# A STUDY ON FOUR-FERMIONIC TENSORIAL INTERACTIONS

**M.Sc. Thesis**


By

**UTTIYA SARKAR**



**DISCIPLINE OF PHYSICS**

**INDIAN INSTITUTE OF TECHNOLOGY INDORE**


**JUNE 2015**

# A STUDY ON FOUR-FERMIONIC TENSORIAL INTERACTIONS

**A THESIS**

*Submitted in partial fulfillment of the requirements for the award of the degree*

***of***

**Master of Science**

*by*

**Uttiya Sarkar**

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**INDIAN INSTITUTE OF TECHNOLOGY INDORE**

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# INDIAN INSTITUTE OF TECHNOLOGY INDORE

## CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **A STUDY ON FOUR-FERMIONIC TENSORIAL INTERACTIONS** in the partial fulfillment of the requirements for the award of the degree of **MASTER OF SCIENCE** and submitted in the **DISCIPLINE OF PHYSICS, Indian Institute of Technology Indore**, is an authentic record of my own work carried out during the time period from July 2014 to June 2015 under the supervision of Dr. Manavendra N. Mahato, Associate Professor, Department of Physics, IIT Indore.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other institute.

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This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

Signature of the Supervisor of
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**(Dr. MANAVENDRA N. MAHATO)**

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**UTTIYA SARKAR** has successfully given his/her M.Sc. Oral Examination held on **JUNE 30 2015**.

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# ACKNOWLEDGEMENTS

Foremost, I would like to express my sincere gratitude to my advisor Prof. Manavendra N. Mahato for his continuous support and guidance of my M.Sc. study and research and his patience, motivation, enthusiasm, and immense depth of knowledge. His guidance helped me in all the time of research and writing of this thesis as well as all other aspects of life. I could not have imagined having a better advisor and mentor for my M.Sc project work.

I thank other faculty members of Physics department, Dr. Ankhi Roy, Dr. Sudeshna Chattopadhyay and Dr. Rajesh Kumar; without their help and support it would not have been possible for me to fulfill my Master studies. I thank my DPGC members, Dr. Antony Vijesh and Dr. Subhendu Rakshit for their continuous support to the project. I would also like to thank Dr. Swadesh K. Sahoo for his helpful contribution to the project.

I thank my classmates, Anupriya Aggarwal, Md. Balal, Rupnayan Borah, Rohit Gupta and Ashish Sehrawat, my seniors in Physics department Najimuddin Khan, Ajay Pratap Singh, Trayambak Bhattacharyya, Siddharth Karmakar, Sudeep ghosh and Sudip Naskar for their help and motivation without which this would not have been possible. I thank my rommies Sourabh Solanki and Jigar Chowdhary, Ravi Kumar and Sahidul. You were the best buddies I have.

Nevertheless, I thank my mother Shrimati Krishna Halder, my father Shri Subimal Sarkar, my sister Shrimati Shreya Sarkar and my brother-in-Law Shri Goutam Dey for their immense help and support throughout my carrier.

**Indore, India** **Uttiya Sarkar**

**June 22nd, 2015**

**Dedicated to:**

*All those trees,*

*who have unknowingly contributed by providing papers*

# A Study on Four-Fermionic Tensorial Interactions

A thesis report

submitted in partial fulfilment of the

requirement for the degree of

**Master of Science in Physics**

submitted by

**Uttiya Sarkar**

under the guidance of

**Dr. Manavendra Mahato**

Discipline of Physics

Indian Institute of Technology

Indore

June 2015

# Abstract

The study of four-fermionic interactions is a field in particle physics which drew a lot of attention of the physicists for the past few decades. But due to the limitations of these theories, the models that are based on four-fermionic theories are generally called toy models. The existing toy models in late twentieth century like Nambu-Jona-Lasinio model and the famous Gross-Neveu model gave some start up to these kind of interactions. Although there were many set backs of the 1st model, but the 2nd one was well founded and gave nice results and conclusions. However the Gross-Neveu model was shown to be integrable only in two dimensions and the theory was essentially massless, it gave some enthusiasm to the researchers to formulate similar theories.

After the success of Gross-Neveu, people started looking at it seriously. In literature, it is already seen that, theories involving four-fermion interactions give condensations of the composite fields and this type of field theories or models serve the purpose of explaining color superconductivity in quantum chromodynamics, phase condensations, formation of cooper pairs and chiral condensations. So, by an intelligent guess, such models can give such interesting results.

In our work, the construction of a four-fermion quadratic interacting field is done by promoting the Gross-Neveu model into an interacting tensor field. The chiral symmetry of the theory is verified. The model is rewritten in terms of auxiliary fields as it simplifies the calculations. The possible condensate is guessed and an auxiliary field proportional to it is introduced. In order to study it, the guessing of auxiliary field

is done. Both symmetric and anti-symmetric forms of the auxiliary field are tested for the validity under condensation conditions. The symmetric form of the auxiliary field gave known result of scalar condensation of the field which is like Gross-Neveu condensate in higher dimensions. The anti-symmetric form is also tested. To reduce the number of free parameters in the chosen form of the auxiliary field, self-dual condition is employed. This is done to make the calculations simpler and to formulate a simpler model. From anti-symmetric case, under the self-duality condition, it is shown that if the condensate occurs then it will be imaginary. To check the condensation formation, generally search of other vacua states is performed and study of their stability is required.

# Contents

# List of Figures

# Chapter 1

# Introduction

The aim of the project is to construct a four-fermionic interacting field theory. The existing models like Gross-Neveu model(GN Model),(1.1) is perhaps the simplest interacting, fermionic field theory. Although this model is integrable only in 1+1 dimensions, but this is the first ever integrable model formulated forty years ago. In this model N massless flavors of Dirac fermions interact via a scalar-scalar four-fermion interaction in 1+1 dimensions. The Lagrangian is given in the following form

$$\mathcal{L}_{\mathrm{GN}} = \sum_{i=1}^{N} \bar{\psi}^i i\partial\!\!\!/ \psi^i + \frac{g^2}{2}\left(\sum_{i=1}^{N} \bar{\psi}^i \psi^i\right)^2 . \tag{1.1}$$

The large N limit of this model is physically suggestive of higher dimensions.

After this formulation, a number of generalisations of the GN model have been considered, adding a bare mass term or modifying the interaction. The best known such generalization is presumably the chiral GN model, the 2d version of the NambuJona-Lasinio (NJL) model which is even older than the GN model,

$$\mathcal{L}_{\mathrm{NJL}} = \sum_{i=1}^{N} \bar{\psi}^{(i)} i\partial\!\!\!/ \psi^{(i)} + \frac{g^2}{2}\left[\left(\sum_{i=1}^{N} \bar{\psi}^{(i)} \psi^{(i)}\right)^2 + \left(\sum_{i=1}^{N} \bar{\psi}^{(i)} i\gamma_5 \psi^{(i)}\right)^2\right] . \tag{1.2}$$

Here the discrete $Z_2$ chiral symmetry of (1.1) gets promoted to a continuous U(1) chiral symmetry. Other four-fermion interactions which can be found in the literature interpolate between (1.1) and (1.2) by introducing two different coupling constants

,or have extra terms which give rise to fermion-fermion pairing rather than fermion-antifermion pairing. These models are of particular interest because such models can give explanation to phase transitions or condensation processes, such as phenomenon of color superconductivity in quantum chromodynamics,or cooper pair formations. Accordingly, the emphasis of these works has typically been on the patterns of symmetry breaking and the phase condensations.

For massive GN models, no important conclusion is achieved so far, or even with some variation of interaction terms, not too many models were constructed. The integrability is undoubtedly crucial in these kind of models. Since it is quite exceptional to be able to solve both equilibrium thermodynamics and the time evolution of an interacting quantum field theory exactly, the question arises whether there are other physically relevant integrable four-fermion models. This is the main topic of the thesis.

This is not an easy question; therefore we shall proceed rather heuristically. The main ingredients have proven helpful in our search for integrability; i.e. related to symmetries, since any integrable model can posses infinite number of symmetries. So any model holding symmetry conditions will be more likely to be integrable. Considering symmetry issues, if one wishes to generalize an integrable model by making it more complicated without losing integrability, we find it plausible that it helps if the symmetry of the starting model gets enhanced in this process. Thus for instance, adding a mass term to the GN Lagrangian (1.1) breaks the discrete chiral symmetry and renders the model non-integrable. By contrast, switching on an interaction in the tensor channel as in the scalar channel still holds the chiral symmetry of the model, but we are not sure whether it maintains integrability. Notice also that the known integrable models have only one coupling constant. It is hard to imagine that integrability can be kept if one adds more interactions with arbitrary coupling constants. So keeping the following discussion on mind, we include a new kind of interaction that involves four-fermionic tensor interactions to the GN model. We expect possible condensation from the model which in turn can highlight some physical phenomena. In this thesis, two of the most celebrated models "Nambu-Jona-Lasinio" and "Gross-Neveu" model are discussed with special emphasis on Gross-Neveu model. This chapter is devoted to the discussion of the model by Nambu and Jona-Lasinio. It is very briefly discussed and the construction of the form of the Lagrangian is shown. In the next section, borrowing the idea of Nambu-Jona-Lasinio, a model constructed by Gross and Neveu is discussed in detail by following their original paper. The construction of the model, loop calculations from that model for large N limit, spontaneous symmetry breaking of the theory giving rise to fermionic mass is shown explicitly. After that dynamical symmetry breaking is also discussed.

Since our work is based on Gross-Neveu model so the continuation of calculation for the tensor interacting field is done by following the GN model.

In chapter 2, the construction of the model including the following calculations are shown. The proposed model is discussed. The chiral symmetry of the model is shown. Then the Lagrangian is rewritten in terms of auxiliary fields. Now, we expect condensation if the vacuum expectation value of the auxiliary field is non vanishing. So we try to guess the form of the auxiliary field by choosing it to be both symmetric and anti-symmetric form. We see that the symmetric form of the auxiliary field gives back the known scalar condensation of Gross-Neveu type. Hence the condensation from it does not give any new result. Whereas from the anti-symmetric form of the auxiliary field, under some necessary conditions,we find no acceptable solutions of fermionic field. However time does not permit to evaluate loop level diagrams of the theory, performing functional methods to evaluate the condensation and checking the stability and ground state energy of the auxiliary field.

In chapter 3, results are discussed. Finally, in chapter 4, some conclusive remarks are made and future work that can be done in this topic is discussed.

## 1.1 Nambu-Jona-Lasinio Model

In 1961, Nambu and Jona-Lasinio together proposed a model by which tried to investigate the cause of nucleon mass generation. That time it was suggested that the nucleon mass arises due to the self-energy of primary fermionic field. This idea was put into a mathematical formulation using Hartree-Fock approximation. In their work they have considered a simplified model of quadratic four-fermion interaction. This allows a $\gamma_5$ gauge group and an interesting consequence of this symmetry is that there arises automatically pseudoscalar zero-mass bound states of nucleon-antinucleon pair which may be regarded as an idealized pion. The theory contains two parameters which can be explicitly related to observed nucleon mass and the pion-nucleon coupling constant. However a detailed treatment of this NJL model will not be discussed here.

A dynamical theory of elementary particles in which nucleons and mesons are derived from a spinor field with interaction term of order 4 is developed (see references [1],[2]). In basic physical ideas, it has thus the characteristic features of a compound-particle model, but from most of the existing theories, dynamical treatment of the interaction makes up an essential part of the theory.

The scheme is motivated by the observation of an interesting analogy between the properties of Dirac particles that appear in the theory of superconductivity, which was originated with great success by Bardeen, Cooper, and Schrieffer and subsequently given an elegant mathematical formulation by Bogoliubov. The characteristic feature of the BCS theory is that it produces an energy gap between the ground state and the excited states of a superconductor. Superconductivity is something that has been confirmed experimentally at very low temperatures. The gap is caused due to the fact that the attractive phonon-mediated interaction between electrons produces correlated pairs of electrons known as Cooper pairs with opposite momenta and spin near the Fermi surface, and it takes a finite amount of energy to create it.

Here,while not going into details we will only discuss about the Lagrangian constructed by Nambu and Jona-Lasinio which will be helpful in understanding the Gross-Neveu Model.

### 1.1.1 The Lagrangian

The possible nature of the primary interaction between fermions can be interaction mediated by some fundamental Bose field or due to an inherent non-linearity in the fermion field. According to this postulate, the interactions essentially include chirality conservation in addition to the conservation of nucleon number. The chirality($\chi$)here is defined as the eigenvalue of $\gamma_5$, or in terms of quantized fields,

$$\chi = \int \bar{\psi}\gamma_0\gamma_5\psi d^3x \tag{1.3}$$

And the nucleon number is given by

$$N = \int \bar{\psi}\gamma_0\psi d^3x \tag{1.4}$$

So, we can also write the generators of the $\gamma_5$ and ordinary gauge group transformation as

$$\psi \to e^{i\alpha\gamma_5}\psi, \bar{\psi} \to \bar{\psi}e^{i\alpha\gamma_5} \tag{1.5}$$

$$\psi \to e^{i\alpha}\psi, \bar{\psi} \to \bar{\psi}e^{-i\alpha} \tag{1.6}$$

Now the dynamics of the theory would require that the interaction be attractive between particle and antiparticle in order to make bound-state formation possible. Under the chiral transformation given by (1.5), various tensors transform as follows:

$$\begin{aligned}
Vector &: i\bar{\psi}\gamma_\mu\psi \to i\bar{\psi}\gamma_\mu\psi \\
AxialVector &: i\bar{\psi}\gamma_\mu\gamma_5\psi \to i\bar{\psi}\gamma_\mu\gamma_5\psi \\
Scalar &: \bar{\psi}\psi \to \bar{\psi}\psi\cos 2\alpha + i\bar{\psi}\gamma_5\psi\sin 2\alpha \\
Pseudoscalar &: i\bar{\psi}\gamma_5\psi \to i\bar{\psi}\gamma_5\psi\cos 2\alpha - \bar{\psi}\psi\sin 2\alpha \\
Tensor &: \bar{\psi}\sigma_{\mu\nu}\psi \to \bar{\psi}\sigma_{\mu\nu}\psi\cos 2\alpha + i\bar{\psi}\gamma_5\sigma_{\mu\nu}\psi\sin 2\alpha
\end{aligned}$$

It can be seen that a vector or pseudovector Bose field coupled to the fermion field satisfies the invariance. The vector would satisfy the dynamical requirement since, as in the electromagnetic interaction, the forces would be attractive between opposite nucleon charges. The pseudovector field does not meet the requirement as can be seen by studying the self-consistent mass equation.

The vector field seems to be of particular interest since it can be associated with the nucleon gauge group.

Here the model is dealt with strong interactions, such a field would have to have a finite observed mass in a realistic theory. But in order to hold the invariances of the model it is difficult to add a mass term in the model. Also if the bare mass of both spinor and vector field were zero, the theory would not contain any parameter with the dimensions of mass. The quadratic fermion interaction seems to offer another possibility.

The following Lagrangian density will be assumed with $\hbar = 1$ and c=1

$$\mathcal{L}_{\mathrm{NJL}} = \bar{\psi} i \partial\!\!\!/ \psi + \frac{g^2}{2}[(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\psi)^2] \tag{1.7}$$

The coupling constant $g^2$ is positive, and of dimension to the $(mass)^{-2}$.

The above model gives us a good starting point of such toy models. We can emphasize more and focus ourselves in the model proposed by Gross and Neveu which is considered to be a simpler toy model in 1+1 dimensions. In the next section an overview of the model is discussed step by step.

## 1.2 Gross-Neveu Model

### 1.2.1 Gross-Neveu Model

In 1974, David. Gross and Andre Neveu published a paper based on a two-dimensional massless fermion field theories with quadratic interactions. We will show that these models are asymptotically free. The models are considered in large N limit, where N is the number of components of the fermion field. In such an expansion, one can explicitly sum to all orders in the coupling constants. The dynamical symmetry breaking occurs in this model for any value of the coupling constant.

The resulting theory produces a fermion mass dynamically, which is explicitly shown in here. The search for symmetry breaking is performed using functional methods, and the theory develops a non-vanishing vacuum expectation value. The "potential" of fields is discussed and constructed. General results are derived for arbitrary theories in which all masses are generated dynamically. The model is extended to include gauge fields. It is then found that the gauge vector mesons acquire a mass through a dynamical Higgs mechanism.

