DYNAMICS IN SUPERFLUIDS

M. Sc. Thesis

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DYNAMICS IN SUPERFLUIDS

A THESIS

Submitted in partial fulfillment of the requirements for the award of the degree of Master of Science

> *by* **ANUPRIYA AGGARWAL**



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I hereby certify that the work which is being presented in the thesis entitled **DYNAMICS OF SUPERFLUIDS** in the partial fulfillment of the requirements for the award of the degree of **MASTER OF SCIENCE** and submitted in the **DISCIPLINE OF PHYSICS**, **Indian Institute of Technology Indore**, is an authentic record of my own work carried out during the time period from July, 2013 to June, 2015 under the supervision of Dr. Manavendra N. Mahato, Associate Professor, Department of Physics, IIT Indore. The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other institute.

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Dedicated to

my grandfather

Shri Subhash Chandra Gupta

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Abstract

The Ginzburg Landau equation can also be used to study the dynamics of superfluids apart from the superconductors. The GL equation is a non-linear equation. Some of the simplest solutions of GL equations for superfluids consist of the tan hyperbolic function. Also we can use different methods to find the solution for this kind of differential equation such as the method of separation of variables. By finding the cylindrically symmetric solutions for this equation something can be physically concluded about the geometry of the vortices in superfluids. There must be a specific reason as to why the vortices always appear to be formed in a hexagonal pattern rather than square or any other. This can be concluded by looking at the free energy equations of this system.

1 Introduction to superconductivity

Using iterative Joule-Thomson cooling, Heike Kammerlingh Onnes succeeded in liquefying helium gas at the University of Leiden in the Netherlands on July 10, 1908. It can be considered as the beginning of research on superconductivity and more generally, low-temperature physics. In the following years, Kammerlingh Onnes started investigating the electrical resistance of metals as a function of temperature. Mercury is a liquid at room temperature and distillation techniques can be used to purify it. With the idea of examining properties of a high-purity metal, he took up the measurement of electrical resistivity of mercury in early 1911. Mercury was filled in a U-shaped tube with wires inserted at both ends and its resistance was measured as a function of temperature. The result obtained was quite remarkable in that the resistance decreased with decreasing temperature and then dropped precipitously to zero at about 4.2 K.

The graph shows the variation of resistance as a function of temperature in a superconductor. As temperature decreases, the resistance falls gradually (as for a normal metal). However, at T_c , the resistance drops very sharply to zero and remains so at lower temperatures! Also shown is the variation of heat capacity with temperature in the case of a superconductor. A sharp peak is seen at T_c signifying the phase transition.

This was the first ever observation of superconductivity. He was deciding between gold and mercury for his first measurement and it is fortunate that he chose mercury since gold is not superconducting. Onnes realized the commercial potential of his discovery of superconductivity and began examining other metals. Tin and lead were the next elements to be found superconducting by him. Since the Leiden lab had a near monopoly in the production of liquid helium, Onnes and his co-workers were the leaders in low-temperature physics for many years. Subsequent to the original discovery of Onnes of zero resistance in Hg at 4.2 K, many new superconductors and allied phenomena were discovered. In one of his experiments Onnes started a current in a loop of lead wire cooled to 4 K. Even after a year the current was still flowing



Figure 1: The graph shows the variation of resistance as a function of temperature in a superconductor

without any noticeable change. This was called a persistent current by Onnes. Kammerlingh Onnes was awarded the Nobel prize in 1913.

Finally Onnes was the person who also found that superconductivity could be destroyed by an applied magnetic field called the critical field. The empirical relation for the temperature variation of the critical field was found to be $H_c(T) = H_c(0)[1-(\frac{T}{T_c})^2]$. Here, T_c (the critical temperature) is the temperature below which the substance is superconducting when no magnetic field is applied and $H_c(T)$ is the magnetic field necessary to decrease the critical temperature to T.

The field above which superconductivity is destroyed is called the critical field. The critical field curve separates the normal and the superconducting regions.

A theoretical explanation for any of the observations related to superconductivity was lacking at that time though people believed that it lay in quantum mechanics. Researchers kept working to discover new superconductors and to learn new properties in order to shed more light on this new and astonishing phenomenon. The next important finding was made by Meissner and Ochsenfeld who, in 1933, found that superconductors are not merely perfect conductors (i.e., having a zero resistance) but also exclude a magnetic field from their interior.

This goes by the name of Meissner effect. The response of a superconductor when

it is cooled in a field (Meissner effect) is quite different than that for a hypothetical perfect conductor. The illustration portrays this fact very easily. For a perfect conductor this will result in a field remaining inside the material even after switching off the field after the conductor has been cooled in a field to a perfect conductor state. In contrast, for a superconductor, the field is expelled from the interior even when it is cooled in a field.



Figure 2: Behavior of a perfect conductor



Figure 3: behavior of a superconductor

There were early efforts by brothers London to explain flux expulsion from a superconductor. It did introduce an important physical length scale, namely the penetration depth, associated with superconductors. The microscopic theory of superconductivity was proposed several decades later in 1957 by J. Bardeen (then a Professor of Physics at University of Illinois Urbana Champagne, USA and also a Nobel prize winner for the invention of the point contact transistor) along with his post-doctoral fellow L. Cooper and his PhD student J.R. Schrieffer.

Many of the already observed aspects were explained by the theory. Further, predictions were made which were verified in the years that followed. Bardeen, Cooper, and Schrieffer were awarded the Nobel prize for the theory of superconductivity (also called the BCS theory) in 1972. Even before the BCS theory, a general phenomenology of phase transitions was given by the Ginzburg-Landau theory in 1950. They expressed the free energy as a series in powers of an order parameter which characterized the superconducting state. Using minimization procedures, they obtained further insight into the initially unknown parameters that were included in the theory. With this insight, they obtained a handle on various properties of a superconductor. On the applications front, superconductors (Nb₃Sn and Nb₃Ti) were already being used commercially for solenoid magnets, SQUID sensors, etc.

The superfluid forms condensate at very low temperature and the formation of condensate is the ground state of our system. Thus we construct a Hamiltonian for our system and find its eigenvalues and eigenvectors. The eigenvector with minimum eigenvalue corresponds to the ground state of our system in which condensates are formed. At the ground state the system acquires new exciting properties. To study about these properties and fluctuations we just need to look at the ground state. The Hamiltonian includes all the interactions between the particles. Here I have tried to derive the time independent form of the Gross Pitaevskii equation to explain the system. In order to simplify our problem we use the 'mean field approximation' by assuming that all the interactions are identical in case of every particle.

Then we talk about the symmetry breaking by the formation of Bose Einstein condensates at very low temperatures below the transition temperature. The Bose Einstein condensate is the ground state in a superfluid. This ground state is not invariant with respect to transformation of the original symmetry of Lagrangian of the system. This is called spontaneous symmetry breaking.

2 Normal metals

2.1 Basic Properties Of Metals In The Normal State

Before delving into the properties of superconductors, it is important to give a brief account of the salient features of normal metals. In a non-interacting electron picture, the contribution of the electrons to various properties can be calculated. This is worked out in standard texts on solid state physics and the summary of the results is given below.

2.2 Electrical conductivity

We recall that in the non-interacting-electrons description of metals, one considers electrons moving in a periodic potential of the lattice. In this "Bloch" picture, a perfect crystal is expected to have an infinite electrical conductivity. In real materials, a finite conductivity appears due to the inherent imperfections and defects. Additionally, at non-zero temperatures, lattice vibrations lead to a deviation from periodicity and contribute to electron scattering. In summary, one gets resistivity $\rho \sim T$ for T $\gg \Theta_D$ where Θ_D is the Debye temperature. On the other hand, for T $\ll \Theta_D$ one obtains $\rho \sim T^5$. The residual resistivity as $T \to 0$ decreases with decreasing amounts of impurities. A typical value of resistivity of a good metal at room temperature is of the order of 10^{-6} ohm cm.

As an example, the variation of the resistivity with temperature for silver is shown in the figure below At high temperature, a linear variation of resistivity with temperature is seen while at low temperatures a residual resistivity is present which is linked to the presence of defects and impurities.

For ideal metals, the thermal conductivity due to electrons is given by $\frac{1}{3} v^2 \tau c_v$ where v^2 is the mean square electronic speed, τ is the relaxation time, and c_v is the specific heat capacity of the electrons. The ratio of the thermal conductivity κ to the electrical conductivity times temperature (σ T) of an electron gas is a universal constant (Lorentz number) and this is called Wiedmann-Franz law. The value of the



Figure 4: The resistivity variation with temperature is shown for a normal metal.



Lorentz number is of the order of $10^{-8} \frac{\text{Wattohm}}{\text{K}^2}$ in typical metals.

2.3 Heat capacity

The temperature dependence of the specific heat capacity of an electron gas is given by $\frac{c_v}{k_B} = \gamma T = \frac{\pi^3}{3} g(\epsilon_F) k_B T$ where k_B is the Boltzmann constant and $g(\epsilon_F)$ is the density-of-states at the Fermi level. The value of γ in typical metals is of the order of $1 \frac{\text{mJ}}{\text{mole K}^2}$. In the Debye model, the phonon contribution to the heat capacity per atom is given by $\frac{c_v}{k_B} = 9(\frac{T}{\Theta_D})^3 \int_0^{\Theta_D/T} \frac{x^4 e^x dx}{(e^x - 1)^2}$ where Θ_D is the Debye temperature. The heat capacity variation for a typical metal is shown in the figure. At high temperatures, it tends to a constant value while at low temperatures the heat capacity is a combination of a linear term (due to electrons) and a cubic term (due to lattice vibrations or phonons). When C/T is plotted as function of T^2 , a straight line is



obtained. The intercept on the y-axis gives information about the electron density of states while the slope provides information about characteristic energy associated with lattice vibrations. In the low-temperature limit $T \ll \Theta_D$, the phonon contribution reduces to $\frac{c_v}{k_B} = \frac{12\pi^4}{5} (\frac{T}{\Theta_D})^3$. In a typical metal, the low-temperature heat capacity has, therefore, a combination of a linear and a cubic term in temperature. The low-temperature heat capacity measurement serves as an important probe of the Fermi surface properties of the electron gas.

3 Magnetic susceptibility and Hall effect

3.1 Magnetic susceptibility

The magnetic (spin) susceptibility of a non-interacting electron gas is called the Pauli spin susceptibility and is given by $\chi = \mu_B^2 g(\epsilon_F)$ where μ_B is the Bohr magneton.

Since the temperatures at which the magnetic susceptibility is typically measured are much smaller than $\frac{\epsilon_F}{k_B}$, it is independent of temperature. The Pauli susceptibility has a typical value of $10^{-6} \frac{\text{cm}^3}{\text{mole}}$. In addition, there are other contributions to the susceptibility namely, the Van Vleck orbital paramagnetism, the diamagnetism from the orbital motion of the core electrons, and Landau diamagnetism from the orbital motion of the free electrons. The magnetic susceptibility of conventional metals is independent of temperature.

3.2 Hall effect

The Hall effect is used to determine the concentration and nature of charge carriers in a material. In the standard Hall effect geometry, a magnetic field is applied perpendicular to the direction in which an ohmic current is flowing. Due to the Lorentz force on the charge carriers, a voltage develops along the third orthogonal direction and is called the Hall voltage. The Hall coefficient is defined as $R_H = \frac{E_y}{j_x B_z}$ where j_x is the ohmic current density in the x-direction, B_z is the applied magnetic field in the z-direction, and y is the electric field that is developed in the y-direction. The Hall coefficient is equal to $-\frac{1}{ne}$ (MKS units) if the charge carriers are electrons (of charge e and density n). A typical value of the Hall coefficient in metals is $10^{-10} \frac{\text{m}^3}{\text{C}}$. In contrast to the above, there are qualitative changes that take place in the properties of materials when they become superconducting. These will be elaborated at appropriate places when the properties of superconductors are discussed.

Charge carriers (shown as positive for convenience) flow in the conductor on application of an electrostatic potential and constitute an ohmic current. Switching on a magnetic field perpendicular to the ohmic current results in a Lorentz force on the charge carriers which move in a direction perpendicular to both the ohmic current and the magnetic field. The accumulation of charge carriers as shown will result in an electrostatic potential (Hall voltage) which counteracts the Lorentz force.

4 Superconductivity phenomenon

Superconductivity is a phenomenon of exactly zero electrical resistance and expulsion of magnetic fields occurring in certain materials when cooled below a characteristic critical temperature. It was discovered by Dutch physicist Heike Kamerlingh Onnes on April 8, 1911 in Leiden. Like ferromagnetism and atomic spectral lines, superconductivity is a quantum mechanical phenomenon. It is characterized by the Meissner effect, the complete ejection of magnetic field lines from the interior of the superconductor as it transitions into the superconducting state. The occurrence of the Meissner effect indicates that superconductivity cannot be understood simply as the idealization of perfect conductivity in classical physics. The electrical resistivity of a metallic conductor decreases gradually as temperature is lowered. In ordinary conductors, such as copper or silver, this decrease is limited by impurities and other defects. Even near absolute zero, a real sample of a normal conductor shows some resistance. In a superconductor, the resistance drops abruptly to zero when the material is cooled below its critical temperature. An electric current flowing through a loop of superconducting wire can persist indefinitely with no power source. In 1986, it was discovered that some cuprate-perovskite ceramic materials have a critical temperature above 90 K (183 C). Such a high transition temperature is theoretically impossible for a conventional superconductor, leading the materials to be termed high-temperature superconductors. Liquid nitrogen boils at 77.2 K, and superconduction at higher temperatures than this temperature facilitates many experiments and applications that are less practical at lower temperatures.

4.1 Two fluid model for superconductivity and London equations

4.1.1 Phenomenon of superconductivity

As mentioned before, the resistance of metallic materials decreases with a decrease in temperature and displays a T^5 - behavior at low-temperatures.



As temperature decreases, the resistance falls gradually (as for a normal metal). However, at T_c , the resistance drops very sharply to zero and remains so at lower temperatures. Also shown is the variation of heat capacity with temperature in the case of a superconductor. A sharp peak is seen at T_c signifying the phase transition. While the amount of residual resistivity (i.e., $\rho(T \to 0)$) does depend on the purity of the material, the temperature variation of ρ is smooth. In contrast, for superconductors and as found by Onnes for the first time for the case of mercury, the dc resistivity decreases sharply to zero at the critical temperature T_c . Note that in a typical measurement of dc resistivity, a constant dc current is applied to the sample and the voltage developed across it is measured as a function of temperature. On the other hand, for measuring the ac resistivity (or conductivity), an ac current source has to be used.

