To Investigate Number of Zeros of a Quaternion Polynomial Inside a Given Region Over The Skew Field

M.Sc. Thesis

By

Suraj Bhan

DISCIPLINE OF MATHEMATICS

INDIAN INSTITUTE OF TECHNOLOGY INDORE

December 2017

To Investigate Number of Zeros of a Quaternion Polynomial Inside a Given Region Over The Skew Field

A THESIS

Submitted in partial fulfillment of the requirements for the award of the degree

of

Master of Science

by

Suraj Bhan

DISCIPLINE OF MATHEMATICS

INDIAN INSTITUTE OF TECHNOLOGY INDORE

December 2017

INDIAN INSTITUTE OF TECHNOLOGY INDORE

CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **To Investigate Number of Zeros of a Quaternion Polynomial Inside a Given Region Over The Skew Field** in the partial fulfillment of the requirements for the award of the degree of **MASTER OF SCIENCE** and submitted in the **DISCIPLINE OF MATHEMATICS Indian Institute of Technology Indore**, is an authentic record of my own work carried out during the time period from **June 2015 to December 2017** under the supervision of **Dr Sk. Safique Ahmad, Associate Professor, Discipline of Mathematics, IIT Indore.**

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other institute.

Signature of the student with date
Suraj Bhan

This is to certify that the above statement made by the candidate is correct to the best of my/our knowledge.

Signature of the Supervisor of
M.Sc. thesis #1 (with date)
S.k. Safique Ahmad

Suraj Bhan has successfully given his/her M.Sc. Oral Examination held on 26\11\2017.

Signature(s) of Supervisor(s) of MSc thesis
Date:

Convener, DPGC
Date:

Signature of PSPC Member #1
Date:

Signature of PSPC Member #1
Date:

ACKNOWLEDGEMENT

I owe my gratitude to the Discipline of Mathematics, IIT Indore for allowing me to undertake the M.Sc. research project work.

I am very much indebted to my thesis supervisor Dr. Sk. Safique Ahmad, Associate Professor of Discipline of Mathematics, IIT Indore for his valuable guidance and supervision. I express my deepest sense of gratitude and obligation for his restless effort in every possible way for my project.

I would like to thank Prince Kanhya, Research Scholar, IIT Indore who always helped me throughout the research work. I am also grateful to Dr. Istkhar Ali and Gyan Swaroop Nag, Research Scholars, IIT Indore for their valuable suggestions.

I would like to thank my PSPC member Dr. Niraj Shukla, Assistant Professor, Discipline of Mathematics and Dr. M. Tanveer, Assistant Professor, Discipline of Mathematics, IIT Indore for their valuable comments and suggestions at various stages of my research work.

A big thank all my classmates for supporting me.

Abstract

This work deal with the behaviour of quaternion function, like continuity, differentiability and the necessary condition for differentiability. We are discussing some important behaviour of quaternion polynomials and trying to convert Rouche's Theorem of complex version into quaternion form. For this, we are trying to convert Argument Theorem of complex form into quaternion form. To prove Argument Theorem we are trying to prove Cauchy-Gaursat Theorem into quaternion form. We are trying to find Necessary-Condition for differentiability into quaternion form of a given quaternion function. We are trying to establish the quaternion version of Green's theorem.

LIST OF TABLES

Chapter – 1

1 Introduction

First, we are trying to understand quaternion :

1.1 Quaternion:

In mathematics, when we extend a complex number in four-dimensional space we get a quaternion number. There was an Irish mathematician William Rowan Hamilton in 1843[7][8] who described quaternion numbers. At that time he was applying it to mechanics in three-dimensional space. A quaternions have a special property that is, multiplication of two quaternion numbers are noncommutative. At that time, Hamilton defined a quaternion in the form of the quotient of two directed lines in the space of three-dimensional or in other words it is defined as the quotient of two vectors in three-dimensional space. Quaternions are represented in the form: $a + bi + cj + dk$, where a, b, c, and d are real numbers, and i, j and k are the fundamental quaternion units. We can defined it in the following way:

Definition 1.1 Quaternion [7]
Let $\mathbb{H}$ be a four dimensional vector space over $\mathbb{R}$. A real quaternion, simply called quaternion, is a vector denoted by η(say) and is of the form $\eta = a_0 + a_1 i + a_2 j + a_3 k \in \mathbb{H}$ with real coefficients a_0, a_1, a_2 and a_3. Here a_0 is called real part of η and remaining part is called imaginary part of η and satisfying the following properties:
$i^2 = j^2 = k^2 = -1 = i.j.k$
$i.j = k, \qquad j.i = -k$
$j.k = i, \qquad k.j = -i$
$k.i = j, \qquad i.k = -j$
If real part is zero in the quaternion number then it is called pure quaternion.
It is called the Division ring or Skew field. We can defined it in the following way:

Definition 1.2 Skew field

It is also called division ring. In division ring all the properties of the field satisfied except for commutativity. In this field inverse exists for all the non zero elements.

Definition 1.3 Division ring

A division ring is a set D equipped with two binary operations, $(+)$ and $(.)$, satisfying the following three set of axioms called the ring axioms:

1. $a + b = b + a$, $\forall a, b \in D$ *(that is, + is commutative).*

2. $\exists 0 \in D$ *such that* $a + 0 = a$, $\forall a \in D$ *(that is,* 0 *is the additive identity).*

3. $\forall a \in D$ *there exists* $-a \in D$ *such that* $a + (-a) = 0$ *(that is,* $-a$ *is the additive inverse of* a*).*

4. $(a + b) + c = a + (b + c)$, $\forall a, b, c \in D$ *(that is, + is associative).*

5. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$, $\forall a, b, c \in R$ *(that is,* $\cdot$ *is associative).*

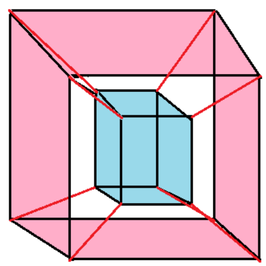
6. $\exists 1 \in D$ *such that* $a.1 = a$ *and* $1.a = a$, $\forall a \in D$ *(that is,* 1 *is the multiplicative identity).*

7. $\forall (a \neq 0) \in D$, $\exists b \in D$ *such that* $a.b = 1$

8. *Multiplication is distributive with respect to addition:* $a \cdot (b+c) = (a \cdot b) + (a \cdot c)$, $\forall a, b, c \in D$ *(left distributivity).*
   $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$, $\forall a, b, c \in D$ *(right distributivity).*
   Note:
   It is not necessary $a.b = b.a$, $\forall\ a, b \in D$

In present time, generally, quaternions form a four-dimensional associative normed division algebra over the real numbers, and therefore this is also a domain. In fact, the first noncommutative division algebra to be discovered was quaternions. Generally, the algebra of quaternions is denoted by $\mathbb{H}$ (for Hamilton), The algebra $\mathbb{H}$ holds a special place in the analysis since, according to the Frobenius theorem[13], it is one of only two finite-dimensional division rings containing the real numbers as a proper subring, the other being the complex numbers. These rings are also Euclidean Hurwitz algebras [14], of which quaternions are the largest associative algebra. Since we are living in three-dimensional space, so we can not easily see the figure of four dimensional space in two-dimensional blackboard. because we can not see this fourth dimensional.

But when we think in a mathematical way then we can find that surely it exists. Generally, we use this fourth-dimension as a time dimension. For example, when we see any flying aeroplane in the sky, then after some times like half an hour, we can not find the exact location of the moving aeroplane because at the same time our earth also moves. So here we can see the importance of this time dimension as fourth-dimensional space. We can undrestand a fourth-dimensional figure in 3-D rotation as: So rotation is important to understand quaternion. This figure is made of two cubes. We can see this figure as:



In fourth chapter, we will study about fourth-dimensional figure.

1.2 History:

In 1843 Hamilton developed the theory of quaternions. Rotation is important to understand quaternion. With the help of Euler angle (1748) and Olinde Rodrigues' parameterization of general rotations by four parameters (1840), they are trying to developed it, but neither of these writers treated the four-parameter rotations as an algebra[9]. Carl Friedrich Gauss had also discovered quaternions in 1819, but this work was not published until 1900 [10]. Hamilton tried to extend complex number in three-dimensional space. He knew that the complex numbers could be interpreted as points in a plane, and he was looking for a way to do the same for points in three-dimensional space. Since with the help of coordinates a point in space can be searched. In three dimensional a point have coordinates, which are triples of numbers, and for many years he had known how to add and subtract triples of numbers. However, Hamilton had been stuck on the problem of multiplication and division for a long time. He could not figure out how to calculate the quotient of the coordinates of two points in space. The great discovered in quaternions finally came on Monday 16 October 1843 in Dublin, when he was going to preside at a council meeting to the Royal Irish Academy. As he walked with his wife along the towpath of the Royal Canal, then at that time the concepts behind quaternions were taking shape in his mind. When the answer dawned on him, Hamilton could not resist the urge to carve the formula for the quaternions, $i^2 = j^2 = k^2 = ijk = -1$, [15] into the stone of Brougham Bridge as he paused on it. On the following day, Hamilton wrote a letter to his friend and fellow mathematician, John T. Graves, describing the train of thought that led to his discovery. This letter was later published in the London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science, vol. In the letter, Hamilton states, And here there dawned on me the notion that we must admit, in some sense, the fourth dimension of space for calculating with triples ... An electric circuit seemed to close, and a spark flashed forth. Hamilton called a quadruple with these rules of multiplication a quaternion, and he devoted most of the remainder of his life to studying and teaching them. Hamilton's treatment is more geometric than the modern approach, which emphasizes quaternions' algebraic properties. He founded a school of "quaternionists", and he tried to popularize quaternions in several books. The last and longest of his books, Elements of Quaternions, was 800 pages long; it was edited by his son William Edwin Hamilton and published shortly after his death. After Hamilton's death, his student Peter Tait continued promoting quaternions. At this time, quaternions were a mandatory examination topic in Dublin. Topics in physics and geometry that would now be described using vectors, such as kinematics in space and Maxwell's equations, were described entirely in terms of quaternions. There was even a professional research association, the Quaternion Society, devoted to the study of quaternions and other hypercomplex number systems. From the mid-1880s, quaternions began to be displaced by vector analysis, which had been developed by Josiah Willard Gibbs, Oliver Heaviside, and Hermann von Helmholtz. Vector analysis described the same phenomena as quaternions, so it borrowed some ideas and terminology liberally from the literature of quaternions. However, vector analysis was conceptually simpler and notationally cleaner, and eventually quaternions were relegated to a minor role in mathematics and physics. A side-effect of this transition is that Hamilton's work is difficult to comprehend for many modern readers. Hamilton's original definitions are unfamiliar and his writing style was wordy and difficult to understand. However, quaternions have had a revival since the late 20th century, primarily due to their utility in describing spatial rotations. The representations of rotations by quaternions are more compact and quicker to compute than the representations by matrices. Also, unlike Euler angles, they are not susceptible to gimbal lock. For this reason, quaternions are used in computer graphics, computer vision, robotics, control theory, signal processing, attitude control, physics, bioinformatics, molecular dynamics, computer simulations, and orbital mechanics. For example, it is common for the attitude control systems of spacecraft to be commanded in terms of quaternions. Quaternions have received another boost from number theory because of their relationships with the quadratic forms. Since 1989, the Department of Mathematics of the National University of Ireland, Maynooth has organized a pilgrimage, where scientists (including the physicists Murray Gell-Mann in 2002, Steven Weinberg in 2005, and the mathematician Andrew Wiles in 2003) walk from Dunsink Observatory to the Royal Canal bridge. Hamilton's carving is no longer visible.