The usual method of generating spontaneous symmetry breaking in quantum field theory is to introduce an elementary scalar field which develops a non-vanishing vacuum expectation value. However, this mechanism is, of course, not necessary in every case. In more general way of spontaneous symmetry breaking, such as the Goldstone theorem, it is independent of whether the Goldstone particle is associated with an elementary or composite field. In the model of Nambu and Jona-Lasinio, the origin of the spontaneous chiral symmetry breaking is discussed in detail. The specific field theoretic model was analysed, which indicated dynamical symmetry breaking. But unfortunately this model, involving quadratic fermion interaction in four dimensions, was un-renormalizable. Thus it was necessary to introduce a cut-off and the validity of the approximations made to solve the model was very unclear.

Here, the two-dimensional model field theories are examined, which involve fermions with quartic interactions. These models are essentially equivalent to the Nambu-Jona-Lasinio models, but it is in two dimensions, so they are renormalisable.

The reason for choosing such kind of Lagrangian is that these are physical asymptotically free theories. In order to perform calculations of the N-component fermion fields the large N limit condition is imposed. In such an expansion one can sum, in each order of 1/N, to all orders in the coupling constant. This is a good approximation, as the lowest order provides us a very non-trivial theory. In four-dimensional gauge theories, on the other hand, no small expansion parameter appears to exist in the small momentum region. We find that the increasing attractive interaction at long distances invariably produces bound states and dynamical symmetry breaking. The resulting theories produce a fermion mass, a scalar bound state, and if the broken symmetry is continuous, the bound-state is a Goldstone boson. All dimensionless parameters are calculable, and the theory ends up involving no adjustable parameters. This is in agreement with asymptotically free theories. The model is also extended to include gauge fields in the Lagrangian. Then it is shown that the gauge mesons acquire a mass through a dynamical Higgs mechanism, as one might expect.

Now, let us proceed in discussing the paper which will be extremely helpful in developing our own Lagrangian.

### 1.2.2 The Model

The models that are considered in this paper will contain N-component fermion fields with quartic interactions in two space-time dimensions. These models are without the exception of non-Abelian gauge theories in four dimensions, the only physically sensible asymptotically free theories. Now, if we let N to be very large, these theories can be solved in an expansion in powers of 1/N . The spontaneous symmetry breaking is also studied from this model. The model is given in the form of the Lagrangian density

$$\mathcal{L}_{GN} = \bar{\psi} i \partial\!\!\!/ \psi + \frac{g^2}{2}(\bar{\psi}\psi)^2 \tag{1.8}$$

where $\psi$ is the N-component, massless fermion field. This Lagrangian is invariant under the discrete $\gamma_5$ transformation $\psi \to \gamma_5 \psi$ which ensures the masslessness of the fermion to any order of perturbation theory. It can be easily noted that if mass term is added in the Lagrangian, it will not remain to be invariant under $\gamma_5$ transformation.

In the large-N limit, as we shall see, g will vanish like 1/N so that we define

$$\lambda = g^2 N$$

which is the t-Hooft condition. For large N, $\lambda$ must be finite, so g will be very small. This theory is renormalisable in two dimensions and will require only wave-function and coupling-constant renormalisation. It is easy to verify that no new interactions, such as $(\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma^\mu\psi)$ are generated, at least to order $g^8 = \lambda^4(1/N)^4$. Thus one can restrict himself in calculating only one loop calculations of this theory. The resulting theory is then characterized by a single dimensionless parameter, $g^2$.

Now, let us concentrate on a Yukawa interaction, in which the mass of the scalar and its coupling become infinite in such a way so as to reproduce in the limit our local quadratic coupling. Consider the Yukawa Lagrangian

$$\mathcal{L}' = \bar{\psi} i\partial\!\!\!/\psi + \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 + gm\bar{\psi}\psi\phi \tag{1.9}$$

Here g is a positive constant. Now if m becomes infinite, the above Yukawa interaction becomes equivalent to the original interaction of Eq.(1.8), the combination of vertices $(igm)^2$ and scalar propagator $i/(P^2 - m^2)$ yielding for infinite m the local coupling $+ig^2$. Since the resulting theory is asymptotically free, it is expected that the the theory will survive in the large-m limit. The Lagrangian $\mathcal{L} = -g^2\left(\bar{\psi}\psi\right)^2$ corresponds to the local limit of a Yukawa theory with imaginary coupling.

The m$\rightarrow\infty$ limit can be taken in $\mathcal{L}$ by rescaling the scalar field

$$\sigma = m\phi \tag{1.10}$$

and letting $m \rightarrow \infty$ The resulting Lagrangian

$$\mathcal{L}_\sigma = \bar{\psi}(i\partial\!\!\!/)\psi - \frac{1}{2}\sigma^2 - g\bar{\psi}\psi\sigma \tag{1.11}$$

Now this gives the exact fermion Green's functions as does $\mathcal{L}_{GN}$ This can also be seen by examining the generating functional for these Green's functions in the pathintegral formulation:

$$
\begin{aligned}
Z(\eta,\bar{\eta}) &= (constant)\int d\psi d\bar{\psi} exp\Big[i\left(\bar{\psi}\partial\!\!\!/\psi+\frac{g^2}{2}(\bar{\psi}\psi)^2+\bar{\eta}\psi+\bar{\psi}\eta\right)\Big] \\
&= (constant)\int d\psi d\bar{\psi} d\sigma exp\Big[i\left(\bar{\psi}\partial\!\!\!/\psi-\frac{1}{2}(\sigma)^2-g\bar{\psi}\psi+\bar{\eta}\psi+\bar{\psi}\eta\right)\Big]
\end{aligned}
\tag{1.12}
$$

and performing the $\sigma$ integration in the latter expression. Now for simplicity, one can always consider the theory generated by $\mathcal{L}_\sigma$ since both of them has the same form of Green's function and we will be always involved in calculating some amplitude related to Green's function. So any one of the theory will give the same result. The discrete symmetry which prevents g from acquiring a mass in perturbation theory is

$$
\begin{aligned}
\psi &\to \gamma_5\psi \\
\sigma &\to \sigma
\end{aligned}
\tag{1.13}
$$

We can also do the same treatment and rewrite the NJL model Lagrangian $\mathcal{L}_{NJL}$ in terms of $\mathcal{L}_\sigma$

$$
\mathcal{L}_{\rm NJL}=\bar{\psi}i\partial\!\!\!/\psi+\frac{g^2}{2}[(\bar{\psi}\psi)^2-(\bar{\psi}\gamma_5\psi)^2] \tag{1.14}
$$

in the form

$$
\mathcal{L}_\sigma=\bar{\psi}i\partial\!\!\!/\psi-\frac{1}{2}\left(\sigma^2+\pi^2\right)+g\left(\sigma\bar{\psi}\psi+i\pi(\bar{\psi}\gamma_5\psi)\right) \tag{1.15}
$$

where $\sigma$ and $\pi$ are the auxiliary fields.

We have already discussed that this model is chirally invariant. We will use this $\mathcal{L}_\sigma$ to show the dynamical symmetry breaking and generation of mass of the gauge field.

### 1.2.3 Loop Calculation

Now the model is solved in the large-N limit. The dominant graphs in this limit will be those containing the maximal number of fermion loops, since each of these yields a factor of $g^2N$. Keeping $\lambda$ fixed, as $N\to\infty$. It is easy to see that in the $\sigma$

formulation and from equation (1.14), the only radiative corrections of order 1 to the 4-point function are those that are given by $\sigma$ propagator.

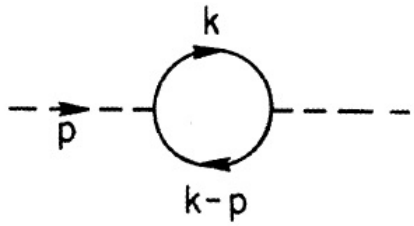


Figure 1.1: self-energy of $\sigma$

The lowest-order of self-energy graph (1.1) is simply

$$\begin{aligned} \Pi(P) &= -(g)^2 N \int \frac{d^2k}{(2\pi)^2} \frac{Tr[\not{k}(\not{k}-\not{P})]}{k^2(k-P)^2} \\ \Pi(P^2) &= \frac{+i\lambda}{2\pi} \int_0^1 d\alpha \left[ \ln\left( \frac{-\lambda^2}{\alpha(1-\alpha)P^2} \right) - 2 \right] \end{aligned} \tag{1.16}$$

where $\lambda$ is an ultraviolet cut-off. We have used the Feynman formula in Eq.(1.16) to perform the integration

$$\frac{1}{ab} = \int_0^1 \frac{d\alpha}{[a\alpha + b(1-\alpha)]^2} \tag{1.17}$$

The renormalisation requires that the propagator,

$$D(P) = \frac{i}{[1 + i\pi(P)]}$$

satisfy $D_R(P^2) = -i$ at $P^2 = -\mu^2$.

This means that we must subtract $\Pi(P^2)$ at $P^2 = -\mu^2$ :

$$\begin{aligned} \Pi_R(P^2,\mu^2) &= -\frac{i\lambda}{2\pi} \ln(-P^2/\mu^2) \\ D_R(P^2,\mu^2) &= \frac{-i}{1 + (\lambda/2\pi)\ln(-P^2/\mu^2)}. \end{aligned} \tag{1.18}$$

All other radiative corrections are of order 1/N. Thus, the 4-point function is given by the graphs in Fig.1.2(which are equivalent to the graphs of Fig.1.3) and is equal to

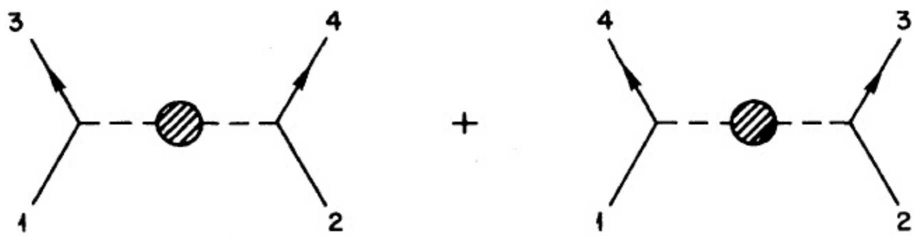


Figure 1.2: 4-point Fermion interaction graph

$$G(P_1P_2;P_3P_4) = ig^2\left[\frac{1}{1+(\lambda/2\pi)\ln(s/\mu^2)} + \frac{1}{1+(\lambda/2\pi)\ln(u/\mu^2)}\right], \tag{1.19}$$

where $s = -(P_3 - P_1)^2$, $u = -(P_4 - P_1)^2$ are the Mandelstam variables such that positive s and u mean space-like energy squared and momentum transfer squared.

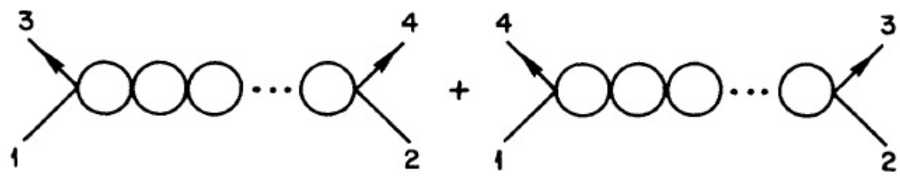


Figure 1.3: Leading order graphs which are equivalent to Fig.1.2

To evaluate the renormalisation- group parameters we note that to order 1/N there is no wave-function renormalisation of the $\psi$ field, nor is the vertex $\bar{\psi}\psi\sigma$ renormalised. Therefore the renormalised coupling $g_R$ is related to the bare coupling $g_0$ by $g_R = g_0\sqrt{Z_\sigma}$, where Z is the wave-function renormalisation constant of the $\sigma$ field. Thus the $\beta$ function and the anomalous dimension of $\sigma(\gamma_\sigma)$ are related by

$$\begin{aligned} \beta(g) &= \mu\frac{\partial}{\partial\mu}g_R \\ &= g\mu\frac{\partial}{\partial\mu}\sqrt{Z_\sigma} \\ &= g\gamma_\sigma(g) \end{aligned} \tag{1.20}$$

Now the $\sigma$ propagator must satisfy the renormalisation group equation given

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + 2\gamma_\sigma(g)\right] D_R(P,\mu) = 0 \tag{1.21}$$

Now, putting Equation (118) back in this equation and solving for $\beta(g)$it is deduced that

$$\begin{aligned}\beta(g) &= \frac{\lambda g}{2\pi} \\ \gamma_\sigma(g) &= \frac{\lambda}{2\pi}\end{aligned} \tag{1.22}$$

The negative sign of $\beta(g)$ means that the theory is asymptotically free. The effective coupling constant satisfies

$$\begin{aligned}\frac{dg(\bar{g},t)}{dt} &= \beta(\bar{g}) \\ \bar{g}(g,0) &= g.\end{aligned} \tag{1.23}$$

Thus it is given by

$$g^2(\bar{g},t) = \frac{g}{1+(\lambda/\pi)t} \tag{1.24}$$

The effective coupling, $\bar{g}^2$, vanishes for large momenta ($t \to \infty$), logarithmically (as 1/t). This is common to all asymptotically free theories. If we see carefully that Equation.(1.24) holds for all t. Thus the small-momentum behaviour of the theory can also be explored. Now $\bar{g}^2$ develops a pole at

$$\begin{aligned}t &= \frac{\pi}{\lambda} \\ P^2 &= -\mu^2 exp(-2\pi/\lambda)\end{aligned} \tag{1.25}$$

This pole is present for any value of $\lambda$ , approaching zero when $\lambda \to 0$.

The existence of this tachyon pole could mean one of two things. First, the theory could be simply nonsense at least in the leading 1/N approximation. Another possibility is that we could simply be constructing the theory about the "wrong" vacuum state. When we separate the fermion-antifermion pairs by a large space-like separation, the Green's function does not fall off exponentially. This can be explained if the vacuum about which we have been perturbing, the normal vacuum which is invariant under $\psi \to \gamma_5\psi$, $\sigma \to -\sigma$ is not the ground state. In the following we shall show that this indeed is the case and the pole in $\bar{g}^2$ is simply the signal for spontaneous symmetry breaking. The symmetry breaking will generate a fermion mass and prevent us from concluding from the pole at some small space-like momenta that the fermion amplitudes develop tachyon poles.

### 1.2.4 Spontaneous Symmetry Breaking

In the previous section it is shown that in the large-N limit the model necessarily develops tachyon poles for any value of the coupling. For the theory to be consistent in this approximation, the normal symmetric vacuum is not in fact the ground state. So we may have perturbed the model about wrong vacuum. If this is the case, we would expect that the true ground state $\bar{\psi}\psi$ has a non-vanishing vacuum expectation value. This is shown here by looking at the "potential" of $\bar{\psi}\psi$. Let us first consider our theories with the addition of a constant external source coupled to $\bar{\psi}$ . In this case, an external source is simply a mass term. Thus the Lagrangian becomes

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - M)\psi + \frac{g^2}{2}(\bar{\psi}\psi)^2 \tag{1.26}$$

Or

$$\mathcal{L}_\sigma = \bar{\psi}(i\partial\!\!\!/ - M)\psi - \frac{1}{2}\sigma^2 - g\bar{\psi}\psi\sigma \tag{1.27}$$

By just following the previous treatment, the 4-point fermion Green's function can be constructed which is now given by

$$\begin{aligned} G(P_1P_2;P_3P_4) &= ig^2\left[\frac{1}{1+(\lambda/2\pi)[B(s,M^2)-B(\mu^2,M^2)]}\right. \\ &+ \left.\frac{1}{1+(\lambda/2\pi)[B(u,M^2)-B(\mu^2,M^2)]}\right] \end{aligned} \tag{1.28}$$

where B is essentially the massive fermion loop of Fig. 1.4

$$B(s,M^2) = \left(\frac{s+4M^2}{s}\right)^{1/2} \ln\left(\frac{(s+4M^2)^{1/2}+\sqrt{s}}{(s+4M^2)^{1/2}-\sqrt{s}}\right) \tag{1.29}$$

From this equation we can see that $B(s, M^2)$ is monotonically increasing function of s or the momentum whose minimum value is at $B(0, M^2) = 2$. Therefore, as long as

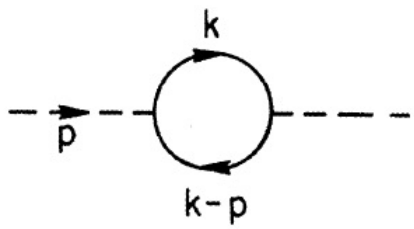


Figure 1.4: self-energy of $\sigma$ for massive fermion loop

$M > 0$, we do not expect any pole for space-like momentum ($s > 0$).