4.1.2 Frequency dependent conductivity in the Drude model

The Drude model treats the electron gas with the methods of the kinetic theory of a dilute neutral gas with the following assumptions: (i) between collisions, the motion of electrons is considered to be independent of the static ions or the other electrons, (ii) collisions are thought to be instantaneous events which abruptly change the velocities of the electrons, (iii) a relaxation time τ is introduced which is the mean time between collisions, and (iv) electrons achieve equilibrium with their surroundings only via collisions. The conductivity in a spatially uniform and time-independent electric field

is given by $\sigma_0 = \frac{ne^2\tau}{m}$. For a time-dependent field represented by $\mathbf{E}(t) = \Re(\mathbf{E}(\omega)e^{-i\omega t})$, one obtains the current density $\mathbf{j}(\omega) = \sigma(\omega)\mathbf{E}(\omega)$ where the frequency dependent conductivity is given by

$$\sigma(\omega) = \frac{\sigma_0}{1 - i\omega\tau} = \frac{\sigma_0}{1 + \omega^2\tau^2} + i\frac{\sigma_0\omega\tau}{1 + \omega^2\tau^2} \tag{1}$$

4.1.3 Two fluid model

Since the response of superconductors to ac fields is known to be dissipative, a simple two-fluid model is introduced to explain the ac conductivity of superconductors. Consider a material with total electron density n, which comprises of a superconducting part n_s and a normal part n_n having relaxation times τ_s and τ_n , respectively. Since infinite conductivity is obtained for a superconductor in the dc case, it is natural to consider $\tau_s = \infty$. The shape of the $\frac{\sigma_0}{1+\omega^2\tau^2}$ part of the conductivity as a function of frequency is bell shaped. At a given temperature, the height of the peak at zero frequency (σ_0) grows as τ increases with the area under the curve remaining constant. Note that $\int \frac{d\omega}{1+\omega^2\tau^2} = \frac{\pi}{2\tau}$ and hence the area under the curve is $\frac{ne^2\pi}{2m}$. As $\tau \to \infty$, the width of the curve approaches zero and the height diverges with the area remaining constant (i.e., a Dirac-delta function). Similarly, the imaginary part of the conductivity approaches $\frac{ne^2}{m\omega}$ as $\tau \to \infty$.

Therefore, at non-zero frequencies, the superconducting fraction of the electrons contribute only to the imaginary part of the conductivity which is $\frac{n_s e^2}{m\omega}$. Further, in the limit $\omega \tau_n \ll 1$, the real part of the conductivity is approximately $\frac{n_n e^2 \tau}{m}$ which is finite. This illustrates the dissipative behavior of a superconductor in an ac field. Since a dc current can flow in a superconductor without dissipation, it is possible to set up currents in superconducting loops which do not decay with time. These are called persistent currents. Indeed, solenoid magnets made of superconducting wires are commercially available which can routinely provide magnetic fields as large as 100-200 kOe. The highest persistent field (268 kOe) has been achieved using cuprate based (YBCO) superconductors. In reality, there is a small decay of the circulating current due to something called flux flow resistance. This is however extremely small and only ppm level changes in the magnetic fields are seen over decades.

4.1.4 Diamagnetism

Another defining feature of superconductors is diamagnetism. We note here that the response of a superconductor to an applied magnetic field is distinctly different from what one might expect of a hypothetically perfect or ideal conductor ($\sigma = \infty$). Imagine that a normal, non-superconducting material were cooled in the absence of a magnetic field. Further consider that this material becomes a perfect conductor below some temperature. If a magnetic field is now turned on, the magnetic field (B) inside the conductor should remain unchanged at zero (this follows from Maxwell equations and is shown below). On the other hand, if the material were cooled in the presence of a magnetic field, the field would penetrate the material in the finite conductivity state and will continue to do so when the material is cooled to temperatures where it has $\sigma = \infty$. In contrast to the above behavior expected of a perfect or ideal conductor, a superconductor (for $T < T_c$) always has B = 0 inside it, i.e., whether it is cooled in a magnetic field or the field is turned on after cooling the sample (issues such as penetration depth and critical fields will be discussed later). The expulsion of field from the superconductor when it is cooled in a field is called the Meissner effect while a similar consequence which takes place when the sample is cooled in zero field and then a field is applied for $T < T_c$ is called perfect diamagnetism. Let us start from the Maxwell equation also referred to as the Faraday law

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \tag{2}$$

Newton's law for electrons under the influence of an electric field is given by (note that this is strictly valid only for spatially uniform fields which is not the case at interfaces)

$$m\frac{d\vec{v}}{dt} = -e\vec{E} \tag{3}$$

$$\Rightarrow m \frac{\partial (-ne\vec{v})}{\partial t} = ne^2 \vec{E} \tag{4}$$

$$\Rightarrow \vec{\nabla} \times \vec{E} = \frac{m}{ne^2} \vec{\nabla} \times \frac{\partial \vec{J}}{\partial t}$$
(5)

Consequently

$$\vec{\nabla} \times \frac{\partial \vec{J}}{\partial t} = \frac{ne^2}{mc} \frac{\partial \vec{B}}{\partial t} \tag{6}$$

Making use of another of Maxwell equations (neglecting the displacement current and taking $\vec{B} = \vec{H}$ for the superconductor we get $\vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{J}$), we obtain

$$\vec{\nabla} \times \frac{\partial (\vec{\nabla} \times \vec{B})}{\partial t} = \frac{4\pi n e^2}{m c^2} \frac{\partial \vec{B}}{\partial t}$$
(7)

Further, using the vector identity $\overrightarrow{\nabla} \times \overrightarrow{\nabla} \times \overrightarrow{B} = \overrightarrow{\nabla} (\overrightarrow{\nabla} \cdot \overrightarrow{B}) - \nabla^2 \overrightarrow{B}$ along with no-magnetic-monopole condition $\vec{\nabla} \cdot \vec{B} = 0$,

$$\nabla^2 \frac{\partial \overrightarrow{B}}{\partial t} = \frac{4\pi n e^2}{m c^2} \frac{\partial \overrightarrow{B}}{\partial t}$$
(8)

Applying the above equation to a slab of a perfect conductor implies that $\frac{\partial B}{\partial t}$ must fall off exponentially inside the conductor with a characteristic length scale (London penetration depth λ) of $\sqrt{\frac{mc^2}{4\pi ne^2}}$. Therefore, changes in the field are attenuated exponentially inside a perfect conductor.

4.1.5 London Equations

In order to explain the Meissner effect in superconductors, Fritz and Heinz London proposed the following equations in 1935.

$$\overrightarrow{\nabla} \times \overrightarrow{J} = -\frac{ne^2}{mc} \overrightarrow{B} \Rightarrow \overrightarrow{J} = -\frac{ne^2 \overrightarrow{A}}{mc}$$
(9)

$$\frac{\partial \vec{J}}{\partial t} = -\frac{ne^2}{m}\vec{E} \tag{10}$$

The second of the above equations is, in any case, valid for a perfect conductor while the first one helps explain Meissner effect. Note that the first equation did not follow from any microscopic theory but was merely an attempt to describe empirical observations. The penetration depth varies with temperature in a qualitative manner as indicated in the figure below. The empirical variation has been found to follow

$$\lambda(T) = \frac{\lambda(0)}{\sqrt{1 - (\frac{T}{T_c})^4}} \tag{11}$$

The values of $\lambda(0)$ range from tens of A^o to thousands of A^o



Figure 5: The figure illustrates the penetration of magnetic field inside a superconductor. The schematic variation of penetration depth with temperature is also shown.

4.1.6 Critical field

We will now talk about the effect of a magnetic field on the T_c of a superconductor. The T_c is found to decrease when a magnetic field is present and superconductivity can be completely suppressed by applying a sufficiently strong magnetic field (called the critical field) H_c . As will be discussed later, Type I superconductors have a single critical field (H_c) while Type II superconductors have a lower and an upper critical field (H_{c1} and H_{c2} , respectively) between which the substance is said to be in the mixed state. Typical values of H_c are in the region of 0.01 Tesla while those of H_{c2} can be as high as 100 Tesla. Clearly, the Type II superconductors with high H_{c2} are useful from the point of view of applications. The typical variation of H_c with temperature is shown in the figure below. It follows

$$H_{\rm c}(T) = H_{\rm c}(0) \left(1 - \left(\frac{T}{T_{\rm c}}\right)^2\right) \tag{12}$$

Here, at a temperature T a field of $H_c(T)$ is needed to suppress superconductivity completely.



Figure 6: The schematic variation of critical field with temperature for Type-I and Type-II superconductors is shown in the figure.

4.1.7 Critical current

Likewise, there is a limit to the current that one can pass through a superconductor beyond which it is normal (non-superconducting). For a cylindrical conductor, one can estimate it in the following manner. For a current I flowing uniformly through a cylindrical conductor of radius a, the field at the surface is $H = \frac{I}{2\pi a}$. Clearly, if this exceeds H_c the whole wire will become normal. The critical current is therefore $I_c = 2\pi a H_c$. In practice, a current of hundreds of Amperes can be passed through superconducting wires without dissipation.

4.1.8 critical temperature

The critical temperature for superconductors is the temperature at which the electrical resistivity of a metal drops to zero. The transition is so sudden and complete that it appears to be a transition to a different phase of matter; this superconducting phase is described by the BCS theory. Several materials exhibit superconducting phase transitions at low temperatures. The highest critical temperature was about 23 K until the discovery in 1986 of some high temperature superconductors.

Materials with critical temperatures in the range 120 K have received a great deal of attention because they can be maintained in the superconducting state with liquid nitrogen (77 K).

4.1.9 Isotope effect

Among the parameters which affect T_c , the isotopic mass M plays an important role and changes T_c as $T_c \propto M^{-\alpha}$ with $\alpha \approx 0.5$. The isotope effect was discovered in 1950 in Hg and Sn where different isotopes could be synthesized using methods of nuclear physics. This must have been an important clue in the development of the BCS theory since it indicated the importance of the lattice for the mechanism of superconductivity. The most important consequence of London equations is the screening of a magnetic field from the bulk of a superconductor.

5 Solution of London equations and free energy calculations

5.0.10 Solution of London Equations for sample cases

Flat slab in a magnetic field Consider a flat superconducting slab of thickness d in a magnetic field H_a parallel to the slab. The boundary condition is that field matches at $x = \pm d/2$. Subject to this condition, the solution to the London equation $\nabla^2 h = \frac{1}{\lambda^2} h$, where h is the microscopic value of the flux density, is easily determined to be a superposition of two exponential terms. The result can be written as

$$h = H_{\rm a} \frac{\cosh(x/\lambda)}{\cosh(d/2\lambda)} \tag{13}$$

Clearly, the minimum value of the flux density is attained at the mid-plane of the slab where it has a value $H_a/\cosh(d/2\lambda)$.

Averaging this internal field over the sample thickness one gets



Figure 7: The field variation inside a superconducting slab is shown.

$$B \equiv \bar{h} = H_{\rm a} \frac{2\lambda}{d} \tanh\left(\frac{d}{2\lambda}\right) \tag{14}$$

Let us consider the limit $d \gg \lambda$. This leads to B = 0 deep inside the superconductor. In the other limit, i.e., $d \ll \lambda$, we expand $\tanh(x) = x - x^3/3$. Therefore, B approaches $H_{\rm a}(1 - \frac{d^2}{12\lambda^2})$. Since $B = H_{\rm a} + 4\pi M$, we get

$$M = -\frac{H_{\rm a}}{4\pi} \left(\frac{d^2}{12\lambda^2}\right) \tag{15}$$

As a consequence, magnetization measurements can be made on thin films of known thicknesses and the penetration depth can be estimated from such measurements. Since the magnetization is reduced below its Meissner value, the effective critical field for a thin sample is greater than that for bulk. The difference in the free energy between the normal state and the superconducting state is

$$(F_n - F_s)|_{H=0} = -\int_{0}^{H_c} M dH$$
(16)

For the case of complete flux expulsion, the above difference in free energies is

$$\Delta F = -\int_{0}^{H_{c}} \frac{H}{4\pi} dH = \frac{H_{c}^{2}}{8\pi}$$
(17)

This energy, which stabilizes the superconducting state is called the condensation energy and H_c is called the thermodynamic critical field. For a thin film sample (with a field applied parallel to the plane) we get,

$$\Delta F = \frac{1}{4\pi} \frac{d^2}{12\lambda^2} \int_0^{H_{cpar}} H dH = \frac{H_{cpar}^2}{8\pi} \frac{d^2}{12\lambda^2}$$
(18)

In terms of the bulk thermodynamic critical field

$$\frac{H_{c\,\text{par}}^2}{8\pi} \frac{d^2}{12\lambda^2} = \frac{H_c^2}{4\pi} \tag{19}$$

$$H_{cpar} = \frac{\sqrt{12\lambda}}{d} H_{c} \tag{20}$$

5.0.11 Critical current of a wire

Consider a long superconducting wire having a circular cross-section of radius a. Also, assume that $\lambda \gg a$. A current I is passed through the wire. This gives rise to a circumferential magnetic field at the surface of the wire $H = \frac{2I}{ca}$. In a simple minded picture, when this field reaches H_c , the wire will become normal. Therefore, the critical current $I_c = \frac{ca}{2}H_c$ depends linearly on the radius and not on the area. The current flows only in a surface layer of thickness λ . Hence, the current density $J_c \approx \frac{I_c}{2\pi a\lambda} = \frac{H_c}{2\pi a\lambda} \frac{ca}{2}$. Therefore, $J_c = \frac{c}{4\pi} \frac{H_c}{\lambda}$.

5.0.12 Free energy calculations

Now consider the case of a type I superconductor in a relatively large field. First we will carry out some calculations assuming zero demagnetisation factor. For a normal sample of volume V in a magnetic field H_a , the Helmholtz free energy is given by

$$F_n = V f_{n0} = V \frac{H_a^2}{8\pi} + V_{\text{ext}} \frac{H_a^2}{8\pi}$$
(21)

Here f_{n0} is the free energy density in zero applied field. V_{ext} is the volume external to the sample volume where the field H_a exists. On the other hand, for a superconductor, the field is excluded from its interior and hence its free energy is given by

$$F_s = V f_{s0} + V_{\text{ext}} \frac{H_{\text{a}}^2}{8\pi}$$
(22)

Here we have ignored the fact that the field actually penetrates in a layer of depth $\sim \lambda$ from the surface. The difference between the above two free energies is then

$$F_n - F_s = V(f_{n0} - f_{s0}) + V \frac{H_a^2}{8\pi}$$
(23)

Since the condensation energy density is the stabilization energy

$$F_n - F_s = V \frac{H_c^2}{8\pi} + V \frac{H_a^2}{8\pi}$$
(24)

For $H_a = H_c$

$$(F_n - F_s) \mid_{H_c} = V \frac{H_c^2}{4\pi}$$

$$\tag{25}$$

This is the energy increase (sample plus the surroundings) when a sample becomes normal at $H_a = H_c$. The increase comes about because the energy source (generator) maintaining the constant field does work against the back emf. This emf is induced as the flux threading the sample changes (starts entering the bulk of the superconductor). Actually, discussion in terms of a Helmholtz free energy is appropriate for a situation where B is held constant (i.e., no induced emf). Here, we are holding H constant so the appropriate thermodynamic potential is the Gibbs free energy G. Recall that the Gibbs free energy density g is related to the Helmholtz free energy density f as follows

$$g = f - \frac{hH}{4\pi} \tag{26}$$

In the normal state, the local flux density h is equal to the average flux density B which is the same as the applied field $H_{\rm a}$. Therefore we get

$$G_n = V f_{n0} - V \frac{H_a^2}{8\pi} - V_{ext} \frac{H_a^2}{8\pi}$$
(27)

In the superconducting state, flux is excluded from the superconductor, so h = B = 0. This gives

$$G_s = V f_{s0} - V_{\text{ext}} \frac{H_a^2}{8\pi}$$

$$\tag{28}$$

The difference between the two free energies is then

$$G_n - G_s = V(f_{n0} - f_{s0}) - V \frac{H_a^2}{8\pi}$$
(29)

For an applied field equal to the thermodynamic critical field, we get $G_n = G_s$ i.e., there will be a phase equilibrium between the normal and the superconducting phase at $H_a = H_c$.

