, we have developed the concept of perturbation bounds for right eigenvalues/generalized right eigenvalues of a quaternionic matrix/quaternionic matrix pencil. In particular, Bauer-Fike type theorems for right eigenvalues/generalized right eigenvalues of a diagonalizable quaternionic matrix/diagonalizable quaternionic matrix pencil have derived. We have provided a relative perturbation bound for right eigenvalues of an invertible diagonalizable quaternionic matrix. We have discussed perturbation bounds for the zeros of quaternionic polynomials.

Finally, we have given the perturbation theory on matrices and polynomials over the skew field of quaternions. Specifically, the Bauer-Fike type theorems for right eigenvalues/generalized right eigenvalues of a diagonalizable quaternionic matrix/diagonalizable quaternionic matrix pencil have derived. In addition, perturbation bounds for right eigenvalues of a quaternionic matrix are discussed via block diagonal decomposition and Jordan canonical form of a quaternionic matrix. The location of right eigenvalues of a quaternionic matrix and a sufficient condition for the stability of a perturbed quaternionic matrix have given. We have introduced perturbation bounds for zeros of quaternionic polynomials.

In the future work, we extend some of the localization theorems of Chapter 2 for the left and right eigenvalues of quaternionic block matrices. Another open question is investigate modified Gerschgorin and Ostrowski balls for the zeros of quaternionic polynomials by applying the localization theorems (proved in Chapter 2). One can think about the backward error for right eigenvalues and right eigenvector of structured quaternionic matrix polynomials. Bauer-Fike type theorem for quaternionic matrix polynomials is left for future investigation. As we have developed a general framework for quaternionic matrices and quaternionic matrix polynomials. Consequently we can be extended many results from the complex field to the skew field of quaternions.

APPENDIX-A

In this appendix, we find the powers of quaternionic companion matrices. Substituting $t = 2$ in Theorem 2.35, we have the following expressions:

$$C_{p_l}^2 = \begin{array}{c} \\ m-2 \\ 2 \end{array} \!\!\!\begin{array}{c} \quad 2 \qquad m-2 \\ \left[\begin{array}{c|c} 0 & I \\ \hline C & D \end{array}\right] \end{array}, \text{ where } C = \begin{bmatrix} C_{p_l}(m, 1:2) \\ C_{p_l}^2(m, 1:2) \end{bmatrix} = \begin{bmatrix} -q_0 & -q_1 \\ q_{m-1}q_0 & q_{m-1}q_1 - q_0 \end{bmatrix}$$

and

$$D = \begin{bmatrix} C_{p_l}(m, 3:m) \\ C_{p_l}^2(m, 3:m) \end{bmatrix} = \begin{bmatrix} -q_2 & -q_3 & \dots & -q_{m-1} \\ q_{m-1}q_2 - q_1 & q_{m-1}q_3 - q_1 & \dots & (q_{m-1})^2 - q_{m-2} \end{bmatrix}.$$

$$C_{\tilde{p}_l}^2 = \begin{array}{c} \\ m-2 \\ 2 \end{array} \!\!\!\begin{array}{c} \quad 2 \qquad m-2 \\ \left[\begin{array}{c|c} 0 & I \\ \hline C & D \end{array}\right] \end{array}, \text{ where } C = \begin{bmatrix} C_{\tilde{p}_l}(m, 1:2) \\ C_{\tilde{p}_l}^2(m, 1:2) \end{bmatrix} = \begin{bmatrix} -\overline{q_0} & -\overline{q_1} \\ \overline{q_{m-1}}\;\overline{q_0} & \overline{q_{m-1}}\;\overline{q_1} - \overline{q_0} \end{bmatrix}$$

and

$$D = \begin{bmatrix} C_{\tilde{p}_l}(m, 3:m) \\ C_{\tilde{p}_l}^2(m, 3:m) \end{bmatrix} = \begin{bmatrix} -\overline{q_2} & -\overline{q_3} & \dots & -\overline{q_{m-1}} \\ \overline{q_{m-1}}\;\overline{q_2} - \overline{q_1} & \overline{q_{m-1}}\;\overline{q_3} - \overline{q_1} & \dots & (\overline{q_{m-1}})^2 - \overline{q_{m-2}} \end{bmatrix}.$$

$$C_{q_l}^2 = \begin{array}{c} \\ m-2 \\ 2 \end{array} \!\!\!\begin{array}{c} \quad 2 \qquad m-2 \\ \left[\begin{array}{c|c} 0 & I \\ \hline C & D \end{array}\right] \end{array}, \text{ where } C = \begin{bmatrix} -q_0^{-1} & -q_0^{-1}q_{m-1} \\ q_0^{-1}q_1q_0^{-1} & q_0^{-1}q_1q_0^{-1}q_{m-1} - q_0^{-1} \end{bmatrix}$$