So, for no poles occurring under these conditions, the Green's function must always hold the inequality

$$\frac{\lambda}{2\pi} < \frac{1}{B(\mu^2, M^2) - B(0, M^2)} \tag{1.30}$$

as follows from equation (1.28). But if M is decreased, for fixed $\lambda$ and $\mu$, a bound-state pole is developed whose mass decreases as M decreases. At the point when inequality in Equation.(1.30) is just violated the bound-state mass is zero.

Just like the previous case, using Eq.(1.29) and Equation.(1.30) $M^2$ is given by

$$M^2 \approx \mu^2 exp(-2\pi/\lambda) \tag{1.31}$$

for small $\lambda$ .

From the above relation, it seems that zero-mass fermion-antifermion bound state is formed when the mass is reduced below this critical value. When the mass is decreased even further, the bound state would appear to become a tachyon which is obtained before. However, when its mass vanishes, the vacuum can be unstable tachyon. It is a consequence of constructing the amplitude by perturbing about an unstable vacuum.

The $\mathcal{L}_\sigma$ Lagrangian is used to investigate the "potential" as a function of the classical g field, since $\sigma$ is essentially equal to $g\bar{\psi}\psi$. However, this potential is not exactly equal to the potential of the composite operator $\bar{\psi}\psi$ of the GN model. But they are closely related.

The ground state must occur at a minimum of this potential, let us consider the following fact $\sigma_c =< 0|\sigma|0 >=< 0|g\bar{\psi}\psi|0 >$. Now the vacuum-to-vacuum amplitude in the presence of an external source coupled to $\sigma$ is given by

$$exp(iW(J)) \equiv \int d\psi d\bar{\psi} d\sigma exp\left(i\left[\mathcal{L}_\sigma(\sigma,\bar{\psi},\psi)+J\sigma\right]\right) \tag{1.32}$$

W(J) is the generator of the connected Green's functions of the $\sigma$ field. $\sigma_c$ is related to W(J) by

$$\begin{aligned} \sigma_c(x) &= \frac{\delta W}{\delta J(x)} \\ &= < 0|\sigma(x)|0 >_J \end{aligned} \tag{1.33}$$

Doing Legendre transformation of W(J),

$$\Gamma(\sigma_c) = \int d^4x \sigma_c(x) J(x) - W(J) \tag{1.34}$$

Again from the above equation

$$\Gamma = \int d^4x V(\sigma_c) \tag{1.35}$$

So, for the equilibrium point where energy will be minimum, the following conditions are satisfied

$$\begin{aligned} \frac{\partial V}{\partial \sigma_c} &= 0 \\ \frac{\partial^2 V}{\partial^2 \sigma_c} &> 0. \end{aligned} \tag{1.36}$$

Spontaneous symmetry breaking will occur if this two conditions boils down to give a non-zero expectation value of $\sigma$. $\Gamma$ is the generating functional of the one-particle irreducible (1PI) n-point functions of the $\sigma$ field. So V($\sigma$) is given by

$$V(\sigma_c) = \sum \frac{1}{n!}(\sigma_c)^n \Gamma(0,....,0) \tag{1.37}$$

where $\Gamma(0,....,0)$ is the sum of all 1PI Green's functions with n external $\sigma$ lines carrying zero four -momentum.

So the tree level diagram will contribute to this potential as

$$V(\sigma_c) = \frac{{\sigma_c}^2}{2} \tag{1.38}$$

For the tree level approximation, the minimum expectation value of the potential is at $\sigma_c =< 0|\sigma|0 >$.

So, essentially there are other contributions to V. In fact the leading terms in V for large N are given by the tree graphs plus all one-loop graphs given in Fig. 1.5. Summing over all the one-loop graphs with an ultraviolet cut-off $\Lambda$, the potential is

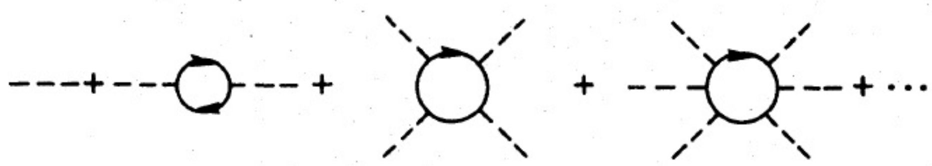

Figure 1.5: Feynman Graphs contributes to V($\sigma$)

given by

$$
\begin{aligned}
V &= \frac{{\sigma_c}^2}{2} - Ni \sum \int^{\Lambda} \frac{d^2k}{(2\pi)^2} \frac{1}{2n} \frac{(g^2 {\sigma_c}^2)^n}{k^2} \\
&= \frac{{\sigma_c}^2}{2} - \frac{\lambda}{4\pi} {\sigma_c}^2 [\ln \Lambda^2 + 1 - \ln(g^2 {\sigma_c}^2)]
\end{aligned} \tag{1.39}
$$

For the above calculation (see reference [13]). This potential V must be renormalised now, which can be performed by using Coleman and Weinberg treatment (see reference [4]), by subtracting (1.39) at some value,$\sigma_0$, of the classical field. If we define $(\partial^2 V/\partial\sigma^2)|_{\sigma=0} = 1$ it is equivalent to subtracting the $\sigma$ propagator at zero momentum. That in turn will give inferred divergences at zero momentum, or zero fields. Therefore the renormalisation is done by following Coleman and Weinberg method and by demanding that

$$(\partial^2 V/\partial\sigma^2)|_{\sigma=\sigma_0} = 1 \tag{1.40}$$

We then have

$$V(\sigma_c, \sigma_0, g) = \frac{{\sigma_c}^2}{2} + \frac{\lambda}{4\pi} {\sigma_c}^2 \left[ \ln \left( \frac{\sigma_c}{\sigma_0} \right)^2 - 3 \right] \tag{1.41}$$

The potential V obeys the renormalisation-group equation

$$\left[ \sigma_0 \frac{\partial}{\partial \sigma_0} + \bar{\beta}(g) \frac{\partial}{\partial g} - \bar{\gamma}(g) \sigma_c \frac{\partial}{\partial \sigma_c} \right] V(\sigma_c, \sigma_0, g) = 0 \tag{1.42}$$

Since this renormalisation procedure followed in this process is different from the previous method; $\beta(g)$ and $\gamma(g)$ will be different from the previous values. Therefore,

$$\begin{aligned}\beta(g) &= g\bar{\gamma}(g) \\ &= -\frac{\lambda g/2\pi}{1+(\lambda/2\pi)}\end{aligned} \tag{1.43}$$

It is now seen that the symmetric point, $\sigma$ =0, is not a minimum of the potential. The true potential is given by Fig. 1.6. The one-loop corrections give rise to a

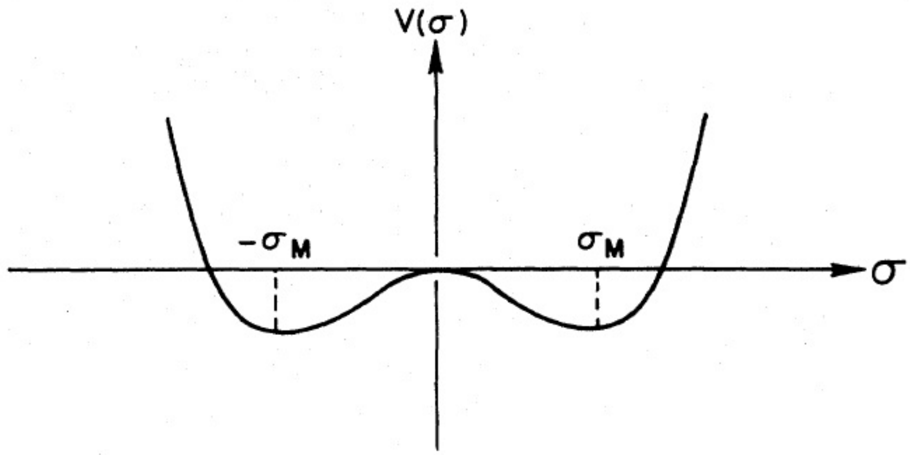


Figure 1.6: Form of V($\sigma$) to the leading order in 1/N)

negative term which dominates, for small $\sigma_c$. For large $\sigma_c$ , the potential is positive and increasing, and thus the theory is stable. The minimum of the potential occurs at $\sigma_c = \sigma_M$ where

$$\begin{aligned}V'(\sigma_M,\sigma_0,g)=\frac{\partial V}{\partial \sigma} &= 0 \\ \sigma_M\left[1+\frac{\lambda}{2\pi}\left[\ln\left(\frac{\sigma_M}{\sigma_0}\right)^2-2\right]\right] &= 0\end{aligned} \tag{1.44}$$

and

$$V''(\sigma_M,\sigma_0,g)=1+\frac{\lambda}{2\pi}\ln\left(\frac{\sigma_M}{\sigma_0}\right)^2=\frac{\lambda}{\pi} \tag{1.45}$$

i.e., when

$$|\sigma_M|=\sigma_0 exp\left(1-\frac{\pi}{\lambda}\right) \tag{1.46}$$

Now it can clearly be seen that why a tachyon pole was found previously, because we were perturbing about a local minima of the potential. In the true ground state $\sigma$ has a non-vanishing vacuum expectation value. By shifting the $\sigma$ field to its true ground state and then carrying out perturbation theory about the asymmetric vacuum gives us right result. Essentially by choosing $|\sigma_M| = \sigma_0 exp(1 - \frac{\pi}{\lambda})$, the discrete symmetry $\sigma \to -\sigma$ and chiral symmetry are broken and the fermion acquires a mass

$$M_F = g\sigma_0 exp\left(1 - \frac{\pi}{\lambda}\right) \tag{1.47}$$

### 1.2.5 Dynamical Higgs Mechanism

In this section, we will show that if a Lagrangian involving gauge mesons, is chosen in such a way, that it is invariant under a specific choice of gauge group, the dynamical symmetry breaking gives the gauge meson a mass. The chiral symmetry will be true for this choice of Lagrangian. The gauge invariant Lagrangian with the addition of gauge fields is given by

$$\mathcal{L} = \bar{\psi}\left(i\partial\!\!\!/ + eB\!\!\!/\gamma_5\right)\psi + \frac{g^2}{2}\left[\left(\bar{\psi}\psi\right)^2 - \left(\bar{\psi}\gamma_5\psi\right)^2\right] + \frac{1}{4}\left(\partial_\mu B_\nu - \partial_\nu B_\mu\right)^2 \tag{1.48}$$

Here, B is our gauge field. One can rewrite this Lagrangian in terms of $\sigma$ and $\pi$ fields, where $\pi$ is an auxiliary field. By writing it in terms of $\sigma$ and $\pi$ fields, the dynamical mass generation of B-field can be shown. The equivalent Lagrangian of Eq.(1.48) is thus given by

$$\mathcal{L}_\sigma = \bar{\psi}(i\partial\!\!\!/ + eB\!\!\!/\gamma_5)\psi + \frac{1}{4}(\partial_\mu B_\nu - \partial_\nu B_\mu)^2 - \frac{1}{2}(\sigma^2 + \pi^2) + g\bar{\psi}(\sigma + i\gamma_5\pi)\psi \tag{1.49}$$

Notice that this transformation is similar to the transformation performed previously for the NJL model in section 1.1, but here we have the gauge terms.

In 1+1 dimensions the coupling constant e has the dimension of mass. This e is defined as the finite bare coupling constant.So, there will not be any effect on the asymptotic freedom and the renormalisation-group properties of the theory.

The model is studied in large N limit. The $\sigma$ potential is unaffected by the new interaction terms. So, as it is discussed in the previous section, $\sigma$ acquires a vacuum expectation value, similarly fermion mass is generated. We now focus into the $B_\mu$ and $\pi$ fields and rewrite the Lagrangian (1.49) in terms of these fields.

The one loop graphs are given in Fig.1.7. Here the self-energy of B-field, B-$\pi$ inter-


Figure 1.7: The Feynman graphs that contribute to the vector-meson and pseudoscalar-meson self-energies

action and $\pi$-field self-energy Feynman Diagrams are shown. Now one can write the effective Lagrangian in terms of the bilinear products in limit $N \to \infty$ as

$$\begin{aligned}\mathcal{L}_{eff}(B_\mu, \pi) &= -\frac{1}{2}B_\mu P^2 B_\mu + B_\mu P_\mu B_\nu P_\nu + \frac{\alpha}{2\pi}{B_\mu}^2 + \frac{\alpha}{2\pi}B_\mu \frac{P_\mu P_\nu}{P^2}B_\nu \\ &+ \alpha B_\mu M^2 U \frac{P_\mu P_\nu}{P^2}B_\nu + igemN\pi U P_\mu B_\mu + \frac{1}{4}\alpha\pi P^2 U\pi \qquad (1.50)\end{aligned}$$

where

$$U = \frac{1}{\pi[P^2(P^2 - 4M_F^2)]^{1/2}} \ln \frac{(-P^2 + 4M_F^2)^{1/2} - (-P^2)^{1/2}}{(-P^2 + 4M_F^2)^{1/2} + (-P^2)^{1/2}} \qquad (1.51)$$

As previously stated that this effective Lagrangian contains only bilinear forms of the fields. Here $B_\mu$ and $P_\mu$ are the fields and $\alpha$ is a constant with dimensions of mass. Also it is interesting to note that this Lagrangian is not chiral-invariant by itself. This is why, the symmetry of the theory is broken and the mass for the meson field is generated. It is still necessary to introduce a gauge-fixing term. We make the convenient choice

$$\mathcal{L}_c = \frac{1}{2}\left[(P^\mu B_\mu)\left(1 - \frac{\alpha}{\pi P^2} - \frac{2\alpha M^2 U}{P^2}\right)^{1/2} - \frac{iegNU}{\left(1 - \frac{\alpha}{\pi P^2} - \frac{2\alpha M^2 U}{P^2}\right)}\right]^2 \qquad (1.52)$$

which diagonalizes the $B_\mu$ and $\pi$ propagators. With this choice the propagators are defined as

$$\begin{aligned} B_\mu\, propagator &: \frac{-ig_{\mu\nu}}{(P^2 + \alpha/\pi)} \\ \pi\, propagator &: \frac{4i}{U(-P^2 + \alpha/\pi P^2 - 2\alpha U M_F^2/P^2)} \qquad (1.53)\end{aligned}$$

Hence the vector meson has acquired a mass $\lambda/\pi$. From the effective Lagrangian it is clearly observable.

The fact that $\pi$ propagator is not present in $\mathcal{L}_c$ , tells that it is an auxiliary field.

Now we will discuss that by using this model, how some condensation can be explained. These discussions are found in the literature. Here we will discuss two cases, one related to fermion-fermion and fermion-anti fermion pairing giving explanations to Cooper pair formation and another related to time-crystals.

## 1.3 Gross-Neveu model:Fermion-Fermion and Fermion-AntiFermion pairing

The massless Gross-Neveu and chiral Gross-Neveu models are well known examples of integrable quantum field theories in 1+1 dimensions. But whether integrability is preserved if one either replaces the four-fermion interaction in fermion-antifermion channels by a dual interaction in fermion-fermion channels, or if one adds such a dual interaction to an existing integrable model. The relativistic Hartree-Fock-Bogoliubov approach is adequate to deal with the large N limit of such models. In this way, construction and solution of three integrable models with Cooper pairing is performed. This type of field theories can serve as exactly solvable toy models for color superconductivity in quantum chromodynamics. So far, from our discussion, it is already known about the Gross-Neveu Model; it is the simplest interacting, fermionic field theory. The model is given by

$$\mathcal{L}_{\text{GN}} = \bar{\psi} i \partial\!\!\!/ \psi + \frac{g^2}{2} (\bar{\psi}\psi)^2. \tag{1.54}$$

On the other hand the Nambu-Jona-Lasinio model is given by

$$\mathcal{L}_{\text{NJL}} = \bar{\psi} i \partial\!\!\!/ \psi + \frac{g^2}{2} [(\bar{\psi}\psi)^2 + (\bar{\psi} i \gamma_5 \psi)^2]. \tag{1.55}$$

The main motivation of this section is to see whether there is any integrable four-fermion model other than the known ones.

This is not an easy question. Two main ingredients have proven helpful in the search for integrability: The first one is related to symmetries, the second one to the concept of duality between fermion-fermion and fermion-antifermion pairing.

From the symmetry conditions, if one wishes to generalize an integrable model by making it more complicated without losing its integrability, it helps if the symmetry of the starting model gets enhanced in this process. Thus for instance, adding a mass term to the GN Lagrangian breaks the discrete chiral symmetry and makes the model non-integrable. On the other hand switching on an interaction enhances the chiral symmetry and maintains integrability. Notice also that the known integrable models have only one coupling constant. It is hard to imagine that integrability can be kept if one adds more interactions with arbitrary coupling constants.