5.0.13 Field variation for a non-zero demagnetization factor

Consider a spherical sample of radius R. Outside the sphere, we have $\nabla \cdot \vec{B} = 0$ and $\vec{\nabla} \times \vec{B} = 0$. Consequently $\nabla^2 \vec{B} = 0$. Clearly, B approaches H_a as $r \to \infty$. Also, the perpendicular component of B is zero at r = R. The solution to to $\nabla^2 B = 0$ is then

$$B_{\rm out} = H_{\rm a} + \frac{H_{\rm a}R^3}{2} \overrightarrow{\nabla} \left(\frac{\cos\theta}{r^2}\right) \tag{30}$$

Therefore, the tangential component of B at the surface of the sphere can be calculated and is given below

$$B_{\theta}(r=R) = \frac{3}{2}H_{\rm a}\sin\theta \tag{31}$$

This exceeds the applied field at the equator. Even when the applied field is less than H_c , so long as it is greater than $\frac{2}{3}H_c$, B can attain a value of H_c at the equator. Therefore, for $\frac{2}{3}H_c < H_a < H_c$ there will be a coexistence of normal and superconducting regions. This has been called the "intermediate" state. Note that this is different from the "mixed" state which occurs at applied magnetic fields between H_{c1} and H_{c2} , even in the absence of demagnetization effects. In general, for ellipsoidal samples (i.e., where a demagnetization factor is well defined), when the applied field is in the range $1 - \eta < \frac{H_a}{H_c} < 1$ (where η is the demagnetization factor) an intermediate state will occur. The value of η for a sphere is 1/3, for a flat plate with field perpendicular to it is 1, for a long cylinder with the field along the axis it is 0, and for a long cylinder

with a field perpendicular to its axis it is 1/2.

Contrast of exterior-field pattern (a) when demagnetizing coefficient is nearly zero and (b) when it is 1/3 for a sphere. In (b) the equatorial field is three halves the applied field for the case of full Meissner effect, which is shown. The field pattern inside



Figure 8: Contrast of exterior-field pattern (a) when demagnetizing coefficient is nearly zero and (b) when it is 1/3 for a sphere. In (b) the equatorial field is three halves the applied field for the case of full Meissner effect, which is shown.

a spherically shaped superconducting sample is shown. The larger concentration of field lines near the equator is a result of the demagnetization factor.

6 Thermodynamics of the superconducting transition

The variation of specific heat with temperature is often a good probe of phase transitions in matter. Historically, it is Ehrenfest who first classified phase transitions based on the variation of the thermodynamic free energy with some state variable such as temperature. The order of a transition was defined as the lowest derivative of free energy (with respect to some variable) that was discontinuous at the transition. If the first derivative of free energy is discontinuous (such as the case of a solid-liquid transition where the density is discontinuous), then the transition is called first order. In the case of ferromagnetic transition of Fe for example, the susceptibility (i.e., the second derivative of free energy with field) is discontinuous and one would classify it as a second order phase transition. However, there are many cases in nature where rather than discontinuous jumps in thermodynamic variables, there is a divergence such as in the heat capacity of a superconductor. Over the decades, changes in these criteria have been proposed to accommodate such cases. The modern classification of phase transitions is based on the existence or lack thereof of a latent heat. If a phase transition involves a latent heat, i.e., the substance absorbs or releases heat without a change in temperature, then it is called a 1^{st} order phase transition. In the absence of a latent heat, the phase transition is a 2^{nd} order transition. Landau gave a theory of 2^{nd} order phase transitions and its application to superconductors.

For a type I superconductor, in general, there is an entropy change at the transition temperature (and therefore a latent heat) making the transition 1^{st} order. However, in zero magnetic field, the entropy change is zero and hence the transition is 2^{nd} order. In the normal state, the electronic contribution to the heat capacity is linear in temperature, as explained in a previous chapter. The heat capacity exhibits a jump at T_c and at lower temperatures, it falls with an exponential temperature dependence. The exponential dependence is due to the opening up of a gap in the excitation spectrum. Signatures of a gap are seen in various other properties such as thermal conductivity, current-voltage characteristics, etc.



Figure 9: The figure shows the schematic variation of enthalpy in the case of a transition involving a latent heat

The accompanying figures contrast the variation with temperature of some basic thermodynamic quantities such as the entropy S, the internal energy U, the heat capacity C and the Helmholtz free energy F.

6.0.14 Basic thermodynamics and magnetism

We will first review the basic thermodynamics related to magnetic materials. Consider a solenoid having N/L turns per unit length. A long cylindrical sample (for this choice the dimagnetisation field is negligible) is placed in the solenoid. The field in the sample is $\overrightarrow{H} = \frac{NI}{L}\hat{z}$ if a current I flows in the solenoid. The total work done on increasing the current from I to I + dI is (finally, a positive work is done by the sources in increasing the energy of the sample and the vacuum)

$$dW = -N\mathcal{E}Idt \tag{32}$$



$$= N \frac{d\phi}{dt} I dt \tag{33}$$

$$= NId\phi \tag{34}$$

$$= NAIdB \tag{35}$$

$$=\frac{V}{4\pi}\overrightarrow{H}\cdot d\overrightarrow{B}$$
(36)

$$= V(\overrightarrow{H} \cdot d\overrightarrow{M} + \overrightarrow{H} \cdot d\overrightarrow{H})$$
(37)

The first term is the magnetic work done on the sample. If the sample were not present, work would still have been done to increase the electromagnetic field energy and that is the second term. Our convention will be to include ONLY the magnetic work done on the sample and not on the vacuum. The first law of thermodynamics then reads



Figure 10: 1. Entropy 2. Heat capacity 3. Internal Energy 4. Free Energy

$$dU = TdS + V\overrightarrow{H} \cdot d\overrightarrow{M} \tag{38}$$

This is analogous to the gas equation where the work done on the gas is -PdV. The field \overrightarrow{H} is similar to -P and the magnetization is similar to the volume of the gas. Therefore we can think of the internal energy as a function of the entropy and the magnetization with

$$T = \frac{\partial U}{\partial S} \tag{39}$$

$$H = \frac{1}{V} \frac{\partial U}{\partial M} \tag{40}$$

However, S and M are not very convenient variables to work with from a practical viewpoint since what we control externally is the current and hence H and rather the temperature T than the entropy. Therefore, it will be useful to write down an energy equation expressed in terms of changes in applied magnetic field and temperature.
For a magnetic system, the Helmholtz and the Gibbs free energy can then be written down as follows

$$F(T,M) = U - TS \tag{41}$$

(10) and

$$G(T,H) = U - TS - V\overrightarrow{H} \cdot \overrightarrow{M}$$
(42)

Therefore the appropriate free energy to consider is the Gibbs free energy (which is expressed in terms of changes in temperature and magnetic field) and the entropy and magnetization can be calculated from its derivatives.

$$dG = dU - TdS - SdT - V\overrightarrow{H} \cdot d\overrightarrow{M} - V\overrightarrow{M} \cdot d\overrightarrow{H}$$
(43)

$$= -SdT - V\vec{M} \cdot d\vec{H} \tag{44}$$

$$S = -\frac{\partial G}{\partial T} \tag{45}$$

$$M = -\frac{1}{V}\frac{\partial G}{\partial H} \tag{46}$$

From the Gibbs free energy we can also write the Helmholtz free energy $F = G + V \overrightarrow{H} \cdot \overrightarrow{M}$ as also the internal energy U.

6.0.15 Application to superconductors

We will now calculate the Gibbs free energy difference between the normal state and superconducting state. At a fixed temperature, the applied magnetic field is varied to a critical field H_c , so that superconductivity is destroyed and the substance becomes normal. Therefore, we integrate the dG equation from 0 to H_c . This yields

$$G_{\rm s}(T, H_{\rm c}) - G_{\rm s}(T, 0) = -V \int \vec{M} \cdot d\vec{H}$$
(47)

Since, in the Meissner state, $H = -4\pi M$, the integral gives $V \frac{H_c^2}{8\pi}$. At the critical field, of course, the free energy of the superconductor is equal to that of the normal phase

i.e., $G_{\rm s}(T, H_{\rm c}) = G_{\rm n}(T, H_{\rm c})$. Also, since the normal state has negligible magnetization, $G_{\rm n}(T, H_{\rm c}) = G_{\rm n}(T, 0)$. Therefore, in zero magnetic field, the superconducting state is lower in energy (Gibbs free energy) than the normal state by an amount $V \frac{H_{\rm c}^2}{8\pi}$. Clearly in zero field, the Gibbs and the Helmholtz free energy are the same and so the above statement is also valid for Helmholtz free energy. The above difference in the energy density between the normal and the superconducting state (the normal state being of higher energy) is called the condensation energy. The entropy difference can be calculated by taking the derivative of the Gibbs free energy with temperature. The equation below gives the difference in the entropy per unit volume between the superconducting and the normal state.

$$S_s(T, H_c) - S_n(T, H_c) = -\frac{H_c}{4\pi} \frac{dH_c}{dT}$$

$$\tag{48}$$

Therefore, when the external field is zero, the entropy difference is zero and there is no latent heat. The transition is then second-order. On the other hand, in the presence of a field there is a non-zero entropy change and hence the transition is of first-order.

7 Ginzburg-Landau phenomenological theory

Ginzburg Landau theory The next several lectures will discuss the phenomenological theory of superconductivity proposed by Ginzburg and Landau. This starts with the proposal of a free energy functional involving an expansion in terms of an order parameter. A minimization procedure leads to the Ginzburg-Landau equations. These can be solved and various predictions made regarding the properties of the superconducting state. Introduction The Ginzburg-Landau (GL) theory while not being a microscopic theory (but rather a macroscopic theory) provides a physically intuitive picture of the properties of superconductors. Broadly speaking, it starts by proposing a "wave function" $\psi(r)$ as a complex order parameter, characteristic of the superconducting state. Consequently, $|\psi(r)|^2$ is considered as the density of superconducting electrons $n_s(r)$. Next, it is assumed that the free-energy density of a superconductor can be expanded in powers of the supercarrier density as also in powers of $|\nabla \psi|^2$. After doing so, the free-energy density is minimized with respect to the order parameter to finally obtain the energy stabilization of the superconducting state compared to the normal state. Note that the mechanism which drives superconductivity is not addressed by this theory but rather a practical approach is taken whereby given the fact that superconductivity exists, equations are provided to model their properties and in the process, new length scales such as the penetration depth and coherence length emerge. Functional description in different limiting cases is obtained. Further the existence of Type II superconductors can be hypothesized. The GL theory was proposed in 1950 (after Landau had given a general theory of second order phase transitions) which was before the BCS theory and in 1959, after the publication of the BCS theory, Gor'kov showed that the GL theory could in fact be rigorously obtained as a limiting case of the microscopic BCS theory. Free energy formulation We now introduce the GL free energy density and then carry out a derivation of the GL equations followed by their application to specific geometries.

$$f_s = f_{n0} + \alpha |\psi|^2 + \frac{\beta}{2} |\psi|^4 + \frac{1}{2m^*} |(p - \frac{qA}{c})\psi|^2 + \frac{h^2}{8\pi}$$
(49)

Let us now elaborate on what the various terms in the RHS above are. Here, f_{n0} is the (Helmholtz) free energy density in the absence of an applied magnetic field. Since the theory is considered to be applicable at temperatures close to the transition (where $n_s(r)$ is expected to be small), higher order terms in $|\psi(r)|^2$ and $|\nabla \psi|^2$ are neglected. One might ask as to why can we not expand in powers of the complex order parameter $\psi(r)$ as opposed to the square of its magnitude? The answer to this is that since the free energy is a real quantity, we need the RHS to be real as well. In that case, why can we not expand in powers of $|\psi|$ which is real and would then give a cubic term? The answer to this is that $|\psi|^3$ is not analytic at the origin and hence not admissible (derivative is different as one approaches the origin from different directions). The kinetic energy term takes into account the effect of applied external magnetic fields (here \vec{p} is the operator $-i\hbar \vec{\nabla}$ and $(\vec{p} - \frac{q\vec{A}}{c})$ is the operator for the kinetic momentum; $(p_{kin} + \frac{qA}{c})$ is the canonical momentum). The mass and charge have been taken as m^* and q since at this point a connection with the electronic (or of pairs) mass and charge is not evident. The total kinetic energy would be given by

$$\int d^3r \frac{1}{2m^*} \psi^* (\overrightarrow{p} - \frac{q\overrightarrow{A}}{c})^2 \psi \tag{50}$$

Integrating this by parts, the first term will vanish at the extrema since the wave function should be well behaved. The second term gives

$$\int dr \frac{1}{2m^*} |(\overrightarrow{p} - \frac{q\overrightarrow{A}}{c})\psi|^2 \tag{51}$$

The integrand is then the energy density and this explains the electromagnetic term in the GL energy density. The last term is simply the magnetic self energy of the normal phase while f_{n0} is free energy of the normal phase in zero applied field. Free energy in the absence of field, currents and gradients Let's us first consider the simplest situation where there are no magnetic fields nor are there any gradients of the wave function (this would correspond to a region well inside a superconductor). Here the free energy density reduces to

$$f_s = f_n + \alpha |\psi|^2 + \frac{\beta}{2} |\psi|^4 \tag{52}$$

It is evident that both the coefficients α and β can not be positive since that would not give rise to an energy lowering for finite $|\psi|^2$ and would in fact describe the normal state. The only physically meaningful possibility is for α to be negative and β to be positive (either (i) the inverse or (ii) both having a negative sign would give lowest free energy for arbitrarily high values of $|\psi|^2$ and hence are not admissible). For a negative α , taking the derivative of $f_s - f_n$ with respect to $|\psi|^2$, we get

$$|\psi|^2 = -\frac{\alpha}{\beta} \tag{53}$$

The schematic variation of the free energy stabilization as a function of $|\psi|^2$ is shown in figure for both positive and negative α .

Now let us study general case of nonuniform Superconductor in magnetic field.



Figure 11: The schematic variation of the free energy stabilization as a function of $|\psi|^2$ is shown in figure for both positive and negative α .