1.3 Use of Quaternion

Quaternions find uses in both theoretical and applied mathematics, in particular for calculations involving three-dimensional rotations such as in three-dimensional computer graphics, computer vision and crystallographic texture analysis [11]. In practical applications, they can be used alongside other methods, such as Euler angles and rotation matrices, or as an alternative to them, depending on the application. Quaternions are useful as a way of describing many natural phenomena, including Newtonian mechanics, as was Hamiltons intention. Today, quaternions are mainly used to compute three-dimensional rotations for computer graphics.In Space technology and to find location of electrons in the atom, we use quaternion division ring algebra. Quaternion works in 3-D so here Wolfram—Alpha provides several representations of the corresponding rotation:

1.4 Aim

Here we want to find number of zeros of a given quaternion polynomial inside the given closed curve in which quaternion polynomial belongs. Number of theories have been developed to find the zeros of a complex polynomial. Here we are working on quaternion division ring algebra. So we are discussing about roots of quaternion polynomial and we want to find the region of a root of a given quaternion polynomial, also, we want to find how many roots of a given polynomial exist in a given region. In complex analysis, Rouche's theorem is helpful to find roots inside the given region of a given complex polynomial. Similarly, we are trying to investigate number of zeros inside a given region of a quaternion polynomial over the skew field. In other words we want to extend Rouche's theorem of complex form into quaternion form.

Chapter – 2

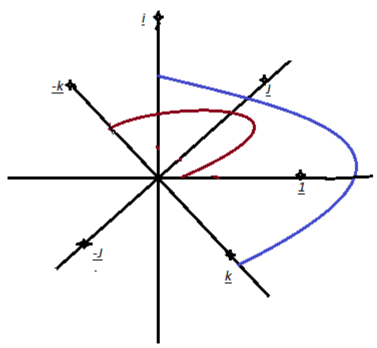
2 Properties of Quaternion

(a): Multiplication of quaternion number is not commutative, We can understand it in the following way:

Multiplication	1	i	j	k
1	1	i	j	k
i	i	-1	k	-j
j	j	-k	-1	i
k	k	j	-i	-1

In the above figure, $i.j \neq j.i$, so it is not commutative.

We can also see graphically multiplication of imaginary units as rotation in the following figure:



(b): $\|\eta\| = \sqrt{\eta.\eta*} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}$, here $\eta*$ denotes the conjugate of η.

Proof: Let $\eta = x_0 + x_1 i + x_2 j + x_3 k$, then $\eta^* = x_0 - x_1 i - x_2 j - x_3 k$

$\eta.\eta^* =(x_0 + x_1 i + x_2 j + x_3 k).(x_0 - x_1 i - x_2 j - x_3 k)$

$= x_0^2 + x_1^2 + x_2^2 + x_3^2 - x_0.x_1 i - x_0.x_2 j - x_0.x_3 k + x_1.x_0 i - x_1 x_2 i.j - x_1 x_3 i.k + x_2.x_0 j - x_2.x_1 ji - x_2 x_3 jk + x_3.x_0 k - x_3 x_1 ki - x_3 x_2 kj = x_0^2 + x_1^2 + x_2^2 + x_3^2$,

this shows that $\|\eta\| \geq 0$

Note: $\eta^{-1} = \frac{\eta*}{\|\eta\|^2}$ denotes the inverse of η, where $\eta \neq 0$

It is a normed vector space because it is satisfying all the properties of norm.

(c): $\|\eta\| \geq 0$**,** $\|\eta\| = 0 \iff \eta = 0$**,** $\forall \eta \in \mathbb{H}$

(d): $\|\eta_1 + \eta_2\| \leq \|\eta_1\| + \|\eta_2\|$**,** $\forall \eta \in \mathbb{H}$

(e): $(\eta_1.\eta_2)^* = (\eta_2)^*(\eta_1)^*$

Proof: Let $\eta_1 = x_0 + x_1 i + x_2 j + x_3 k$ and $\eta_2 = y_0 + y_1 i + y_2 j + y_3 k$

$\eta_1.\eta_2 = (x_0 + x_1i + x_2j + x_3k).\ (y_0 + y_1i + y_2j + y_3k)$

$=(x_0.y_0 + x_0y_1i + x_0y_2j + x_0y_3k + x_1y_0i - x_1y_1 + x_1.y_2ij + x_1y_3ik + x_2y_0j + x_2y_1ji - x_2y_2 + x_2y_3jk + x_3y_0k + x_3y_1ki + x_3y_2kj - x_3y_3)$

then

$(\eta_1.\eta_2)^* =(x_0.y_0 + x_0y_1i + x_0y_2j + x_0y_3k + x_1y_0i - x_1y_1 + x_1.y_2ij + x_1y_3ik + x_2y_0j + x_2y_1ji - x_2y_2 + x_2y_3jk + x_3y_0k + x_3y_1ki + x_3y_2kj - x_3y_3)^* =(x_0.y_0 - x_0y_1i - x_0y_2j - x_0y_3k - x_1y_0i - x_1y_1 - x_1.y_2ij - x_1y_3ik - x_2y_0j - x_2y_1ji - x_2y_2 - x_2y_3jk - x_3y_0k - x_3y_1ki - x_3y_2kj - x_3y_3)$

$\eta_2^*.\eta_1^*=(y_0 - y_1i - y_2j - y_3k).(x_0 - x_1i - x_2j - x_3k)=y_0x_0 - y_1x_0i - y_2x_0j - y_3x_0k - y_0x_1i - y_1x_1 - y_2x_1ij - y_3x_1ik - y_0x_2j - y_1x_2ji - y_2x_2 - y_3x_2jk - y_0x_3k - y_1x_3ki - y_2x_3kj - y_3x_3$

Hence $(\eta_1.\eta_2)^* = (\eta_2)^*(\eta_1)^*$

(f): A quaternion number commutes with its conjugate.

Proof: clearly $\eta = x_0 + x_1i + x_2j + x_3k$ and $\eta* = x_0 - x_1i - x_2j - x_3k$.

Then $\eta.\eta^* =(x_0 + x_1i + x_2j + x_3k).(x_0 - x_1i - x_2j - x_3k) = x_0^2 + x_1^2 + x_2^2 + x_3^2$ $\eta^*.\eta = (x_0 - x_1i - x_2j - x_3k)(x_0 + x_1i + x_2j + x_3k) = x_0^2 + x_1^2 + x_2^2 + x_3^2$

Hence, $\eta.\eta^* = \eta^*.\eta$.

(g): Associativity holds in quaternion division ring.

Proof: Let $\eta_1 = x_0 + x_1i + x_2j + x_3k$, $\eta_2 = y_0 + y_1i + y_2j + y_3k$, and $\eta_3 = z_0 + z_1i + z_2j + z_3k$. $(\eta_1.\eta_2)\eta_3 = ((x_0 + x_1i + x_2j + x_3k).(y_0 + y_1i + y_2j + y_3k))(z_0+z_1i+z_2j+z_3k)= (x_0.y_0+x_0y_1i+x_0y_2j + x_0y_3k + x_1y_0i - x_1y_1 + x_1.y_2ij + x_1y_3ik + x_2y_0j + x_2y_1ji - x_2y_2 + x_2y_3jk + x_3y_0k + x_3y_1ki+x_3y_2kj-x_3y_3)(z_0+z_1i+z_2j+z_3k) = x_0.y_0z_0+x_0y_1iz_0+x_0y_2z_0j+x_0y_3z_0k+x_1y_0iz_0 - x_1y_1z_0 + x_1.y_2z_0ij + x_1y_3z_0ik + x_2y_0z_0j + x_2y_1z_0ji - x_2y_2z_0 + x_2y_3z_0jk + x_3z_0y_0k + x_3y_1z_0ki + x_0.y_0z_1i - x_0y_1z_1 + x_0y_2z_1ji + x_0y_3z_1ki - x_1y_0z_1 - x_1y_1z_1i + x_1.y_2z_1iji + x_1y_3z_1iki + x_2y_0z_1ji - x_2y_1z_1j - x_2y_2z_1i + x_2y_3z_1jki + x_3y_0z_1ki - x_3y_1z_1k + x_3y_2z_1kji - x_3y_3z_1i + x_0.y_0z_2j + x_0y_1z_2ij - x_0y_2z_2 + x_0y_3z_2kj + x_1y_0z_2ij - x_1y_1z_2j - x_1.y_2z_2ij + x_1y_3z_2ikj - x_2y_0z_2j + x_2y_1z_2jij - x_2y_2z_2j + x_2y_3z_2jkj + x_3y_0z_2kj + x_3y_1z_2kij - x_3y_2z_2k - x_3y_3z_2j)x_0.y_0.z_3k + x_0y_1z_3ik + x_0y_2z_3jk - x_0y_3z_3 + x_1y_0z_3ik - x_1y_1z_3 + x_1.y_2z_3ijk - x_1y_3z_3i + x_2y_0z_3jk + x_2y_1z_3jik - x_2y_2z_3k - x_2y_3z_3j - x_3y_0z_3 + x_3y_1z_3kik + x_3y_2z_3kjk - x_3y_3z_3k$

Since $x_i.y_j.z_k = y_j.x_i.z_k = z_k.y_j.z_k$, $\forall\ (i, j, k) = (1, 2, 3, 4)$