Now coming back to duality, the duality transformation discussed here consists in replacing fields by their complex conjugates, separately for left-handed and right-handed fermions. This is very different from charge conjugation. After this transformation, one can relate the transformed models with fermion-antifermion pairing, i.e. chiral symmetry breaking and fermion-fermion pairing, i.e. superconductivity.

So starting with a free, massless fermions and performing the Pauli-Gürsey symmetry operation to 1+1 dimensions, the duality transformation is applied to GN model. This yields two distinct integrable models related to Cooper pairing. The concept of "self-dual" field theories are then introduced. So the GN model is casted into self-dual form.

### 1.3.1 Pauli-Gürsey symmetry in 1+1 dimensions

The Pauli-Gürsey symmetry is a symmetry group of massless Dirac fermions. It combines the chiral transformations and charge conjugation. For GN model Lagrangian we only need the special case of Pauli-Gürsey symmetry in 1+1 dimensions. The following chiral representation of the Dirac matrices are used

$$\begin{aligned}
\gamma^0 = \sigma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
\gamma^1 = i\sigma_2 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
\gamma^5 = \gamma^0\gamma^1 = -\sigma_3 &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned} \tag{1.56}$$

The upper and lower components of the Dirac spinor are defined as

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} . \tag{1.57}$$

The light cone coordinates in the following convention is used in this work

$$z = x - t, \quad \bar{z} = x + t, \quad \partial_0 = \bar{\partial} - \partial, \quad \partial_1 = \bar{\partial} + \partial \,, \tag{1.58}$$

so that the free, massless Dirac Lagrangian becomes

$$\mathcal{L}_0 = \bar{\psi} i \partial\!\!\!/ \psi = -2i\psi_1^* \partial \psi_1 + 2i\psi_2^* \bar{\partial} \psi_2 \,. \tag{1.59}$$

So the Pauli-Gürsey group in 1+1 dimensions can be generated by four basic (canonical) transformations,

$$\begin{aligned}
\psi_1 &\rightarrow \mathrm{e}^{\mathrm{i}\alpha} \psi_1 \,, \\
\psi_2 &\rightarrow \mathrm{e}^{\mathrm{i}\beta} \psi_2 \,, \\
\psi_1 &\rightarrow \psi_1^* \,, \\
\psi_2 &\rightarrow \psi_2^* \,.
\end{aligned} \tag{1.60}$$

The first two lines are essentially the chiral transformations and the last two lines are discrete transformations which are not a symmetry of the classical action. The charge conjugation of this convention is given by

$$\psi_c = \gamma_5 \psi^* = \begin{pmatrix} -\psi_1^* \\ \psi_2^* \end{pmatrix} \tag{1.61}$$

So, as a conclusion one can say that the discrete Pauli-Gürsey transformations can be thought of as combinations of chiral transformations and charge conjugation. The group structure behind Eq(58) is O(2)$_R$⊗O(2)$_L$, if we decompose $\psi_1, \psi_2$ into real and imaginary parts, an extension of the chiral symmetry group is given by SO(2)$_R$⊗SO(2)$_L$.

### 1.3.2 Canonical transformation of GN and NJL models

Let us consider the original GN model with discrete chiral symmetry first. Performing the transformation given by Equation (1.60), we obtain

$$\mathcal{L}_{\rm GN} = -2i\psi_1^*\partial\psi_1 + 2i\psi_2^*\bar{\partial}\psi_2 + \frac{g^2}{2}\left(\psi_1^*\psi_2 + \psi_2^*\psi_1\right)^2 . \tag{1.62}$$

So we can see that the conservation of fermion number, the Z$_2$ chiral subgroup ($\psi_1 \to \pm\psi_1, \psi_2 \to \pm\psi_2$) and charge conjugation are unbroken by the interaction term. If we perform the canonical transformation $\psi_1 \to \psi_1^*$ which leaves only the free part of the Lagrangian invariant, we generate a new interacting theory which will also be integrable. Under thesr transformation the Lagrangian of Eq.(1.62) thus becomes

$$\tilde{\mathcal{L}}_{\rm GN} = -2i\psi_1^*\partial\psi_1 + 2i\psi_2^*\bar{\partial}\psi_2 + \frac{g^2}{2}\left(\psi_1^*\psi_2^* + \psi_2\psi_1\right)^2 . \tag{1.63}$$

One can easily see from this Lagrangian that it gives rise to fermion-fermion pairing instead of fermion-antifermion pairing, i.e., the feature resembling superconductivity rather than chiral symmetry breaking. The residual Pauli-Gürsey symmetries are now U(1)$_A$ (conservation of axial charge), Z$_2$ chiral symmetry, and charge conjugation. The Cooper pair condensate in this model is real. Previously in GN model it was the chiral condensation. Now from the NJL model, the continuous chiral symmetry of the Pauli-Gürsey group holds true, so by performing the transformation discussed in the previous subsection, on the NJL Lagrangian, it takes the form

$$\mathcal{L}_{\rm NJL} = -2i\psi_1^{(1)*}\partial\psi_1^{(1)} + 2i\psi_2^{(1)*}\bar{\partial}\psi_2^{(1)} + 2g^2\left(\psi_1^{(1)*}\psi_2^{(1)}\right)\left(\psi_2^{(2)*}\psi_1^{(2)}\right) . \tag{1.64}$$

The discrete part of the Pauli-Gürsey group breaks down to charge conjugation. Applying the duality transformation to this Lagrangian yields the following four-fermion theory,

$$\tilde{\mathcal{L}}_{\rm NJL} = -2i\psi_1^{(1)*}\partial\psi_1^{(1)} + 2i\psi_2^{(1)*}\bar{\partial}\psi_2^{(1)} + 2g^2\left(\psi_1^{(1)*}\psi_2^{(1)*}\right)\left(\psi_2^{(2)}\psi_1^{(2)}\right) . \qquad (1.65)$$

where 1 and 2 in the superscript stands for the fact that they are different fermionic components.

This is yet another field theory with fermion-fermion pairing, with $\mathrm{U}(1)_A$ symmetry. It is in fact identical to the Cooper pair Lagrangian proposed by Chodos, Minakata and Cooper (CMC),

$$\mathcal{L}_{\rm CMC} = \bar{\psi}^{(i)} i\partial\!\!\!/ \psi^{(i)} + 2G^2\left(\bar{\psi}^{(i)}\gamma_5\psi^{(j)}\right)\left(\bar{\psi}^{(i)}\gamma_5\psi^{(j)}\right) , \qquad (1.66)$$

for the choice $g^2 = 2G^2$. Since the duality transformation is a canonical transformation, there is no need to solve the Cooper pair model anew if the NJL model has been solved already. All one has to do is translate the physical observables into the dual language.

In the next section we will discuss about fermion-fermion and fermion-antifermion pairing.

### 1.3.3 Self-dual GN model

Since the known GN model breaks the discrete part of the Pauli-Gürsey group down to charge conjugation, it enables us to generate a new integrable model by applying the transformation ($\psi_1 \to \psi_1^*$) to the GN Lagrangian. We now try to construct another integrable model by "self-dualizing" the GN Lagrangian. Note that in our theory also we will try to construct the Lagrangian following these steps. This means that we add the interaction term of the dual GN model to the GN model Lagrangian, so that the full Lagrangian will have a part of the Pauli-Gürsey symmetry with the free, massless theory,

$$
\begin{aligned}
\mathcal{L}_{\mathrm{sdGN}} &= -2i\psi_1^*\partial\psi_1 + 2i\psi_2^*\bar{\partial}\psi_2 \\
&\quad + \frac{g^2}{2}\left[\left(\psi_1^*\psi_2 + \psi_2^*\psi_1\right)^2 + \left(\psi_1^*\psi_2^* + \psi_2\psi_1\right)^2\right] . \qquad (1.67)
\end{aligned}
$$

This is the self-dual Gross-Neveu (sdGN) model. The symmetries like $\mathrm{U}(1)_V$ or $\mathrm{U}(1)_A$ symmetries are broken in this Lagrangian. It is interesting to note here that the interaction terms can give rise to both fermion-fermion and fermion-antifermion pairing and both condensation are real.

Now for the large N limit of this model, standard Hubbard-Stratonovich transformation(see reference [11],[12]) is performed on the sdGN Lagrangian(1.67). The Lagrangian is thus given by

$$
\begin{aligned}
\mathcal{L}'_{\mathrm{sdGN}} &= \mathcal{L}_{\mathrm{sdGN}} - \frac{1}{2g^2}\left[\mathcal{S} + g^2\left(\psi_1^*\psi_2 + \psi_2^*\psi_1\right)\right]^2 \\
&\quad - \frac{1}{2g^2}\left[\mathcal{B} + g^2\left(\psi_1^*\psi_2^* + \psi_2\psi_1\right)\right]^2 . \qquad (1.68)
\end{aligned}
$$

where $\mathcal{S}$ and $\mathcal{B}$ are the two real, scalar, flavor singlet fields. Now one can expand the above Lagrangian to

$$
\begin{aligned}
\mathcal{L}'_{\mathrm{sdGN}} &= -2i\psi_1^*\partial\psi_1 + 2i\psi_2^*\bar{\partial}\psi_2 - \mathcal{S}\left(\psi_1^*\psi_2 + \psi_2^*\psi_1\right) \\
&\quad - \mathcal{B}\left(\psi_1^*\psi_2^* + \psi_2\psi_1\right) - \frac{1}{2g^2}\left(\mathcal{S}^2 + \mathcal{B}^2\right) . \qquad (1.69)
\end{aligned}
$$

Now from the Euler-Lagrange equation, equations of motion for $\mathcal{S}$ and $\mathcal{B}$ equation of motion can be obtained and they are given by

$$
\begin{aligned}
\mathcal{S} &= -g^2\left(\psi_1^*\psi_2 + \psi_2^*\psi_1\right) , \\
\mathcal{B} &= -g^2\left(\psi_1^*\psi_2^* + \psi_2\psi_1\right) . \qquad (1.70)
\end{aligned}
$$

This shows that $\mathcal{S}$ and $\mathcal{B}$ are the auxiliary fields. In the large $N$ limit, the auxiliary fields can be replaced by their expectation values, according to the expectation values of (1.70). Now for the canonical quantisation of this model the Hamiltonian density corresponding to $\mathcal{L}'_{\rm sdGN}$ is

$$\begin{aligned}\mathcal{H} &= i\psi_1^*\partial_1\psi_1 - i\psi_2^{(*}\partial_1\psi_2 + \mathcal{S}\left(\psi_1^*\psi_2 + \psi_2^{(*}\psi_1\right) \\ &\quad + \mathcal{B}\left(\psi_1^*\psi_2^{(*} + \psi_2\psi_1\right) + \frac{1}{2g^2}\left(\mathcal{S}^2 + \mathcal{B}^2\right) \ . \end{aligned} \tag{1.71}$$

This Hamiltonian can be written in the form given below

$$H = \frac{1}{2}\int dx \left(\psi_1^\dagger, \psi_2^\dagger, \psi_1, \psi_2\right) \begin{pmatrix} i\partial_1 & \mathcal{S} & 0 & \mathcal{B} \\ \mathcal{S} & -i\partial_1 & -\mathcal{B} & 0 \\ 0 & -\mathcal{B} & i\partial_1 & -\mathcal{S} \\ \mathcal{B} & 0 & -\mathcal{S} & -i\partial_1 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_1^\dagger \\ \psi_2^\dagger \end{pmatrix} , \tag{1.72}$$

This is our total Hamiltonian and the $\psi_i$'s are the different fermionic components. The 4×4 matrix appearing in (1.72) is denoted as $h$ afterwards.This is invariant under charge conjugation. This transformation can be represented through a unitary matrix,

$$\begin{pmatrix} \psi^\dagger \\ \psi \end{pmatrix}_c = \begin{pmatrix} 0 & \gamma_5 \\ \gamma_5 & 0 \end{pmatrix} \begin{pmatrix} \psi^\dagger \\ \psi \end{pmatrix} . \tag{1.73}$$

From this unitary matrix $U_c$ it is easy to verify that

$$h = U_c h U_c^\dagger \, . \tag{1.74}$$

The matrix $h$ can be block-diagonalized by a constant, unitary transformation $V$,

$$h = V^\dagger h_{\rm bd} V, \quad h_{\rm bd} = \begin{pmatrix} h_I & 0 \\ 0 & h_{II} \end{pmatrix} , \tag{1.75}$$

with

$$V = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} , \quad VV^\dagger = 1, \quad h_{I,II} = \begin{pmatrix} i\partial_1 & S_{I,II} \\ S_{I,II} & -i\partial_1 \end{pmatrix} , \tag{1.76}$$

and $S_I = \mathcal{S} - \mathcal{B}, S_{II} = \mathcal{S} + \mathcal{B}$. Now we plug (1.75) into (1.72) to obtain

$$H = \frac{1}{2} \int dx \left(\psi^\dagger, \psi\right) V^\dagger h_{\rm bd} V \begin{pmatrix} \psi \\ \psi^\dagger \end{pmatrix} = \frac{1}{2} \int dx \Psi^\dagger h_{\rm bd} \Psi \,. \tag{1.77}$$

In the last step unitarily transformed fermion field operators are introduced

$$\Psi = V \begin{pmatrix} \psi \\ \psi^\dagger \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_1 + \psi_1^\dagger \\ \psi_2 - \psi_2^\dagger \\ \psi_1 - \psi_1^\dagger \\ \psi_2 + \psi_2^\dagger \end{pmatrix} := \begin{pmatrix} \chi_1 \\ i\chi_2 \\ -i\chi_3 \\ \chi_4 \end{pmatrix} . \tag{1.78}$$

Now one can clearly see that the things are written in terms of Majorana fields. Thus block-diagonalization of the Hamiltonian matrix $h$ reveals that the natural degrees of freedom are four independent Majorana fields per flavor ($\chi_a^\dagger = \chi_a$) obeying the anticommutation relations

$$\{\chi_a^{(i)}(x), \chi_b^{(j)}(y)\} = \delta_{ab}\delta_{ij}\delta(x-y)\,. \tag{1.79}$$

We can separate the components and rewrite the Hamiltonian into a sum of two commuting Hamiltonians,

$$H = H_I + H_{II}\,, \tag{1.80}$$

with

$$\begin{aligned} H_I &= \frac{1}{2} \int dx \left(\chi_1, \chi_2\right) \begin{pmatrix} i\partial_1 & iS_I \\ -iS_I & -i\partial_1 \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} , \\ H_{II} &= \frac{1}{2} \int dx \left(\chi_3, \chi_4\right) \begin{pmatrix} i\partial_1 & iS_{II} \\ -iS_{II} & -i\partial_1 \end{pmatrix} \begin{pmatrix} \chi_3 \\ \chi_4 \end{pmatrix} . \end{aligned} \tag{1.81}$$

There is possibility that these two terms can be coupled via scalar field $S_{I,II}$ but it is shown that this is not the case. The inverse relations to (1.78) are,

$$\begin{aligned} \psi_1 &= \frac{1}{\sqrt{2}} \left(\chi_1 - i\chi_3\right) , \\ \psi_2 &= \frac{1}{\sqrt{2}} \left(\chi_4 + i\chi_2\right) , \end{aligned} \tag{1.82}$$

Using this, we can write $\mathcal{S},\mathcal{B}$ in Eq. (1.70) in terms of Majorana fields,

$$\begin{aligned}\mathcal{S} &= -ig^2\left(\chi_3\chi_4+\chi_1\chi_2\right) \,, \\ \mathcal{B} &= -ig^2\left(\chi_3\chi_4-\chi_1\chi_2\right) \,. \end{aligned} \tag{1.83}$$

Hence, in the large $N$ limit,

$$\begin{aligned} S_I &= -2ig^2\langle\chi_1\chi_2\rangle \,, \\ S_{II} &= -2ig^2\langle\chi_3\chi_4\rangle \,, \end{aligned} \tag{1.84}$$

As we have previously stated that in large N limit the auxiliary fields can be replaced by their expectation values.

So the full problem now separates into two simpler, independent problems. As a matter of fact, $H_{I,II}$ and the self-consistency conditions (1.84) are the same as in the standard GN model, but with Majorana instead of Dirac fields (the O($N$) symmetric model, rather than the U($N$) or O($2N$) symmetric model with Dirac fermions). This shows at once that the sdGN model is integrable and that its solution can be reduced to solutions of the standard GN model.