At T tends to T_c the Gibbs free energy density can be written as

$$G_s(H) = G_n + \alpha |\psi|^2 + \frac{\beta}{2} |\psi|^4 + \frac{1}{2m^*} |(-i\hbar\nabla\psi - 2e\mathbf{A})\psi|^2 + \frac{B^2}{2\mu_0} - \mathbf{B}.\mathbf{H}$$
(54)

 G_n is the free energy of the system in the normal state. ψ is the complex order parameter. B is the magnetic field and H is the magnetic field strength. where the kinetic energy density is written using the fact that 2mv = p2eA and knowing that $\hat{p} = i\nabla$. The quantity m^* means the double electron mass. The Gibbs free energy is

$$G_{s}(H) = G_{n} + \int \alpha |\psi|^{2} + \frac{\beta}{2} |\psi|^{4} + \frac{1}{2m^{*}} |(-i\hbar\nabla\psi - 2e\mathbf{A})\psi|^{2} + \frac{\nabla \times A^{2}}{2\mu_{0}} - \nabla \times A.H$$
(55)

Our task is to find the functions $\psi(r)$ and A(r). Taking into account that $|\psi|^2 = \psi \psi^*$, we, first, write variation of energy with respect to ψ , assuming that ψ and A are constant and then using the identity:

$$\nabla(\delta\psi^*.v) = v.\nabla\delta\psi^* + \delta\psi^*.\nabla.v \tag{56}$$

$$\int dV (\nabla \delta \psi^*) v = -\int \delta \psi^* . \nabla v . dV + \int \nabla (\delta \psi^* . v) dV$$
(57)

$$= -\int \delta\psi^* \cdot \Delta v \cdot dV + \oint \delta\psi^* \cdot v \cdot dS \tag{58}$$

where we have used the Gauss theorem.

substituting equation (5) in equation (2); Now the second term becomes

$$\int dV [(i\hbar(\nabla\delta\psi^*) - 2e\mathbf{A})v]$$
(59)

$$\int dV [i\hbar v (\nabla \delta \psi^*) - 2e\mathbf{A}v] \tag{60}$$

$$\int dV (-i\hbar\delta\psi^*\nabla v - 2e\mathbf{A}v) + \oint_s ...d\mathbf{S}$$
(61)

$$\int dV \delta \psi^* (-i\hbar \nabla - 2e\mathbf{A})^2 \psi + \oint_s \dots d\mathbf{S}$$
(62)

$$0 = \int dV [\alpha \psi + \beta \psi |\psi|^2 + \frac{1}{4m} (-i\hbar \nabla - 2eA)^2 \psi] \delta\psi^* + \oint_s [-i\hbar \nabla \psi - 2eA\psi] \delta\psi^* dS \quad (63)$$

This expression can be equal to zero for arbitrary variation $\delta \psi^*$ only if the expressions in the square brackets are both equal to zero. Thus, we arrive to the first Ginzburg-Landau equation and to the boundary condition which supplements it:

$$\alpha\psi + \beta\psi|\psi^2| + \frac{1}{4m}(-i\hbar\nabla - 2eA)^2\psi = 0$$
(64)

$$(-i\hbar\nabla\psi - 2eA\psi).\mathbf{n} = 0 \tag{65}$$

where **n** is the vector normal to the surface of the SC. It is easy to check that variation of the Gibbs energy with respect to ψ will give an equation complex conjugate. Let us do the variation with respect to A.

$$\delta_A G_S = \int dV [\frac{1}{4m} \delta_A [(i\hbar\nabla\psi^*)(-i\hbar\nabla\psi - 2e\mathbf{A}\psi)] + \frac{\nabla\times\mathbf{A}\nabla\times\delta\mathbf{A}}{\mu_0} - \mathbf{H}\nabla\times\delta\mathbf{A}] \quad (66)$$
$$= \int dV [\frac{1}{4m} (2e\psi^*\delta\mathbf{A})(-i\hbar\nabla\psi - 2e\mathbf{A}\psi) + \frac{1}{4m} (i\hbar\nabla\psi^* - 2e\psi^*)(-e\psi\delta\mathbf{A}) + (\frac{\nabla\times\mathbf{A}}{\mu_0} - \mathbf{H})\nabla\times\delta\mathbf{A}] \quad (67)$$

now we shall use the identity

$$\mathbf{a} \cdot \nabla \times \mathbf{b} = \mathbf{b} \nabla \times \mathbf{a} - \nabla \cdot [\mathbf{a} \nabla \times \mathbf{b}]$$
(68)

so the last term becomes

$$\int dV \left(\frac{\nabla \times \mathbf{A}}{\mu_0} - \mathbf{H}\right) \nabla \times \delta \mathbf{A} = \int dV \delta \mathbf{A} \frac{\nabla \times \mathbf{A} \nabla \times \mathbf{A}}{\mu_0} - \oint dS \left[\delta \mathbf{A} \times \left(\frac{\nabla \times \mathbf{A}}{\mu_0} - \mathbf{H}\right)\right]$$
(69)

The last integral vanishes because $\delta \mathbf{A}|_S = 0$ Finally, after some transformations,

$$\int dV \left[\frac{i\hbar e}{2m}(\psi^* \nabla \psi - \psi \nabla \psi^*) + \frac{2e^*}{m} \mathbf{A} |\psi|^2 + \frac{\nabla \times \nabla \times \mathbf{A}}{\mu_0}\right] \delta \mathbf{A} = 0$$
(70)

This integral can be equal to zero for any variation $\delta \mathbf{A}$ only if the expression in the square brackets is equal to zero. This leads us to the second Ginzburg-Landau equation:

$$\frac{\nabla \times \nabla \times \mathbf{A}}{\mu_0} = -\frac{i\hbar e}{2m} (\psi^* \nabla \psi - \psi \nabla \psi^*) - \frac{2e^*}{m} \mathbf{A} |\psi|^2$$
(71)

We introduce the dimensionless wave function

$$\psi(r) = \frac{\psi(r)}{\psi_0} \tag{72}$$

where $\psi_0 = \frac{n_s}{2} = \frac{|\alpha|}{\beta}$.

Further, we introduce the following new quantities

$$\xi^2 = \frac{\hbar^2}{4m|\alpha|};\tag{73}$$

$$\lambda^2 = \frac{m\beta}{2e^2|\alpha|\mu_0} \tag{74}$$

Then both GL equations can be rewritten in the more convenient and compact form

$$\xi^{2}(i\nabla + \frac{2\pi}{\phi_{0}}\mathbf{A})^{2}\psi - \psi + \psi|\psi|^{2} = 0$$
(75)

$$\lambda^2 \nabla \times \nabla \times \mathbf{A} = -i\phi_0(\psi^* \nabla \psi - \psi \nabla \psi^*) - |\psi|^2 \mathbf{A}$$
(76)

The boundary condition leads to the following boundary condition for ψ

$$(i\nabla + \frac{2\pi}{\phi_0}\mathbf{A})\mathbf{n}\psi = 0 \tag{77}$$

Actually even more general boundary condition

$$(i\nabla + \frac{2\pi}{\phi_0}\mathbf{A})\mathbf{n}\psi = i\alpha\psi \tag{78}$$

this also gives $j_S = 0$ at the boundary.

7.0.16 Determination of coefficients a and b in the absence of fields and gradients

The cross terms in the product above cancel out and some simple algebra yields for the energy $\frac{1}{2m^*}(\hbar^2 \vec{\nabla} (|\psi|)^2 + (\hbar \vec{\nabla} \varphi - \frac{q\vec{A}}{c})^2 |\psi|^2)$. The first term gives the kinetic energy from the regions where there is a variation of the magnitude of the wave function such as at the boundary or in a domain wall. The second term involves the gradient of the phase and is associated with the supercurrent. This can be seen as follows. The quantum mechanical probability current density for a charged particle in a magnetic field is given by

$$\frac{\hbar}{2m^*i}(\psi^*\overrightarrow{\nabla}\psi - \psi\overrightarrow{\nabla}\psi^*) - \frac{q}{m^*c}\overrightarrow{A}|\psi|^2 = \frac{1}{m^*}(\hbar\overrightarrow{\nabla}\varphi - \frac{q\overrightarrow{A}}{c})^2|\psi|^2$$
(79)

Note that to get the electrical current density, one has to multiply this by the charge. Therefore the last term in the kinetic energy is $\frac{J^2m^*}{2q^2|\psi|^2}$. We know that the vector potential is not unique and we can add the gradient of a scalar and still get the same magnetic field. We choose a gauge here (which is called the London Gauge) which cancels the $\hbar \vec{\nabla} \varphi$ term. Then the kinetic energy term (in the absence of a gradient of the magnitude of the wave function) becomes $\frac{q^2A^2|\psi|^2}{2m\epsilon^2} = \frac{A^2}{8\pi} \frac{4\pi q^2|\psi|^2}{mc^2}$. We have earlier seen that in the London model $\lambda_L^2 = \frac{mc^2}{4\pi nc^2}$. The kinetic energy term now has the identification $\frac{A^2}{8\pi\lambda_{GL}^2}$ where $|\psi|^2$ is proportional to the superconducting carrier density, q is the effective charge and m^* the effective mass. While it was not known at the time the theory was formulated as to what the effective mass and charge were (they were thought to be the bare electrons, $m^* = 2m$ and $q^* = 2q$. Consequently, the density of superconducting pairs n_s^* is half the density of electrons n_s . As a result, the expression for the GL penetration depth is the same as the London penetration depth.

$$\alpha = -\frac{H_c^2}{4\pi n_s} = -\frac{q^2}{mc^2} H_c^2(T) \lambda_{GL}^2(T)$$
(80)

We can now try to give physical meaning to the coefficients α and β . Since $\frac{H_c^2}{8\pi} = \frac{\alpha^2}{2\beta}$

and $\beta = -\frac{\alpha}{n_s}$, we have

$$\alpha = -\frac{H_c^2}{4\pi n_s} = -\frac{q^2}{mc^2} H_c^2(T) \lambda_{GL}^2(T)$$
(81)

$$\beta = \frac{4\pi q^4}{m^2 c^4} H_c^2(T) \lambda_{GL}^4(T).$$
(82)

Of course, $|\psi_{\infty}|^2 = n_s^* = \frac{mc^2}{4\pi q^2 \lambda_{GL}^2(T)}$. Here, $H_c(T)$ and $\lambda_{GL}(T)$ are experimentally measurable quantities. Using the empirical temperature dependence for the critical field and the penetration depth i.e., $H_c(T) \propto (1 - (\frac{T}{T_c})^2)$ and $\frac{1}{(\lambda_{GL}(T))^2} \propto (1 - (\frac{T}{T_c})^4)$, we obtain the temperature dependence of α and β .

$$\alpha(T) \propto \frac{(1 - (\frac{T}{T_c})^2)^2}{(1 - (\frac{T}{T_c})^4)^2} = \frac{(1 - (\frac{T}{T_c})^2}{(1 + \frac{T}{T_c})^2)^2} \simeq 1 - \frac{T}{T_c}$$
(83)

$$\beta(T) \propto \frac{(1 - (\frac{T}{T_c})^2)^2}{(1 - (\frac{T}{T_c})^4)^2} = \frac{1}{(1 + \frac{T}{T_c})^2)^2} \simeq \text{constant}$$
(84)

Here we have expanded in powers of $1 - \frac{T}{T_c}$ and kept only the leading terms. In all the above treatment, we have ignored any spatial variation of the order parameter inside the superconductor.

7.0.17 GL equations in presence of fields currents and gradients

We have presently solved for a situation where there were no fields, currents or gradients. In case these are present we need to minimize the free energy containing the contribution of these terms,

$$F = \int f_s d^3x = \int (f_{n0} + \alpha |\psi|^2 + \frac{\beta}{2} |\psi|^4 + \frac{1}{2m^*} |(-\iota\hbar\vec{\nabla} - q\vec{A}c)\psi|^2 + \frac{\hbar^2}{8\pi}) d^3x \quad (85)$$

This is really a free energy functional which is a scalar number but it depends on $\psi(\text{and } \psi^* \text{and } A)$ at all coordinate points in the system. The free energy has to be minimized with respect to variations of ψ , ψ^* and A. This means the following.

$$dF = \int \left[\frac{\partial f_s}{\partial \psi} d\psi + \frac{\partial f_s}{\partial \psi^*} d\psi^* + \frac{\partial f_s}{\partial A} dA\right] d^3x = 0$$
(86)

To evaluate the above, let us write the free energy density in an expanded form

$$f_s - f_{n0} = \alpha |\psi|^2 + \frac{\beta}{2} |\psi|^4 + \frac{1}{2m^*} |(-\iota\hbar\vec{\nabla} - q\vec{A}c)\psi|^2 + \frac{\hbar^2}{8\pi}$$
(87)

$$= \alpha \psi^* \psi + \frac{\beta}{2} \psi^2 \psi^{*2} + \frac{1}{2m^*} [\hbar^2 |\vec{\nabla}\psi|^2 + \frac{q^2 \psi^* \psi A^2}{c^2} + \frac{\iota \hbar q A}{c} (\psi^* \vec{\nabla}\psi - \psi \vec{\nabla}\psi^*)] + \frac{(\vec{\nabla} \times \vec{A})^2}{8\pi}$$
(88)

Now we take the partial derivative with respect to ψ^* . Note that $\int \frac{1}{2m^*} \frac{\partial}{\partial \psi^*} [\hbar^2 | \vec{\nabla} \psi |^2] d\psi^* d^3 x$ can be integrated by parts (write $| \vec{\nabla} \psi |^2 = \vec{\nabla} \psi^* \cdot \vec{\nabla} \psi$ and use $\int \vec{\nabla} u \cdot \vec{v} d^3 x =_S \int u \vec{v} \cdot dS - \int u \vec{\nabla} \cdot \vec{v} d^3 x$) to give $\frac{-\hbar^2}{2m^*} \int \nabla^2 \psi d\psi^* d^3 x +$ a surface integral (let us call this S_1). Similarly

$$\int \frac{1}{2m^*} \frac{\partial}{\partial \psi^*} \left[\frac{q^2 \psi^* \psi A^2}{c^2} \right] d\psi^* d^3 x = \int \frac{q^2 \psi A^2}{2m^* c^2} d\psi^* d^3 x \tag{89}$$

and

$$\int \frac{1}{2m^*} \frac{\partial}{\partial \psi^*} \left[\frac{\iota \hbar q A}{c} (\psi^* \vec{\nabla} \psi) \right] d\psi^* d^3 x = \int \frac{\iota \hbar q A}{2m^* c} (\vec{\nabla} \psi) d\psi^* d^3 x \tag{90}$$

Further $\int \frac{1}{2m^*} \frac{\partial}{\partial \psi^*} \left[\frac{i\hbar q \vec{A} \cdot}{c} (-\psi \vec{\nabla} \psi^*)\right] d\psi^* d^3 x$ can be integrated by parts. This integration yields the above volume integral + a surface integral (let us call this S2). Combine the two surface integrals S1 and S2, and impose the following boundary condition to make them zero. (normal component of $(-i\hbar \vec{\nabla} - \frac{q \vec{A}}{c})\psi$ on the bounding surface is zero). Putting the integrand corresponding to $\frac{\partial f_s}{\partial \psi^*} = 0$, we get

$$\alpha\psi + \beta|\psi|^2\psi + \frac{1}{2m^*}|(-\iota\hbar\vec{\nabla} - \frac{q\vec{A}}{c})^2|\psi = 0$$
(91)

This is called the first GL equation. Minimizing the energy with respect to ψ just yields the complex conjugate of this equation. Next we need to minimize the free energy wrt A. For this we need to consider only the terms in the free energy which depend on A.