$\eta_1.(\eta_2\eta_3) = (x_0.y_0 + x_0y_1i + x_0y_2j + x_0y_3k + x_1y_0i - x_1y_1 + x_1.y_2ij + x_1y_3ik + x_2y_0j + x_2y_1ji - x_2y_2 + x_2y_3jk + x_3y_0k + x_3y_1ki + x_3y_2kj - x_3y_3)(z_0 + z_1i + z_2j + z_3k) = x_0.y_0z_0 + x_0y_1iz_0 + x_0y_2z_0j + x_0y_3z_0k + x_1y_0iz_0 - x_1y_1z_0 + x_1.y_2z_0ij + x_1y_3z_0ik + x_2y_0z_0j + x_2y_1z_0ji - x_2y_2z_0 + x_2y_3z_0jk + x_3z_0y_0k + x_3y_1z_0ki + x_0.y_0z_1i - x_0y_1z_1 + x_0y_2z_1ji + x_0y_3z_1ki - x_1y_0z_1 - x_1y_1z_1i + x_1.y_2z_1iji+x_1y_3z_1iki+x_2y_0z_1ji-x_2y_1z_1j-x_2y_2z_1i+x_2y_3z_1jki+x_3y_0z_1ki-x_3y_1z_1k+x_3y_2z_1kji - x_3y_3z_1i + x_0.y_0z_2j + x_0y_1z_2ij - x_0y_2z_2 + x_0y_3z_2kj + x_1y_0z_2ij - x_1y_1z_2j - x_1.y_2z_2ij + x_1y_3z_2ikj - x_2y_0z_2j+x_2y_1z_2jij-x_2y_2z_2j+x_2y_3z_2jkj+x_3y_0z_2kj+x_3y_1z_2kij-x_3y_2z_2k-x_3y_3z_2j)x_0.y_0.z_3k + x_0y_1z_3ik+x_0y_2z_3jk-x_0y_3z_3+x_1y_0z_3ik-x_1y_1z_3+x_1.y_2z_3ijk-x_1y_3z_3i+x_2y_0z_3jk+x_2y_1z_3jik-$

$x_2y_2z_3k - x_2y_3z_3j - x_3y_0z_3 + x_3y_1z_3kik + x_3y_2z_3kjk - x_3y_3z_3k$

Hence $(\eta_1.\eta_2)\eta_3$=$\eta_1(\eta_2.\eta_3)$.

(h):$\|\eta_1.\eta_2\| = \|\eta_2.\eta_1\|$

Proof: Clearly η commutes with its conjugate. Therefore $\|\eta_1.\eta_2\| = \|\eta_2.\eta_1\|$ **(i):** $\eta^* = \eta \iff \eta \in \mathbb{R}$

proof: $\implies$

Let η^*=η

$\eta = x_0 + x_1i + x_2j + x_3k$ and $\eta^* = x_0 - x_1i - x_2j - x_3k$

Then, $x_0 + x_1i + x_2j + x_3k = x_0 - x_1i - x_2j - x_3k$

$\Rightarrow 2x_1i + 2x_2j + 2x_3k = 0$

$\Rightarrow x_1i + x_2j + x_3k = 0$

$\Rightarrow x_1 = 0, x_2 = 0, x_3 = 0$

$\eta = x_0$, which is a real number so $\eta \in \mathbb{R}$

$\impliedby$

Let $\eta \in \mathbb{R}$, then $x_1 = 0, x_2 = 0, x_3 = 0$

Hence η^*=η

(j): A quaternion number can be uniquely expressed as $A + kB$, where A and B are complex number.

Proof: Let $\eta = a_0 + a_1i + a_2j + a_3k = a_0 + a_1i + a_2ki + a_3k = a_0 + a_1i + k(a_2i + a_3) = A + kB$, where $A = a_0 + a_1i$ and $B = a_3 + a_2i$ both A and B are complex number. **k**:$(\eta_1 + \eta_2)^*$=$\eta_1^* + \eta_2^*$

Proof: Let $\eta_1 = x_0 + x_1i + x_2j + x_3k$ and $\eta_2 = y_0 + y_1i + y_2j + y_3k$, then

$\eta_1 + \eta_2$=$x_0 + y_0 + (x_1 + y_1)i + (x_2 + y_2)j + (x_3 + y_3)k$

$(\eta_1 + \eta_2)^* = x_0 + y_0 - (x_1 + y_1)i - (x_2 + y_2)j - (x_3 + y_3)k$

$= (x_0 - x_1i - x_2j - x_3k) + (y_0 - y_1i - y_2j - y_3k)$

$= {\eta_1}^* + {\eta_2}^*$

(l):$\eta^{**} = \eta$

Let $\eta = x_0 + x_1i + x_2j + x_3k$, then $\eta^* = x_0 - x_1i - x_2j - x_3k$

$\Rightarrow \eta^{**} = x_0 + x_1i + x_2j + x_3k$

$\Rightarrow \eta^{**} = \eta$

(l): $\mathbb{C} \simeq \mathbb{H}$

There is an isomorphism between quaternion division ring and complex field. We can defined, it in the following way:

Since associativity holds in quaternion algebra so they can be considered in term of matrix $f : \mathbb{H} \longrightarrow \mathbb{C}$ such that:

$a+ib+cj+dk \longrightarrow \begin{bmatrix} a+ib & c+di \\ -c+di & a-bi \end{bmatrix} = aI_2+bI+cJ+dK$, where $I_2 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I := \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$,

$J := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $K := \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$, clearly $IJK = -I_2 = I^2 = J^2 = k^2$

Proof: Well define: Let $\eta_1 = \eta_2$

Where, $\eta_1 = x_0 + x_1 i + x_2 j + x_3 k$ and $\eta_2 = y_0 + y_1 i + y_2 j + y_3 k$,

$\implies x_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + x_1 \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} + x_2 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$

$= y_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + y_1 \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} + y_2 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + y_3 \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$

$\implies f(\eta_1){=}f(\eta_2)$

One-One: Let $f(\eta_1) = f(\eta_2)$

then

$x_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + x_1 \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} + x_2 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$

$= y_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + y_1 \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} + y_2 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + y_3 \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$

Then $(x_0 - y_0) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (x_1 - y_1) \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} + (x_2 - y_2) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + (x_3 - y_3) \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = 0$

$\implies (x_0 - y_0 = 0), (x_1 - y_1 = 0), (x_2 - y_2 = 0), (x_3 - y_3 = 0)$

$\implies x_0 = y_0, x_1 = y_1, x_2 = y_2, x_3 = y_3$

$\implies \eta_1 = \eta_2$

Onto: Let $Z = x+iy$ be a complex number then we can write z in matrix form as $\begin{bmatrix} x & -y \\ y & x \end{bmatrix}$ this function(matrics) is a 1-1 correspondence between complex numbers and real numbers. Then $x + i0 - jy + k0$ is the number such that $f(x + i0 - jy + k0) \longrightarrow \begin{bmatrix} x & -y \\ y & x \end{bmatrix}$

Homomorphism: Let $\eta_1 = x_0 + x_1 i + x_2 j + x_3 k$ and $\eta_2 = y_0 + y_1 i + y_2 j + y_3 k$

Then $\eta_1 + \eta_2 = x_0 + y_0 + (x_1 + y_1)i + (x_2 + y_2)j + (x_3 + y_3)k$ and

$f(\eta_1 + \eta_2) = f\{x_0 + y_0 + (x_1 + y_1)i + (x_2 + y_2)j + (x_3 + y_3)k\} = (x_0 + y_0) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + x_1 \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} +$
$(x_2 + y_2) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + (x_3 + y_3) \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = x_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + x_1 \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} + x_2 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} +$

$y_0\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+y_1\begin{bmatrix}i & 0\\ 0 & -i\end{bmatrix}+y_2\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}+y_3\begin{bmatrix}0 & i\\ i & 0\end{bmatrix}=f(\eta_1)+f(\eta_2)$

$f(\eta_1.\eta_2)=f\{(x_0y_0-x_1y_1-x_2y_2-x_3y_3)+i(x_0y_1+x_1y_0+x_2y_3-x_3y_2)+(x_0y_2+x_2y_0+x_3y_1-x_1y_3)j+(x_0y_3+y_3x_0+x_1y_2-x_2y_1)\}k$

$=(x_0y_0-x_1y_1-x_2y_2-x_3y_3)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+(x_0y_1+x_1y_0+x_2y_3-x_3y_2)\begin{bmatrix}i & 0\\ 0 & -i\end{bmatrix}+(x_0y_2+x_2y_0+x_3y_1-x_1y_3)\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}+(x_0y_3+y_3x_0+x_1y_2-x_2y_1)\begin{bmatrix}0 & i\\ i & 0\end{bmatrix}$

$\Longrightarrow f(\eta_1).f(\eta_2)=\{x_0\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+x_1\begin{bmatrix}i & 0\\ 0 & -i\end{bmatrix}+x_2\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}+x_3\begin{bmatrix}0 & i\\ i & 0\end{bmatrix}\}.\{y_0\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+y_1\begin{bmatrix}i & 0\\ 0 & -i\end{bmatrix}+y_2\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}+y_3\begin{bmatrix}0 & i\\ i & 0\end{bmatrix}\}$

$=(x_0y_0-x_1y_1-x_2y_2-x_3y_3)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+(x_0y_1+x_1y_0+x_2y_3-x_3y_2)\begin{bmatrix}i & 0\\ 0 & -i\end{bmatrix}+(x_0y_2+x_2y_0+x_3y_1-x_1y_3)\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}+(x_0y_3+y_3x_0+x_1y_2-x_2y_1)\begin{bmatrix}0 & i\\ i & 0\end{bmatrix}$

$\Longrightarrow f(\eta_1.\eta_2)=f(\eta_1).f(\eta_2)$

Hence there is a 1-1 correspondence between quaternion number and complex number.

We can also defined an isomorphism ϕ such that:

$\phi:(\mathbb{H},+,.)\longrightarrow(M_{4\mathbb{R}},+,.)$ such that

$\phi(a+ib+jc+kd)=\begin{bmatrix}a & -b & -c & -d\\ b & a & -d & c\\ c & d & a & -b\\ a & c & b & a\end{bmatrix}$, where $M_{4\mathbb{R}}$ is the space of 4×4 matrics over the real numbers.

2.1 Basic of quaternion

A quaternion may be represented in the Cartesian form as :

$q=a+ix+jy+kz$, where i,j and k are mutually perpendicular unit vector obeying the multiplication rule defined by Hamilton. We can use a quaternion to do rotations in 3-D.

The quarernion product is the same as the cross product of vectors : $i\times j=k$, $j\times k=i$, $k\times i=j$ except for the cross product $i\times i=j\times j=k\times k=0$, while for quarenion this is -1.