and

$$D = \begin{bmatrix} -q_0^{-1}q_{m-2} & \dots & -q_0^{-1}q_1 \\ q_0^{-1}q_1q_0^{-1}q_{m-2} - q_0^{-1}q_{m-1} & \dots & (q_0^{-1}q_1)^2 - q_0^{-1}q_2 \end{bmatrix}.$$

$$C_{\tilde{q}_l}^2 = \begin{array}{c} \\ m-2 \\ 2 \end{array} \!\!\!\begin{array}{c} \quad 2 \qquad m-2 \\ \left[\begin{array}{c|c} 0 & I \\ \hline C & D \end{array}\right] \end{array}, \text{ where } C = \begin{bmatrix} -\overline{q_0^{-1}} & -\overline{q_0^{-1}q_{m-1}} \\ \\ \overline{q_0^{-1}q_1}\;\overline{q_0^{-1}} & \overline{q_0^{-1}q_1}\;\overline{q_0^{-1}q_{m-1}} - \overline{q_0^{-1}} \end{bmatrix}$$

and

$$D = \begin{bmatrix} -\overline{q_0^{-1} q_{m-2}} & \dots & -\overline{q_0^{-1} q_1} \\ \\ \overline{q_0^{-1} q_1} \;\; \overline{q_0^{-1} q_{m-2}} - \overline{q_0^{-1} q_{m-1}} & \dots & \left(\overline{q_0^{-1} q_1}\right)^2 - \overline{q_0^{-1} q_2} \end{bmatrix}.$$

By Theorem 2.37 for $t = 2$, we obtain the following expressions:

$$C_{p_r}^2 = \begin{array}{c} \\ 2 \\ m-2 \end{array} \begin{array}{c} \begin{array}{cc} m-2 & 2 \end{array} \\ \left[\begin{array}{c|c} 0 & C \\ \hline I & D \end{array}\right] \end{array},$$

where

$$C = \left[C_{p_r}(1:2,m) \quad C_{p_r}^2(1:2,m)\right] = \begin{bmatrix} -q_0 & q_0 q_{m-1} \\ -q_1 & q_1 q_{m-1} - q_0 \end{bmatrix}$$

and

$$D = \left[C_{p_r}(3:m,m) \quad C_{p_r}^2(3:m,m)\right] = \begin{bmatrix} -q_2 & q_2 q_{m-1} - q_1 \\ -q_3 & q_3 q_{m-1} - q_2 \\ \vdots & \vdots \\ -q_{m-1} & (q_{m-1})^2 - q_{m-2} \end{bmatrix}.$$

$$C_{\tilde{p_r}}^2 = \begin{array}{c} \\ 2 \\ m-2 \end{array} \begin{array}{c} \begin{array}{cc} m-2 & 2 \end{array} \\ \left[\begin{array}{c|c} 0 & C \\ \hline I & D \end{array}\right] \end{array}, \text{where } C = \begin{bmatrix} -\overline{q_0} & \overline{q_0}\,\overline{q_{m-1}} \\ -\overline{q_1} & \overline{q_1}\,\overline{q_{m-1}} - \overline{q_0} \end{bmatrix} \text{ and}$$

$$D = \begin{bmatrix} -\overline{q_2} & \overline{q_2}\,\overline{q_{m-1}} - \overline{q_1} \\ -\overline{q_3} & \overline{q_3}\,\overline{q_{m-1}} - \overline{q_2} \\ \vdots & \vdots \\ -\overline{q_{m-1}} & (\overline{q_{m-1}})^2 - \overline{q_{m-2}} \end{bmatrix}.$$

$$C_{q_r}^2 = \begin{array}{c} \\ 2 \\ m-2 \end{array} \begin{array}{c} \begin{array}{cc} m-2 & 2 \end{array} \\ \left[\begin{array}{c|c} 0 & C \\ \hline I & D \end{array}\right] \end{array}, \text{ where } C = \begin{bmatrix} -q_0^{-1} & q_0^{-1}\, q_1 q_0^{-1} \\ -q_{m-1} q_0^{-1} & q_{m-1} q_0^{-1} q_1 q_0^{-1} - q_0^{-1} \end{bmatrix} \text{ and}$$

$$D = \begin{bmatrix} -q_{m-2} q_0^{-1} & q_{m-2} q_0^{-1} q_1 q_0^{-1} - q_{m-1} q_0^{-1} \\ \vdots & \vdots \\ -q_1 q_0^{-1} & (q_1 q_0^{-1})^2 - q_2 q_0^{-1} \end{bmatrix}.$$