The variation of the above terms due to a variation δA to A is $\left[\frac{q^2|\psi|^2 A}{m^* c^2} + \frac{i\hbar q}{2m^* c} (\psi^* \overrightarrow{\nabla} \psi - \psi \overrightarrow{\nabla} \psi^*)\right] \delta A$. Now we need to do the same for the $\frac{(\overrightarrow{\nabla} \times \overrightarrow{A})^2}{8\pi}$ term.

$$(\overrightarrow{\nabla} \times (\overrightarrow{A + \delta A}))^2 \tag{92}$$

$$= \overrightarrow{\nabla} \times (\overrightarrow{A + \delta A}) \cdot \overrightarrow{\nabla} \times (\overrightarrow{A + \delta A})$$
(93)

The change in energy due to variation of A (keeping only the term linear in δA) is $\frac{(\overrightarrow{\nabla} \times \overrightarrow{A}) \cdot (\overrightarrow{\nabla} \times \overrightarrow{\delta A})}{4\pi}$.

The two GL equations are now summarized below:

$$\alpha\psi + \beta|\psi|^2\psi + \frac{1}{2m^*} \left(-\iota\hbar\overrightarrow{\nabla} - \frac{q\overrightarrow{A}}{c}\right)^2\psi = 0 \tag{94}$$

$$\overrightarrow{J} = \frac{q|\psi|^2}{m^*} \left(\hbar \overrightarrow{\nabla} \varphi - \frac{q \overrightarrow{A}}{c}\right) \tag{95}$$

The first equation is very similar to the Schroedinger equation (except for the $\beta |\psi|^2 \psi$ term). The non-linear term tends to favor wave functions which have small or no spatial variations. In the case of the appearance of an interface (such as between a superconductor and a normal metal), clearly the wave function (or the order parameter) will become zero at the surface while deep inside the superconductor it will have its maximum value.

8 Coherence length and penetration length

Consider a Superconductor semispace x > 0 and a thin film of normal metal at -t < x < 0. It is clear that close to the Normal state boundary the concentration of Superconducting electrons (order parameter ψ) will be a bit smaller than in the bulk of Superconductor ($|\psi| = 1$). Let us find the characteristic length scale on which ψ recovers. Since we have 1D problem, $\psi = \psi(x)$. We choose such a calibration of A that ψ is real. In this case the GL equation reads

$$-\xi^2 \frac{d^2 \psi}{dx^2} - \psi + \psi^3 = 0$$
(96)

Assuming that the order parameter changes just a little, i.e., that $\psi = 1 - \epsilon(x)$ we can rewrite this equation leaving only terms linear in ϵ :

$$\xi^2 \frac{d^2 \epsilon}{dx^2} - 2\epsilon = 0 \tag{97}$$

The solution is

$$\epsilon = \epsilon(0)e^{-\frac{\sqrt{2}x}{\xi}} \tag{98}$$

 $\xi~$ is a characteristic length on which the order parameter changes.

 ξ is the so-called coherence length.

The quantity λ is known London penetration depth (penetration depth of magnetic field).

now

$$\lambda^2 = \frac{m}{2e^2|\psi_0|^2\mu_0} = \frac{m}{e^2n_S\mu_0}$$
(99)

Also the temperature dependencies are

$$\lambda \propto (T_c - T)^{\frac{-1}{2}} \tag{100}$$

$$\xi \propto (T_c - T)^{\frac{-1}{2}}$$
 (101)

Using $\lambda(T)$ and $\xi(T)$ we introduce the Ginzburg-Landau parameter

$$\kappa = \frac{\lambda}{\xi} \tag{102}$$

$$\kappa = \sqrt{2} \frac{2\pi}{\psi_0} \lambda^2 H_{cm} \tag{103}$$

 H_{cm} is the critical magnetic field.

$$\sqrt{2}\mu_0 H_{cm} = \sqrt{2}B_c = \frac{\phi_0}{2\pi\lambda\xi} \tag{104}$$

1. If $\lambda \ll \xi$, the magnetic field can penetrate only a small distance into the surface region of a superconductor but not into its interior. Such a superconductor has only one critical magnetic field and is called type-I superconductor.

If $\lambda >> \xi$, we expect that the magnetic field can penetrate deep into the interior of a superconductor and that small normal regions of size ξ will be formed within the superconductor while it remains in the superconducting phase. Such a superconductor has two critical magnetic fields and is called the type-II superconductor.

9 vortex formation in type-II superconductor

In type II superconductor, the magnetic field can penetrate into the interior of the superconductor while it still remains to be superconducting. In this case, the magnetic flux passes through only the normal regions of the superconductor. The normal regions are tube shaped and are referred to as vortices. also the magnetic flux contained in each vortex is quantized. This demonstrates that superconductivity is a macroscopic quantum phenomenon.

10 Field and order parameter variation inside a vortex

Magnetic properties of classic Type-II superconductors As seen earlier, superconductors with $\kappa > \frac{1}{\sqrt{2}}$ have solutions of the GL equations with non-zero order parameter for fields less than H_{c2} . The Abrikosov solution has a regular array of vortices. Each unit cell of the flux line array has total flux equal to $\phi_0 = \frac{hc}{2e}$. We will now examine the solution to the GL equation at H_{c1} where the first vortex enters the superconductor. Field and order parameter variation inside a vortex Consider a situation where the applied field is small such that the vortex density is small. No interaction between vortices is considered. As argued earlier, the term in a magnetic system, analogous to the -PdV term of a hydrostatic system, is HdM. Therefore, when H is the control parameter (held constant), the relevant free energy is the Gibbs free energy. At $H = H_{c1}$

$$G_s \mid_{no\,flux} = G_s \mid_{first\,vortex} \tag{105}$$

Note that $G = F - \frac{H}{4\pi} \int h d^3 r$ where h is the B-field at the location r. For the case of no flux in the superconductor, $G_s = F_s$. The LHS of equation (1) is then F_s . Imagine that the vortex has an extra free energy per unit length ϵ_1 . For a vortex of length L, the energy contribution is $\epsilon_1 L$. Therefore, RHS of equation (1) is $F_s - \frac{H_{c1}}{4\pi} \int h d^3 r + \epsilon_1 L$ Remember that h is the local B-field. Hence $\int h d^3 r$ (which is integrated over the volume of the vortex) is $\phi_0 L$ where ϕ_0 is the flux quantum. The energy lowering by the $\int h d^3 r$ term is matched by the increase due to the vortex energy $\epsilon_1 L$ such that the net change in G is zero. This leads to

$$H_{c1} = \frac{4\pi\epsilon_1}{\phi_0} \tag{106}$$

Now consider the extreme Type-II limit $\kappa \gg 1$, i.e., $\lambda \gg \xi$. The core of the vortex can be thought to have a radius ξ . Since the B-field falls over a length λ (starting from the center of the vortex) which is much greater than ξ , over most of the vortex the order parameter is at its full value and the superconductor can be treated as a London superconductor.

Let $\psi = \psi_{\infty} f(r) e^{i\theta}$ represent the vortex wave function (axial symmetry is built into this). Now the vector potential

$$\overrightarrow{A} = A(r)\hat{\theta} \tag{107}$$

and

$$A(r) = \frac{1}{r} \int_{0}^{r} rh(r) dr$$
 (108)

In the London gauge $\overrightarrow{\nabla} \cdot \overrightarrow{A} = 0$. Here that is not the case since A_{∞} here is $\frac{\phi_0}{2\pi r}$ while in the London gauge $A_{\infty} = 0$. Also, near the center of the vortex $A(r) \simeq \frac{h(0)r}{2}$. This is because $A(r) = \frac{1}{r} \int_0^r r h(r) dr = \frac{1}{r} \frac{r^2}{2} h(0)$. Now substitute $\psi = \psi_{\infty} f(r) e^{i\theta}$ in the first GL equation given below

$$\alpha\psi + \beta|\psi|^2\psi + \frac{1}{2m^*} \left| \left(-\iota\hbar\overrightarrow{\nabla} - \frac{q\overrightarrow{A}}{c} \right)^2 \right|\psi = 0$$
(109)

Writing the gradient in cylindrical coordinates and then simplifying one gets

$$f - f^{3} - \xi^{2} \left[\left(\frac{1}{r} - \frac{2\pi A}{\phi_{0}} \right)^{2} f - \frac{1}{r} \frac{d}{dr} \left(r \frac{df}{dr} \right) \right] = 0$$

$$(110)$$

Similarly, write the equation for the current density $\overline{J} = \frac{c}{4\pi} \overline{\nabla} \times \overline{h}$.

We need simultaneous solutions to the two GL equations. This requires numerical methods for a general case. Consider the limiting situation $r \to 0$ i.e., near the center

of the vortex. Here $A(r) = \frac{h(0)r}{2}$. Substitute this in the first GL equation obtained above. Further, assume a power law solution $f = cr^n$ where n is positive. For small r, the leading term will be $-\xi^2 cr^{n-2}(1-n^2)$. this should go to zero for small r. Therefore n = 1 and f varies linearly near the center of the vortex. To determine higher order corrections, consider that f has terms quadratic and cubic in r in addition to the linear one. This leads to

$$f \approx crbigg \left[1 - \frac{r^2}{8\xi^2} \left(1 + \frac{h(0)}{H_{c2}}\right)\right]$$
 (111)

where c is a normalization constant. This shows that the rise of f(r) begins to saturate at $r \approx 2\xi$. The approximate general solution is $f \approx \tanh \frac{\nu r}{\xi}$ where ν is a constant of order 1.

11 BEC and symmetries

11.1 Bose-Einstein condensation

From thermodynamics we know that, within a gas, all the particles behave in the same manner and in principle they can occupy certain quantum states, that is certain energy states. If these particles are fermions, two or more of them cannot occupy the same quantum state as they obey the Pauli exclusion principle. However, if they are bosons, any number of them can occupy the same quantum state. When we put these particles in a given configuration, they will get distributed in the energy levels of such a configuration, with an increasing occupation of the states of minimum energy as we lower the temperature. For a collection of bosons, and in the limit where the temperature goes to zero, all the particles are going to occupy the ground state of the system. Therefore, for a sufficiently low temperature, the majority of the particles are in the same quantum state, and have the same velocity. In this way, the collection of bosons behaves like a macroscopic fluid with new properties, such as superfluidity. In order to study these properties, it is only necessary to concentrate on the ground state.

which is an eigenvector of the Hamiltonian associated to the physical situation under consideration, and whose eigenvalue corresponds to the energy of that level. Thus, we have to find the eigenvectors with minimum eigenvalue for the Hamiltonian that describes our system. This Hamiltonian includes all the interactions between every pair of particles. In order to simplify the problem, we make use of the mean-field approximation, which means that the action felt by a given particle due to the rest is substituted by the mean action of the fluid over the particle. This approximation is good if one can neglect the correlations in the gas, that is, if the gas is diluted, which is the case for the condensates obtained experimentally. In this way, we have changed a complex model for the interaction among bosons by a very simplified one that is valid for diluted gases. The interactions between bosons are not necessary for condensation to take place; however, they play a very important role in the properties of the condensate as we will see later on. In this sense, using the usual Schrodinger equation is not enough; instead we require the use of the Gross-Pitaevskii equation (GPE).

11.2 The two fluid model for BECs

When helium gas (isotope ${}^{4}He$) is cooled down, it is possible to observe a transition into a liquid phase at a temperature of 4.2 K and at a pressure of 1 atmosphere. If we continue cooling down the system well under 2.17 K, this liquid phase acquires highly unusual properties: it becomes a superfluid. Superfluidity is manifested, among other effects, by the lack of viscosity; in other words, the liquid flows without friction. Thus, if we launch some of this superfluid helium into a ring-shaped channel for example, it will not stop. If we try to move an object up to a certain velocity across the surface of the liquid, it will not experience any resistance. Another interesting feature of the system is the creation of vortices, which can be seen as small twisters inside the liquid, that behave quite differently from the ones observed in water for example. These vortices have certain quantisation properties (the velocity of the liquid cannot take any arbitrary value). These properties, observed in liquid helium ${}^{4}He$ in 1937, were also seen in helium ${}^{3}He$ at a lower temperature and more recently in Bose-Einstein condensates. As discussed above, a BEC is a particular state of matter achieved at ultra cold temperatures. These phenomena are the macroscopic manifestation of quantum effects. In the 1950s, Landau and Ginzburg proposed to model the electrons that give rise to superconductivity as a superfluid with the aid of an equation of the simplified form

$$i\hbar\frac{\partial u}{\partial t} = -\frac{\hbar^2}{2}\nabla^2 u + gu(|u|^2 - 1), \qquad (112)$$

in the frame of a two-fluid model, where $|u|^2$ represents the superfluid density, which flows without friction, whereas the rest of the matter is supposed to be in a normal fluid state, and g is a proportionality constant with dimensions of energy. This equation renders satisfactory predictions, but its usage is limited due to the fact that the interactions in a liquid such as helium are fairly strong. However, the Landau-Ginzburg model is a particular case of an equation that emerges quite naturally when we study the behavior of Bose-Einstein condensates up to a first order approximation. This equation is known as the Gross-Pitaevskii equation (GPE).

$$i\hbar\frac{\partial\psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_{ext} + g|\psi|^2\right)\psi,\tag{113}$$

where m is the mass of the atoms of the condensate, $|\psi|^2$ is the atomic density, V_{ext} represents an external potential and g is a parameter that measures the atomic interactions. The GPE has the same mathematical form as the nonlinear Schrodinger equation (NLSE), which is basically the Schrodinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_{ext}\right)\psi,\tag{114}$$

plus a nonlinear term that in this case takes into account the interaction between the particles. In the case of Bose-Einstein condensates, the interactions are so weak that the predictions made with this equation are very good.