In fact we can think a quaternion as having a scalar (number) and a vector part:

$v_0+v_1i+v_2j+v_3k=(v_0,\mathbf{v})$, where v_0 is scalar number and $\mathbf{v}$ is vector part.

We can use the cross product and the dot product $\mathbf{v}.\mathbf{w}=(v_1w_1+v_2w_2+v_3w_3)$

where $\mathbf{w}=w_0+iw_1+jw_2+kw_3$. To define the product of quaternion yet another way:

If a quaternion is divided up into a scalar part and a vector part, i.e. then the formulas for addition and multiplication are: $(v_0, \mathbf{v})(w_0, \mathbf{w}) = (v_0 w_0 - \mathbf{v}\mathbf{w}, v_0 w + w_0 v + \mathbf{v} \times \mathbf{w})$[12][9]

2.1.1 By Niven's Algorithm [2]

We can apply fundamental theorem of algebra in quaterenion form:

1. Every quaternion polynomial has at least one zero in $\mathbb{H}$.
2. If the number of zeros of a polynomial of degree m is finite then there are exactly m zeros counting possible repetition.

Hence from above two important points we shall try to prove the following statement: We are trying to prove the following statement.

Let $f(\eta)$ and $g(\eta)$ be two analytic functions inside and on a simple closed curve H. If $|f(\eta)| > |g(\eta)|$ on H and also $f(\eta)$ and $g(\eta)$ both have finite number of zeros then $f(\eta)$ and $f(\eta) + g(\eta)$ both have the same number of zeros inside H.

2.2 Some important definitions and theorems in quaternion division ring algebra which will help to reach our aim

Definition 2.1 Closed curve in $\mathbb{H}$
If $\phi_1(t), \phi_2(t)$ and $\phi_3(t)$ are real valued functions of the real variable t is assumed to be continuous in $t_1 \le t \le t_2$. Then the parametric equation
$\eta = x_1 + x_2 i + x_3 j + x_4 k = \phi_1(t) + i\phi_2(t) + j\phi_3(t) + k\phi_4(t)$
defined a continuous curve on an arc in the quaternion plane joining the point $a = \eta(t_1)$ and $b = \eta(t_2)$.

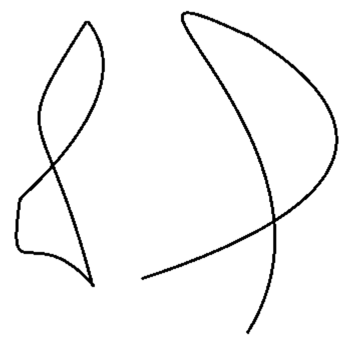
Definition 2.2 Connected Set[4]
An open set S is said to be connected if any two points of S can be joined by a path consisting of straight line segments all points of which are in S.
In other words, A connected set is a set that can not be partitioned into two nonempty subsets which are open in the relative topology induced on the set.
Equivalently, it is a set which can not be partitioned into two nonempty subsets such that each subset has no points in common with the set closure of the other.

Definition 2.3 *[4] A connected set is called domain.*

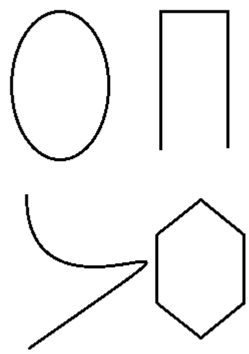
Definition 2.4 Simple Closed Curve and Open Curve

A connected curve that does not cross itself and ends at the same point where it begins. Examples are circles, ellipse, and polygons. In a simple closed curves the shapes are closed by line-segments or by a curved line. Triangle, quadrilateral, circle, etc., are examples of closed curves The shape which is not closed by line-segments or a curve is called an open curve. A closed curve which does not cross itself is called a simple closed curve. Here we are giving some figures which is helpful to understand these definitions.

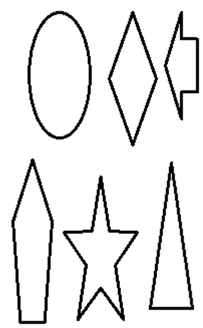
Non Simple Curves:



Simple Curves:



Closed Curves:



Meromorphic Function:

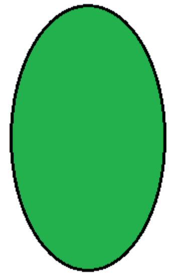
A function is meromorphic on a domain D iff f is analytic in D except at finitely many poles.

Definition 2.5 Simple connected region

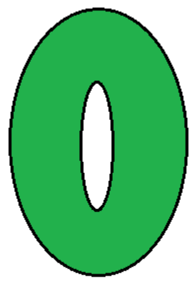
A region R is called simply connected if any simple closed curve which lies in R can be shrunk to a point without leaving R

A region R which is not simply connected is called multiple connected. In other words, A simply connected domain is a path-connected domain where one can continuously shrink any simple closed curve into a point while remaining in the domain.

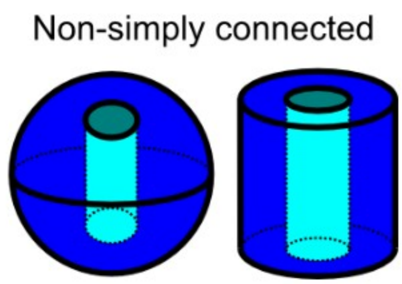
Simple Connected Region:



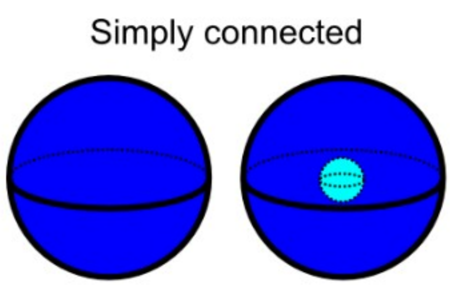
Multiple Connected Region:



For three dimensional domains, the concept of simply connected domain is subtle, however the following figure is helpful to understand it.



(Here the source of figure is from google)

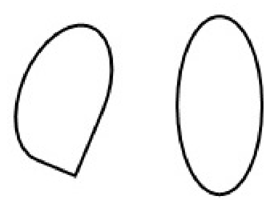


For more than thre dimensions the concept of simply connected domain is more subtle, still We know a simply connected domain is one without holes going all the way through it. Simply, a domain with just hole in the middle(like a ball whose center is hollow) is a simple connected, as we can continuously shrunk any closed curve to a point by going around the hole and remaining in the middle. On the other hand, a ball with a hole drilled all the way through it, or a spool with a hollow central axis is not simply connected

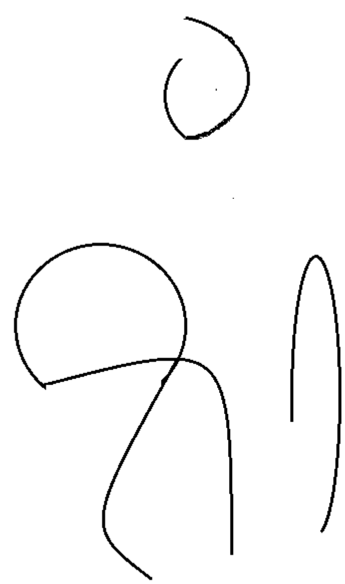
Definition 2.6 Jorden curve[4]
Any continuous closed curve which does not intersect itself and which may or may not have a finite length is called a Jordan curve.

We can undrstand it from the following figure:

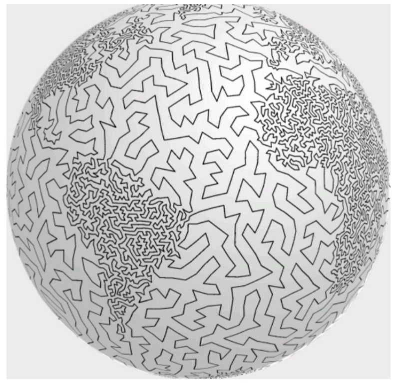
Jordan Curves:



Non-Jordan Curves:



In topology, a Jordan curve is a non-self-intersecting continuous loop in the plane, and another name for a Jordan curve is a plane simple closed curve. Another example of Jordan curve on three- dimensional figure is the following: (Here the source of figure is from google.)



Here curves on sphere are simply closed. We can obtain Jordan curve on more than three dimensions.

2.3 Jordan-Curve Theorem[4]

A Jordan curve divides the plane into two regions having the curve as a common boundary. That region which is bounded is called the interior or inside of the curve, while the other region is called the exterior or outside of the curve. Using the Jordan curve theorem, it can be shown that the region inside a simple closed curve is a simply connected region whose boundary is the simple closed curve.

2.4 Cauchy-Riemann Equation in complex form

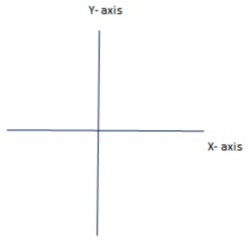
[4]. Let f be a function such that

$f : D \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be differentiable function Then if complex number z be defined in this way $z = x + iy$ and then let $f(z) = u(x, y) + iv(x, y)$, Here u and v both are differentiable function because f is differentiable. then Cauchy Riemann equation in complex form is defined as:

$\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0$

Proof: Since f is differentiable so limit coming from all direction should be same:

Limit along x-axis is defined as:



$f'(z) = \lim_{h \to 0} \dfrac{u(x+h,y) - u(x,y)}{h} + i \lim_{h \to 0} \dfrac{v(x+h,y) - v(x,y)}{h} = \dfrac{\partial u}{\partial x} + i \dfrac{\partial v}{\partial x}$
Limit along y-axis is defined as:
$f'(z) = \lim_{h \to 0} \dfrac{u(x,y+h) - u(x,y)}{ih} + i \lim_{h \to 0} \dfrac{v(x,y+h) - v(x,y)}{ih} = \dfrac{1}{i} \dfrac{\partial u}{\partial y} + \frac{i}{i} \frac{\partial v}{\partial y}$
$= -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$

Limit should be same from all direction so $\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$

On comparison real and imaginary part we will get

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y}$ = -$\frac{\partial v}{\partial x}$ Since $f(z) = u(x,y) + iv(x,y)$,
$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$ and $\frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y}$, then
$\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} + i(\frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y})$
$= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} + i \frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}$
$= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} - i \frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}$
$= 0$

2.5 Cauchy-Riemann-Fueter equation[1]

In a simialar way, let f be a function such that

$f : D \subseteq \mathbb{H} \longrightarrow \mathbb{H}$ be differentiable function, then if quaternion number η be defined in this way $\eta = x_0 + ix_1 + jx_2 + kx_3$ and then let $f(\eta) = u_0(x_0, x_1, x_2, x_3) + iu_1(x_0, x_1, x_2, x_3) + ju_1(x_0, x_1, x_2, x_3) + ku_1(x_0, x_1, x_2, x_3)$, Here u_0, u_1, u_2 and u_3 all are differentiable function because f is differentiable, then Cauchy-Riemann Equation in quaternion form is defined as:
$\frac{\partial f}{\partial x_0} + i \frac{\partial f}{\partial x_1} + j \frac{\partial f}{\partial x_2} + k \frac{\partial f}{\partial x_3} = 0$

2.6 Cauchy Integral formula in quaternion division algebra [1]

If the function is continuously differentiable and satisfied equation number 2 then Gauss's theorem can be used to show that

$$\oint_{\partial c} D_q f = 0,$$

where C is any smooth closed 3-manifold in H and that if q_0 lies inside C then $f(q_0) = \frac{1}{2\pi^2} \oint_{\partial c} \frac{(q-q_0)^(-1)}{(|q-q_0|)^2} D_q f(q)$.