We have dropped the purely bosonic part from the Hamiltonian, which contains the coupling constant $g^2$ of the sdGN model. This coupling constant does not have to coincide with $G^2$, the one of the pair of standard GN models. We shall determine $G^2$ by demanding that the bosonic part of the Hamiltonian be also additive,

$$\frac{\mathcal{S}^2+\mathcal{B}^2}{2g^2}=\frac{S_I^2+S_{II}^2}{2G^2}, \qquad S_{I,II}=\mathcal{S}\mp\mathcal{B}\,. \tag{1.85}$$

This fixes the GN coupling constant to the value $G^2=2g^2$. We will confirm this choice via the self-consistency conditions of the GN and sdGN models below.

We are interested to see the same theory now in terms of Majorana fields and from that we will show that it will indeed reduce into two independent models. Now let us go back to the Lagrangian (1.67) and express the Dirac fields in terms of Majorana fields right away, using (1.82),

$$\begin{aligned}\mathcal{L}_{\rm sdGN} &= -i\chi_1\partial\chi_1 + i\chi_2\bar{\partial}\chi_2 - g^2\left(\chi_1\chi_2\right)^2 \\ &\quad -i\chi_3\partial\chi_3 + i\chi_4\bar{\partial}\chi_4 - g^2\left(\chi_3\chi_4\right)^2 \,. \end{aligned} \tag{1.86}$$

This is indeed a sum of two independent O($N$) GN Lagrangians. This simple exercise shows that they have nothing to do with one another, but can be exposed already at the level of the Lagrangian.

Thus the conclusion is that the O($N$) symmetric GN model with $N$ Majorana fields is equivalent to the U($N/2$) symmetric GN model with $N/2$ Dirac fields. The solutions of the GN model are usually formulated for Dirac fields and one must transform expressions (1.81,1.84) into Dirac language. By choice one can write

$$\begin{aligned}\psi_{I,1} &= \frac{1}{\sqrt{2}}\left(\chi_1 - i\chi_1\right)\,, \\ \psi_{I,2} &= \frac{1}{\sqrt{2}}\left(\chi_2 + i\chi_2\right)\,, \\ \psi_{II,1} &= \frac{1}{\sqrt{2}}\left(\chi_3 - i\chi_3\right)\,, \\ \psi_{II,2} &= \frac{1}{\sqrt{2}}\left(\chi_4 + i\chi_4\right)\,, \end{aligned} \tag{1.87}$$

for $i = 1, ..., N/2$. Following the same formulation as before in the case of Dirac fields

$$H_I = \int dx \sum_{i=1}^{N/2} \left(\psi_{I,1}^{(i)\dagger}, \psi_{I,2}^{(i)\dagger}\right) \left(\begin{array}{cc} i\partial_1 & S_I \\ S_I & -i\partial_1 \end{array}\right) \left(\begin{array}{c} \psi_{I,1}^{(i)} \\ \psi_{I,2}^{(i)} \end{array}\right)\,, \tag{1.88}$$

and a similar equation with all subscripts $I$ replaced by $II$. The condensation operators, assuming that the two standard GN models have coupling constant $G^2$ are given by

$$\begin{aligned} S_I &= -G^2\left(\psi_{I,1}\dagger\,\psi_{I,2} + \psi_{I,2}\dagger\,\psi_{I,1}\right) = -iG^2\chi_1\chi_2\,, \\ S_{II} &= -G^2\left(\psi_{II,1}\dagger\,\psi_{II,2} + \psi_{II,2}^{(i)\dagger}\psi_{II,1}\right) = -iG^2\chi_3\chi_4\,, \end{aligned} \tag{1.89}$$

This agrees with (1.84) provided if we set $G^2 = 2g^2$. From the 't Hooft condition

$$\frac{N}{2}G^2 = Ng^2 = \text{const.} \tag{1.90}$$

Thus the sdGN model with $N$ Dirac flavors and coupling constant $g^2$ is equivalent to the two independent GN models with $N/2$ Dirac flavors for each one with coupling constant $2g^2$. The value of the 't Hooft coupling, $Ng^2$, is the same in the sdGN model and the two GN models.

Now to construct a self-consistent HFB solution of the sdGN model from any pair of self-consistent HF solutions of the standard GN model, the time dependent Hartree-Fock equations(TDHF) for the two independent GN models can be written into the form

$$\begin{pmatrix} 2i\partial & S_I & 0 & 0 \\ S_I & -2i\bar{\partial} & 0 & 0 \\ 0 & 0 & 2i\partial & S_{II} \\ 0 & 0 & S_{II} & -2i\bar{\partial} \end{pmatrix} \begin{pmatrix} \varphi_{I,1} \\ \varphi_{I,2} \\ \varphi_{II,1} \\ \varphi_{II,2} \end{pmatrix} = 0\,. \tag{1.91}$$

Here, the spinors are solutions of the Dirac equation describing the single particle levels. The self-consistency conditions are thus given by

$$\begin{aligned} S_I &= -Ng^2 \sum^{\text{occ}} \left(\varphi^*_{I,1}\varphi_{I,2} + \varphi^*_{I,2}\varphi_{I,1}\right)\,, \\ S_{II} &= -Ng^2 \sum^{\text{occ}} \left(\varphi^*_{II,1}\varphi_{II,2} + \varphi^*_{II,2}\varphi_{II,1}\right)\,, \end{aligned} \tag{1.92}$$

where the sum runs over all occupied states. The time-dependent Hartree-Fock-Bogoliubov (TDHFB) equation for the sdGN model on the other hand can be written as the following system of four coupled equations,

$$\begin{pmatrix} 2i\partial & \mathcal{S} & 0 & \mathcal{B} \\ \mathcal{S} & -2i\bar{\partial} & -\mathcal{B} & 0 \\ 0 & -\mathcal{B} & 2i\partial & -\mathcal{S} \\ \mathcal{B} & 0 & -\mathcal{S} & -2i\bar{\partial} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = 0\,, \tag{1.93}$$

supplemented by the self-consistency conditions

$$\begin{aligned} \mathcal{S} &= -\frac{Ng^2}{2} \sum^{\text{occ}} \left(\phi^*_1\phi_2 + \phi^*_2\phi_1 - \phi^*_4\phi_3 - \phi^*_3\phi_4\right)\,, \\ \mathcal{B} &= -\frac{Ng^2}{2} \sum^{\text{occ}} \left(\phi^*_1\phi_4 + \phi^*_4\phi_1 - \phi^*_2\phi_3 - \phi^*_3\phi_2\right)\,. \end{aligned} \tag{1.94}$$

The unitary transformation $V$, Eq. (1.76), transforms equations (1.93) into (1.91) and the self-consistency condition (1.94) into (1.92), since $S_{I,II} = \mathcal{S} \mp \mathcal{B}$. So finally we can relate this to

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = V^\dagger \begin{pmatrix} \varphi_{I,1} \\ \varphi_{I,2} \\ \varphi_{II,1} \\ \varphi_{II,2} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \varphi_{I,1} + \varphi_{II,1} \\ \varphi_{I,2} + \varphi_{II,2} \\ \varphi_{I,1} - \varphi_{II,1} \\ \varphi_{II,2} - \varphi_{I,2} \end{pmatrix} . \tag{1.95}$$

setting $\varphi_{II,1} = \varphi_{II,2} = 0$ gives the sdGN model spinors

$$\Phi_I = \frac{1}{\sqrt{2}} \begin{pmatrix} \varphi_{I,1} \\ \varphi_{I,2} \\ \varphi_{I,1} \\ -\varphi_{I,2} \end{pmatrix} . \tag{1.96}$$

and also from the GN model labelled $II$ corresponds to setting $\varphi_{I,1} = \varphi_{I,2} = 0$ in (1.91) and consequently to the sdGN spinors

$$\Phi_{II} = \frac{1}{\sqrt{2}} \begin{pmatrix} \varphi_{II,1} \\ \varphi_{II,2} \\ -\varphi_{II,1} \\ \varphi_{II,2} \end{pmatrix} . \tag{1.97}$$

The contribution to both $\mathcal{S}$ and $\mathcal{B}$ is $S_{II}/2$, so that the relations $S_{I,II} = \mathcal{S} \mp \mathcal{B}$ are indeed satisfied. Notice also that the quasi-particle spinors $\Phi_{I,II}$ are eigenstates of the charge conjugation matrix $U_c$ from Eq. (1.73),

$$U_c \Phi_I = -\Phi_I , \quad U_c \Phi_{II} = \Phi_{II} \, . \tag{1.98}$$

From the above calculation we can say that the energy is the sum of the energies of both constituent solutions, since this also holds for the Hamiltonians. Since the massless GN model is integrable and its complete large-$N$ solution is known analytically, the same is true for the self-dual variation of the GN model.

In the last subsection of this Self-dual Model we will look into these solutions from more physical point of view.

**1) Vacua**

The GN model with spontaneously broken $Z_2$ chiral symmetry has two degenerate vacua with $S = \pm m = \pm 1$ (dynamical fermion mass in natural units), Consequently there are four degenerate vacua in the self-dual GN model, (Fig 1.8) From the figure, we see that there are four vacua. The ground state is either a superconductor ($\mathcal{S} = 0, \mathcal{B} = \pm 1$) or a chirally broken state ($\mathcal{S} = \pm 1, \mathcal{B} = 0$). All four states are physically indistinguishable, as they differ only in the convention for the fermion operators.

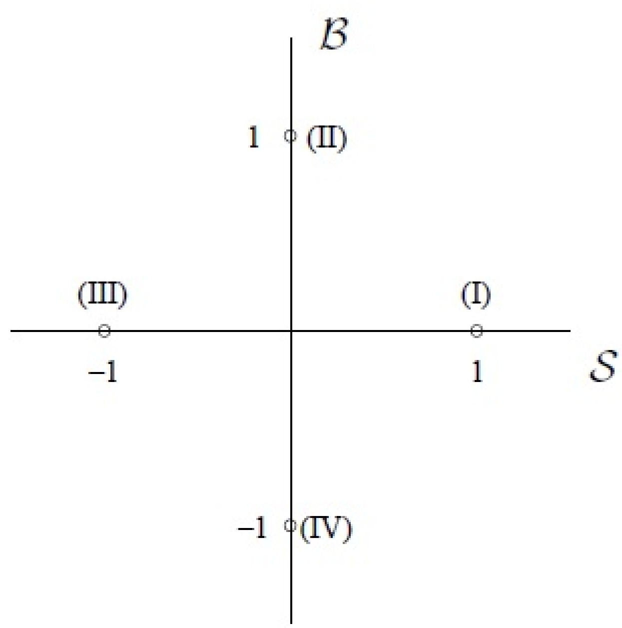


Figure 1.8: Four Vacua of sdGN Model

**2) Kinks**

From the known GN model Lagrangian, the kink interpolates between the two vacua with $S = \pm 1$. In the self-dual GN model there are six types of "domain walls" separating two out of the four vacua. We can get the kinks between two neighbouring vacua (I and II, II and III, III and IV, IV and I),(Fig. 1.9) whose mass is half of

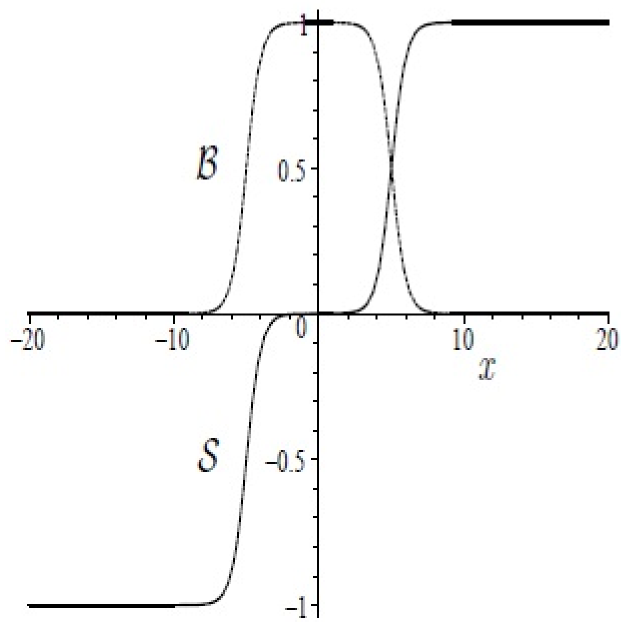


Figure 1.9: Static kink joining vacua II and I

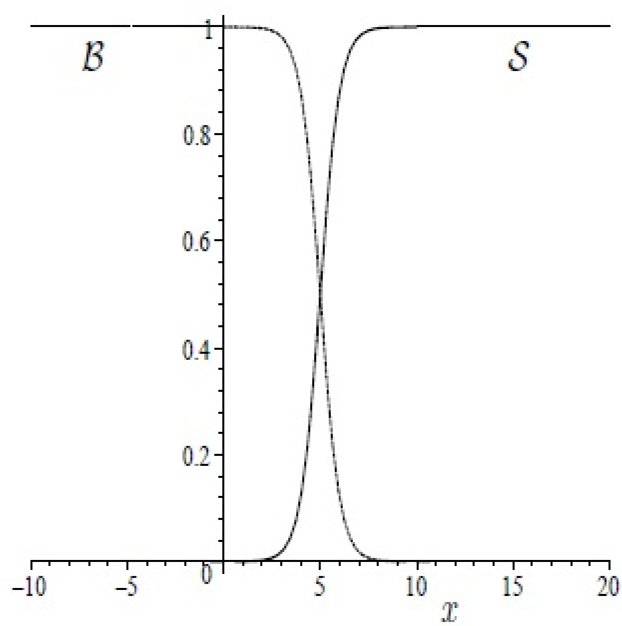


Figure 1.10: Static kink joining vacua III and I

the mass of a GN kink with $N$ Dirac flavors, $N/2\pi$. If we choose two kinks which are shifted relative to each other, we can say that we are between two opposite vacua (I and III, II and IV), (Fig. 1.10). Here, the mass is equal to the mass of the GN kink with $N$ Dirac flavors, $N/\pi$. The width of this kink can be made arbitrarily large by pulling two of the constituent kinks apart. In the transition region, there is a localized zone where the system is in the dual vacuum. This can be used for instance to manufacture a domain wall between the $\mathcal{B} = 1$ and $\mathcal{B} = -1$ superconducting vacua, separated by a normal (chirally broken) region — a kind of Josephson junction.

### Multi-Kink

Dynamical solutions result if we choose time-dependent kink solutions of the GN model as ingredients for $S$. A snapshot of such a solution may be described as an arbitrary succession of regions of vacua I IV, separated by the kind of kinks described before (Figs.1.9 and 1.10). Under time evolution, these domain walls move and collide, the details depending on the input parameters. The only static kink solutions are the single domain walls, like in the GN model.

Here we will stop discussing further about this model and pay attention to the work conducted during the masters thesis. We will mainly focus on the construction of the model by introducing interacting tensor fields of the fermions. Then following the Gross-Neveu model we will see that it is chirally invariant. After that we will write the model in terms of auxiliary field. Then by choosing the symmetric and anti-symmetric form of the auxiliary field, we will show that in case of symmetric form, it leads us to condensation of Gross-Neveu type. So, the symmetric model does not give us any new result. In case of anti-symmetric form we do not find possible condensation under certain assumptions.

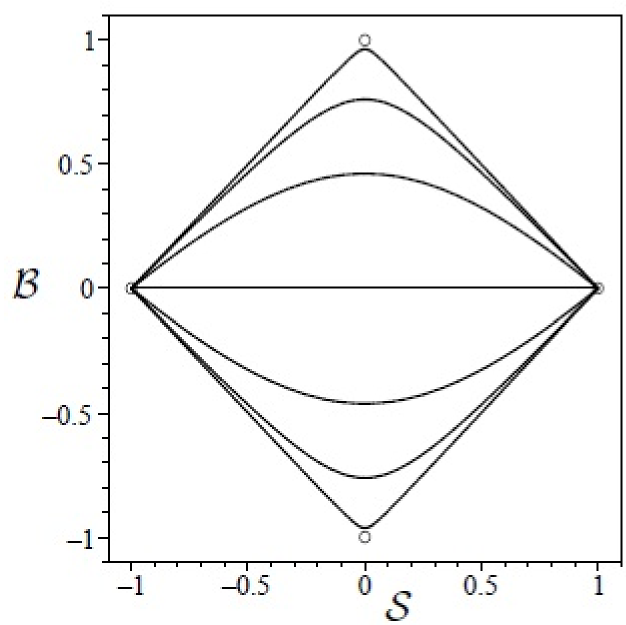


Figure 1.11: (S,B)-plot of static kink joining vacua III and I of the sdGN model. From top to bottom: a = 2, 1, 0.5, 0, 0.5, 1, 2

# Chapter 2

# My Work

The main aim of the project is to construct the form of a Lagrangian with quadratic terms involving tensorial terms. One should essentially chose the Lagrangian such that it is invariant under certain transformations. During this one year of the project several kinds of the form of the Lagrangian were proposed. We start by a standard form of the model involving gamma matrices. The model is explained briefly in 1+1 dimensions. But it is shown mathematically that it is reducing to the known Nambu-Jona-Lasinio and Gross-Neveu form. So we looked at it in higher dimensions like in 3+1 dimensions. We find that it is chirally symmetric. So our model has to be massless. Then the form of auxiliary field is evaluated. These are discussed below.