11.3 The Gross-Pitaevskii equation

Bose-Einstein condensate is obtained from a collection of bosons in the ground state at very low temperatures. We can therefore ask about the energy of the ground state and use this to provide us with information about the system as we can indeed do for any other gas. The general Hamiltonian that describes the system is given by

$$\hat{H} = \sum_{i=1}^{N} \left(\frac{\mathbf{p}_{i}^{2}}{2m} + V_{ext}(\mathbf{r}_{i}) \right) + \frac{1}{2} \sum_{i=1}^{N} \sum_{j \neq i}^{N} V\left(|\mathbf{r}_{i} - \mathbf{r}_{j}| \right),$$
(115)

where the first term on the right-hand-side is the kinetic energy of the *i*-th particle, the following term represents the external effects introduced by the external potential V_{ext} , and the final term represents the interactions between the N particles. The ground state corresponds to the minimum energy and thus we can find it by minimizing it. Now we need to minimize the free energy $F = E - \mu N$, where E is the energy and ' μ ' is the chemical potential. Given a Hamiltonian \hat{H} and a wave function ψ , we can obtain the energy as follows:

$$E(\psi) = \frac{\langle \psi | \hat{H} | \psi \rangle}{\langle \psi | \psi \rangle},\tag{116}$$

and as such we can use this expression to minimize the free energy F. In the condensate we have N particles and we can thus associate a wave function ψ_i to every one of them. However, in order to capture the essential aspects of the problem we make a meanfield approximation. This means that for one particle, all the rest have the same status as they all are in the same independent state $|\psi\rangle$ and thus we can drop the labeling of the wave functions. In this way we need to minimize the free energy over a space of functions of the type $|\Psi\rangle = |\psi\rangle \otimes |\psi\rangle \otimes \cdots \otimes |\psi\rangle$, where \otimes represents the tensor product and thus $|\Psi\rangle$ is the N-particle tensor product wave function; we are considering the following normalization $\langle \Psi | \Psi \rangle = 1$. This approximation is valid if the condensate is not very dense; otherwise, the interactions with the closer neighbors would be much stronger than with the particles that are farther apart. Our problem is thus reduced to minimizing $F(\Psi) = \langle \Psi | \hat{H} | \Psi \rangle - \mu \langle \Psi | \Psi \rangle$. Let us now compute each of the terms involved in this calculation. For the kinetic energy term, we have that

$$\langle \Psi | \sum_{i=1}^{N} \frac{\mathbf{p}^{2}}{2m} | \Psi \rangle = \sum_{i=1}^{N} \frac{\hbar^{2}}{2m} \int \nabla \psi^{*}(\mathbf{r}_{i}) \nabla \psi(\mathbf{r}_{i}) d\mathbf{r}_{i},$$

$$= N \frac{\hbar^{2}}{2m} \int |\nabla \psi(\mathbf{r})|^{2} d\mathbf{r},$$

$$= -N \frac{\hbar^{2}}{2m} \int \psi^{*}(\mathbf{r}) \nabla^{2} \psi(\mathbf{r}) d\mathbf{r},$$

$$(117)$$

where as stated above $|\Psi\rangle$ is the N-particle tensor product wave function and $\psi(\mathbf{r})$ is the single particle wave function. The potential term can readily be written as

$$\langle \Psi | \sum_{i=1}^{N} V_{ext}(\mathbf{r}_i) | \Psi \rangle = N \int \psi^*(\mathbf{r}) V_{ext} \psi(\mathbf{r}) d\mathbf{r}.$$
 (118)

For the interaction term we have

$$\langle \Psi | \frac{1}{2} \sum_{i=1}^{N} \sum_{j \neq i}^{N} V \left(|\mathbf{r}_{i} - \mathbf{r}_{j}| \right) |\Psi \rangle$$

$$= \frac{1}{2} \sum_{i=1}^{N} \sum_{j \neq i}^{N} \int d\mathbf{r}_{i} \int \psi^{*}(\mathbf{r}_{i}) \psi^{*}(\mathbf{r}_{j}) V \left(|\mathbf{r}_{i} - \mathbf{r}_{j}| \right) \psi(\mathbf{r}_{i}) \psi(\mathbf{r}_{j}) d\mathbf{r}_{j},$$

$$= \frac{N(N-1)}{2} \int d\mathbf{r} \int \psi^{*}(\mathbf{r}) \psi(\mathbf{r}') V \left(|\mathbf{r} - \mathbf{r}'| \right) \psi(\mathbf{r}) \psi(\mathbf{r}') d\mathbf{r}'.$$
(119)

where we have written the expression in order to facilitate the rest of our calculations. Given the expressions above, we now need to minimize them. We will consider a small variation in the wave function $\psi(\mathbf{r})$, its real and imaginary parts, we take ψ and ψ^* as independent variables.

$$\frac{\delta}{\delta\psi^*} \langle \Psi | \frac{1}{2} \sum_{i=1}^N \sum_{j\neq i}^N V\left(|\mathbf{r}_i - \mathbf{r}_j| \right) |\Psi\rangle$$
$$= N(N-1) \int \delta\psi^*(\mathbf{r}) \left(\int |\psi(\mathbf{r})|^2 V(|\mathbf{r} - \mathbf{r}'|) d\mathbf{r}' \right) \psi(\mathbf{r}) d\mathbf{r}.$$
(120)

Similarly, for the chemical potential we have

$$\frac{\delta \langle \Psi | \Psi \rangle}{\delta \psi^*} = N \left(\int \psi^*(\mathbf{r}) \psi(\mathbf{r}) d\mathbf{r} \right)^{N-1} \int \delta \psi^*(\mathbf{r}) \psi(\mathbf{r}) d\mathbf{r}$$
$$= N \int \delta \psi^*(\mathbf{r}) \psi(\mathbf{r}) d\mathbf{r}.$$
(121)

Putting together all the different terms for the free energy F, we have that the variation is given by:

$$\frac{\delta F}{\delta \psi^*} = 0 = N \int \left[-\frac{\hbar^2}{2m} \nabla^2 \psi(\mathbf{r}) + V_{ext}(\mathbf{r}) \psi(\mathbf{r}) + (N-1) \left(\int |\psi(\mathbf{r})|^2 V(|\mathbf{r}-\mathbf{r}'|) d\mathbf{r}' \right) \psi(\mathbf{r}) - \mu \psi(\mathbf{r}) \right] \delta \psi^*(\mathbf{r}) d\mathbf{r}, \quad (122)$$

and therefore the quantity inside the square brackets in the above expression must vanish. It is quite common to choose an interaction potential such that $V(|\mathbf{r} - \mathbf{r}'|) =$

 $\frac{4\pi\hbar^2}{m}a\delta(\mathbf{r}-\mathbf{r}')$ where *a* is the *s*-wave scattering length and using the approximation that $N-1\simeq N$ we end up with

$$-\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r}) + V_{ext}(\mathbf{r})\psi(\mathbf{r}) + N\frac{4\pi\hbar^2}{m}a|\psi(\mathbf{r})|^2\psi(\mathbf{r}) = \mu\psi(\mathbf{r}), \qquad (123)$$

which is the time-independent Gross-Pitaevskii equation. The scattering length a measures the intensity of the interactions between the bosons. Its sign indicates whether the interactions are attractive (a < 0) or repulsive (a > 0); we will later discuss the importance of this distinction. In this way, we have to do minimization of the free energy $F = E - \mu N$.

11.4 superfluidity and symmetries

1. U(1) symmetry: Lagrangian of our model will be invariant under this symmetry. And the spontaneous breaking of this symmetry will be a necessary condition for superfluidity.

2. Conserved charge: A conserved charge is a consequence of the U(1) symmetry by the Noether's theorem. This is essential for all superfluids because this charge is transported by a superflow.

3.Bose- Einstein condensation: Bose-Einstein condensation is another way of saying that the U(1) symmetry is broken.

4.Spontaneous symmetry breaking: the Bose Einstein Condensate is the ground state in a superfluid. This ground state is not invariant with respect to transformations of the original symmetry of the Lagrangian of the system. This is called spontaneous symmetry breaking.

5. Goldstone mode: If the spontaneously broken symmetry is global, a massless mode arises for all the temperatures below the critical temperature.

6. Symmetry restoration and critical temperature: All the superfluids that I am going to discuss are at very small temperatures. There is a critical temperature at which the ground state becomes symmetric under the full symmetry of the Lagrangian.

11.5 Lagrangian and conserved charge

We start from the following Lagrangian \mathcal{L} for a complex scalar field $\phi(X)$ depending on space-time, $X = (x_0, \mathbf{x})$,

$$\mathcal{L} = \partial_{\mu}\varphi^*\partial^{\mu}\varphi - m^2|\varphi|^2 - \lambda|\varphi|^4 \tag{124}$$

The Lagrangian describes spin-0 bosons with mass m which interact repulsively with each other with a coupling constant $\lambda > 0$ We first observe that \mathcal{L} is invariant under U(1) rotations of the field,

$$\varphi = e^{-i\alpha}\varphi\tag{125}$$

Since α is constant one talks about a global transformation or a global symmetry, as opposed to a local symmetry where α would be allowed to depend on space-time. In order to account for Bose-Einstein condensation, we need to separate the condensate from the fluctuations. This is done by writing

$$\varphi(X) \to \phi(X) + \varphi(X)$$
 (126)

where $\phi(X)$ is the condensate and $\varphi(X)$ are the fluctuations. The point of this decomposition is that the fluctuations are a dynamical field.

Condensate describes a macroscopic occupation of the bosons in the ground state of the system, usually the zero-momentum state. Analogously, in field theory, we may Fourier decompose the field and separate the state with zero four-momentum $K = (k_0; \mathbf{k})$. More generally, if we want to allow for a nonzero superfluid velocity v_s , condensation takes place in a state with nonzero four-momentum, say P. Therefore, a uniform condensate that moves with a constant velocity determined by P can be written as $\phi(X) = \varphi(P)e^{-iPX}$. At this point, however, we do not need to make any assumption for $\phi(X)$, although later we shall mostly talk about a uniform condensate, or, even simpler, about a condensate at rest, P = 0. We write the complex condensate in terms of its modulus ρ and its phase ϕ ,

$$\phi(X) = \frac{\rho(X)}{\sqrt{2}} e^{i\psi X} \tag{127}$$

The fluctuations $\varphi(X)$ will later be needed to compute the dispersion of the Goldstone mode. As a first step, we neglect the fluctuations. In this case, the Lagrangian only depends on the classical field,

$$\mathcal{L} = \mathcal{L}^{(0)} + fluctuations \tag{128}$$

where

$$\mathcal{L} = \frac{1}{2} \partial_{\mu} \rho \partial^{\mu} \rho + \frac{\rho^2}{2} (\partial_{\mu} \psi \partial^{\mu} \psi - m^2) - \frac{\lambda}{4} \rho^4$$
(129)

Next we write down the equations of motion for ρ and ψ . Notice that the phase ψ only appears through its space-time derivative, and thus the Euler-Lagrange equations are

$$0 = \frac{\partial \mathcal{L}}{\partial \rho} - \partial_{\mu} \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \rho)}$$
(130)

$$0 = \partial_{\mu} \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\psi)} \tag{131}$$

From this general form, we compute

$$\partial_{\mu}(\rho^2 \partial^{\mu} \psi) = 0 \tag{132}$$

where we have abbreviated

$$\sigma = \sqrt{\partial_{\mu}\psi\partial^{\mu}\psi} \tag{133}$$

11.6 Spontaneous symmetry breaking

Let us now for simplicity assume that ρ and $\partial^{\mu}\psi$ are constant in space and time. As a consequence, the current and the stress-energy tensor also become constant, and the equations $\partial_{\mu}j^{m}u = \partial_{\mu}T^{\mu\nu} = 0$ are trivially fulfilled. Therefore, with this assumption we will not be able to discuss complicated hydrodynamics, but we shall be able to

discuss the basic concepts of spontaneous symmetry breaking and the physics of a uniform superflow. In this case, the Lagrangian becomes

$$\mathcal{L}^{(0)} = -U \tag{134}$$

$$U = -\frac{\rho^2}{2}(\sigma^2 - m^2) + \frac{\lambda}{4}\rho^4$$
(135)

where U is called tree-level potential. This equation is equivalent to finding extremal points of U with respect to ρ

$$0 = \frac{\partial U}{\partial \rho} = \rho(\rho^2 - m^2 - \lambda \rho^2) \tag{136}$$

which has the solutions

$$\rho = 0; \rho^2 = \frac{\sigma^2 - m^2}{\lambda} \tag{137}$$

We need $\sigma^2 > m^2$ for the nontrivial solution to exist. In order to understand this condition, remember that relativistic Bose- Einstein condensation occurs when the chemical potential is larger than the mass. Let us try to understand how this comes about. Usually, a chemical potential μ is introduced via $\mathcal{H} - \mu N$ with the Hamiltonian \mathcal{H} and the conserved charge density $N = j^0$. This is equivalent to adding the chemical potential to the Lagrangian as if it were the temporal component of a gauge field,

$$\mathcal{L} = |(\partial_0 - i\mu)\varphi|^2 - |\nabla\varphi|^2 - m^2|\varphi|^2 - \lambda|\varphi|^4$$
(138)

One can show that this modified Lagrangian with kinetic term $|(\partial_0 - i\mu)\varphi|^2$ and a condensate with a trivial phase, $\psi = 0$, is identical to the original Lagrangian with kinetic term $\partial_{\mu}\varphi\partial^{\mu}\varphi^*$ and a condensate with a time-dependent phase $\psi = \mu t$. We conclude that it does not matter whether we put the chemical potential directly into the Lagrangian or whether we introduce it through the phase of the condensate. Consequently, $\mu = \partial_0 \psi$ is the chemical potential associated with the conserved charge. However, this is not exactly what we were expecting. We had conjectured that $\sigma = \sqrt{\partial_{\mu}\psi\partial^{\mu}\psi}$ plays the role of a chemical potential. This is only identical to $\partial_0\psi$ if we set $\nabla\psi = 0$. We shall discuss the case of a nonzero $\nabla\psi$ below and find that σ is the chemical potential in the rest frame of the superfluid, while $\partial_0\psi$ is the chemical potential in the frame where the superfluid moves with a velocity determined by $\nabla\psi$.



Here, we first discuss the simpler case $\nabla \psi = 0$ where indeed $\mu = \sigma = \partial_0 \psi$. For chemical potentials $|\mu| < m$, the minimum is at $\phi = 0$, i.e., there is no condensation. For condensation, one needs a negative coefficient in front of the ϕ^2 term, i.e., the modulus of the chemical potential must be larger than the mass. In this case, the potential has a 'Mexican hat' or 'bottom of a wine bottle' shape. (Since we consider a repulsive interaction for which $\lambda > 0$ the potential is bounded from below, otherwise the system would be unstable.) The U(1) symmetry of the Lagrangian is reflected in the rotationally symmetric wine bottle potential. The nontrivial minimum, at a given angle ψ of the condensate, is not invariant under U(1) because a U(1) transformation rotates the condensate along the bottom of the wine bottle. This mechanism, where the Lagrangian has a symmetry which is not respected by the ground state, is called spontaneous symmetric phase, here, the condensate ϕ , is called the order parameter.