2.7 Integration in Quaternion Division Algebra

The boundary H of a region is said to be traversed in the positive sense or direction if an observer traveling in this direction has the region to the left. This convention leads to the direction indicates by an arrow.

we use the symbol

$$\oint_H f(\eta)$$

to denote integration of $f(\eta)$ around the boundary H in the positive sense. The integral around H may be called contour integral.

2.8 Argument Theorem in complex form [4]

Let $f(\eta)$ be analytic inside and on a simple closed curve C except for a finite number of poles inside C then

$$N - P = \frac{1}{2\pi i} \oint_c \frac{f'(z)}{f(z)} dz$$

where N and P are respectively number of zeros and poles of $f(z)$ inside C.

2.9 Cauchy Integral formula in complex form[4]

Let $f(z)$ is analytic inside and on the boundary C of a simple connected region R, then $f(a) = \frac{1}{2\pi i} \oint_c \frac{f(z)}{(z-a)} dz$, where $a \in C$.

2.10 Green's Theorem

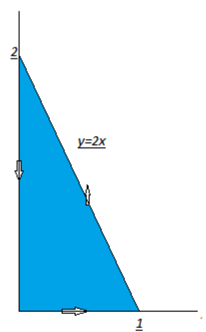
In mathematical, Green's theorem gives the relationship between a line integral around a simple closed curve C and a double integral over the plane D bounded by C.

2.10.1 Green's Theorem in the plane

Let C be a positively oriented, piecewise smooth, simple, closed curve and let D be the region enclosed by the curve. If P and Q have continuous first order partial derivatives on D then,
$\int_c Pdx + Qdy = \iint_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dA$

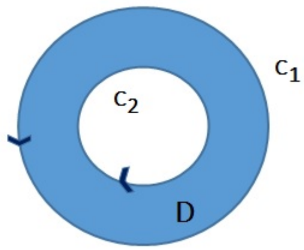
Example 2.7 *Use Green's Theorem to evaluate* $\oint xydx + x^2y^3dy$ *where* C *is the triangle with vertices* $(0,0),(1,0),(1,2)$ *with positive orientation.*

Solution: First, we are describing figure in such a way that the condition for C and D in Green's theorem will be true



Clearly curve C satisfied Green,s theorem condition.
Here $0 \leq x \leq 1$ and $0 \leq y \leq 2x$
Here $P = xy$ and $Q = x^2y^3$ So using Green's Theorem the line integral becomes, $\oint xydx + x^2y^3dy = \iint_D (2xy^3 - x)dA = \int_0^1 \int_0^{2x} (2xy^3 - x)dydx$
$= \int_0^1 (xy^4 - xy)_0^{2x}$
$=\int_0^1 (8x^5 - 2x^2)dx$
$=(4x^6/3 - 2/3x^3)_0^1$
$=2/3$

Green's Theorem directly does not work in the region which has the hole. But still, we can apply Green's Theorem on the region which have hole. For this let us take an example suppose a region is as the following figure:



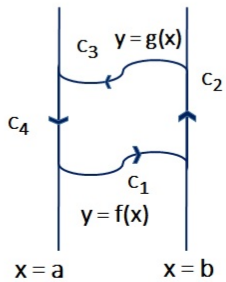
Clearly this region has a hole. Now to calculate line integral around D as showing in figure by yellow colour.

For this first we apply Green's Theorem for C_1 after that We apply Green's Theorem for C_2 curve after that $\oint_{c_1} - \oint_{c_2}$ will be the required answer. Since it can not apply directly to multiple connected region, so in statement of Rouche's Theorem there is simple closed curve, because with the help of Green's Theorem we prove Cauchy's-Gaursat theorem.

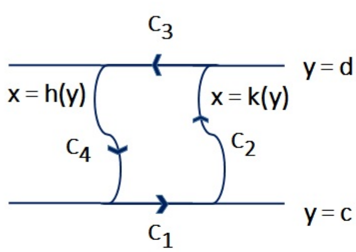
Proof of Green's Theorem:

the following is a proof of half of the theorem for the simplified area D, a type I region where C_1 and C_3 are curves connected by vertical lines (possibly of zero length). A similar proof exists for the other half of the theorem when D is a type II region where C_2 and C_4 are curves connected by horizontal lines (again, possibly of zero length). Putting these two parts together, the theorem is thus proven for regions of type III (defined as regions which are both type I, and type II). The general case can then be deduced from this special case by decomposing D into a set of type III regions.

The following figure is of type I region



The following figure is of type II region.



When D is of I region, then we shall show:

$\oint_C P\,dx = \iint_D \left(-\frac{\partial P}{\partial y}\right)\,dA \qquad (1)$

When D is of II region then we shall show:

$\oint_C Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x}\right)\,dA \qquad$ (2) after prove 1 and 2, then Green's theorem follows immediately for the region D. We can prove (1) easily for regions of type I, and (2) for regions of type II. Green's theorem then follows for regions of type III.

Assume region D is a type I region and can thus be characterized, as pictured on the right, by i.e.

If $D = \{(x,y)|a \le x \le b\}$, $f(x) \le y \le g(x)$ with $f(x), g(x)$ continuous on $a \le x \le b$, we have $C = C_1 + C_2 + C_3 + C_4$ where C_1, C_2, C_3 and C_4 are as shown in figure of type I, Since the region is of type I, we have

$$\iint_D \frac{\partial P}{\partial y}\, dA = \int_a^b \int_{f(x)}^{g(x)} \frac{\partial P}{\partial y}(x,y)\, dy\, dx$$
$$= \int_a^b \Big\{P(x,g(x)) - P(x,f(x))\Big\}\, dx. \qquad (3)$$

Now compute the line integral in (1). C can be rewritten as the union of four curves: C_1, C_2, C_3, C_4.

With C_1, use the parametric equations: $x = x$, $y = f(x)$, $a \le x \le b$. Then

$$\int_{C_1} P(x,y)\, dx = \int_a^b P(x,f(x))\, dx,$$

and $\int_a^b P(x,g(x)\, dx = -\int_{C_3} P(x,y))\, dx$.

We thus obtain

$\iint_D \frac{\partial P}{\partial y} dxdy = \int_{C_1} Pdx + \int_{C_3} Pdx = \int_C Pdx$

Since the integration of Pdx is zero on C_2 and C_4 as x is constant there.

If D is a region of type II then

If $D = \{(x,y)|h(y) \le x \le k(y)\}$, $c \le y \le d$ with $h(y), k(y)$ continuous on $c \le y \le d$, we have $C = C_1 + C_2 + C_3 + C_4$ where C_1, C_2, C_3, C_4 are as shown in figure of type II, Since the region is of type II, we have

$$\iint_D \frac{\partial Q}{\partial x}\, dA = \int_c^d \int_{h(y)}^{k(y)} \frac{\partial Q}{\partial x}(x,y)\, dy\, dx$$
$$= \int_c^d \Big\{Q(k(y),y) - Q(h(y),y))\Big\}\, dy. \qquad (3)$$

Using the standard parametrizations of C_2 and C_4, we have

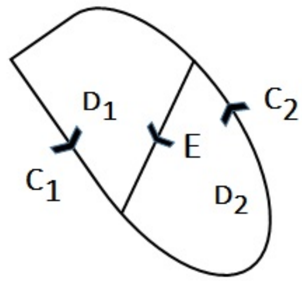
$\int_c^d Q(k(y),y)dy = \int_{c_2} Qdy$, and $\int_c^d Q(h(y),y)dx = -\int_{c_4} Qdy$

We thus obtain $\iint_D \frac{\partial Q}{\partial x} dxdy = \int_{C_2} Qdy + \int_{C_4} Qdy = \int_C Qdy$

Since the integration of Qdy is zero on C_1 and C_3 as y is constant there.

Now if D is a region of the plain that is simultaneously type I and II then

$\iint_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dxdy = \int_C Pdx + Qdy$. Let us take a simple closed region as following figure.



Let $D = D_1 + D_2$ is the region of the given simple closed curve,
$\iint_D(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dxdy = \int_C Pdx + Qdy$, Here $C = C_1 + C_2$
Clearly $\iint_D(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dxdy = \iint_{D_1+D_2}(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dxdy = \iint_{D_1}(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dxdy + \iint_{D_2}(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dxdy$ and $\iint_{D_1}(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dxdy = \int_{C_1-E}(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dxdy = \int_{C_1}(Pdx+Q)dy - \int_E(Pdx+Qdy)$
$\iint_{D_2}(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dxdy = \int_{C_2+E} Pdx + Qdy = \int_{C_2} Pdx + Qdy + \int_E Pdx + Qdy$
Hence $\iint_D(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dxdy = \int_{C_1}(Pdx + Qdy) - \int_E(Pdx + Qdy) + \int_{C_2}(Pdx + Qdy) + \int_E(Pdx + Q)dy = \int_{C_1}(Pdx + Qdy) + \int_{C_2}(Pdx + Qdy) = \int_C(Pdx + Qdy)$.

2.10.2 Complex form of Green's Theorem[4]

Let $F(z, \overline{z})$ be continuous function and have continuous partial derivative in a region R and on its boundary C, where $z = x + iy$, $\overline{z} = x - iy$ are conjugate coordinates. Then Green's theorem can be written in the complex form $\oint_c F(z, \overline{z})dz = 2i \int\int \frac{\partial F}{\partial \overline{z}} dA$, where dA represents the element of area $dxdy$.
We want to convert this form into quaternion form.