## 2.1 The Proposed Model

Following the idea of Gross-Neveu Model, we attempted to write a Lagrangian of the form given by

$$\mathcal{L} = \bar{\psi} i \partial\!\!\!/ \psi + \frac{g^2}{2} \left( \bar{\psi} \gamma_\mu \gamma_\nu \psi)(\bar{\psi} \gamma^\mu \gamma^\nu \psi \right) \tag{2.1}$$

This is basically four-fermionic tensorial interactions. Let's first study this model in 1+1 dimensions, since it is obvious from the previous discussions of GN model that in 1+1 dimensions this model will be renormalisable.

All these models include Four-Fermion interaction terms. The initial focus of the work at first is to study them in 1+1 dimensions. We have chosen the Lagrangian to be massless. We can also add mass terms. One can also think of a Lagrangian with more than one coupling constant. However these approaches are not a part of the current discussion.

In 1+1 dimensions if we look at the Lagrangian Eq.2.1, we note the following commutation relations between gamma matrices

$$\begin{aligned} \{\gamma_0, \gamma_1\} &= 0 \\ [\gamma_0, \gamma_1] &= 2\gamma_5 \end{aligned} \tag{2.2}$$

Therefore, $\gamma_0, \gamma_1$ and $\gamma_5 = i\gamma_0\gamma_1$ form a SU(2) algebra.

Using the above relations we find that our model yields nothing new but reduces to the same model as before to the Gross-Neveu one by giving $\bar{\psi}\psi$ kind of condensation. So one must go onto doing a higher dimensional theory in our model, especially in 3+1 dimensions.

We can look into the chiral symmetry of the model. It will indicate the masslessness of the model. It is discussed below.

### 2.1.1 Chiral Symmetry of $\mathcal{L}_\psi$

Starting from the Lagrangian given by

$$\mathcal{L}_\psi = \bar{\psi} i\partial\!\!\!/ \psi + \frac{g^2}{2}\left(\bar{\psi}\gamma^\mu\gamma^\nu\psi)(\bar{\psi}\gamma_\mu\gamma_\nu\psi\right) \tag{2.3}$$

To check that if this Lagrangian is chiral symmetric or not we have to perform $\gamma_5$ transformation. We will show that our Lagrangian $\mathcal{L}_\psi$ is invariant under $\gamma_5$ transformation. So performing the transformation

$$\psi \to \gamma_5\psi$$

and therefore

$$\bar{\psi} \to -\bar{\psi}\gamma_5$$

So,

$$
\begin{aligned}
\mathcal{L}'_\psi &= \bar{\psi}\gamma_5 i\partial\!\!\!/\gamma_5\psi + \frac{g^2}{2}\left(-\bar{\psi}\gamma_5\gamma^\mu\gamma^\nu\gamma_5\psi\right)\left(-\bar{\psi}\gamma_5\gamma_\mu\gamma_\nu\gamma_5\psi\right) \\
&= -\bar{\psi}\gamma_5 i\partial_\mu\gamma^\mu\gamma_5\psi + \frac{g^2}{2}\left(-\bar{\psi}\gamma_5\eta^{\mu\rho}\gamma_\rho\eta^{\nu\sigma}\gamma_\sigma\gamma_5\psi\right)\left(-\bar{\psi}\gamma_5\gamma_\mu\gamma_\nu\gamma_5\psi\right) \\
&= \bar{\psi}\gamma_5 i\partial_\mu\gamma_5\gamma^\mu\psi + \frac{g^2}{2}\left(-\bar{\psi}\eta^{\mu\rho}\gamma_\rho{\gamma_5}^2\eta^{\nu\sigma}\gamma_\sigma\psi\right)\left(-\bar{\psi}\gamma_\mu{\gamma_5}^2\gamma_\nu\psi\right) \\
&= \bar{\psi}i\partial\!\!\!/\psi + \frac{g^2}{2}\left(\bar{\psi}\gamma^\mu\gamma^\nu\psi)(\bar{\psi}\gamma_\mu\gamma_\nu\psi\right) \\
&= \mathcal{L}_\psi
\end{aligned}
\tag{2.4}
$$

where we have used the relations ${\gamma_5}^2 = 1, \{\gamma_\mu, \gamma_5\} = 0$.

Hence, we conclude that $\mathcal{L}_\psi$ is chiral symmetric. This means our Lagrangian 2.1 has certain symmetry properties, i.e. it is invariant under chiral transformation. This invariance helped the lack of mass terms in our model, since they generally break the chiral symmetry.

## 2.2 Finding Fermionic Condensate

It is very useful to write our chosen Lagrangian $\mathcal{L}_\psi$ in terms of auxiliary fields as it will give us a hint to find the condensation. By following the Gross-Neveu model, we see that the guessing of the form of the auxiliary scalar field and comparing the Gross-Neveu Lagrangian with the Yukawa Lagrangian makes it easier to obtain the condensation. It can be easily concluded that the condensate forms if the vacuum expectation value of the auxiliary field is found to be non-zero. If the v-e-v of the auxiliary field is found to be zero then it only means that we are already in the vacuum state and no symmetry breaking is possible. So any kind of condensation is unexpected. Now, let us begin with the standard Lagrangian and try to write it in terms of auxiliary field

$$
\mathcal{L}_\psi = \bar{\psi}i\partial\!\!\!/\psi + \frac{g^2}{2}\left(\bar{\psi}\gamma^\mu\gamma^\nu\psi)(\bar{\psi}\gamma_\mu\gamma_\nu\psi\right) \tag{2.5}
$$

By modifying this Lagrangian, and introducing auxiliary field $\sigma_{\mu\nu}$ one can write

$$\mathcal{L}_\sigma = \bar{\psi} i \partial\!\!\!/ \psi - \frac{1}{2}\sigma^{\mu\nu}\sigma_{\mu\nu} + \frac{g}{2}\left(\bar{\psi}\gamma^\mu\gamma^\nu\psi\right)\sigma_{\mu\nu} + \frac{g}{2}\sigma^{\mu\nu}\left(\bar{\psi}\gamma_\mu\gamma_\nu\psi\right) \tag{2.6}$$

Now, we should obtain the exact form of the auxiliary field $\sigma_{\mu\nu}$. Since $\sigma_{\mu\nu}$ has no kinetic energy part in the Lagrangian therefore, using the Lagrange's equation of motion for $\sigma_{\mu\nu}$, we have;

$$\begin{aligned} \partial_\zeta \left(\frac{\partial \mathcal{L}_\sigma}{\partial_\zeta \sigma_{\mu\nu}}\right) - \left(\frac{\partial \mathcal{L}_\sigma}{\partial \sigma_{\mu\nu}}\right) &= 0 \\ \partial_\zeta \left(\frac{\partial \mathcal{L}_\sigma}{\partial_\zeta \sigma^{\mu\nu}}\right) - \left(\frac{\partial \mathcal{L}_\sigma}{\partial \sigma^{\mu\nu}}\right) &= 0 \end{aligned} \tag{2.7}$$

Putting $\mathcal{L}_\sigma$ from Equation (2.6) we obtain

$$\sigma^{\mu\nu} = g\left(\bar{\psi}\gamma^\mu\gamma^\nu\psi\right)$$

and

$$\sigma_{\mu\nu} = g\left(\bar{\psi}\gamma_\mu\gamma_\nu\psi\right) \tag{2.8}$$

## 2.3 Determination of the Solution of $\psi$ consistent with the ansatz chosen for $\sigma_{\mu\nu}$

So we have the form of the auxiliary field, but in terms of fermionic field itself. We don't exactly know the form of $\sigma_{\mu\nu}$. It can be either symmetric or anti-symmetric. So in order to obtain the exact form of $\sigma_{\mu\nu}$ we should solve the equation of motion for $\psi$ which in turn will be consistent with the form of auxiliary field. Let's first proceed with the general treatment of finding $\psi$.

### 2.3.1 Equation of motion for $\psi$

The equation of motion for $\psi$ is given by

$$\begin{aligned} \partial_\mu \left(\frac{\partial \mathcal{L}_\sigma}{\partial_\mu \bar{\psi}}\right) - \left(\frac{\partial \mathcal{L}_\sigma}{\partial \bar{\psi}}\right) &= 0 \\ or, \left(i\gamma^\mu\partial_\mu + g\gamma^\mu\gamma^\nu\sigma_{\mu\nu}\right)\psi &= 0 \end{aligned} \tag{2.9}$$

Equation (2.9) has translation symmetry, which means that if $x^\mu \to x^\mu + a^\mu$ then $\partial_\mu \to \partial_\mu$. So, $\psi$ can be simply written in the form of

$$\psi = \int dPu(P)e^{iP.x}$$

So, Equation (2.9) becomes

$$\left(-\gamma^\mu P_\mu + g\gamma^\mu\gamma^\nu\sigma_{\mu\nu}\right)u(P) = 0 \tag{2.10}$$

Now from relativistic Schrodinger's equation of motion we can also write the same equation in the form

$$\begin{aligned}\left(\vec{\alpha}.\vec{P} - \beta g\gamma^\mu\gamma^\nu\sigma_{\mu\nu}\right)\psi &= i\beta\frac{\partial\psi}{\partial t}\\ or, \left(\vec{\alpha}.\vec{P} - g\gamma^\mu\gamma^\nu\sigma_{\mu\nu}\right)\psi &= E\psi\end{aligned} \tag{2.11}$$

where the same equation is written in Dirac representation and the definition of $\beta$ and $\alpha$ are given by

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{2.12}$$

and

$$\alpha = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \tag{2.13}$$

The solution for $\psi$ from Equation (2.11)can be obtained as follows

General Treatment to solve Equation (2.11) for $\psi$:-

$$\left(\vec{\alpha}.\vec{P} - g\gamma^\mu\gamma^\nu\sigma_{\mu\nu}\right)\psi = E\psi \tag{2.14}$$

Thus writing this in 2*2 form and $\psi = \begin{pmatrix} \chi \\ \varphi \end{pmatrix}$ we obtain

$$
\begin{aligned}
\begin{pmatrix} -g\gamma^\mu\gamma^\nu\sigma_{\mu\nu} & \vec{\alpha}.\vec{P} \\ \vec{\alpha}.\vec{P} & -g\gamma^\mu\gamma^\nu\sigma_{\mu\nu} \end{pmatrix} \begin{pmatrix} \chi \\ \varphi \end{pmatrix} &= E \begin{pmatrix} \chi \\ \varphi \end{pmatrix} \\
\begin{pmatrix} E + g\gamma^\mu\gamma^\nu\sigma_{\mu\nu} & -\vec{\alpha}.\vec{P} \\ \vec{\alpha}.\vec{P} & E + g\gamma^\mu\gamma^\nu\sigma_{\mu\nu} \end{pmatrix} \begin{pmatrix} \chi \\ \varphi \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (2.15)
\end{aligned}
$$

implies

$$(E + g\gamma^\mu\gamma^\nu\sigma_{\mu\nu})\,\chi - \left(\vec{\alpha}.\vec{P}\right)\varphi = 0 \qquad (2.16)$$

$$-\left(\vec{\alpha}.\vec{P}\right)\chi + (E + g\gamma^\mu\gamma^\nu\sigma_{\mu\nu})\,\varphi = 0 \qquad (2.17)$$

From equation 2.17 we find

$$
\begin{aligned}
(E + g\gamma^\mu\gamma^\nu\sigma_{\mu\nu})\,\varphi &= \left(\vec{\alpha}.\vec{P}\right)\chi \\
(E + g\gamma^\mu\gamma^\nu\sigma_{\mu\nu})^{-1}(E + g\gamma^\mu\gamma^\nu\sigma_{\mu\nu})\,\varphi &= (E + g\gamma^\mu\gamma^\nu\sigma_{\mu\nu})^{-1}\left(\vec{\alpha}.\vec{P}\right)\chi \\
\varphi &= (E + g\gamma^\mu\gamma^\nu\sigma_{\mu\nu})^{-1}\left(\vec{\alpha}.\vec{P}\right)\chi
\end{aligned}
\qquad (2.18)
$$

substituting $\varphi$ into equation 2.16, we obtain

$$(E + g\gamma^\mu\gamma^\nu\sigma_{\mu\nu})\,\chi = \left(\vec{\alpha}.\vec{P}\right)(E + g\gamma^\mu\gamma^\nu\sigma_{\mu\nu})^{-1}\left(\vec{\alpha}.\vec{P}\right)\chi \qquad (2.19)$$

So, by solving Equations (2.18) and (2.19) the solution of $\psi$ can be obtained.

We will follow these steps in order to solve for fermionic field ,but the form of auxiliary field is completely unknown initially. However the solution for $\psi$ and the form of $\sigma_{\mu\nu}$ must be consistent with each other. So we will start by specific choice of the auxiliary field.

### 2.3.2 Choice of ansatz for $\sigma_{\mu\nu}$

In our work we have performed calculations with the symmetric and anti-symmetric case of the auxiliary field. these are discussed below.

**Symmetric form**

Suppose we guess the the form of $\sigma_{\mu\nu}$ to be symmetric and proportional to space-time metric $g_{\mu\nu}$ like

$$\begin{aligned} \sigma_{\mu\nu} &= cg_{\mu\nu} \\ &= c \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \end{aligned} \tag{2.20}$$

Starting from the equation of motion for $\psi$

$$\left(\vec{\alpha}.\vec{P} - g\gamma^{\mu}\gamma^{\nu}\sigma_{\mu\nu}\right)\psi = E\psi \tag{2.21}$$

Now,

$$\begin{aligned} g\gamma^{\mu}\gamma^{\nu}\sigma_{\mu\nu} &= gc\left\{\left(\gamma^0\right)^2 - \left(\gamma^1\right)^2 - \left(\gamma^2\right)^2 - \left(\gamma^3\right)^2\right\} \\ &= 4gcI_{4\times4} \end{aligned} \tag{2.22}$$

So the Equation (2.21) becomes

$$\left(\vec{\alpha}.\vec{P} - 4gcI_{4\times4}\right)\psi = E\psi \tag{2.23}$$

$$\begin{pmatrix} E+4gc & -\vec{\sigma}.\vec{p} \\ -\vec{\sigma}.\vec{p} & E+4gc \end{pmatrix}\begin{pmatrix} \chi \\ \varphi \end{pmatrix} = 0 \tag{2.24}$$

$$\begin{aligned} (E+4gc)\,\chi - (\vec{\sigma}.\vec{p})\,\varphi &= 0 \\ -(\vec{\sigma}.\vec{p})\,\chi + (E+4gc)\,\varphi &= 0 \end{aligned} \tag{2.25}$$

Following the similar procedure discussed in 2.3.1 to determine the solution of $\chi$ ; at first, we obtain

$$\begin{aligned} \varphi &= (E+4gc)^{-1}\,(\vec{\sigma}.\vec{p})\,\chi \\ (E+4gc)\,\chi &= (\vec{\sigma}.\vec{p})\,(E+4gc)^{-1}\,(\vec{\sigma}.\vec{p})\,\chi \end{aligned} \tag{2.26}$$

For simplification, we replace E+4gc as E'

$$\begin{pmatrix} E' & 0 \\ 0 & E' \end{pmatrix} \chi = \frac{1}{E'^2} (\vec{\sigma}.\vec{p}) \begin{pmatrix} E' & 0 \\ 0 & E' \end{pmatrix} (\vec{\sigma}.\vec{p})\, \chi \tag{2.27}$$

We choose $\chi = \begin{pmatrix} a \\ b \end{pmatrix}$ where a, b are free parameters.