12 Solutions of GL equation

12.1 Simplest solution for vortex

We begin with writing a simple second order equation of the form

$$-\frac{d^2f}{dx^2} + \alpha f + \beta f^3 = 0$$
 (139)

We put such a function in this equation which is a simple solution such as the tan hyperbolic function. Here we are considering the tan hyperbolic function because it has the following identity, which will help us to solve this equation easily

$$\frac{d^2}{dx^2} \tanh x = 2 \tanh^3 x - 2 \tanh x \tag{140}$$

$$\frac{d^2 f}{dx^2} = a'b'[1 - \tanh^2 b] + a'' \tanh b + a'b'[1 - \tanh^2 b] + ab''(1 - \tanh^2 b) + ab'^2(-2 \tanh b)(1 - \tanh^2 b) - \frac{\hbar^2}{2m}[2a'b'(1 - \tanh^2 b) + a'' \tanh b + ab''(1 - \tanh^2 b) - 2ab'^2(\tanh b - \tanh^3 b)] + \alpha a \tanh b + \beta a^3 b$$
(141)

$$-\frac{\hbar^2}{2m}(2a'b'+ab'') = 0 \tag{142}$$

$$2a'b' + ab'' = 0 (143)$$

$$-\frac{\hbar^2}{2m}(2ab'^2) + \beta a^3 = 0 \tag{144}$$

Let us take

$$\beta = \frac{\hbar^2}{2m}\tilde{\beta} \tag{145}$$

$$-2ab^{\prime 2} + \tilde{\beta}a^2 = 0 \tag{146}$$

$$b^{\prime 2} = \frac{\tilde{\beta}a^2}{2} \tag{147}$$

$$b' = \sqrt{\frac{\tilde{\beta}}{2}}a\tag{148}$$

$$\frac{db}{dx} = \sqrt{\frac{\tilde{\beta}}{2}a} \tag{149}$$

$$b = \sqrt{\frac{\tilde{\beta}}{2}}dx + c \tag{150}$$

where c is a constant.

$$b'' = \sqrt{\frac{\tilde{\beta}}{2}}a' \tag{151}$$

$$2a'\sqrt{\frac{\tilde{\beta}}{2}}a + a\sqrt{\frac{\tilde{\beta}}{2}}a' = 0 \tag{152}$$

$$3\sqrt{\frac{\tilde{\beta}}{2}a'a} = 0 \tag{153}$$

$$a'a = 0 \tag{154}$$

$$a(x)\frac{da}{dx} = 0\tag{155}$$

So, either a(x)=0 or $\frac{da}{dx}=0$. This implies that a is a constant.

$$2a\frac{\tilde{\beta}}{2}a^2 + \tilde{\alpha}a = 0 \tag{156}$$

$$a^2 = -\frac{\tilde{\alpha}}{\tilde{\beta}} \tag{157}$$

$$a = i\sqrt{\frac{\tilde{\alpha}}{\tilde{\beta}}} = i\sqrt{\frac{\alpha}{\beta}} \tag{158}$$

$$b = \frac{a}{\hbar}\sqrt{m\beta}x + c \tag{159}$$

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}[a\tanh b(x) + \alpha a \tanh b(x) + \beta a^2 \tanh^3 b(x)] = 0$$
(160)

$$-\frac{\hbar^2}{2m}\frac{d}{dx}[1-\tanh^2 b(x)]b' + \alpha \tanh b(x) + \beta a^2 \tanh^3 b(x) = 0$$
(161)

$$-\frac{\hbar^2}{2m}\frac{a}{dx}[1-\tanh^2 b(x)]b' + \alpha \tanh b(x) + \beta a^2 \tanh^3 b(x) = 0$$
(161)
$$-\frac{\hbar^2}{2m}[-2\tanh b(1-\tanh^2 b)b'^2 + (1-\tanh^2 b)b''] + \alpha \tanh b + \beta a^2 \tanh^3 b = 0$$
(162)

$$-\frac{\hbar^2}{2m}[-2(1-\tanh^2 b)\frac{\tilde{\beta}}{2}a^2] + \alpha + \beta a^2 \tanh^3 b = 0$$
(163)

$$a^2 = -\frac{\tilde{\alpha}}{\tilde{\beta}} \tag{164}$$

Thus we get a simple solution of the GL equation of the form of a tan hyperbolic function, with a being a function of x and b being a constant.

12.2 Tan hyperbolic solution of the GL equation in a nonzero magnetic field

$$\psi = a \tanh b \tag{165}$$

$$-\frac{\hbar^2}{2m}\nabla^2\psi + \frac{q^2A^2}{2m}\psi + \alpha\psi + \psi\psi^2 = 0$$
(166)

$$\frac{\partial \psi}{\partial r} = a' \tanh b + rab'^2 b \tag{167}$$

$$\frac{\hbar^2}{2m} \left[\left(\frac{a'}{r} + a'' - 2ab' \right) \tanh b + \left(2a'b' + \frac{ab'}{r} + ab'' \right) (1 - \tanh^2 b) + 2ab'^2 \tanh^3 b \right] + \frac{q^2k^2r^2}{2m}a \tanh b + \alpha a \tanh b + \beta a^3 \tanh^3 b = 0$$

Coefficient of $(1 - \tanh^2 b)$:

$$2a'b' + \frac{ab'}{r} + ab'' = 0 \tag{168}$$

Coefficient of $\tanh b$:

$$-\frac{\hbar^2}{2m} \left[\left(\frac{a'}{r} + a'' - 2ab' \right) \right] + \frac{q^2 k^2 r^2 a}{2m} + \alpha a = 0$$
(169)

Coefficient of $\tanh^3 b$:

$$-\frac{\hbar^2}{2m}(2ab')^2 + \beta a^3 \tanh^3 b = 0$$
(170)

$$\frac{\hbar^2}{m}b^{\prime 2} = \beta a^2 \tag{171}$$

$$b^{\prime 2} = \frac{m\beta}{\hbar^2} a^2 \tag{172}$$

$$b' = \frac{m\beta}{\hbar}a\tag{173}$$

$$b'' = \frac{m\beta}{\hbar}a' \tag{174}$$

$$b = \frac{m\beta}{\hbar} \int a dr \tag{175}$$

$$2a'\frac{m\beta}{\hbar}a^2 + \frac{am\beta a^2}{\hbar r} + \frac{a\beta m}{\hbar}.2aa' = 0$$
(176)

$$2a' + \frac{a}{r} + 2a' = 0 \tag{177}$$

$$4a' + \frac{a}{r} = 0 \tag{178}$$

$$4a' = -\frac{a}{r} \tag{179}$$

$$4\frac{da}{dr} = -\frac{a}{r} \tag{180}$$

$$4\frac{da}{a} = -\frac{dr}{r} \tag{181}$$

$$4\ln a = -\ln r + constant \tag{182}$$

$$\ln a^4 = \ln(kr)^{-1} \tag{183}$$

$$a^4 = \frac{1}{kr} \tag{184}$$

$$a = \frac{1}{(kr)^{\frac{1}{4}}}$$
(185)

$$a^2 = \frac{1}{\sqrt{kr}} \tag{186}$$

$$b' = \beta \frac{1}{\sqrt{kr}} \frac{m}{\hbar^2} \tag{187}$$

$$\frac{db}{dr} = \frac{\beta m}{\sqrt{kr}\hbar^2} \tag{188}$$

$$b = \frac{\beta m}{\sqrt{k}\hbar^2} \int \frac{dr}{\sqrt{r}} \tag{189}$$

$$b = \frac{2\beta m}{\hbar^2} \sqrt{\frac{r}{k}} + constant$$
(190)

$$a' = \left(\frac{-1}{4}\right) \left(\frac{-5}{4}\right) \frac{1}{k^{\frac{1}{4}}} r^{\frac{9}{4}}$$
(191)

Solving we finally get

$$\tilde{\alpha'}r^{\frac{3}{2}} + \tilde{\alpha''}r^4 + \tilde{\alpha}r^2 = constant$$
(192)

where $\tilde{\alpha} = 2m\alpha$.

12.3 Equations in the presence of magnetic field

Now let us see what happens when a magnetic field is applied to the lattice. The vortices have a cylindrical symmetry. We need to choose a magnetic field which satisfies the following two conditions.

Condition 1: It should satisfy the symmetry requirement.

$$\mathbf{A} = f(r)\hat{\phi} \tag{193}$$

Condition 2: It should be equal to the curl of a physically acceptable vector potential.

$$\mathbf{B} = \nabla \times \vec{A} \tag{194}$$

$$\nabla \times \vec{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{e_r} + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{e_\phi} + \left(\frac{1}{r}\frac{\partial}{\partial r}(rA_\phi) - \frac{1}{r}\frac{\partial A_r}{\partial \phi}\right)\hat{e_z} \quad (195)$$

$$\vec{B} = -\frac{\partial f(r)}{\partial z}\hat{e_r} + \left(\frac{1}{r}\frac{\partial}{\partial r}(rf(r))\hat{e_z}\right)$$
(196)

$$B_r = -\frac{\partial f(r)}{\partial z} = 0 \tag{197}$$

$$B_z = \frac{1}{r} [f(r) + rf'(r)]$$
(198)

$$B_{z} = \frac{f(r)}{r} + f'(r)$$
(199)

$$B_z = \frac{f(r)}{r} + \frac{\partial f(r)}{\partial r}$$
(200)

if B_z was zero

$$\frac{\partial f(r)}{f(r)} = -\frac{\partial r}{r} \tag{201}$$

$$\ln f(r) = -\ln r + k \tag{202}$$

where k is a constant.

$$\ln f(r) = \ln \frac{k}{r} \tag{203}$$

$$f(r) = \frac{k}{r} \tag{204}$$

But this gives

$$B_z = \frac{k}{r^2} - \frac{k}{r^2} = 0 \tag{205}$$

This is a valid case, but we need a non-zero magnetic field. So, let us take some other function. If $\vec{A} = kr\hat{\phi}$

$$\nabla \times \vec{A} = \frac{1}{r} \frac{\partial}{\partial r} (kr^2) \hat{e_z} = 2k \hat{e_z}$$
(206)

$$\vec{B} = 2k\hat{e_z} \tag{207}$$

So, at last we have determined the value of f(r) i.e. f(r) = kr

12.4 Solution of GL equation by the method of separation of variables

Let us start with writing the simplest form of the Ginzburg-Landau equation i.e.

$$-\frac{\hbar^2}{2m}\nabla^2\psi + \alpha\psi + \beta\psi^3 = 0$$
(208)

For simplicity let us assume that the value of α is so small that it can be neglected. We take $\alpha = 0$

$$-\frac{\hbar^2}{2m}\nabla^2\psi + \beta\psi^3 = 0 \tag{209}$$

Let us assume that ψ depends upon the radial distance r and the angle ϕ . So we write $\psi = R(r)\Phi(\phi)$

$$\nabla^2 \psi = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \phi^2}$$
(210)

$$\nabla^2 \psi = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} (R\Phi) \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} (R\Phi)$$
(211)

Let us substitute the value of ψ in the GL equation

$$-\frac{\hbar^2}{2m} \left[\Phi \frac{1}{r} \left(\frac{\partial R}{\partial r} + r \frac{\partial^2 R}{\partial r^2} \right) + \frac{R}{r^2} \frac{\partial^2 \Phi}{\partial \phi^2} \right] + \beta R^3 \Phi^3 = 0$$
(212)

Divide this equation by $R\Phi$

$$-\frac{\hbar^2}{2m} \left[\frac{1}{rR} \left(\frac{\partial R}{\partial r} + r \frac{\partial^2 R}{\partial r^2} \right) + \frac{1}{\Phi r^2} \frac{\partial^2 \Phi}{\partial \phi^2} \right] + \beta R^2 \Phi^2 = 0$$
(213)

$$-\frac{\hbar^2}{2m} \left[\frac{1}{rR} \left(\frac{\partial R}{r} + r \frac{\partial^2 R}{r^2} \right) \right] + \beta R^2 \Phi^2 = 0$$
(214)

Assuming Φ is a constant

$$-\frac{\hbar^2}{2m} \left[\frac{1}{rR^3} \left(\frac{R}{\partial r} + r \frac{\partial^2 R}{\partial r^2} \right) \right] = -\beta \Phi^2 = -k$$
(215)

$$\Phi^2 = -\frac{k}{\beta} \tag{216}$$

This implies

$$\Phi = \pm \sqrt{\frac{k}{\beta}} \tag{217}$$

and it is a constant.

$$\frac{1}{rR^3} \left(\frac{\partial R}{\partial r} + r \frac{\partial^2 R}{\partial r^2} \right) = \frac{2mk}{\hbar^2} = p$$
(218)

$$\frac{1}{rR^3}\frac{\partial R}{\partial r} + \frac{1}{R^3}\frac{\partial^2 R}{\partial r^2} = p \tag{219}$$

$$\frac{\partial^2 R}{\partial r^2} + \frac{1}{r} \frac{\partial R}{\partial r} - pR^3 = 0$$
(220)

Solving this differential equation we get

$$R = \frac{i\sqrt{2}}{\sqrt{\frac{k}{4}r^2 + c_0r + c_1}} \tag{221}$$

Where c_0 and c_1 are constants.

12.5 Cylindrically symmetric solution of the GL equation

The GL equation is

$$-\frac{\hbar^2}{2m}\nabla^2\psi + \alpha\psi + \beta\psi^3 = 0$$
(222)

Let $\alpha = 0$

So, the equation now becomes

$$-\frac{\hbar^2}{2m}\nabla^2\psi + \beta\psi^3 = 0 \tag{223}$$

$$-\frac{\hbar^2}{2m}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{r}\right) + \beta R^3 = 0$$
(224)

$$\frac{\hbar^2}{2m} \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial R}{r} \right) = \beta R^3 \tag{225}$$

$$\frac{1}{rR^3}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) = \frac{\beta 2m}{\hbar^2} = p \tag{226}$$

$$\frac{1}{rR^3}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) = p \tag{227}$$

$$\frac{\partial R}{\partial r} + r \frac{\partial^2 R}{\partial r^2} = p r R^3 \tag{228}$$

$$r\frac{dR}{dr} = \frac{dR}{d\rho} \tag{229}$$

$$\frac{dR}{dr} + r\frac{d^2R}{dr^2} = \frac{d^2R}{drd\rho}$$
(230)

$$\frac{d^2R}{d\rho^2} = \frac{dr}{d\rho}\frac{dR}{dr} + r\frac{d^2R}{d\rho dr}$$
(231)

$$\frac{d^2R}{d\rho^2} = r\frac{dR}{dr} + r^2\frac{d^2R}{dr^2}$$
(232)

$$\frac{1}{r^2}\frac{d^2R}{d\rho^2} - \frac{1}{r}\frac{dR}{dr} = \frac{d^2R}{d\rho^2}$$
(233)

$$\frac{1}{r^2}\frac{d^2R}{d\rho^2} - \frac{1}{r}\frac{dR}{dr} + \frac{1}{r}\frac{dR}{dr} = pR^3$$
(234)

where p is a constant.

$$\exp^{-2\rho} \frac{d^2 R}{d\rho^2} = pR^3 \tag{235}$$

$$\int \frac{d^2 R}{R^3} = p \int \frac{d\rho^2}{\exp^{-2\rho}}$$
(236)

$$\frac{d^2R}{dr^2} = -\exp^{-\rho}\frac{dR}{d\rho}\frac{d\rho}{dr} + \exp^{-\rho}\frac{d^2R}{drd\rho}$$
(237)

$$\frac{d^2R}{dr^2} = -\exp^{-\rho}\frac{dR}{rd\rho}\frac{d\rho}{d\rho} + \exp^{-\rho}\frac{d^2R}{rd\rho d\rho}$$
(238)

$$\frac{d^2R}{dr^2} = -\frac{\exp^{-\rho}}{\rho}\frac{dR}{d\rho} + \frac{\exp^{-\rho}}{\rho}\frac{d^2R}{d\rho^2}$$
(239)

$$\frac{d^2 R}{dr^2} = -\exp^{-2\rho} \frac{dR}{d\rho} + \exp^{-2\rho} \frac{d^2 R}{d\rho^2}$$
(240)

$$\exp^{-2\rho} \frac{d^2 R}{d\rho^2} = pR^3$$
 (241)

$$\frac{d^2 R}{R^3} = p \frac{d^2 \rho}{\exp^{-2\rho}}$$
(242)

$$\exp^{-2\rho} \frac{d}{d\rho} \left(\frac{dR}{d\rho} \right) = pR^3 \tag{243}$$

$$\int d\left(\frac{dR}{d\rho}\right) = \int p \exp^{2\rho} R^3 d\rho + c_0 \tag{244}$$

$$\frac{dR}{d\rho} = \frac{p'R^3}{2} \exp^{2\rho} \tag{245}$$

$$\int \frac{dR}{R^3} = \int \frac{p'}{2} \exp^{2\rho} d\rho \tag{246}$$

$$\frac{1}{R^2} = -\frac{p'}{2} \exp^{2\rho} -constant \tag{247}$$

$$R = \iota \sqrt{\frac{2}{p''}} \exp^{-\rho} = p''' \exp^{-\rho}$$
(248)

or we can also write

$$R = k \exp^{-\rho} \tag{249}$$

where **k** is a constant i.e. equal to p'''

12.6 Solution of the GL equation in Cartesian coordinates

we know from the preceding section that the solution of the GL equation in cylindrical coordinates is written as

$$R = k \exp^{-\rho} \tag{250}$$

where $\rho = \ln r$.

$$-\frac{\hbar^2}{2m} \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \left(\frac{k}{r} \right) \right) \right] + \beta \frac{k^3}{r^3} = 0$$
(251)

$$-\frac{\hbar^2}{2m} \left[\frac{1}{r} \frac{k}{r^2} \right] + \beta \frac{k^3}{r^3} = 0$$
 (252)

This equation will be satisfied if $\beta = \frac{\hbar^2}{2mk^2}$.