Definition 2.8 Quaternion function

A function $f : D \subseteq \mathbb{H} \to \mathbb{H}$ is called a quaternion function, where D is a domain.

Example 2.9 $f(\eta) = \eta^2$*, where* $\eta \in D \subseteq \mathbb{H}$

let $\eta = a + ib + jc + kd$*, then*

$f(\eta) = u_1 + iu_2 + ju_3 + ku_4$*, where* u_1, u_2, u_3, u_4 *are all of the form* (a, b, c, d)*.*

Definition 2.10 Distance of two quaternion numbers

Let η_1 *and* η_2 *be two quaternion numbers then distance between these numbers is*

$\|\eta_1 - \eta_2\| = \sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2 + (c_1 - c_2)^2 + (d_1 - d_2)^2}$

where $\eta_1 = a_1 + ib_1 + jc_1 + kd_1$ *and* $\eta_2 = a_2 + ib_2 + jc_2 + kd_2$

2.11 Limit in Quaternion

Let $f(\eta)$ be a quaternion function $f : D \subseteq \mathbb{H} \to \mathbb{H}$. Let $\eta_0 \in D'$, we say that the number $l \in \mathbb{H}$ is the limit of $f(\eta)$ as η approaches η_0 if for every $\{\epsilon > 0\} \in \mathbb{R}$ we can find some positive number $\{\delta > 0\} \in \mathbb{R}$ depending on ϵ and η such that $\|f(\eta) - l\| < \epsilon$ whenever $0 < \|\eta - \eta_0\| < \delta$ and denoted by

$$\lim_{\eta \to \eta_0} f(\eta) = l.$$

Let $l = l_1 + il_2 + jl_3 + kl_4$ and $\eta_0 = x_1 + ix_2 + jx_3 + kx_4$, then

$$\|f(\eta) - l\| = \|u_1 + iu_2 + ju_3 + ku_4 - l_1 - il_2 - jl_3 - kl_4\| < \epsilon \Longleftarrow \|a + ib + jc + kd - x_1 - ix_2 - jx_3 - kx_4\| < \delta, \tag{1}$$

where u_1, u_2, u_3, u_4 are all the functions of $\{a, b, c, d\}$ and $\{a, b, c, d\} \in \mathbb{R}$, equation 1 is possible only when following four equations will be satisfied

$$|u_1 - l_1| < \epsilon \text{ whenever } |a - x_1| < \delta \tag{2}$$

$$|u_2 - l_2| < \epsilon \text{ whenever } |b - x_2| < \delta \tag{3}$$

$$|u_3 - l_3| < \epsilon \text{ whenever } |c - x_3| < \delta \tag{4}$$

$$|u_4 - l_4| < \epsilon \text{ whenever } |d - x_2| < \delta \tag{5}$$

$$OR$$

$$\lim_{(a,b,c,d) \to (x_1,x_2,x_3,x_4)} u_1(a, b, c, d) = l_1$$

$$\lim_{(a,b,c,d) \to (x_1,x_2,x_3,x_4)} u_2(a, b, c, d) = l_2$$

$$\lim_{(a,b,c,d)\to\ (x_1,x_2,x_3,x_4)} u_3(a,b,c,d) = l_3$$

$$\lim_{(a,b,c,d)\to\ (x_1,x_2,x_3,x_4)} u_4(a,b,c,d) = l_4$$

Definition 2.11 Continuity of a quaternion function:
If $\eta = a + ib + jc + kd$ then
let $f(\eta) = u_1(a,b,c,d) + iu_2(a,b,c,d) + ju_3(a,b,c,d) + ku_4(a,b,c,d)$ if we want to check continuity at $\eta_0 = x_1 + ix_2 + jx_3 + kx_4$ then we can check the continuity of u_1, u_2, u_3, u_4 at x_1, x_2, x_3, x_4 respectively.

If all functions u_1, u_2, u_3, u_4 at x_1, x_2, x_3, x_4 respectively are continuous then we can say quaternion function is continuous. If at least one of them is not continuous then quaternion function is not continuous.

Example 2.12 Quaternion differentiable function:
A quaternion valued function $f(\eta)$ is said to be differentiable function at $\eta = \eta_0$ if $\lim\limits_{\eta\to\eta_0}(f(\eta) - f(\eta_0))/(\eta - \eta_0) = \lim\limits_{h\to 0}(f(\eta_0 + h) - f(\eta_0))/h$ exist.

This differentiability is of two types

Definition 2.13 Left-hand derivative :
When we multiply h^{-1} in $(f(\eta_0 + h) - f(\eta_0))$ from left side then we call it left hand derivative

$$\lim_{h\to 0} h^{-1}(f(\eta_0 + h) - f(\eta_0)).$$

Definition 2.14 Right-hand derivative:
When we multiply h^{-1} in $(f(\eta_0+h)-f(\eta_0))$ from right side, then we call it right- hand derivative

$$\lim_{h\to 0}(f(\eta_0 + h) - f(\eta_0))h^{-1}.$$

2.12 Analyticity in quaternion division ring algebra

If the derivative $f'(\eta)$ exist at all point η of a region R then $f(\eta)$ is analytic in R.

2.12.1 Necessary Condition for differentiabilty:

Let $f(\eta)$ be differentiable at $\eta_0 = a + ib + jc + kd$.
$f(\eta_0) = u_1(a,b,c,d) + iu_2(a,b,c,d) + ju_3(a,b,c,d) + ku_4(a,b,c,d)$.

$f'(\eta_0) = \lim\limits_{\eta \to \eta_0} (f(\eta) - f(\eta_0))/(\eta - \eta_0) = \lim\limits_{h \to 0} (f(\eta_0 + h) - f(\eta_0))/h.$
Since limit along all direction at η_0 should be same. So first we are taking

limit along real x-axis, then h is of the form h:=(h,0,0,0) and

$f'(\eta_0)$=$\lim\limits_{h \to 0}(u_1(a+h,b,c,d) - u_1(a,b,c,d)/h$+i$\lim\limits_{h \to 0}(u_2(a+h,b,c,d) - u_2(a,b,c,d)/h$
+j$\lim\limits_{h \to 0}(u_3(a+h,b,c,d) - u_3(a,b,c,d)/h$
+k$\lim\limits_{h \to 0}(u_4(a+h,b,c,d) - u_4(a,b,c,d)/h$ We can write it in this form

$$\mathbf{f'}(\eta) = \mathbf{u_1 a} + \mathbf{iu_2 a} + \mathbf{ju_3 a} + \mathbf{ku_4 a} \tag{6}$$

taking limit along imaginary iy-axis, then h is of the form h:=(0,h,0,0) and

$f'(\eta)$=$\lim\limits_{h \to 0}(u_1(a,b+h,c,d) - u_1(a,b,c,d)/ih$
+i$\lim\limits_{h \to 0}(u_2(a,b+h,c,d) - u_2(a,b,c,d)/ih$
+j$\lim\limits_{h \to 0}(u_3(a,b+h,c,d) - u_3(a,b,c,d)/ih$
+k$\lim\limits_{h \to 0}(u_4(a,b+h,c,d) - u_4(a,b,c,d)/ih)$ This equation can be written in this form.
$f'(\eta) = (1/i)u_1 b + (i/i)u_2 b + (j/i)u_3 b + (k/i)u_4 b$

on solving the above equation from left side we get.

$$\mathbf{f'}(\eta) = \mathbf{u_2 b} - \mathbf{iu_1 b} + \mathbf{ju_4 b} - \mathbf{ku_3 b}. \tag{7}$$

Remark:

Here we are multiplying (i, j, k) from left side. We can multiply it from right side also. **Limit along imaginary jz-axis**, then h is of the form h:=(0,0,h,0) and

$f'(\eta)$=$\{\lim\limits_{h \to 0}(u_1(a,b,c+h,d) - u_1(a,b,c,d)/jh + i\lim\limits_{h \to 0}(u_2(a,b,c+h,d) - u_2(a,b,c,d)/jh + j\lim\limits_{h \to 0}(u_3(a,b,c+h,d) - u_3(a,b,c,d)/jh + k\lim\limits_{h \to 0}(u_4(a,b,c+h,d) - u_4(a,b,c,d)/jh)\}$
then $f'(\eta) = (1/j)u_1 c + (i/j)u_2 c + (j/j)u_3 c + (k/j)u_4 c$

on solving the above equation from left side we get.

$$\mathbf{f'}(\eta) = \mathbf{u_3 c} - \mathbf{iu_4 c} - \mathbf{ju_1 c} + \mathbf{ku_2 c}. \tag{8}$$

Limit along kw-axis, then h is of the form h:=(0,0,0,h) and

$f'(\eta)$=$\{(\lim\limits_{h \to 0}(u_1(a,b,c,d+h) - u_1(a,b,c,d)/kh + i\lim\limits_{h \to 0}(u_2(a,b,c,d+h) - u_2(a,b,c,d)/kh + j\lim\limits_{h \to 0}(u_3(a,b,c,d+h) - u_3(a,b,c,d)/kh + k\lim\limits_{h \to 0}(u_4(a,b,c,d+h) - u_4(a,b,c,d)/kh)$
then $f'(\eta) = (1/k)u_1 d + (i/k)u_2 d + (j/k)u_3 d + (k/k)u_4 d)\}$

On solving the above equations from left side we get.

$$f'(\eta) = u_4 d + iu_3 d - ju_2 d - ku_1 d. \tag{9}$$

equation $6, 7, 8, 9$ are all equal so:

$f'(\eta)$=$u_{1a} + iu_{2a} + ju_{3a} + ku_{4a} = u_{2b} - iu_{1b} + ju_{4b} - ku_{3b} = u_{3c} - iu_{4c} - ju_{1c} + ku_{2c} = u_{4d} + iu_{3d} - ju_{2d} - ku_{1d}$

Hence for differentiability

$(u_1 a = u_2 b = u_3 c = u_4 d)$,

$(-u_1 a = u_2 a = u_3 d = -u_1 c)$,

$(u_3 a = u_4 b = -u_1 c = -u_2 d)$,

$(u_4 a = -u_3 b = u_2 c = -u_1 d)$

Conclusion:

Among these above equations if one of them does not satisfied then given function can not be be differential.