So, from Equation (2.27) we have

$$\begin{aligned}
\begin{pmatrix} E' & 0 \\ 0 & E' \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} &= \frac{1}{E'^2} \begin{pmatrix} P_3 & P_1 - iP_2 \\ P_1 + iP_2 & -P3 \end{pmatrix} \begin{pmatrix} E' & 0 \\ 0 & E' \end{pmatrix} \\
&\times \begin{pmatrix} P_3 & P_1 - iP_2 \\ P_1 + iP_2 & -P3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \\
\begin{pmatrix} E' & 0 \\ 0 & E' \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} &= \frac{1}{E'^2} \begin{pmatrix} P_3 & P_1 - iP_2 \\ P_1 + iP_2 & -P3 \end{pmatrix} \\
&\times \begin{pmatrix} E'P_3 & E'(P_1 - iP_2) \\ E'(P_1 + iP_2) & -E'P_3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \\
\begin{pmatrix} E' & 0 \\ 0 & E' \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} &= \frac{1}{E'^2} \begin{pmatrix} E'\left({P_1}^2 + {P_2}^2 + {P_3}^2\right) & 0 \\ 0 & E'\left({P_1}^2 + {P_2}^2 + {P_3}^2\right) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}
\end{aligned}$$

$$\begin{pmatrix} E' - \frac{{P_i}^2}{E'} & 0 \\ 0 & E' - \frac{{P_i}^2}{E'} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0 \tag{2.28}$$

Now we will solve this equation for $\chi$. Eigenvalue of the matrix is $\lambda = E' - \frac{{P_i}^2}{E'}, E' - \frac{{P_i}^2}{E'}$ and

$$\begin{aligned}
\left(E' - \frac{{P_i}^2}{E'}\right) a &= 0 \\
\left(E' - \frac{{P_i}^2}{E'}\right) b &= 0
\end{aligned} \tag{2.29}$$

From here we see that a and b are still free parameters and for $a, b \neq 0$ we have

$$\begin{aligned}
\left(E' - \frac{{P_i}^2}{E'}\right) &= 0 \\
\left\{(E+4gc)^2 - {P_i}^2\right\} &= 0 \\
(E^2 + 8gcE + 16g^2c^2 - {P_i}^2) &= 0 \\
(8gcE + 16g^2c^2) &= 0 \\
E = -2gc & \qquad (2.30)
\end{aligned}$$

This indicates that for positive energy solutions our chosen constant c should be negative.

Now, since we have two arbitrary parameters a and b, so by choice we can define $\chi$ and $\varphi$ by using (2.16) and (2.17). Then we can replace the solution into our known form of $\sigma_{\mu\nu}$ (2.8) to check the consistency with the chosen ansatz of $\sigma_{\mu\nu}$ (2.20).

Let the solutions of $\chi$ and $\varphi$ be

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix}, \qquad \varphi = \frac{E'}{E'^2}\left(\vec{\sigma}.\vec{P}\right)\begin{pmatrix} a \\ b \end{pmatrix} \qquad (2.31)$$

and $\psi = \begin{pmatrix} \chi \\ \varphi \end{pmatrix}$

Therefore,

$$\begin{aligned}
\sigma_{\mu\nu} &= g\,(\bar{\psi}\gamma_\mu\gamma_\nu\psi) \\
&= g\begin{pmatrix} \chi^\dagger & -\varphi^\dagger \end{pmatrix}\gamma_\mu\gamma_\nu\begin{pmatrix} \chi \\ \varphi \end{pmatrix} \\
&= g\begin{pmatrix} \chi^\dagger & -\varphi^\dagger \end{pmatrix}\gamma_\mu\gamma_\nu\begin{pmatrix} \chi \\ \varphi \end{pmatrix} \qquad (2.32)
\end{aligned}$$

Now in order to find a solution for $\chi$ and $\varphi$ or a and b consistent with $\sigma_{\mu\nu}$(2.20), we can look into the following conditions

$\sigma_{00} = c$, $\sigma_{11} = \sigma_{22} = \sigma_{33} = -c$, $\sigma_{0i} = 0$ and $\sigma_{ij} = 0$ where $i, j = 1, 2, 3$ $i \neq j$.

Starting from $\sigma_{ij} = 0$

$$\begin{aligned}
\sigma_{ij} &= g\left(\bar{\psi}\gamma_i\gamma_j\psi\right) \\
&= g\begin{pmatrix} \chi^\dagger & -\varphi^\dagger \end{pmatrix}\gamma_i\gamma_j\begin{pmatrix} \chi \\ \varphi \end{pmatrix} \\
&= g\begin{pmatrix} \chi^\dagger & -\varphi^\dagger \end{pmatrix}\begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix}\begin{pmatrix} \chi \\ \varphi \end{pmatrix} \\
&= g\begin{pmatrix} \varphi^\dagger\sigma_i & \chi^\dagger\sigma_i \end{pmatrix}\begin{pmatrix} \sigma_j\varphi \\ -\sigma_j\chi \end{pmatrix} \qquad (2.33)
\end{aligned}$$

$$\begin{aligned}
\sigma_{ij} &= g\left(\varphi^\dagger\sigma_i\sigma_j\varphi - \chi^\dagger\sigma_i\sigma_j\chi\right) \\
&= g\left(\frac{1}{E'^2}\chi^\dagger\left(\vec{\sigma}.\vec{P}\right)\sigma_i\sigma_j\left(\vec{\sigma}.\vec{P}\right)\chi - \chi^\dagger\sigma_i\sigma_j\chi\right) \\
&= ig\left(\frac{1}{E'^2}\chi^\dagger\left(\vec{\sigma}.\vec{P}\right)\sigma_k\left(\vec{\sigma}.\vec{P}\right)\chi - \chi^\dagger\sigma_k\chi\right) + g\left(\frac{1}{E'^2}\chi^\dagger\left(\vec{\sigma}.\vec{P}\right)^2\chi - \chi^\dagger\chi\right) \\
&= ig\left(\frac{1}{E'^2}\chi^\dagger\left(\vec{\sigma}.\vec{P}\right)\sigma_k\left(\vec{\sigma}.\vec{P}\right)\chi - \chi^\dagger\sigma_k\chi\right) + g\left(\frac{3P_i^2}{E'^2}\chi^\dagger\chi - \chi^\dagger\chi\right) \qquad (2.34)
\end{aligned}$$

Looks very complicated! Let's then turn back to other components and evaluate them first.

Now, when $\sigma_{oi} = 0$ we have

$$\begin{aligned}
\sigma_{0i} &= g\left(\bar{\psi}\gamma_0\gamma_i\psi\right) \\
&= g\begin{pmatrix} \chi^\dagger & -\varphi^\dagger \end{pmatrix}\gamma_0\gamma_i\begin{pmatrix} \chi \\ \varphi \end{pmatrix} \\
&= g\begin{pmatrix} \chi^\dagger & -\varphi^\dagger \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}\begin{pmatrix} \chi \\ \varphi \end{pmatrix} \\
&= g\begin{pmatrix} \chi^\dagger & -\varphi^\dagger \end{pmatrix}\begin{pmatrix} \sigma_i\varphi \\ \sigma_i\chi \end{pmatrix} \\
&= g\left(\chi^\dagger\sigma_i\varphi - \varphi^\dagger\sigma_i\chi\right) \\
&= g\left[\chi^\dagger\sigma_i\frac{1}{E'}\left(\vec{\sigma}.\vec{P}\right)\chi - \frac{1}{E'}\chi^\dagger\left(\vec{\sigma}.\vec{P}\right)\sigma_i\chi\right] \\
&= \frac{g}{E'}\chi^\dagger\left[\sigma_i\left(\vec{\sigma}.\vec{P}\right) - \left(\vec{\sigma}.\vec{P}\right)\sigma_i\right]\chi \qquad (2.35)
\end{aligned}$$

This looks easier to handle and making it more simple by choosing $\vec{P}$ to be constant $P$, we get

$$\sigma_{0i} = \frac{gP}{E'}\chi^{\dagger}\left[\sigma_i\left(\sigma_1+\sigma_2+\sigma_3\right)-\left(\sigma_1+\sigma_2+\sigma_3\right)\sigma_i\right]\chi \tag{2.36}$$

and for one component of the tensor $\sigma_{0i}$,where i = 1 $\sigma_{01} = 0$.

So,

$$\begin{aligned}
\sigma_{01} &= \frac{gP}{E'}\chi^{\dagger}\left[\sigma_1\left(\sigma_1+\sigma_2+\sigma_3\right)-\left(\sigma_1+\sigma_2+\sigma_3\right)\sigma_1\right]\chi \\
&= 0 \\
\chi^{\dagger}\left(2i\sigma_3-2i\sigma_2\right)\chi &= 0 \qquad (2.37)
\end{aligned}$$

$$\begin{aligned}
\begin{pmatrix} a^{\dagger} & b^{\dagger} \end{pmatrix}\begin{pmatrix} 1 & i \\ -i & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} &= 0 \\
\begin{pmatrix} a^{\dagger} & b^{\dagger} \end{pmatrix}\begin{pmatrix} a+ib \\ -ia-b \end{pmatrix} &= 0 \\
a^{\dagger}a+ia^{\dagger}b-iab^{\dagger}-bb^{\dagger} &= 0 \\
\left(a^{\dagger}a-bb^{\dagger}\right)+i\left(a^{\dagger}b-ab^{\dagger}\right) &= 0 \qquad (2.38)
\end{aligned}$$

This follows that

$$a^{\dagger}a = bb^{\dagger} \tag{2.39}$$

$$a^{\dagger}b = ab^{\dagger} \tag{2.40}$$

Now, from another component $\sigma_{02} = 0$ we have

$$\begin{aligned}
\chi^{\dagger}\left(-2i\sigma_3+2i\sigma_1\right)\chi &= 0 \\
\begin{pmatrix} a^{\dagger} & b^{\dagger} \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} &= 0 \\
\begin{pmatrix} a^{\dagger} & b^{\dagger} \end{pmatrix}\begin{pmatrix} -a+b \\ a+b \end{pmatrix} &= 0 \\
-a^{\dagger}a+a^{\dagger}b+ab^{\dagger}+bb^{\dagger} &= 0
\end{aligned}$$

$$\tag{2.41}$$

Now, using 2.39, we get

$$\begin{aligned} a^\dagger b + ab^\dagger &= 0 \\ a^\dagger b &= -ab^\dagger \end{aligned} \tag{2.42}$$

Now, one can see that (2.40) and (2.42) are contradictory to each other. So the only possible solution to a and b and hence $\chi$ or $\varphi$ is null solution.

We do not get any possible result from this calculation. However if we look into the form of $\sigma_{\mu\nu}$, we can directly see the following

$$\begin{aligned} \sigma_{\mu\nu} &= g\left(\bar{\psi}\gamma_\mu\gamma_\nu\psi\right) \\ \sigma_{\mu\nu} &= g\left[\left(\bar{\psi}2\eta_{\mu\nu}\psi\right) - \left(\bar{\psi}\gamma_\nu\gamma_\mu\psi\right)\right] \\ \sigma_{\mu\nu} &= g\left[2\eta_{\mu\nu}\left(\bar{\psi}\psi\right) - \sigma_{\nu\mu}\right] \end{aligned} \tag{2.43}$$

by using the symmetric property of $\gamma_\mu$ and $\gamma_\nu$ given by $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$. Again we see from the symmetry property of $\sigma_{\mu\nu}$ that $\sigma_{\mu\nu} = \sigma_{\nu\mu}$.

So the equation (2.43) gives

$$\begin{aligned} \sigma_{\mu\nu} &= g\left[2\eta_{\mu\nu}\left(\bar{\psi}\psi\right) - \sigma_{\mu\nu}\right] \\ \sigma_{\mu\nu} &= g\eta_{\mu\nu}\left(\bar{\psi}\psi\right) \end{aligned} \tag{2.44}$$

We see that this is nothing but the scalar$\left(\bar{\psi}\psi\right)$ condensation like the Gross-Neveu model. So, in case of a symmetric auxiliary field, our model reduces to Gross-Neveu like model in 3+1 dimensions with a condensate.

### Anti-Symmetric

Now, let's start by an anti-symmetric case of the auxiliary field. Suppose we guess the form of $\sigma_{\mu\nu}$ to be anti-symmetric in the form given by

$$\sigma_{\mu\nu} = \begin{pmatrix} 0 & l & m & n \\ -l & 0 & p & q \\ -m & -p & 0 & r \\ -n & -q & -r & 0 \end{pmatrix} \tag{2.45}$$

We see that there is a lot of free parameters in this form and it will be difficult to compute the solution with it. So, for convenience we choose the form to be self-dual. Then by self-dualizing $\sigma_{\mu\nu}$, i.e. by using the following relation

$$\sigma_{\mu\nu} = \sum_{\rho}\sum_{\eta}\frac{1}{2}\epsilon_{\mu\nu\rho\eta}\sigma^{\rho\eta} \tag{2.46}$$

We obtain $\sigma_{\mu\nu}$ to be

$$\sigma_{\mu\nu} = \begin{pmatrix} 0 & l & m & n \\ -l & 0 & n & -m \\ -m & -n & 0 & l \\ -n & m & -l & 0 \end{pmatrix} \tag{2.47}$$

Starting from the equation of motion for $\psi$

$$\left(\vec{\alpha}.\vec{P} - g\gamma^{\mu}\gamma^{\nu}\sigma_{\mu\nu}\right)\psi = E\psi \tag{2.48}$$

Now,

$$g\gamma^{\mu}\gamma^{\nu}\sigma_{\mu\nu} = 2g\begin{pmatrix} -(l+m+n) & l\sigma_1 + m\sigma_2 + n\sigma_3 \\ l\sigma_1 + m\sigma_2 + n\sigma_3 & -(l+m+n) \end{pmatrix} \tag{2.49}$$

So the equation becomes

$$\left(\vec{\alpha}.\vec{P} - g\gamma^{\mu}\gamma^{\nu}\sigma_{\mu\nu}\right)\psi = E\psi \tag{2.50}$$

Writing this equation in $2 \times 2$ form,

$$\begin{pmatrix} E - 2g(l+m+n) & 2g(l\sigma_1 + m\sigma_2 + n\sigma_3) - \vec{\sigma}.\vec{P} \\ 2g(l\sigma_1 + m\sigma_2 + n\sigma_3) - \vec{\sigma}.\vec{P} & E - 2g(l+m+n) \end{pmatrix}\begin{pmatrix} \chi \\ \varphi \end{pmatrix} = 0 \tag{2.51}$$

So this equation gives us

$$\begin{aligned} \{E - 2g(l+m+n)\}\chi + \left\{2g(l\sigma_1 + m\sigma_2 + n\sigma_3) - \left(\vec{\sigma}.\vec{P}\right)\right\}\varphi &= 0 \\ \left\{2g(l\sigma_1 + m\sigma_2 + n\sigma_3) - \left(\vec{\sigma}.\vec{P}\right)\right\}\chi + \{E - 2g(l+m+n)\}\varphi &= 0 \end{aligned} \tag{2.52}$$

Further we can simplify by writing the following substitutions $E' = E - 2g(l+m+n)$ and $B = -2g(l\sigma_1 + m\sigma_2 + n\sigma_3) + \vec{\sigma}.\vec{P}$. Following the similar procedure discussed in 2.3.1 to determine the solution of $\chi$, we obtain

$$\begin{aligned} \varphi &= (E)^{-1} B\chi \\ (E')\,\chi &= B\,(E)^{-1} B\chi \\ (E'^2 - B^2)\chi &= 0 \end{aligned} \tag{2.53}$$

Now,

$$\begin{aligned} B &= -2g(l\sigma_1 + m\sigma_2 + n\sigma_3) + \left(\vec{\sigma}.\vec{P}\right) \\ B^2 &= \left(\vec{\sigma}.\vec{P}\right)^2 - 4g^2(l\sigma_1 + m\sigma_2 + n\sigma_3)^2 \\ &- 2g\left(\vec{\sigma}.\vec{P}\right)(l\sigma_1 + m\sigma_2 + n\sigma_3) - 2g(l\sigma_1 + m\sigma_2 + n\sigma_3)\left(\vec{\sigma}.\vec{P}\right) \\ &= \sum P_i^2 - 4g^2(l^2 + m^2 + n^2) - 4g(P_3 n + P_2 m + P_1 l) \end{aligned} \tag{2.54}$$

Just like before we choose $\chi = \begin{pmatrix} a \\ b \end{pmatrix}$ where a, b are free parameters.

So,

$$(E'^2 - B^2)\chi = 0$$

From this we get

$$\left(E^2 + 4g^2c^2 - 2gcE - P_i^2 + 4g^2(l^2 + m^2 + n^2) + 4g(P_3 n + P_2 m + P_1 l)\right)\chi = 0 \tag{2.55}$$

Now, for the solution of $\chi$ to be not equal to zero; the condition has to be satisfied is given by

$$E = 4g^2c^2 + 4g^2(l^2 + m^2 + n^2) + 4g(P_3 n + P_2 m + P_1 l) \tag{2.56}$$

(where c = l + m+ n); a,b will remain free parameters.