Now let us suppose that there is a system of two vortices, where one of the vortices is at the origin and one is at a distance 'a' on the x-axis.

Let us write the wave functions of the two vortices in Cartesian coordinates and see whether they satisfy the GL equation individually.

$$\psi_1 = \frac{k}{\sqrt{x^2 + y^2}}; \psi_1 = \frac{k}{\sqrt{(x-a)^2 + y^2}}$$
(253)

Let us first solve for ψ_1 .

$$-\frac{\hbar^2}{2m}\nabla^2 \left[\frac{k}{\sqrt{x^2+y^2}}\right] + \beta \left[\frac{k}{\sqrt{x^2+y^2}}\right]^3 = 0$$
(254)

$$-\frac{\hbar^2}{2m} \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] \left[\frac{k}{\sqrt{x^2 + y^2}}\right] + \beta \frac{k^3}{(x^2 + y^2)^{\frac{3}{2}}} = 0$$
(255)

$$-\frac{\hbar^2 k}{2m} \left[\frac{\partial}{\partial x} \left(\frac{-1}{2} \frac{2x}{(x^2 + y^2)^{\frac{3}{2}}} \right) + \frac{\partial}{\partial y} \left(\frac{-1}{2} \frac{2y}{(x^2 + y^2)^{\frac{3}{2}}} \right) \right] + \beta \frac{k^3}{(x^2 + y^2)^{\frac{3}{2}}} = 0$$
(256)

$$\frac{\hbar^2 k}{2m} \left[\frac{1}{(x^2 + y^2)^{\frac{3}{2}}} - \frac{3x^2}{(x^2 + y^2)^{\frac{5}{2}}} + \frac{1}{(x^2 + y^2)^{\frac{3}{2}}} - \frac{3y^2}{(x^2 + y^2)^{\frac{5}{2}}} \right] + \beta \frac{k^3}{(x^2 + y^2)^{\frac{3}{2}}} = 0 \quad (257)$$

$$\frac{\hbar^2 k}{2m} \left[\frac{2}{(x^2 + y^2)^{\frac{3}{2}}} - \frac{3(x^2 + y^2)}{(x^2 + y^2)^{\frac{5}{2}}} \right] + \frac{\hbar^2}{2mk^2} \left[\frac{k^3}{(x^2 + y^2)^{\frac{3}{2}}} \right] = 0$$
(258)

$$-\frac{\hbar^2 k}{2m} \left[\frac{1}{(x^2 + y^2)^{\frac{3}{2}}} \right] + \frac{\hbar^2 k}{2m} \left[\frac{1}{(x^2 + y^2)^{\frac{3}{2}}} \right] = 0$$
(259)

the equation is completely satisfied. Hence ψ_1 is an exact solution of the GL equation. Now we shall solve the GL equation for ψ_2 .

$$-\frac{\hbar^2}{2m}\nabla^2 \left[\frac{k}{\sqrt{(x-a)^2+y^2}}\right] + \beta \left[\frac{k}{\sqrt{(x-a)^2+y^2}}\right]^3 = 0$$
(260)

$$-\frac{\hbar^2}{2m} \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] \left[\frac{k}{\sqrt{(x-a)^2 + y^2}}\right] + \beta \frac{k^3}{((x-a)^2 + y^2)^{\frac{3}{2}}} = 0$$
(261)

$$-\frac{\hbar^{2}k}{2m} \left[\frac{\partial}{\partial x} \left(\frac{-1}{2} \frac{2(x-a)}{((x-a)^{2}+y^{2})^{\frac{3}{2}}} \right) + \frac{\partial}{\partial y} \left(\frac{-1}{2} \frac{2y}{((x-a)^{2}+y^{2})^{\frac{3}{2}}} \right) \right] + \beta \frac{k^{3}}{((x-a)^{2}+y^{2})^{\frac{3}{2}}} = 0 \quad (262)$$

$$\frac{\hbar^2 k}{2m} \left[\frac{1}{((x-a)^2 + y^2)^{\frac{3}{2}}} - \frac{3(x-a)^2}{((x-a)^2 + y^2)^{\frac{5}{2}}} + \frac{1}{((x-a)^2 + y^2)^{\frac{3}{2}}} - \frac{3y^2}{((x-a)^2 + y^2)^{\frac{5}{2}}} \right] + \beta \frac{k^3}{((x-a)^2 + y^2)^{\frac{3}{2}}} = 0 \quad (263)$$

$$\frac{\hbar^2 k}{2m} \left[\frac{2}{((x-a)^2 + y^2)^{\frac{3}{2}}} - \frac{3((x-a)^2 + y^2)}{((x-a)^2 + y^2)^{\frac{5}{2}}} \right] + \frac{\hbar^2}{2mk^2} \left[\frac{k^3}{(x-a)^2 + y^2} \right] = 0 \quad (264)$$

$$-\frac{\hbar^2 k}{2m} \left[\frac{1}{((x-a)^2 + y^2)^{\frac{3}{2}}} \right] + \frac{\hbar^2 k}{2m} \left[\frac{1}{((x-a)^2 + y^2)^{\frac{3}{2}}} \right] = 0$$
(265)

Thus ψ_2 is also an exact solution of the GL equation.

Now we put $\psi_1 + \psi_2$ in the GL equation,

we can write

$$\psi = \frac{k}{\sqrt{x^2 + y^2}} + \frac{k}{\sqrt{(x-a)^2 + y^2}}$$
(266)

$$\nabla^2 \psi = \left(\frac{2}{\partial x^2} + \frac{2}{\partial y^2}\right) \left[\frac{k}{\sqrt{x^2 + y^2}} + \frac{k}{\sqrt{(x-a)^2 + y^2}}\right]$$
(267)

$$=k\frac{\partial}{\partial x}\left(-\frac{x}{(x^2+y^2)^{\frac{3}{2}}}\frac{(x-a)}{((x-a)^2+y^2)^{\frac{3}{2}}}\right)+k\frac{\partial}{\partial y}\left(-\frac{y}{(x^2+y^2)^{\frac{3}{2}}}\frac{y}{((x-a)^2+y^2)^{\frac{3}{2}}}\right)$$
(268)

$$= -k\left(\frac{2}{(x^2+y^2)^{\frac{3}{2}}} - \frac{3}{(x^2+y^2)^{\frac{3}{2}}} + \frac{2}{((x-a)^2+y^2)^{\frac{3}{2}}} - \frac{3}{((x-a)^2+y^2)^{\frac{3}{2}}}\right)$$
(269)

$$= k \left(\frac{1}{(x^2 + y^2)^{\frac{3}{2}}} + \frac{1}{((x - a)^2 + y^2)^{\frac{3}{2}}} \right)$$
(270)

Now the GL equation becomes

$$= -\frac{\hbar^2 k}{2m} \left[\frac{1}{(x^2 + y^2)^{\frac{3}{2}}} + \frac{1}{((x-a)^2 + y^2)^{\frac{3}{2}}} \right] + \beta \left[\frac{k}{\sqrt{x^2 + y^2}} + \frac{k}{\sqrt{(x-a)^2 + y^2}} \right]^3 = 0$$
(271)

ERROR:

$$\frac{\hbar^2}{2m} \frac{3}{\sqrt{x^2 + y^2}\sqrt{(x-a)^2 + y^2}} \left[\frac{k}{\sqrt{x^2 + y^2}} + \frac{k}{\sqrt{(x-a)^2 + y^2}}\right]$$
(272)

Now let us integrate this error with respect x.

By doing so we get a function of the form

$$\begin{pmatrix}
\sqrt[4]{-1}\left(-\sqrt{2y+\iota a}\log\left(\frac{(2(-1)^{\frac{3}{4}}y(a^{2}+\iota ay-ax+y^{2}+\iota yx)}{\sqrt{a}\sqrt{2y-\iota a}(y-\iota x)}\right) - \frac{2\iota y\sqrt{a^{2}-2ax+y^{2}+x^{2}}}{x+\iota m}\right) \\
+ \iota\sqrt{2y-\iota a}\log\left(\frac{2\iota y\sqrt{a^{2}-2ax+y^{2}+x^{2}}}{x-\iota y} - \frac{2\sqrt[4]{-1}y(a^{2}-a(x-\iota y)+y(y+\iota x))}{\sqrt{a}\sqrt{2y-\iota a}(y+\iota x)}\right) \\
+ \sqrt{2y+\iota a}\log\left(\frac{2m\left(-\frac{\sqrt[4]{-1}(ax+y^{2}+\iota yx))}{\sqrt{a}\sqrt{2y-\iota a}} - \iota\sqrt{m^{2}+x^{2}}\right)}{a-\iota y-x}\right) \\
- \iota\sqrt{2y-\iota a}\log\frac{2y\left(\frac{\sqrt[4]{-1}(\iota ax+\iota y^{2}+yx))}{\sqrt{a}\sqrt{2y-\iota a}} + \iota\sqrt{y^{2}+x^{2}}\right)}{a-\iota y-x} - 2\sqrt{a}y\sqrt{2y-\iota a}\sqrt{2y+\iota a} \\$$
(273)

$$\frac{\hbar^2}{2m}\frac{3}{\sqrt{x^2+y^2}\sqrt{(x-a)^2+y^2}}\left[\frac{k}{\sqrt{x^2+y^2}}+\frac{k}{\sqrt{(x-a)^2+y^2}}\right]$$
(274)

for the following equation I have plotted graphs taking a = 5 and $\frac{3\hbar^2 k}{2m} = c = 1$ and varying the value of y from 0 to 1 in steps of 0.1.

As we can see the error difference is maximum when we are taking the measurements



Figure 12: at y=0

on the line joining the two vortices As we move away from the vortices the sharpness



Figure 13: y=0.1

of the peak increases and the graph becomes a continuous function.


Figure 15: y=0.3

13 Why vortices are in hexagonal pattern?

Suppose there are n_v vortex lines per unit area of the bucket all having their cores parallel to the axis of rotation.

each one possesses the same circulation κ . The strength of the array is specified by the vorticity $\vec{\omega}$, defined to be equal to the total circulation within the unit area

$$\vec{\omega} = n_v \vec{\kappa} \tag{275}$$



Figure 17: y=0.5

we know that

$$\vec{\kappa} = \oint_{L_2} \vec{v_s} \cdot \vec{dl} = \oint_A (\vec{\nabla} \times \vec{v_s}) \cdot d\vec{A}$$
(276)

when $(\nabla \times \vec{v_s})$ is finite, it signifies the presence of vortices. In a bucket, the total circulation enclosed by a contour of radius R centered on the axis is $\pi n_v R^2 \kappa$. For the superfluid to appear to rotate with uniform angular velocity $\vec{\Omega}$. The total circulation must be equal to $2\pi R(R\Omega)$.



Figure 19: y=0.7

Comparing the two formulae

$$\pi N_v R^2 \kappa = 2\pi R(R\Omega) \tag{277}$$

The vortex line density is given by

$$n_v = \frac{2\Omega}{\kappa} \tag{278}$$

$$\vec{\omega} = n_v \vec{\kappa} \tag{279}$$

So, the vorticity $\vec{\omega} = 2\vec{\Omega}$.

for the superfluid rotating with velocity of the container, $\vec{\Omega}$. The condition for equi-



Figure 21: y=0.9

librium is that the quantity

$$F' = F - \vec{L}.\vec{\Omega} \tag{280}$$

should be minimum. F is the free energy of the rotating fluid. \vec{L} is the total angular momentum.

ASSUMPTION: The liquid is at a temperature that is low enough so that $\rho_s \approx \rho$. If the radius of the container is R_0 , the total number of vortex lines with circulation $\vec{\kappa}$ is

$$N = \frac{2\pi R_0^2 \Omega}{\kappa} \tag{281}$$



Figure 22: y=1

To minimize F', we need to consider those contributions which depend upon N and κ as F' depends upon Ω .

since the vortices have the same circulation, each one will have energy ϵ_v i.e.

$$\epsilon_v = \frac{\rho_s \kappa^2}{4\pi} \ln \frac{b}{a_0} \tag{282}$$

and the same angular momentum $\vec{l_v}$ whose direction is parallel to $\vec{\Omega}$. Thus we need to minimize the following

$$F'' = N(\epsilon_v - \ln \Omega) \tag{283}$$

The angular momentum of each vortex is given by

$$\vec{l_v} = \frac{\rho_s \kappa}{8\pi\Omega} \vec{\kappa}.$$
(284)

This is additional to the angular momentum possessed by fluid by virtue of its rigid body rotation.

$$F'' = \frac{1}{2}\rho_s R_0^2 \Omega \kappa \left[\ln\left(\frac{b}{a_0}\right) - \frac{1}{2} \right]$$
(285)

For fixed Ω , κ must take minimum value. So, the ground state of rotating He II contains a regular array of vortex lines all having the smallest possible value of circulation, which means the total number lines is a maximum.

Now consider different types of 2-D arrays

1. Square

2. Hexagonal

Let us find out that out of which of the above cases, the lines are closer

CASE 1. :Square area = a^2 number of lines = 4 $n_v = \frac{1}{a^2}$

CASE 2. :Hexagonal

 $area = 6x \frac{1}{6} \times a \times a \sin 60^0$ $\frac{3\sqrt{3}}{2}a^2$

number of lines = 3

$$n_v = \frac{3 \times 2}{3\sqrt{3}a^2}$$
$$= \frac{2}{\sqrt{3}a^2}$$
$$= \frac{1.1547}{a^2}$$

So, from these calculations it is clear that density in case of hexagonal packing is preferred.



Figure 23: density in case of hexagonal packing is preferred

14 Conclusion

The Ginzburg Landau equation can be used to study the dynamics of superfluids apart from the superconductors. Some of the simplest solutions of GL equations for superfluids consist of the tan hyperbolic function. Also we can use different methods to find the solution for this kind of differential equation such as the method of separation of variables. By finding the cylindrically symmetric solutions for this equation something can be physically concluded about the geometry of the vortices in superfluids. There must be a specific reason as to why the vortices always appear to be formed in a hexagonal pattern rather than square or any other. This can be concluded by looking at the free energy equations of this system.

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