Definition 2.15 Holomorphic function*[18]:*

A function f is said to be holomorphic in an open set $U \subseteq \mathbb{C}$ if $f'(z)$ exist $\forall z \in U$

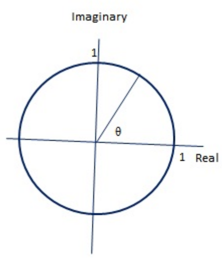
Chapter – 3

3 Four Dimension Visualization

To see four-dimensional space, we should understand 3-D rotaions or we can say to understand Euler angles. For this first we understand rotation about the complex number, in complex numbers, the rotations are done using unit quaternions like

$cos\theta + isin\theta = e^{i\theta}$

It is defined a unit circle in complex form



Here $e^{i\theta}$ denotes the whole circle as difined above.

We can denote any complex vector with the help of unit complex number by multiplying its length in unit complex vector $e^{i\theta}$. For example if Z is a complex vector and x is the length of this vector then we can defined z as $z = xe^{i\theta}$. Similarly we are moving to quaternion, In quaternion three-dimensional space as being pure quaternion:

$R^3 = \{xi + yjj + zk : x, y, z \in \mathbb{R}\}$

The rotations are done using unit quaternions, like $cos\theta + isin\theta$, $cos\theta + jsin\theta$, $cos\theta + ksin\theta$, where i, j, k are imaginary units defined into quaternion number. We will write these as: $e^{i\theta}, e^{j\theta}, e^{k\theta}$ respectively.

But there are many more unit quaternions than these, for example:

i, j, k are just three special unit quaternions.

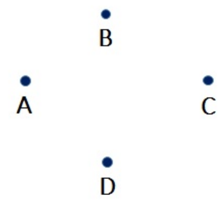
Take any unit imaginary quaternion $u = u_1 i + u_2 j + u_3 k$ that is any unit vector, then

$cos\theta + usin\theta$ is a unit quaternion and we can write it as $e^{u\theta}$. Unit quaternions represent the group of Euclidean rotations in three dimensions in a very straightforward way. The correspondence between rotations and quaternions can be understood by first visualizing the space of rotations itself. Two rotations by different angles and different axes in the space of rotations. The length of the vector is related to the magnitude of the rotation.[17] To visualize the space of rotations, it helps to consider a simpler case. Any rotation in three dimensions can be described by a rotation by some angle about some axis; [17] for our purposes, we will use an axis vector to establish handedness for our angle. Consider the special case in which the axis of rotation lies in the xy plane. We can then specify the axis of one of these rotations by a point on a circle through which the vector crosses, and we can select the radius of the circle to denote the angle of rotation. Similarly, a rotation whose axis of rotation lies in the xy plane can be described as a point on a sphere of fixed radius in three dimensions. Beginning at the north pole of a sphere in three-dimensional space, we specify the point at the north pole to be the identity rotation (a zero angle rotation). Just as in the case of the identity rotation, no axis of rotation is defined, and the angle of rotation (zero) is irrelevant. [17] A rotation having a very small rotation angle can be specified by a slice through the sphere parallel to the xy plane and very near the north pole. The circle defined by this slice will be very small, corresponding to the small angle of the rotation. As the rotation angles become larger, the slice moves in the negative z direction, and the circles become larger until the equator of the sphere is reached, which will correspond to a rotation angle of 180 degrees. Continuing southward, the radii of the circles now become smaller (corresponding to the absolute value of the angle of the rotation considered as a negative number). Finally, as the south pole is reached, the circles shrink once more to the identity rotation, which is also specified as the point at the south pole. Notice that a number of characteristics of such rotations and their representations can be seen by this visualization. The space of rotations is continuous, each rotation has a neighborhood of rotations which are nearly the same, and this neighborhood becomes flat as the neighborhood shrinks. Also, each rotation is actually represented by two antipodal points on the sphere, which are at opposite ends of a line through the center of the sphere. This reflects the fact that each rotation can be represented as a rotation about some axis, or, equivalently, as a negative rotation about an axis pointing in the opposite direction (a so-called double cover). The "latitude" of a circle representing a particular rotation angle will be half of the angle represented by that rotation, since as the point is moved from the north to south pole, the latitude ranges from zero to 180 degrees, while the angle of rotation ranges from 0 to 360 degrees. (the "longitude" of a point then represents a particular axis of rotation.) Note however that this set of rotations is not closed under composition. Two successive rotations with axes in the xy plane will not necessarily give a rotation whose axis lies in the xy plane, and thus cannot be represented as a point on the sphere. This will not be the case with a general rotation in 3-space, in which rotations do form a closed set under composition [17]. The sphere of rotations for the rotations that have a "horizontal" axis (in the xy plane). This visualization can be extended to a general rotation in 3-dimensional space. The identity rotation is a point, and a small angle of rotation about some axis can be represented as a point on a sphere with a small radius. As the angle of rotation grows, the sphere grows, until the angle of rotation reaches 180 degrees, at which point the sphere begins to shrink, becoming a point as the angle approaches 360 degrees (or zero degrees from the negative direction). This set of expanding and contracting spheres represents a hypersphere in four dimensional space (a 3-sphere). Just as in the simpler example above, each rotation represented as a point on the hypersphere is matched by its antipodal point on that hypersphere. The "latitude" on the hypersphere will be half of the corresponding angle of rotation, and the neighborhood of any point will become "flatter" (i.e. be represented by a 3-D Euclidean space of points) as the neighborhood shrinks. This behavior is matched by the set of unit quaternions: A general quaternion represents a point in a four dimensional space, but constraining it to have unit magnitude yields a three-dimensional space equivalent to the surface of a hypersphere. The magnitude of the unit quaternion will be unity, corresponding to a hypersphere of unit radius. The vector part of a unit quaternion represents the radius of the 2-sphere corresponding to the axis of rotation, and its magnitude is the cosine of half the angle of rotation. Each rotation is represented by two unit quaternions of opposite sign, and, as in the space of rotations in three dimensions, the quaternion product of two unit quaternions will yield a unit quaternion. We can draw figures in multi dimension as.

To draw four dimension figure we should understand different-different dimensional figure.

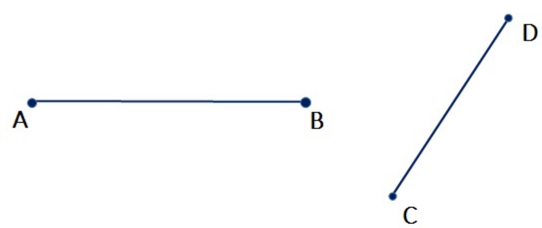
Zero-dimensional figure:

A point has zero dimensions. Ther is no length, height, width, or volume. Its only property is its location. Each corner points of triangle,Each corner points of square are also zero dimensional figure.



In the above figure, all P, Q, R and S denotes zero dimensional figures.

One-dimensional figure:

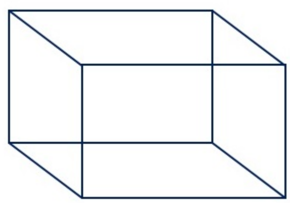


When we connect two points, then we get a one-dimensional object, A line segment is the one-dimensional figure, The above figure denotes one dimensional figure.

Two-dimensional figure:

When we connect two lines segments, then we get a plane which is two-dimensional figure. The above figure denotes two dimensional figure.

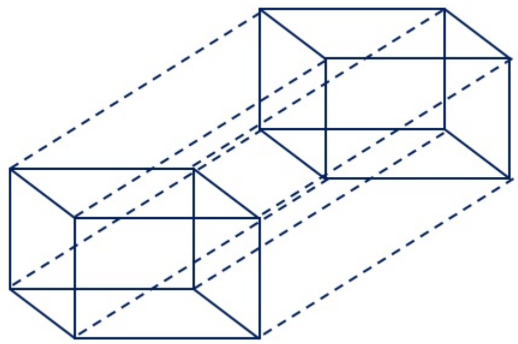
Three-dimensional figure:



When we connect two planes by line segment, i.e. two-dimensional figure connect by line segment then we get three-dimenional figure, The above figure(cube) is a three-dimensional figure.

Four-dimensional figure:

When we connect each corner of two cube i.e three-dimensional figure by line segments then we get a four dimensional figure.



4 Zeros of Polynomials

First we understand about polynomial function.

Polynomial functions

A polynomial function is a mathematical expression constructed with constants and variables using the four operations which are $(+, -, \div, \cdot)$. In general' we can write it as $f(x) = a_0x^n + a_1x^{(n-1)} + ... + a_n$, where x is the variable and $a_0, a_1, a_2, ..., a_n$ these are all constants or say scalars. When scalars come from a set of real numbers then we say $f(x) \in \mathbb{R}[x]$. when scalars come from set of complex number then we say $f(x) \in \mathbb{C}[x]$, and when scalars come from set of quaternion numbers then we say $f(x) \in \mathbb{H}[x]$.

The degree of a polynomial is the highest exponent(except the constant) of the polynomial. We are given the following detail to understand degree of polynomial,

Polynomial	Example
Constant	5
Linear	$5x + 3$
Quadratic	$6x^2 + 3x + 2$
Cubic	$5x^3 + 6x^2 + 3x + 4$
Quartic	$9x^4 + 5x^3 + 4x^2 + 6x + 9$

Here zero degree polynomial is called constant polynomial, two degree polynomial is called quadratic polynomial, three degree polynomial is called cubic polynomial, Four degree polynomial is called quartic polynomial, Since we have to understand quaternion polynomial, so a quaternion polynomial can be defined as

Definition 4.1 Quaternion Polynomial

Let $q_0, q_1, q_2, q_3, \ldots q_m$ be the quaternions then quaternion polynomial of degree m defined in this way:

$$f(x) = q_0 + q_1x + q_2x^2 + q_3x^3 + \ldots + q_mx^m,$$

where $q_m \neq 0$

Zeros of the given polynomial are the same thing as the roots of polynomial i.e. the value of scalar at which given function value goes to zero. To calculate zeros of polynomial there is many method to find zeros of polynomial, like as mean value theorem, Regula-Falsi-Method, Newton

Raphson's Method etc. With the help of these method we can get atleast one real root if it exists of a given function. To find roots of a quaternion polynomial is more difficult in comparison to real or complex polynomial.
Some quaternion polynomials of finite degree have infinite number of zeros.

Example of a quaternion polynomial which has infinite number of zeros.