Let the solution of $\chi$ and $\varphi$ then be given by

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix}, \qquad \varphi = \frac{1}{E'} B \begin{pmatrix} a \\ b \end{pmatrix} \tag{2.57}$$

So, $\psi = \begin{pmatrix} \chi \\ \varphi \end{pmatrix}$.
Now from the condensation form

$$\begin{aligned} \sigma_{\mu\nu} &= g\left(\bar{\psi}\gamma_\mu\gamma_\nu\psi\right) \\ &= g\begin{pmatrix} \chi^\dagger & -\varphi^\dagger \end{pmatrix}\gamma_\mu\gamma_\nu\begin{pmatrix} \chi \\ \varphi \end{pmatrix} \\ &= g\begin{pmatrix} \chi^\dagger & -\varphi^\dagger \end{pmatrix}\gamma_\mu\gamma_\nu\begin{pmatrix} \chi \\ \varphi \end{pmatrix} \end{aligned} \tag{2.58}$$

In order to find a solution for $\chi$ and $\varphi$ or a and b consistent with the anti-symmetric form of the chosen auxiliary field $\sigma_{\mu\nu}$(2.47), we can look into the following conditions

$$\sigma_{ii} = 0 \tag{2.59}$$

$$\sigma_{01} = l = g\left(\bar{\psi}\gamma_0\gamma_1\psi\right) \tag{2.60}$$

$$\sigma_{02} = m = g\left(\bar{\psi}\gamma_0\gamma_2\psi\right) \tag{2.61}$$

$$\sigma_{03} = n = g\left(\bar{\psi}\gamma_0\gamma_3\psi\right) \tag{2.62}$$

$$\sigma_{12} = n = g\left(\bar{\psi}\gamma_1\gamma_2\psi\right) \tag{2.63}$$

$$\sigma_{23} = m = g\left(\bar{\psi}\gamma_2\gamma_3\psi\right) \tag{2.64}$$

$$\sigma_{31} = l = g\left(\bar{\psi}\gamma_3\gamma_1\psi\right) \tag{2.65}$$

From these set of equations we can obtain

$$\begin{aligned} \chi^\dagger\sigma_1\varphi - \varphi^\dagger\sigma_1\chi &= l/g \\ \chi^\dagger\sigma_2\varphi - \varphi^\dagger\sigma_2\chi &= m/g \\ \chi^\dagger\sigma_3\varphi - \varphi^\dagger\sigma_3\chi &= n/g \end{aligned} \tag{2.66}$$

$$\begin{aligned} \chi\sigma_1\chi - \varphi^\dagger\sigma_1\varphi &= il/g \\ \chi^\dagger\sigma_2\chi - \varphi^\dagger\sigma_2\varphi &= im/g \\ \chi^\dagger\sigma_3\chi - \varphi^\dagger\sigma_3\varphi &= in/g \end{aligned} \tag{2.67}$$

Now from equation (2.60), we get the following relation

$$\left(a^\dagger a + b^\dagger b\right)\left(1 - (P_i^2/E'^2) + (4g^2c^2/E'^2) - (E/E'^2)\right) = 0 \tag{2.68}$$

From, equation (2.61) we get the following relation

$$2i(P_2 - 2gm)a^\dagger a + 2(2gn - P_3)a^\dagger b + 2(P_3 - 2gn)b^\dagger a - 2i(P_2 - 2gm)b^\dagger b = E'l/g$$

If we assume here that E'l/g is real then we get the relations

$$\begin{aligned} a^\dagger a &= b^\dagger b \\ b^\dagger a - a^\dagger b &= \frac{E'l}{2g(P_3 - 2gn)} \end{aligned}$$

So, by similar process from these set of equations (2.66) and (2.67) and by using (2.57) we obtain the following relations between a and b

$$a^\dagger a = b^\dagger b \tag{2.69}$$

$$a^\dagger b = -b^\dagger a \tag{2.70}$$

and

$$\begin{aligned} b^\dagger a - a^\dagger b &= \frac{E'l}{2g(P_3 - 2gn)} && (2.71) \\ &= \frac{-E'n}{2g(P_1 - 2gl)} && (2.72) \\ &= \frac{E'^2 l}{g(2P_1P_2 - 4gmP_1 - 4glP_2 + 8g^2lm)} && (2.73) \\ &= \frac{E'^2 m}{g(P_2^2 - P_3^2 - P_1^2 - 4gmP_2 + 4gnP_3 + 4glP_1 + 4g^2m^2 - 4g^2n^2 - 4g^2l^2)} && (2.74) \\ &= \frac{E'^2 n}{g(2P_3P_2 - 4gmP_3 - 4gnP_2 + 8g^2mn)} && (2.75) \end{aligned}$$

Under the following condition

$$E = 4g^2c^2 + 4g^2(l^2 + m^2 + n^2) + 4g(P_3n + P_2m + P_1l) \tag{2.76}$$

(where c = l + m+ n)

and since $a^\dagger a = b^\dagger b$ and a,b is not equal to zero , so from Equation (2.69) we obtain

$$1 - (P_i^2/E'^2) + (4g^2c^2/E'^2) - (E/E'^2) = 0 \tag{2.77}$$

We see that the equations here are quite complicated to solve. In order to obtain a suitable solution for a and b, all these relations together, must hold.

If we try to solve for a and b from equations (2.69), (2.70) and (2.71), we find that only null solution is possible from these equations, as $\frac{E'l}{2g(P_3-2gn)}$ and the rest (Equation (2.72) to (2.75)) are considered to be real. Hence, we say that the only possible solution of $\psi$ which is consistent with the auxiliary field is a null solution, provided that all the parameters are real.

But if we say that $\frac{E'l}{2g(P_3-2gn)}$ and all other equalities are purely imaginary, then by solving (2.69)(2.70),(2.71) we may get acceptable solutions. So we conclude that if we consider $E'l/g$ and other equalities to be purely imaginary then the solutions of a and b may be possible.

We can show the same in a much neater way by choosing the Weyl representation.

In Weyl representation, we can write $\sigma_{\mu\nu} = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma^\mu} & 0 \end{pmatrix}$ where $\sigma^\mu = (1, \sigma^i)$ and $\bar{\sigma^\mu} = (1, -\sigma^i)$.

Now, the form of the condensation is given by

$$\sigma^{\mu\nu} = \bar{\psi}\gamma^\mu\gamma^\nu\psi$$

We note that the form of the auxiliary field is anti-symmetric. So we can write,

$$\begin{aligned}
\gamma^{[\mu}\gamma^{\nu]} &= \frac{1}{2}\left(\begin{bmatrix} 0 & \sigma^\mu \\ \bar{\sigma^\mu} & 0 \end{bmatrix}\begin{bmatrix} 0 & \sigma^\nu \\ \bar{\sigma^\nu} & 0 \end{bmatrix} - \begin{bmatrix} 0 & \sigma^\nu \\ \bar{\sigma^\nu} & 0 \end{bmatrix}\begin{bmatrix} 0 & \sigma^\mu \\ \bar{\sigma^\mu} & 0 \end{bmatrix}\right) \\
&= \frac{1}{2}\left(\begin{bmatrix} \sigma^\mu\bar{\sigma^\nu} & 0 \\ 0 & \bar{\sigma^\mu}\sigma^\nu \end{bmatrix} - \begin{bmatrix} \sigma^\nu\bar{\sigma^\mu} & 0 \\ 0 & \bar{\sigma^\nu}\sigma^\mu \end{bmatrix}\right) \\
&= \begin{bmatrix} \sigma^\mu\bar{\sigma^\nu} - \sigma^\nu\bar{\sigma^\mu} & 0 \\ 0 & \bar{\sigma^\mu}\sigma^\nu - \bar{\sigma^\nu}\sigma^\mu \end{bmatrix}
\end{aligned} \tag{2.78}$$

Since, $\psi = \begin{pmatrix} \chi \\ \varphi \end{pmatrix}$ where $\chi$, $\varphi$ are two component spinors.

Thus

$$\begin{aligned}
\bar{\psi} &= \psi^\dagger \gamma^0 \\
&= \begin{pmatrix} \chi^\dagger & \varphi^\dagger \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \varphi^\dagger & \chi^\dagger \end{pmatrix} \qquad (2.79)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sigma^{\mu\nu} &= \frac{1}{2} \begin{pmatrix} \varphi^\dagger & \chi^\dagger \end{pmatrix} \begin{pmatrix} \sigma^\mu \bar{\sigma^\nu} - \sigma^\nu \bar{\sigma^\mu} & 0 \\ 0 & \bar{\sigma^\mu} \sigma^\nu - \bar{\sigma^\nu} \sigma^\mu \end{pmatrix} \begin{pmatrix} \chi \\ \varphi \end{pmatrix} \\
&= \frac{1}{2} \begin{pmatrix} \varphi^\dagger & \chi^\dagger \end{pmatrix} \begin{pmatrix} (\sigma^\mu \bar{\sigma^\nu} - \sigma^\nu \bar{\sigma^\mu})\chi \\ (\bar{\sigma^\mu} \sigma^\nu - \bar{\sigma^\nu} \sigma^\mu)\varphi \end{pmatrix}
\end{aligned}$$

So

$$\sigma^{\mu\nu} = \frac{1}{2} \left[ \varphi^\dagger (\sigma^\mu \bar{\sigma^\nu} - \sigma^\nu \bar{\sigma^\mu})\chi + \chi^\dagger (\bar{\sigma^\mu} \sigma^\nu - \bar{\sigma^\nu} \sigma^\mu)\varphi \right] \qquad (2.80)$$

Therefore,

$$\begin{aligned}
\sigma^{0i} &= \frac{1}{2} \left[ \varphi^\dagger (\sigma^0 \bar{\sigma^i} - \sigma^i \bar{\sigma^0})\chi + \chi^\dagger (\bar{\sigma^0} \sigma^i - \bar{\sigma^i} \sigma^0)\varphi \right] \\
&= \left[ -\varphi^\dagger \sigma^i \chi + \chi^\dagger \sigma^i \varphi \right] \qquad (2.81)
\end{aligned}$$

and

$$\begin{aligned}
\sigma^{ij} &= \frac{1}{2} \left[ \varphi^\dagger (\sigma^i \bar{\sigma^j} - \sigma^j \bar{\sigma^i})\chi + \chi^\dagger (\bar{\sigma^i} \sigma^j - \bar{\sigma^j} \sigma^i)\varphi \right] \\
&= \frac{1}{2} \left[ -2i\epsilon_{ijk} \varphi^\dagger \sigma^k \chi - 2i\epsilon_{ijk} \chi^\dagger \sigma^k \varphi \right] \\
&= -i\epsilon_{ijk} \left[ \varphi^\dagger \sigma^k \chi + \chi^\dagger \sigma^k \varphi \right] \qquad (2.82)
\end{aligned}$$

Now from the self duality condition(2.46)

$$\sigma^{\mu\nu} = \sum_\rho \sum_\eta \frac{1}{2} \epsilon^{\mu\nu\rho\eta} \sigma^{\rho\eta}$$

Following this we obtain $\sigma^{0i} = \epsilon^{0ijk}\sigma^{jk}$.

Let $\sigma^{0i} = K^i$, then

$$\sigma^{jk} = \epsilon^{ijk} K^i$$

Therefore,

$K^i = \left[-\varphi^\dagger \sigma^i \chi + \chi^\dagger \sigma^i \varphi\right] = -i\left[\varphi^\dagger \sigma^i \chi + \chi^\dagger \sigma^i \varphi\right]$, as $\chi^\dagger \sigma^i \varphi = \varphi^\dagger \sigma^i \chi$. So we say that $K^i$ is purely imaginary number.

Hence, we conclude that the self-duality condition implies the condensation to be imaginary, as found before.

# Chapter 3

# Results and discussion

Starting from our four-fermionic interacting Lagrangian in tensor form, we found several results and conclusions. This type of model is essentially discussed because of its ability to explain color superconductivity in QCD or several other phase condensations. The following important results are noted below.

## Chiral Symmetry of the Theory

By noting the results got from 2.1.1, we see that under the transformation $\gamma_5$, $\mathcal{L}_\psi$ is invariant. This is an important result since we know that such kind of symmetry means the theory is more possible to be integrable. This also ensures the masslessness of the theory. For instance, say that we add a mass term in the Lagrangian, it will break the chiral symmetry of the model.

## Tensor model reducing to Gross-Neveu model

By evaluating the interacting part of our model in 1+1 dimensions, we see that our model is reducing to the known Gorss-Neveu Lagrangian. So, we conclude that no new theory is expected from our model in 2 dimensions. It is easily understandable since our model involves tensor kind of interactions, in 2 dimensions it will essentially give us a scalar kind of interactions.

Now, as we have considered our model in two cases, one for the symmetric form of the auxiliary field, and another for the anti-symmetric form. We see in section 2.3.1 and from equation (2.44) that for the symmetric case of the auxiliary field (2.20), the model gives $(\bar{\psi}\psi)$ condensation back, which is nothing but the scalar condensation of Gross-Neveu type in higher dimensions.

### Results from anti-symmetric form of the auxiliary field

While working with the anti-symmetric form, at first we impose the condition of self-duality (2.46) to the chosen auxiliary field, as it would then reduce the number of free parameters from six to three. Then we see that, while working with the self-dual form the only possible solution can be obtained only if the condensation takes purely imaginary value. However it needs to be further explored if the condensation formation occurs or not. Thus there is a possibility of condensate, which is purely imaginary, but the confirmation requires further exploration.

# Chapter 4

# Conclusions and Scope for Future Work

## 4.1 Conclusions

For many years, integrability of the massless Gross Neveu models seemed like a rather academic issue. The derivation of hadron masses (mesons, baryons, multi-baryon bound states) and of the phase diagrams could equally be done for the massive case as found in some of the literature and as well as for the massless case even to a large extent analytically, although only the massless models are integrable. The study of time dependent problems shows that this perspective has changed in recent years. The scattering of baryons for instance could only be solved in the massless Gross-Neveu and Nambu-Jone-Lasinio models.

These findings in recent studies have incited us to think about other potentially integrable four-fermion models. From the strong interaction physics point of view, models giving rise to Cooper pairing are particularly interesting as toy models for color superconductivity in QCD. Also the models involving fermion-fermion and fermion-anti-fermion pairing give explanation to Cooper fair formation. For a long time, people did not usually looked into a tensorial interacting field. In this work, we tried to reformulate the Gross-Neveu Lagrangian in tensor form. We started with a tensorial interacting Lagrangian, and it is explicitly shown that it is invariant under chiral transformation. We take it as a hint of masslessness of the quantum theory as well. We find the solutions of the model in 2 dimensions, which is reduced to known ones from the Gross-Neveu models. In higher dimensions like in 4 dimensions, the symmetric form of the auxiliary field gave us scalar condensation back and the anti-symmetric form depicts that the condensate solution must be purely imaginary. This could indicate a new vacuum of our model. It remains to be seen whether the methods developed for solving the Gross-Neveu model can be generalized to this situation. Also, one needs to look at the loop level calculations of this theory since it is known to us that in Gross-Neveu model, no condensation is found in tree level approximation.

## 4.2 Scope for the Future Work

### Stability about the condensation solution

In our calculations, we found a possible condensation in symmetric case. We obtained imaginary solution of the condensate in our calculation, for the anti-symmetric case. One can also choose anti self dual condition, or a more general form of the auxiliary field without imposing any self-duality condition. After obtaining the condensation from the model, one can proceed by looking at the stability of any new condensate vacuum and its ground state energy. One can also see the fluctuations of the fermionic field and can obtain equation of motion of the fluctuating field.

### Loop correction of the model

In our project, time did not permit us to look into the loop level calculations of this theory. We were restricted to the condensation solution. In order to perform one loop level calculations one can look into the graphs as given by

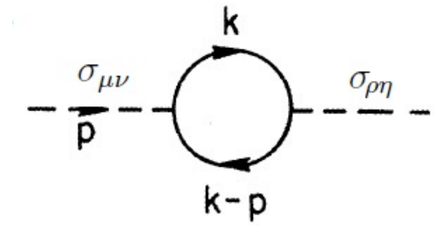


Figure 4.1: self-energy of $\sigma_{\mu\nu}$

By evaluating such graphs and performing functional calculations, following similar procedure to the Gross-Neveu model, one can expect new theories from the model.

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