$$C_{\tilde{q}_r}^2 = \begin{array}{c} \\ 2 \\ m-2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} m-2 & 2 \end{array} \\ \left[\begin{array}{c|c} 0 & C \\ \hline I & D \end{array}\right] \end{array}, \text{ where } C = \begin{bmatrix} -\overline{q_0^{-1}} & \overline{q_0^{-1}}\ \overline{q_1 q_0^{-1}} \\ & \\ -\overline{q_{m-1}q_0^{-1}} & \overline{q_{m-1}q_0^{-1}}\ \overline{q_1 q_0^{-1}} - \overline{q_0^{-1}} \end{bmatrix} \text{ and}$$

$$D = \begin{bmatrix} -\overline{q_{m-2}q_0^{-1}} & \overline{q_{m-2}q_0^{-1}}\ \overline{q_1 q_0^{-1}} - \overline{q_{m-1}q_0^{-1}} \\ \vdots & \vdots \\ -\overline{q_1 q_0^{-1}} & \left(\overline{q_1 q_0^{-1}}\right)^2 - \overline{q_2 q_0^{-1}} \end{bmatrix}.$$

APPENDIX-B

In this appendix, we find the powers of the quaternionic block companion matrices C_V and C_S. Hence from Theorem 5.31 for $t = 2$, we have the following expressions:

$$C_V^2 = \begin{array}{c} \\ {\scriptstyle n(m-2)} \\ {\scriptstyle 2n} \end{array} \begin{array}{c} \begin{array}{cc} {\scriptstyle 2n} & {\scriptstyle n(m-2)} \end{array} \\ \left[\begin{array}{c|c} 0 & I \\ \hline C & D \end{array}\right] \end{array}, \text{where } C = \begin{bmatrix} C_V[m, 1:t] \\ C_V^2[m, 1:t] \end{bmatrix} = \begin{bmatrix} -V_0 & -V_1 \\ V_{m-1}V_0 & V_{m-1}V_1 - V_0 \end{bmatrix}$$

$$\text{and } D = \begin{bmatrix} C_V[m, t+1:m] \\ C_V^2[m, t+1:m] \end{bmatrix} = \begin{bmatrix} -V_2 & -V_2 & \dots & V_{m-1} \\ V_{m-1}V_2 - V_1 & V_{m-1}V_3 - V_2 & \dots & V_{m-1}^2 - V_{m-1} \end{bmatrix}.$$

Also we have

$$C_S^2 = \begin{array}{c} \\ {\scriptstyle n(m-2)} \\ {\scriptstyle 2n} \end{array} \begin{array}{c} \begin{array}{cc} {\scriptstyle 2n} & {\scriptstyle n(m-2)} \end{array} \\ \left[\begin{array}{c|c} 0 & I \\ \hline C & D \end{array}\right] \end{array}, \text{where } C = \begin{bmatrix} C_S[m, 1:t] \\ C_S^2[m, 1:t] \end{bmatrix} = \begin{bmatrix} -S_m & -S_{m-1} \\ S_1S_m & S_1S_{m-1} - S_m \end{bmatrix}$$

$$\text{and } D = \begin{bmatrix} C_S[m, t+1:m] \\ C_S^2[m, t+1:m] \end{bmatrix} = \begin{bmatrix} -S_{m-2} & -S_{m-3} & \dots & -S_1 \\ S_1S_{m-2} - S_{m-1} & S_1S_{m-3} - S_{m-2} & \dots & S_1^2 - S_2 \end{bmatrix}.$$

Now the powers of C_V and C_S are given as follows.

$$C_V^2 := \begin{bmatrix} 0 & 0 & I_n & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & I_n \\ -V_0 & -V_1 & -V_2 & \dots & -V_{m-1} \\ U_0 & U_1 & U_2 & \dots & U_{m-1} \end{bmatrix} \text{ and } C_S^2 := \begin{bmatrix} 0 & 0 & I_n & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & I_n \\ -S_m & -S_{m-1} & -S_{m-2} & \dots & -S_1 \\ \mathcal{L}_m & \mathcal{L}_{m-1} & \mathcal{L}_{m-2} & \dots & \mathcal{L}_1 \end{bmatrix},$$

where $U_j = V_{m-1}V_j - V_{j-1}$ and $\mathcal{L}_{j+1} = S_1S_{j+1} - S_{j+2}, \quad j = 0, 1, \dots, m-1$ with $V_{-1} = S_{m+1} = 0$. Define

$$U := \begin{bmatrix} U_0 & U_1 & \dots & U_{m-1} \end{bmatrix}, \ \mathcal{L} := \begin{bmatrix} \mathcal{L}_m & \mathcal{L}_{m-1} & \dots & \mathcal{L}_1 \end{bmatrix}.$$

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