Example 4.2 *Let* $f(\eta) := \eta^2 + 1 = 0$*, If* $0 \leq p \leq 1$ *be any real number then if we take* $\eta = \sqrt{p}i + \sqrt{(1-p)}j$
then always $\eta^2 = -1$*.*
So it has infinite number of zeros.In another way:
If we take any $\eta \neq 0$*, then*
$\eta i \eta^{-1}$*,* $\eta j \eta^{-1}$*,* $\eta k \eta^{-1}$ *these are all roots of the function.*

Proof:
Putting $\eta = \sqrt{p}i + \sqrt{(1-p)}j$ in $f(\eta) := \eta^2 + 1 = 0$, then
$(\sqrt{p}i + \sqrt{(1-p))}j)^2 + 1 = -P - (1-P) + \sqrt{P}\sqrt{(1-P)}ij - \sqrt{P}\sqrt{(1-P)}ij + 1 = -1 + 1 = 0$
also if we take any $\eta \neq 0$ then
$(\eta i \eta^{-1})^2 + 1 = (\eta i \eta^{-1})(\eta i \eta^{-1}) + 1 = (\eta i(\eta^{-1}\eta) i \eta^{-1}) = (\eta i(1) i \eta^{-1}) + 1 = (\eta(ii)\eta^{-1}) + 1 = -\eta\eta^{-1} + 1 = -1 + = 0$
Similarly we can see for $\eta j \eta^{-1}$ and $\eta k \eta^{-1}$.

4.1 Rouche's theorem in complex form[4]

Let $f(z)$ and $g(z)$ be two analytic functions inside and on a simple closed curve C. If $|g(z)| < |f(z)|$ on C then $f(z)$ and $f(z) + g(z)$ both have same number of zeros inside C.
Proof: Let $F(z) = \frac{g(z)}{f(z)}$ so that $g(z) = F(z).f(z)$ or briefly $g = f.F$ Let N_1 and N_2 are the number of zeros inside "C" of $f + g$ and f respectively.
Since these functions have no poles inside "C"
$N_1 = \frac{1}{2\pi i}\oint_c \frac{f'+g'}{f+g}dz, \qquad N_2 = \frac{1}{2\pi i}\oint_c \frac{f'}{f}dz$
$N_1 - N_2 = \frac{1}{2\pi i}\oint_c (\frac{f'+g'}{f+g} - \frac{f'}{f})dz$ after replacing g in the form f and F we gate
$N_1 - N_2 = \frac{1}{2\pi i}\oint_c \frac{F'}{1+F}dz = 0$
using the given fact that $|F| < 1$ on C so that the series is uniformly decreasing on C and term by term inegration yield the value zero.
Thus $N_1 = N_2$.

Example 4.3 *Suppose* $z^7 - 5z^3 + 12 = 0$*. Let* $f(z) = z^7$ *and* $g(z) = -5z^3 + 12$*. Now on* $|z| = 2$*, at the boundary* $|f(z)| = 2^7$ *and* $|g(z)| = |-5.2^3 + 12| \leq 52$*, clearly* $|f(z)| > |g(z)|$ *on* C*. So*

$f(z)$ and $f(z)+g(z)$ both have same number of zeros inside C.

Hence given polynomial has seven number of zeros because $f(z)$ has seven number of zeros inside C.

5 Result and Discussion

Here we got necessary and sufficient condition for differentiability of the function at a point η in quaternion division ring algebra. That is, let $f : D \subseteq \mathbb{H} \to \mathbb{H}$ be differentiable at $\eta_0 = a + ib + jc + kd$ Suppose $f(\eta) = u_1(a,b,c,d) + iu_2(a,b,c,d) + ju_3(a,b,c,d) + ku_4(a,b,c,d)$, where u_1, u_2, u_3, u_4 these are all functions of a, b, c and d, then function is differentiable at $\eta_0 = a + ib + jc + kd$ only if it satisfied the following condition:

$$(\mathbf{u_1a = u_2b = u_3c = u_4d}),$$

$$(\mathbf{-u_1a = u_2a = u_3d = -u_1c}),$$

$$(\mathbf{u_3a = u_4b = -u_1c = -u_2d}),$$

$$(\mathbf{u_4a = -u_3b = u_2c = -u_1d})$$

If one of them is not satisfied then function can't be differentiable at η_0.

By Niven's Algorithm [2]

We can apply fundamental theorem of algebra into quaterenion form:

1. Every quaternion polynomial has at least one zero in $\mathbb{H}$.

2. If the number of zeros of a polynomial of degree m is finite then there are exactly m zeros counting possible repetition.

Hence from above two important points, we conclude that if theorem will be true then it should be of the following form: i.e., we are trying to prove the following statement:

Let $f(\eta)$ and $g(\eta)$ be two analytic functions inside and on a simple closed curve H. If $|f(\eta)| > |g(\eta)|$ on H and also $f(\eta)$ and $g(\eta)$ both have finite number of zeros then $f(\eta)$ and $f(\eta) + g(\eta)$ both have the same number of zeros inside H.

6 Conclusion and Scope of future work

Rouche's Theorem is usually used to simplify the problem of locating zeros, as follow. Given an analytic function, we write it as the sum of two parts, one of which is simpler and grows faster than the other part. We can then locate the zeros by looking at only the dominating part. For example $z^7 + 3z^3 + 15$ has exactly 7 zeros in the disk $|z| < 2$, Since $|3z^3 + 15| < 128 = |z^7|$ on $|z| = 2$ and z^7, the dominating part, has seven zeros in the disk. Similarly we are trying to apply this theorem into quaternion polynomial, clearly, if this theorem will be true in case of quaternion polynomial, then it will be very easy to guess a number of zeros of a given quaternion polynomial inside the given region. Some finite degree polynomial has infinite number of zeros and to find zeros of a quaternion polynomial is difficult in comparision to complex polynomial. But how many zeros of a quaternion polynomial exist is easy to find with the help of Niven's algorithm.
By Niven's algorithm [2]

1. Every quaternion polynomial has at least one zero in $\mathbb{H}$.
2. If the number of zeros of a polynomial of degree m is finite then there are exactly m zeros counting possible repetition.

Hence from above two important points we reach the conclusion that if theorem will be true then it should be of the following form: We can prove the following form:
Let $f(\eta)$ and $g(\eta)$ be two analytic functions inside and on a simple closed curve H. If $|f(\eta)| > |g(\eta)|$ on H and also $f(\eta)$ and $g(\eta)$ both have finite number of zeros then $f(\eta)$ and $f(\eta) + g(\eta)$ both have same number of zeros inside H.

As we obtaine the Cauchy-Riemann equation in cartesian form for quaternion differentiable function. We can try to extend the following result into polar form of quaternion :
Let the function f(z) $= u(r, \theta) + iv(r, \theta)$ be defined through out some ϵ neighbourhood of a non zero point $z_0 = r_0 e^{i\theta_0}$, and suppose that
(a) the first order partial derivatives of the functions u and v with respect to r and θ exist everywhere in the neighbourhood;
(b) those partial derivatives are continuous at equation (r_0, θ_0). Then $f'(z_0)$ exists, its value being $f'(z_0) = e^{-i\theta}(u_r + iv_r)$, where the right-hand side is to be evaluated at (r_0, θ_0). The above

statement is the polar form of Cauhy-Riemann equation.

We can try to convert it for quaternion polynomial.

Cauchy-Goursat Theorem

If a function f is analytic throughout a simply connected domain D, then $\int_C f(z)dz = 0$, for every closed contour C lying in D.

We are trying to convert this Cauchy-Gaursat Theorem into quaternion form.

We can try to convert it into quaternion form. To prove theorem we are trying to prove Argument theorem of complex form into quaternion form. To prove argument theorem we are trying to prove Cauchy gaursat theorem in quaternion form, for this we are trying to convert Green's theorem into quaternion form and necessary condition for differentiability. Here necessary condition for differentiability have done. Our next target to convert complex version of Green's theorem into quaternion form. We can try to extend many theorem of complex form into quaternion form like as Morera's Theorem, Liouville's theorem,

References

[1] A. Sudbery, *Quaternion analysis*, Math. Proc. Camb. Phil. Soc. (1979), 85, 199-225.

[2] Fuzhen Zhang, *Quaternions and matrices of Quaternions*, Linear algebra and its applications 251:21-57(1997), Elsevier science Inc., 1997 655 Avenue of the Americas, New York NY10010.

[3] R.Serodio, E.pereira and J.vitoria, *Computing the zeros of quaternion polynomial*, Computer and mathematics with applications 42(2001)1229-1237.

[4] Murray R Spiegel, Semour Lipscutz, John J Schiller, Dennis Spellman, *Complex variables.*

[5] Stephen J.Sangwine, *Quaternion Involutions*, Todd A.EII*, View court, Savage, MN 55378-4695, USA.

[6] Gerhard Opfer, *Advance in Applied Clifford Algebra*, 27(2017), 2659-2675, Springer international publishing.

[7] William Rowan Hamilton, *On quaternions, or on anew system of imaginaries in algebra*, (Philosophical Magazine, (1844 - 1850)).

[8] Boris Abramovich Rozen field, *The history of non euclidean geometry, evolution of the concept of a geometric space*, Springer. P. 385 ISBN978038.

[9] Conway, John Horton Smith, Derek Alan, *On quaternions and ocatanions: their geometry arithmetic and symmetry*, P.9 ISBN 1-56881-134-9.

[10] Simon L. Altmann(December 1989), *Hamilton, Rodrigues, and the quaternion. Scandal.* Mathematics Magzine. 62(5):306 doi 10,230712689481. J STORE 2689481.

[11] Karsten Kunze, Helmut Schaeben (November 2004), *The Bingham Distribution of Quaternions and Its Spherical Radon Transform in Texture Analysis".* Mathematical Geology. 8(8) : 917943.

[12] Hamilton, Sir William Rowan (1866). *Hamilton Elements of Quaternions article 285. p. 310.*

[13] Frobenius G., *"ber das Pfaffsche Problem", J. fr Reine und Agnew.* Math., 82 (1877) 230-315.

[14] ULIAN AWRYNOWICZ, *Institute of Mathematics Polish Academy of Sciences* PL-90-136 6dz, Poland Jakub Rembielinski Institute of Physics University of 6dz PL-90-236 6dz, Poland.

[15] STEPHEN J. SANGWIN EDITORS, *Quaternion and Clifford Fourier Transforms and Wavelets.*

[16] AMY BUCHMANN, *A Brief History of Quaternions and the Theory of Holomorphic Functions of Quaternionic Variables* Department of Mathematics and Computer Sciences Schmid College of Science Chapman University Orange, CA 92866 USA.

[17] L. ALTMANN, Rotations, Quaternions, and Double group Oxford University Press, 1986.

[18] AMY BUCHMANN, A Brief History of Quaternions and the Theory of Holomorphic Functions of Quaternionic Variables Department of mathematics and Computer Science. Schmid College of Science Chapmann University Orange, CA 92866 USA, buchm100@mail.chapman.edu.